A multistate habitat-driven Langevin diffusion for inferring behaviour-specific utilization distributions

Brett T. McClintock and Michelle Lander

NOAA Marine Mammal Lab, Seattle, USA

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Langevin diffusion

$$
\begin{aligned}
\mathrm{d}\boldsymbol{\mu}_t &= \frac{\sigma^2}{2} \nabla \log \pi\left(\boldsymbol{\mu}_t \mid \boldsymbol{\beta}\right) \mathrm{d}t + \sigma \mathrm{d}\mathbf{B}_t \\
\pi\left(\boldsymbol{\mu} \mid \boldsymbol{\beta}\right) &= \frac{\exp\left(\sum_{k=1}^{K} x_k\left(\boldsymbol{\mu}\right)\beta_k\right)}{\int_{\mathcal{M}} \exp\left(\sum_{k=1}^{K} x_k\left(\mathbf{z}\right)\beta_k\right) \mathrm{d}\mathbf{z}}
\end{aligned}
$$

Michelot et al. (2019). The Langevin diffusion as a continuous-time model of animal movement and habitat selection, *Methods Ecol Evol*.

Langevin diffusion (discrete approximation of)

$$
\begin{aligned}
\boldsymbol{\mu}_{t+1} &= \boldsymbol{\mu}_t + \frac{\sigma^2 \Delta_t}{2} \nabla \log \pi \left(\boldsymbol{\mu}_t \mid \boldsymbol{\beta} \right) + \boldsymbol{\epsilon}_{t+1} \\
\nabla \log \pi \left(\boldsymbol{\mu} \mid \boldsymbol{\beta} \right) &= \sum_{k=1}^{K} \beta_k \nabla x_k \left(\boldsymbol{\mu} \right)
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\mu}_t &= \text{location at time } t \\
\sigma^2 &= \text{speed parameter} \\
\boldsymbol{\epsilon}_{t+1} &\sim \mathcal{N}\left(\mathbf{0}, \sigma^2 \Delta_t \mathbf{I} \right) \\
\beta_k &= \text{habitat selection coefficient for covariate } k \\
x_k \left(\boldsymbol{\mu} \right) &= \text{covariate } k \text{ evaluated at } \boldsymbol{\mu} \\
\Delta_t &= \text{interval duration between times } t \text{ and } t+1
\end{aligned}
$$

Multistate Langevin diffusion

$$[\boldsymbol{\mu}_{t+1} \mid \boldsymbol{\mu}_t, \boldsymbol{\sigma}, \boldsymbol{\beta}, s_t] \equiv \mathcal{N}\left(\boldsymbol{\mu}_t + \frac{\sigma_{s_t}^2 \Delta_t}{2} \nabla \log \pi_{s_t}(\boldsymbol{\mu}_t \mid \boldsymbol{\beta}_{s_t}), \sigma_{s_t}^2 \Delta_t \mathbf{I}\right)$$

$$s_t = \text{state from time } t \text{ to } t+1 \ (s_t \in \{1, \ldots, S\})$$

$$\pi_s(\boldsymbol{\mu} \mid \boldsymbol{\beta}_s) = \frac{\exp\left(\sum_{k=1}^{K_s} x_k(\boldsymbol{\mu}) \beta_{s,k}\right)}{\int_{\mathcal{M}} \exp\left(\sum_{k=1}^{K_s} x_k(\mathbf{z}) \beta_{s,k}\right) d\mathbf{z}}$$

$$\nabla \log \pi_s(\boldsymbol{\mu} \mid \boldsymbol{\beta}_s) = \sum_{k=1}^{K_s} \beta_{s,k} \nabla x_k(\boldsymbol{\mu})$$

$\longrightarrow$ **This can simply be the observation distribution ($\mathbf{P}_t$) of a (continuous-time) hidden Markov model!**

