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Sanni Hullberger

RELATIVIZATION IN UNIVERSAL STANDARD MATHEMATICS:

A CORPUS-BASED STUDY ON NEWSPAPER PHYSICS

ACADEMIC DISSERTATION

To be presented by the permission of the Faculty of Science and Forestry for public examination in the Auditorium M301 in Metria Building at the University of Eastern Finland, Joensuu, on November 25th, 2016, at 12 o'clock.

University of Eastern Finland
Department of Physics and Mathematics
Joensuu 2016

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Sanni Hullberger
Relativization in universal standard mathematics: a corpus-based study on newspaper physics
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ABSTRACT

This work contains several theoretical and numerical studies on diffractive optics and variable coherence. A generalized Iterative Fourier-Transform Algorithm (IFTA) for spatially partially coherent light is introduced. A diffractive element with a non-paraxial line focus is designed by rigorous diffraction theory. A novel method to control the coherence properties of an optical field by a vibrating grating is derived theoretically. The use of Bragg selectivity of a volume grating to extract a coherent component from a partially coherent field is introduced.

The effect of partial spatial coherence in non-paraxial Talbot imaging and the validity of the paraxial approximation in Lau imaging are analyzed. Non-paraxial Bessel beams generated by diffractive axicons are analyzed by a new quasi-rigorous electromagnetic method.

A new method to encode the complex amplitude of an arbitrary optical field in a zeroth-order diffractive structure is introduced. Waveguide discontinuities and index-modulated waveguide structures are analyzed by applying periodic boundary conditions, which enables the use of rigorous grating-diffraction theories.

Universal Decimal Classification: *53.084.85, 535.3, 535.4, 681.7.02*
OCIS codes: *050.1960, 130.4815*
Keywords: *Diffraction gratings, optical switching devices, partial differential equations, complex analysis*

ACKNOWLEDGEMENTS

Here you should thank your supervisors, colleagues, family etc. You can also add some other appropriate remarks.

Joensuu, November 25, 2016

Sanni Hullberger

LIST OF PUBLICATIONS

This thesis consists of the present review of the author's work in the field of differential equations and the following selection of the author's publications:

I S. Hullberger, J. Käiväräinen, and J. Saarelainen, "Meromorphic functions in modeling of optical fields with singularities," *J. Math. Anal. Appl.* **424**, 1113–1121 (2015).

II S. Hullberger and J. Käiväräinen, "Synthesis of phase gratings for desired coherence states," *Opt. Express* **24**, 1432–1439 (2016).

III L.-T. Koiranen, S. Hullberger, J. Saarelainen, and T. Arola, "Interpretation of polarization patterns," *Opt. Lett.* **41**, 1353–1356 (2016).

IV S. Hullberger, J. Saarelainen, and T. Arola, "Detection of light beams with aid of golden nanorods," *Apl. Phys. Lett.* **109**, 082015 (2016).

Throughout the overview, these papers will be referred to by Roman numerals.

AUTHOR'S CONTRIBUTION

The publications selected in this dissertation are original research papers on differential equations. Here you should explain what is your contribution in each of the papers.

TABLE OF CONTENTS

1 SOME EXAMPLES

1.1 MATHEMATICAL EQUATIONS

As I have shown in my first publication **I** *etc*. Unfortunately, there were some errors so I tried to correct them in my later publications **II**–**IV**.

Other references can (and normally should) be used in the review. Here are some articles [1–4] and some more [5, 6]. Here are some books [7–9]. Symbolic computation system can be referenced like this [10]. Then proceedings or other collective works [11, 12] and thesis [13]. Still some more [14–17] and finally some sited www-pages [18–20].

