Scaling behaviour of developing and decaying networks

2000 Europhys. Lett. 52 33

(http://iopscience.iop.org/0295-5075/52/1/033)

EUROPHYSICS LETTERS 1 October 2000

Europhys. Lett., **52** (1), pp. 33–39 (2000)

Scaling behaviour of developing and decaying networks

S. N. Dorogovtsev [1,2](*) and J. F. F. Mendes [2](**)

[1] *Departamento de Física and Centro de Física do Porto, Faculdade de Ciências Universidade do Porto - Rua do Campo Alegre 687, 4169-007 Porto, Portugal*

[2] *A. F. Ioffe Physico-Technical Institute - 194021 St. Petersburg, Russia*

(received 5 May 2000; accepted in final form 11 August 2000)

PACS. 05.40.-a – Fluctuation phenomena, random processes, noise, and Brownian motion.
PACS. 64.60.Cn – Order-disorder transformations; statistical mechanics of model systems.
PACS. 87.18.Sn – Neural networks.

Abstract. – We find that a wide class of developing and decaying networks has scaling properties similar to those that were recently observed by Barabási and Albert in the particular case of growing networks. The networks considered here evolve according to the following rules: i) Each instant a new site is added, the probability of its connection to old sites is proportional to their connectivities. ii) In addition, a) new links between some old sites appear with probability proportional to the product of their connectivities or b) some links between old sites are removed with equal probability.

The recently observed power laws in a number of networks and the first data on the topology of them [1–14] attracts the special attention to these exciting objects widely studied for a long time [15–19] In fact the scaling properties were found only in a few of a great number of growing networks (the citations of scientific papers, Internet, the collaboration graphs, etc. [2, 4, 7, 20, 21]) but one of them —the Web—is so significant that the topic turned to be really hot.

The simplest model of a scale-free growing network was proposed by Barabási and Albert (BA model) [7, 8]. In this model, each new site is connected with some old site with probability proportional to its connectivity k, *i.e.* to the number of connections with this site. The distribution of the connectivities in the large network has a power law dependence $P(k) \propto k^{-\gamma}$ with the exponent $\gamma = 3$ [7, 8]. In fact, such a network is self-organized into a scale-free structure. As compared with the well-known Erdös-Renýi model [15] of growing networks, which does not produce the power law distributions, the BA model introduces a new important ingredient—the preferential attachment of new links, which makes it suitable for the description of scale-free real networks. A full form of the distribution of the connectivities and some other related properties of the BA model were calculated exactly [22]. Introduction of aging of sites proportional to $\tau^{-\alpha}$, where τ is the age of a site, does not change the scaling properties crucially for $\alpha < 1$, but scaling breaks at higher values of the aging exponent [23]. Several examples of real networks with aging of sites are described in [20]. The simplicity of the BA model makes it a convenient object to study the evolution of networks.

Nevertheless, the BA model describes only a particular type of evolving networks. Of course, reality is much richer. In real networks, *e.g.* in Internet, links are not only added but

(*) E-mail: sdorogov@fc.up.pt
(**) E-mail: jfmendes@fc.up.pt

may break from time to time. That certainly changes the structure of such networks. Note also, that new links between old sites may appear in a different way than links between new and old sites, according to different rules. Therefore, it is tempting to find out whether the observed behaviour is usual for a vast variety of networks or applies only to a very restricted number of invented objects. With that purpose, we extend a set of models of evolving networks starting from the BA model.

In the present letter, we propose models of developing and decaying networks with undirected links which show scaling behaviour. We consider structures which evolve due to the following reasons. First, they grow like in the BA model, *i.e.* in each instant one new site is added and is connected with an old site by an undirected link with a probability proportional to its connectivity k (one may check that the general results —the existence of the scaling and the values of the scaling exponents— do not depend on the number of the connections with a new site). In addition, we introduce a new parallel component of the evolution—the permanent addition of new undirected links between old sites or, on the contrary, the permanent removal of some old links. We consider two different cases. a) A developing network: Each instant, new c links are added between unconnected pairs of old sites i and j with probability proportional to the product of their connectivities $k_i k_j$. (c may be also non-integer. For that one can introduce the probability of addition of a link.) $c \geq 0$. b) A decaying structure (in fact, it is a set of clusters): Each instant, some links between old sites are removed with equal probability. In this case $c \leq 0$.

Note that both processes—the addition of new sites with new links and the addition of new links between old sites (or removing old links) proceed in parallel. Hence, the resulting structures differ from the original BA model all the time.