Continuous-time HMM

$$\mathcal{L}(\boldsymbol{\theta} \mid \boldsymbol{\mu}) = \boldsymbol{\delta}\mathbf{P}_1 \left[\prod_{t=2}^{T} \boldsymbol{\Gamma}_t \mathbf{P}_t \right] \mathbf{1}$$

$$\mathbf{P}_t = \begin{bmatrix} [\boldsymbol{\mu}_{t+1} \mid \boldsymbol{\mu}_t, \boldsymbol{\theta}, s_t = 1] & 0 & \cdots & 0 \\ 0 & [\boldsymbol{\mu}_{t+1} \mid \boldsymbol{\mu}_t, \boldsymbol{\theta}, s_t = 2] & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & [\boldsymbol{\mu}_{t+1} \mid \boldsymbol{\mu}_t, \boldsymbol{\theta}, s_t = S] \end{bmatrix}$$

$$\boldsymbol{\Gamma}_t = \exp(\mathbf{Q}\Delta_t)$$

$$\mathbf{Q} = \begin{array}{cc} \begin{array}{cccc} s_{t+1} = 1 & s_{t+1} = 2 & \cdots & s_{t+1} = S \end{array} & \\ \begin{bmatrix} -q_{1,1} & q_{1,2} & \cdots & q_{1,S} \\ q_{2,1} & -q_{2,2} & \cdots & q_{2,S} \\ \vdots & \vdots & \ddots & \vdots \\ q_{S,1} & q_{S,2} & \cdots & -q_{S,S} \end{bmatrix} & \begin{array}{l} s_t = 1 \\ s_t = 2 \\ \\ s_t = S \end{array} \end{array}$$

Fitting multistate Langevin models in `momentuHMM`

```r
library(remotes)
install_github("bmcclintock/momentuHMM@develop")
```

- Relies on the pseudo-design matrix (`DM`) argument and `langevin()` special function
- Pseudo-design matrix is analogous to **X** in GLMs:

$$g(\boldsymbol{\theta}) = \mathbf{X}\boldsymbol{\beta},$$

  where $g()$ is a link function (e.g. log, logit)
- Rows of **X** correspond to (real-scale) data stream distribution parameters ($\boldsymbol{\theta}$)
- Columns of **X** correspond to (working-scale) data stream distribution parameters ($\boldsymbol{\beta}$)

The pseudo-design matrix

```r
# m is a momentuHMM object (as returned by fitHMM), automatically loaded with the package
m <- example$m

# default identity matrix for step ~ gamma(mean, sd) with no covariates
X <- m$conditions$fullDM$step
X
```

```
##        mean_1:(Intercept) mean_2:(Intercept) sd_1:(Intercept) sd_2:(Intercept)
## mean_1                  1                  0                0                0
## mean_2                  0                  1                0                0
## sd_1                    0                  0                1                0
## sd_2                    0                  0                0                1
```

```r
# working-scale parameters (on log scale)
beta <- m$CIbeta$step$est
beta
```

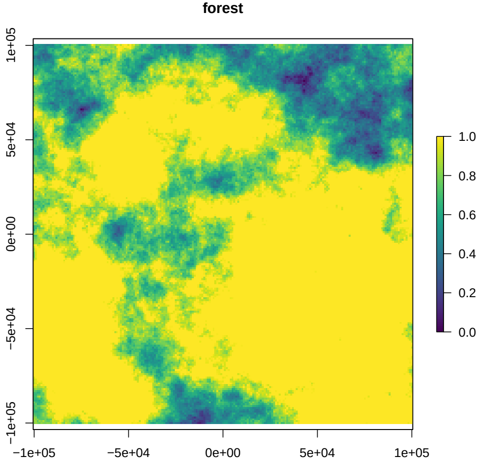
```
##      mean_1:(Intercept) mean_2:(Intercept) sd_1:(Intercept) sd_2:(Intercept)
## [1,]           2.680676           5.014916         2.149273         3.034212
```

The pseudo-design matrix

```
# mean and sd use log link function
exp(X %*% t(beta))

##               [1,]
## mean_1  14.594959
## mean_2 150.643519
## sd_1     8.578616
## sd_2    20.784598

# compare to real-scale parameter estimates
m$CIreal$step$est

##         state 1  state 2
## mean 14.594959 150.6435
## sd    8.578616  20.7846
```

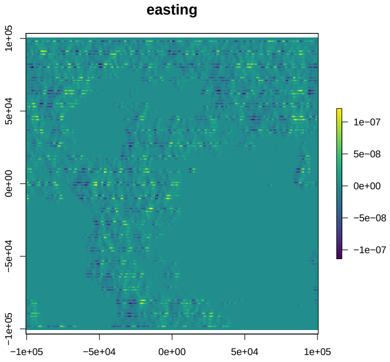
Fitting multistate Langevin models in momentuHMM