1.2 SOME STUFF

Let us consider a time-harmonic electromagnetic field of frequency ω. It is characterized by three-dimensional electric and magnetic fields of the form

$$E(r,t) = \Re\{E(r)\exp(-\mathrm{i}\omega t)\}\,, \tag{1.1}$$
$$H(r,t) = \Re\{H(r)\exp(-\mathrm{i}\omega t)\}\,, \tag{1.2}$$

where $r = (x,y,z)$ is the three-dimensional position vector and $\Re$ means the real part. In a continuous medium, these time-harmonic fields satisfy Maxwell's equations

$$\nabla \times E(r) = \mathrm{i}\omega B(r)\,, \tag{1.3}$$
$$\nabla \times H(r) = J(r) - \mathrm{i}\omega D(r)\,, \tag{1.4}$$
$$\nabla \cdot D(r) = \rho(r)\,, \tag{1.5}$$
$$\nabla \cdot B(r) = 0\,, \tag{1.6}$$

where $D(r)$, $B(r)$, $J(r)$, and $\rho(r)$ are the electric displacement, the magnetic induction, the electric current density and the electric charge density, respectively. In linear isotropic media we have constitutive relations

$$D(r) = \epsilon(r)E(r)\,, \tag{1.7}$$
$$B(r) = \mu(r)H(r)\,, \tag{1.8}$$
$$J(r) = \sigma(r)E(r)\,, \tag{1.9}$$

where $\epsilon(r)$, $\mu(r)$, and $\sigma(r)$ are known as the permittivity, the magnetic permeability, and the conductivity, respectively. The permittivity ϵ is written as $\epsilon(r) = \epsilon_r(r)\epsilon_0$, where ϵ_0 is the permittivity in vacuum and ϵ_r is the relative permittivity. The refractive index of the medium is defined as $n(r) = \sqrt{\epsilon_r(r)}$.

1.2.1 Splitting the equation

Here is one basic example:

$$\begin{aligned} U(x,y,z) = \frac{n\exp(\mathrm{i}kn\Delta z)}{\mathrm{i}\lambda\Delta z}\exp\left[\frac{\mathrm{i}\pi n}{\lambda\Delta z}(x^2+y^2)\right] \\ \times\iint_{-\infty}^{\infty} U(x',y',z_0)\exp\left[\frac{\mathrm{i}\pi n}{\lambda\Delta z}(x'^2+y'^2)\right]dx'dy' \ . \end{aligned} \tag{1.10}$$

The following example is even more complicated:

$$\begin{aligned} &W(x_1,y_1,z_1,x_2,y_2,z_2) \\ &\quad = \iiiint_{-\infty}^{\infty} A(\alpha_1,\beta_1,z_0,\alpha_2,\beta_2,z_0)\exp[-\mathrm{i}2\pi(x_1\alpha_1 - x_2\alpha_2 + y_1\beta_1 - y_2\beta_2)] \\ &\qquad \times\exp\{-\mathrm{i}2\pi[w^*(\alpha_1,\beta_1)\Delta z_1 - w(\alpha_2,\beta_2)\Delta z_2]\}d\alpha_1 d\alpha_2 d\beta_1 d\beta_2 \ . \end{aligned} \tag{1.11}$$

Next example shows how to define one variable in two ways:

$$w = \begin{cases} [(n/\lambda)^2 - (\alpha^2+\beta^2)]^{1/2}\,, & \text{when } \alpha^2+\beta^2 \le (n/\lambda)^2\,, \\ \mathrm{i}[(\alpha^2+\beta^2) - (n/\lambda)^2]^{1/2} & \text{otherwise.} \end{cases} \tag{1.12}$$

How about defining equation group?

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}\,, \qquad \begin{cases} x' = r\tan\phi \\ y' = r\cot\phi \end{cases}\ . \tag{1.13}$$

And finally a matrix vector product:

$$\begin{pmatrix} D_1t_1 & -a_{12}t_2 & \dots & -a_{1n}t_n \\ -a_{21}t_1 & D_2t_2 & \dots & -a_{2n}t_n \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1}t_1 & -a_{n2}t_2 & \dots & D_nt_n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n-2} \\ c_{n-1} \\ c_n \end{pmatrix}\ . \tag{1.14}$$