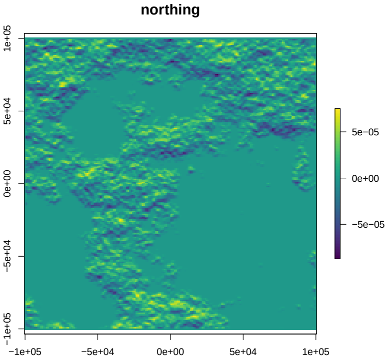
prepData can calculate gradients for raster covariate data



Fitting multistate Langevin models in `momentuHMM`

`prepData` can calculate gradients for raster covariate data





Fitting multistate Langevin models in momentuHMM

prepData can calculate gradients for raster covariate data

```
# prepare data and calculate gradients
tracks <- prepData(data, altCoordNames = "mu",
                   spatialCovs = covlist,
                   gradient = TRUE)
head(tracks)
```

```
##   ID                time      mu.x       mu.y    forest      forest.x
## 1  1 2018-01-01 00:00:00    0.0000     0.0000 0.7108917  1.140799e-04
## 2  1 2018-01-01 07:06:56  108.6667 -1060.6054 0.7155679  1.071387e-04
## 3  1 2018-01-01 14:13:52  349.4669 -1362.3010 0.7155679  1.157587e-04
## 4  1 2018-01-01 21:20:48  718.7713  -940.6311 0.8209750  1.059219e-04
## 5  1 2018-01-02 04:27:44 1014.5414  -930.9015 0.8209750 -1.351290e-05
## 6  1 2018-01-02 11:34:40 3339.4429 -6247.4732 0.9614867  2.727722e-05
##        forest.y
## 1 -2.585472e-05
## 2  5.085680e-05
## 3  4.397668e-05
## 4  1.557643e-06
## 5  4.423903e-06
## 6  3.605136e-05
```

Fitting multistate Langevin models in momentuHMM

langevin() special function in DM argument of fitCTHMM

```
# define observation distribution
dist <- list(mu="rw_mvnorm2") # bivariate normal random walk
```

```
# define single-state pseudo-design matrix
DM1 <- list(mu=matrix(c("mu.x_tm1","langevin(cov1.x)","langevin(cov2.x)",0,0,
                        "mu.y_tm1","langevin(cov1.y)","langevin(cov2.y)",0,0,
                                 0,                 0,                 0,1,0,
                                 0,                 0,                 0,0,1,
                                 0,                 0,                 0,1,0),5,5,byrow=TRUE,
                      dimnames=list(c("mean.x","mean.y","sigma.x","sigma.xy","sigma.y"),
                                    c("mean","cov1","cov2","sigma","sigma.xy"))))
DM1
```

```
## $mu
##          mean       cov1               cov2               sigma sigma.xy
## mean.x   "mu.x_tm1" "langevin(cov1.x)" "langevin(cov2.x)" "0"   "0"
## mean.y   "mu.y_tm1" "langevin(cov1.y)" "langevin(cov2.y)" "0"   "0"
## sigma.x  "0"        "0"                "0"                "1"   "0"
## sigma.xy "0"        "0"                "0"                "0"   "1"
## sigma.y  "0"        "0"                "0"                "1"   "0"
```

```
# fit model
fitLangevin <- fitCTHMM(tracks, nbStates=1, dist=dist, DM=DM1, Par0=Par0)
```

Illustration: Steller sea lions

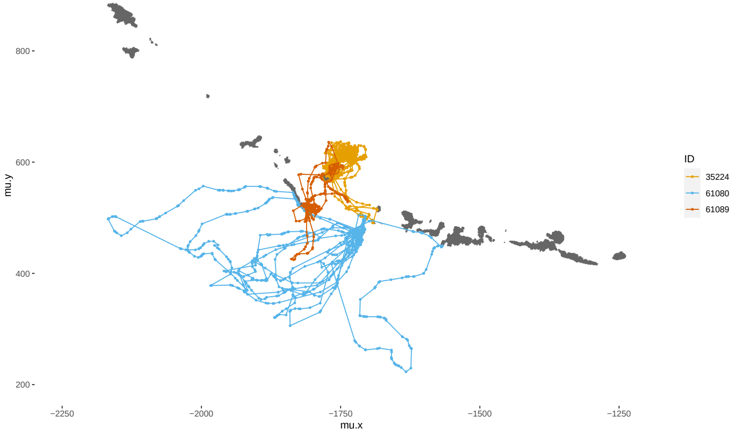
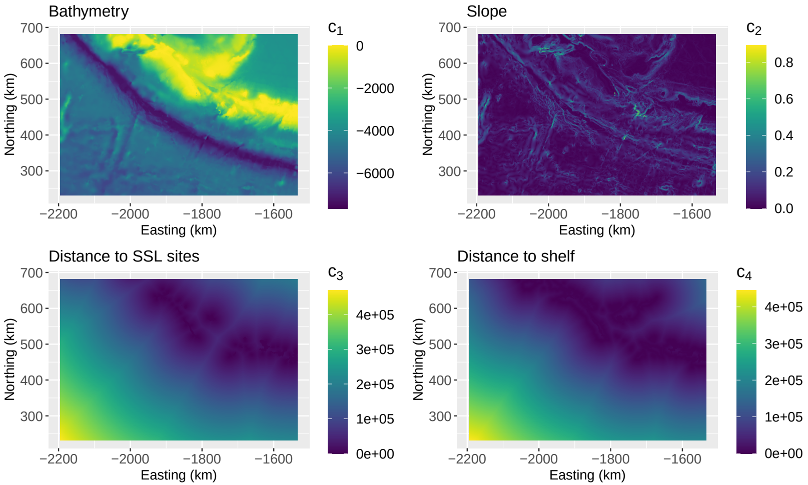
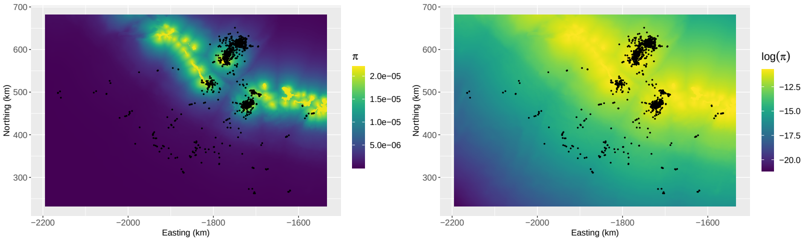


Illustration: Steller sea lions



Michelot et al. (2019). The Langevin diffusion as a continuous-time model of animal movement and habitat selection, *Methods Ecol Evol*.

Illustration: Steller sea lions



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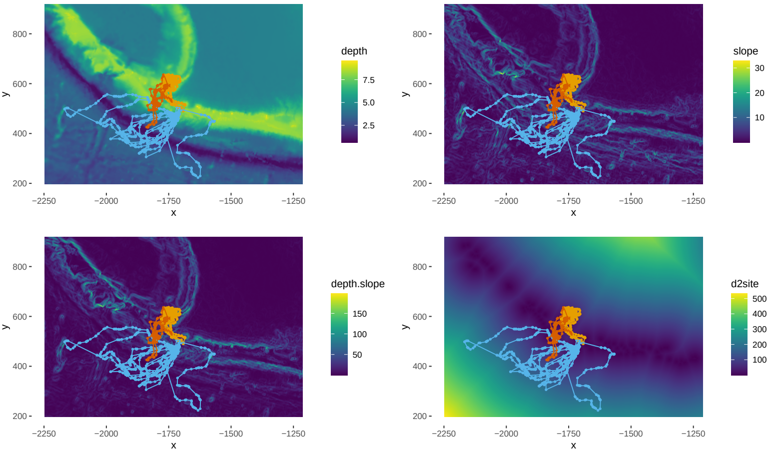


Illustration: Steller sea lions

$$[\boldsymbol{\mu}_{n,t+1} \mid \boldsymbol{\mu}_{n,t}, \boldsymbol{\theta}, s_{n,t}] \equiv$$
$$\begin{cases} \mathcal{N}\left(\boldsymbol{\mu}_{n,t} + \frac{\sigma^2_{n,AS}\Delta_t}{2} \nabla \log \pi_{s_{n,t}}\left(\boldsymbol{\mu}_{n,t} \mid \boldsymbol{\beta}_{s_{n,t}}\right), \sigma^2_{n,AS}\Delta_t \mathbf{I}\right) & \text{if } s_{n,t} \in \{1,2,3\} \\ \mathcal{N}\left(\boldsymbol{\mu}_{n,t}, \sigma^2_{HO}\Delta_t \mathbf{I}\right) & \text{if } s_{n,t} = 4 \end{cases}$$