1.3 THEOREMS, ETC.

Definition 1.1. *Let $\mathbb{Z}$ be the integers.*

Theorem 1.1. *Here is a nice formula:*

$$1+2+\cdots+n = \frac{n(n+1)}{2}\ .$$

Proof. By induction. The details are left to the reader. □

Lemma 1.1. *If $n \in \mathbb{Z}$, then $2n$ is even.*

Proof. This follows easily by Theorem 1.1. □

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$$\bar{x} = \frac{1}{n} \sum_{i=1}^{i=n} x_i = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

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$$\int_0^\infty e^{-\alpha x^2} \mathrm{d}x = \frac{1}{2} \sqrt{\int_{-\infty}^{\infty} e^{-\alpha x^2} \mathrm{d}x} \int_{-\infty}^{\infty} e^{-\alpha y^2} \mathrm{d}y = \frac{1}{2} \sqrt{\frac{\pi}{\alpha}}$$

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$$\sum_{k=0}^{\infty} a_0 q^k = \lim_{n \to \infty} \sum_{k=0}^{n} a_0 q^k = \lim_{n \to \infty} a_0 \frac{1 - q^{n+1}}{1 - q} = \frac{a_0}{1 - q}$$

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$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

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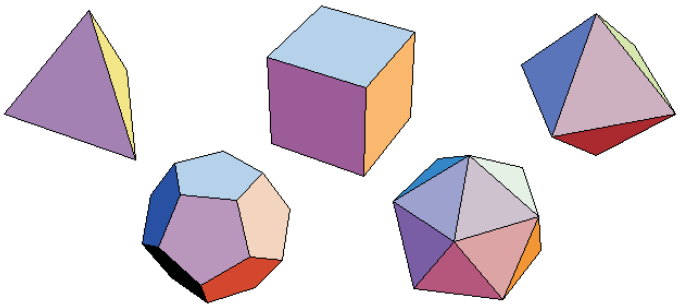

Figure 1.1: A picture.

nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}$$

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

1.4 FIGURES

Recommended way to include figures is to make them first in .eps format or .jpeg format. One picture in Figure 1.1 and many pictures in Figure 1.2.

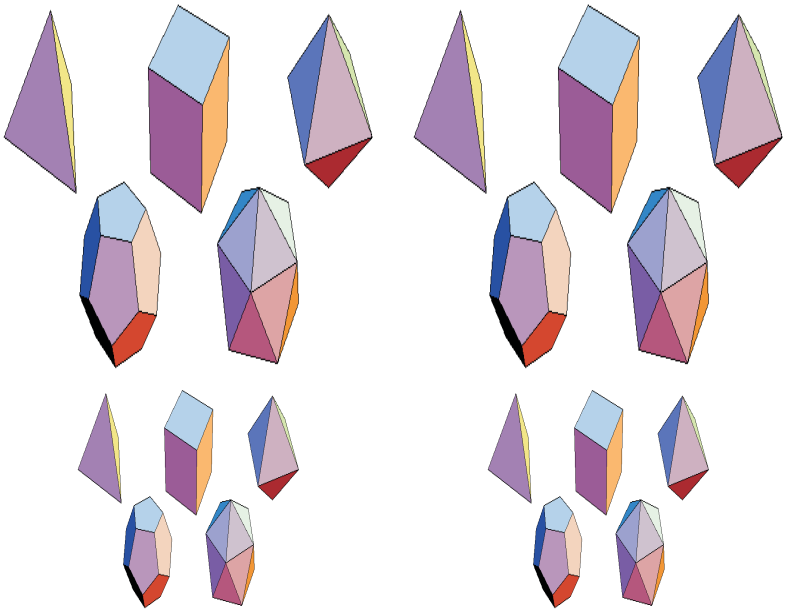

Figure 1.2: Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

1.5 TABLES AND SOMETHING STRANGE

Well, in principle it is easy, but.... Hopefully the tables shown below illustrates a little bit what is going on, and how the tables should look like. First, the basic table: If you have to combine multiple columns, you can do that as shown in Table 1.5.

Table 1.1: A short table caption.

sr_d	sr_o	t
1.0000	1.0000	1.0000
0.2500	0.2499	0.9998

The meaning of $\rightarrow$ in Table 1.5 is obvious, isn't it?