$$[\boldsymbol{\mu}_{n,t+1} \mid \boldsymbol{\mu}_{n,t}, \boldsymbol{\theta}, s_{n,t}] \equiv$$
$$\begin{cases} \mathcal{N}\left(\boldsymbol{\mu}_{n,t} + \frac{\sigma^2_{n,AS}\Delta_{n,t}}{2} \nabla \log \pi_{s_{n,t}}\left(\boldsymbol{\mu}_{n,t} \mid \boldsymbol{\beta}_{s_{n,t}}\right), \sigma^2_{n,AS}\Delta_{n,t} \mathbf{I}\right) & \text{if } s_{n,t} \in \{1,3,4\} \\ \mathcal{N}\left(\boldsymbol{\mu}_{n,t} + \frac{\sigma^2_{n,2}\Delta_{n,t}}{2} \nabla \log \pi_2\left(\boldsymbol{\mu}_{n,t} \mid \boldsymbol{\beta}_2\right), \sigma^2_{n,2}\Delta_{n,t} \mathbf{I}\right) & \text{if } s_{n,t} = 2 \\ \mathcal{N}\left(\boldsymbol{\mu}_{n,t}, \sigma^2_{HO}\Delta_{n,t} \mathbf{I}\right) & \text{if } s_{n,t} = 5 \end{cases}$$

$$\boldsymbol{\beta}_s = \left(\beta_{depth,s}, \beta_{slope,s}, \beta_{depth:slope,s}, \beta_{d2site,s}, \beta_{d2site^2,s}\right)$$

Illustration: Steller sea lions

$$\boldsymbol{\Gamma}_t = \exp\left(\mathbf{Q}_t \Delta_t\right)$$

$$\mathbf{Q}_t = \begin{array}{cc} & \begin{array}{cccc} s_{t+1}=1 & s_{t+1}=2 & \dots & s_{t+1}=S \end{array} \\ & \left[\begin{array}{cccc} -q_{t,1,1} & q_{t,1,2} & \dots & q_{t,1,S} \\ q_{t,2,1} & -q_{t,2,2} & \dots & q_{t,2,S} \\ \vdots & \vdots & \ddots & \vdots \\ q_{t,S,1} & q_{t,S,2} & \dots & -q_{t,S,S} \end{array}\right] \begin{array}{c} s_t=1 \\ s_t=2 \\ \\ s_t=S \end{array} \end{array}$$

$$\log\left(q_{t,i,j}\right) = \alpha_{0,i,j} + \alpha_{1,i,j}\text{d2site}\left(\boldsymbol{\mu}_t\right) \text{ for } i \neq j$$