Table 1.2: Comparison of diffraction efficiencies for three connections, $1 \rightarrow 3$, $1 \rightarrow 5$ and $1 \rightarrow 7$, calculated by the MWCW-theory.

	$1 \rightarrow 3$		$1 \rightarrow 5$		$1 \rightarrow 7$	
i	E_{3i}^m	E_{3i}^r	E_{3i}^m	E_{3i}^r	E_{3i}^m	E_{3i}^r
0	0.3235	0.3237	0.1907	0.1911	0.1413	0.1404
$^+1$	0.3217	0.3217	0.1962	0.1960	0.1412	0.1403
$^+2$	0.0065	0.0066	0.1931	0.1930	0.1408	0.1408
$^+3$	0.0025	0.0024	0.0021	0.0020	0.1397	0.1401
$^+4$	0.0007	0.0008	0.0066	0.0066	0.0007	0.0006
$^+5$	0.0033	0.0034	0.0046	0.0047	0.0038	0.0038
$^+6$	0.0001	0.0001	0.0039	0.0040	0.0016	0.0013
$^+7$	0.0000	0.0000	0.0003	0.0002	0.0002	0.0002

1.5.1 Additional section for more text

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like "Huardest gefburn"? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

2 ANOTHER CHAPTER WITH A SO LONG TITLE THAT IT OCCUPIES AT LEAST TWO LINES

Of course such long titles do not look good so they should be avoided.

2.1 ANOTHER SECTION

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

2.1.1 First case

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2.2 ONE MORE SECTION

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

2.2.1 Second case

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2.2.2 Third case

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3 YET ANOTHER CHAPTER

3.1 ANOTHER SECTION

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

This is the second paragraph. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

And after the second paragraph follows the third paragraph. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

After this fourth paragraph, we start a new paragraph sequence. Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

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words should match the language.

3.1.1 Small subsection

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3.1.2 Another subsection with a very long title just to test TOC formatting

Hello, here is some text without a meaning. This text should show what a printed text will look like at this place. If you read this text, you will get no information. Really? Is there no information? Is there a difference between this text and some nonsense like “Huardest gefburn”? Kjift – not at all! A blind text like this gives you information about the selected font, how the letters are written and an impression of the look. This text should contain all letters of the alphabet and it should be written in of the original language. There is no need for special content, but the length of words should match the language.

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A EQUATIONS

A.1 FIRST EQUATION

$$\eta = \int_{-\infty}^{\infty} f(x) \exp\left(\mathrm{i}2\pi m x / d\right) dx \tag{A.1}$$

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$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

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$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$$

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$$a\sqrt[n]{b} = \sqrt[n]{a^n b}$$

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$$\bar{x} = \frac{1}{n}\sum_{i=1}^{i=n} x_i = \frac{x_1 + x_2 + \ldots + x_n}{n}$$

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$$\int_0^\infty e^{-\alpha x^2}\mathrm{d}x = \frac{1}{2}\sqrt{\int_{-\infty}^{\infty} e^{-\alpha x^2}\mathrm{d}x \int_{-\infty}^{\infty} e^{-\alpha y^2}\mathrm{d}y} = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

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A.2 SECOND EQUATION

$$n_3 \sin\theta_m = n_1 \sin\theta_{\text{in}} \tag{A.2}$$

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$$\sum_{k=0}^{\infty} a_0 q^k = \lim_{n\to\infty} \sum_{k=0}^{n} a_0 q^k = \lim_{n\to\infty} a_0 \frac{1-q^{n+1}}{1-q} = \frac{a_0}{1-q}$$

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$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

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$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2}$$

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Paper **I**

S. Hullberger and T. Bottas
"Synthesis of Bessel beams for
long haul communication"
Optics Letters **31**,
pp. 4432–4436, 2016.

Paper **II**

S. Hullberger, K. Nieminen, and T. Bottas
"Noncoaxial Laguerre-Gauss beams"
Optica **3**
pp. 1064–1071, 2016.