Illustration: Steller sea lions

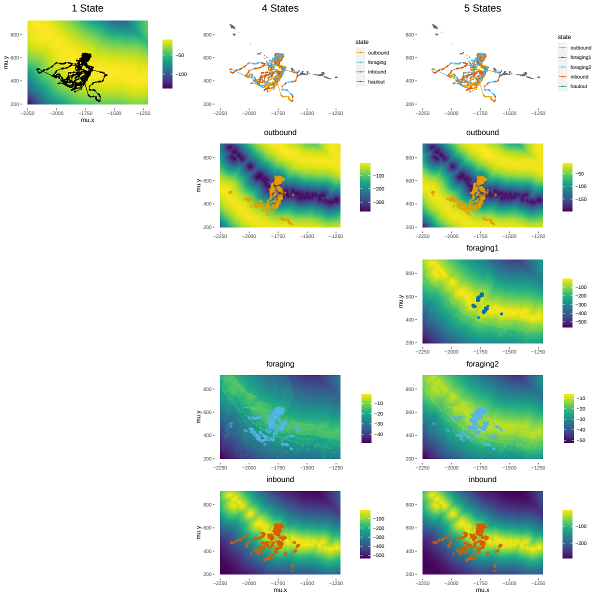
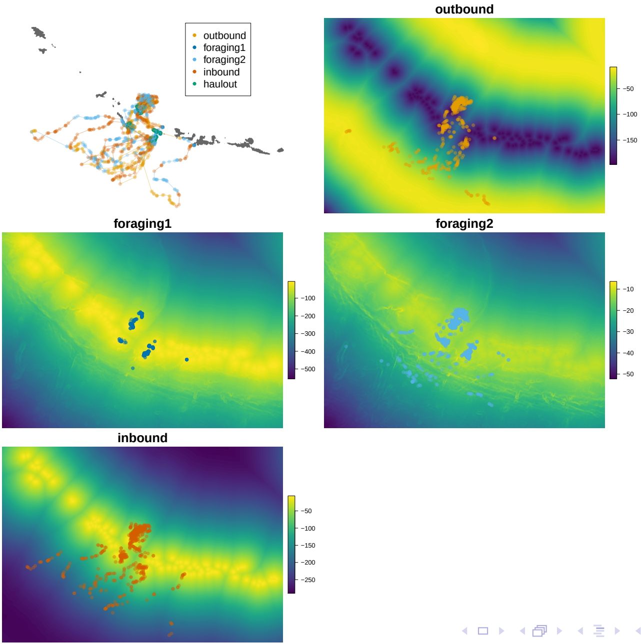
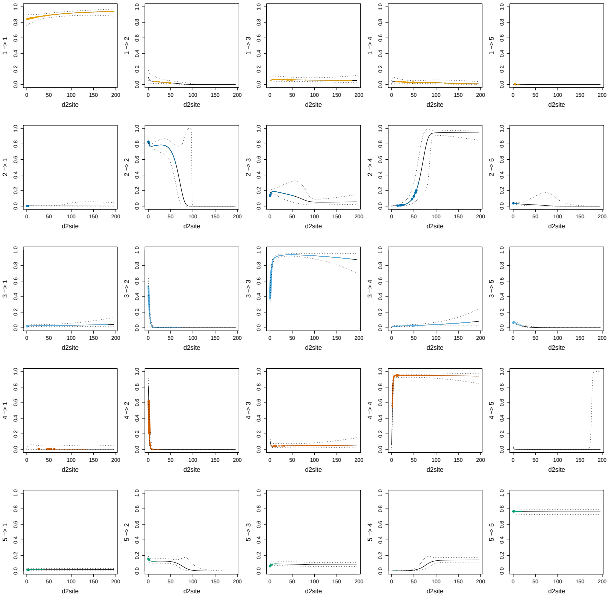


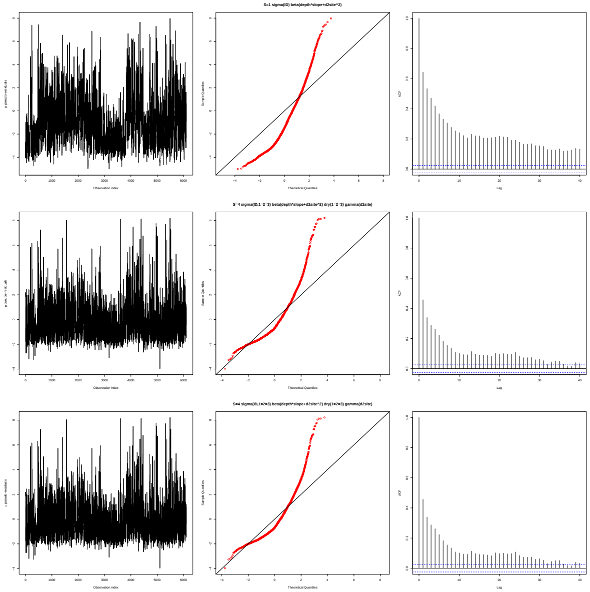
Illustration: Steller sea lions



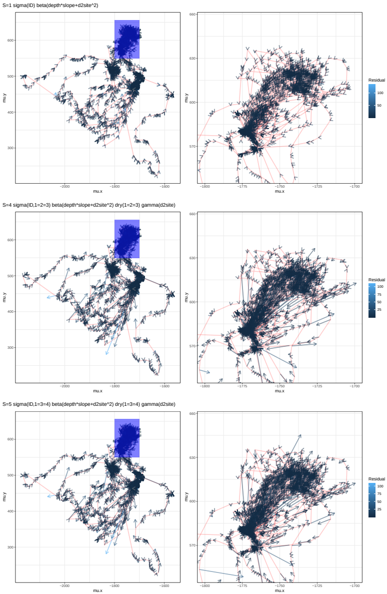
State transition probabilities



Pseudo-residuals



Predicted steps and residuals



Simulation

```r
# simulate from fitted model
simLangevin <- simCTHMM(model=fitLangevin, spatialCovs=covlist)
```