We study the following one-site quantities of the structures: the total distribution of connectivities at long times, $P(k)$, and the average connectivity of a site of an age s at long time t, $\overline{k}(s,t)$, and their scaling exponents $P(k) \propto k^{-\gamma}$ and $\overline{k}(s,t) \propto (s/t)^{-\beta}$.

Below, we demonstrate both analytically and by simulation that the introduced evolving networks show scaling behaviour in a wide range of values of c. Both $P(k)$ and $\overline{k}(s,t)$, for the developing networks, are power law functions for all $c \geq 0$. Nevertheless, only $\overline{k}(s,t)$ demonstrates the power law behaviour in the whole range $-1 < c < 0$ for the decaying structures. In this case, the power law dependence of the distribution $P(k)$ is observed only close to $c = 0$.

In order to study scaling properties of the evolving networks we performed numerical simulations according to the above-introduced rules. Each instant, we add one new site with one link and, in addition, may remove some of the old links (decaying network) or, on the contrary, may add some new links between directly unconnected old sites (developing network) with the relative rate c. Therefore, to study even one-site properties of the structures, one has to keep in memory information about all connections among them. We performed simulations with a total number of sites (*e.g.*, time) $t = 10000$ with 10000 averages for decaying networks, $-1 < c < 0$, and $t = 1000$ with 100000 averages for developing networks, $c \geq 0$. In fig. 1, we present the dependences of the average connectivity $\overline{k}(s,t)$ on the number of a site s at different values of c for both structures, *i.e.* for the decaying network, fig. 1(a), and for the developing one, fig. 1(b).

For both models, in the whole range of c, $\overline{k}(s,t) \propto (s/t)^{-\beta}$. The change of the sign of the exponent β in the developing network at $c = -1/2$ was unexpected (compare with the behaviour of β *vs.* an aging exponent in networks with aging of sites [23]), see fig. 1(a) and the dependence $\beta(c)$ in fig. 2. At this point, the average connectivity turns to be independent of the site age, $\overline{k}(s,t) = 1$. We studied also the distribution $P(k)$ (see fig. 1(c)). It behaves as $k^{-\gamma}$ for all $c \geq 0$ for the developing network but, for the decaying network, the power law

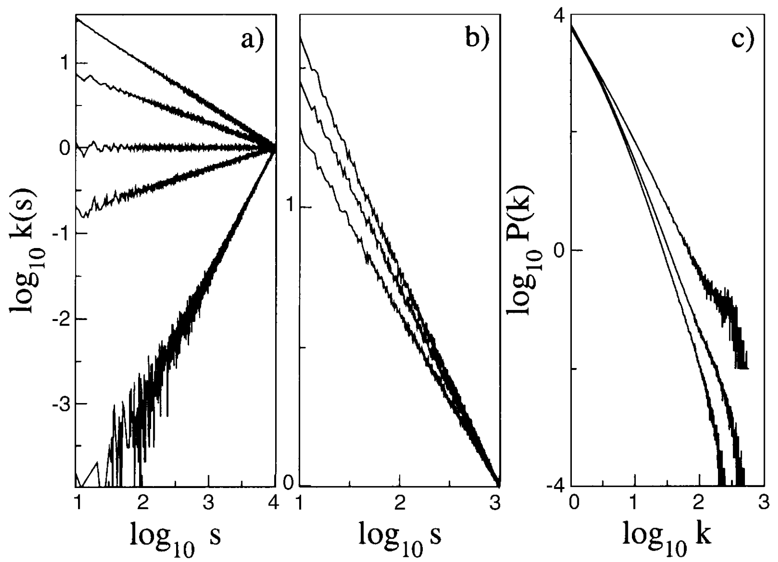

Fig. 1 – Average connectivity *vs.* number s of the site for (a) decaying ($-1 < c < 0$) and (b) developing ($c > 0$) networks, and (c) log-log plot of the distribution of connectivities. Different curves correspond to different values of c. From top to bottom, (a) $c = 0.0, -0.3, -0.5, -0.6, -0.8$; (b) $c = 1.0, 0.5, 0.25$; (c) $c = 1.0, 0.0, -0.2$. The upper curve in plot (c) is displaced along the vertical axis to demonstrate the variation of the slope. The network size is $t = 10000$ for $-1 < c \leq 0$ and $t = 1000$ for $c > 0$.

dependence is found only in a narrow region of c near zero (see fig. 2). The range of the values of k for which we observe such behaviour diminishes with the decrease of c and then disappears. Note that the finite-size effects are strong in this region ($c < 0$). We studied also the models in which, unlike the structures considered above, old links between sites are permanently removed with probability proportional to the product of the connectivities of the sites, or new links between old sites are permanently added with equal probability. The simulation demonstrates that the scaling breaks in both cases.

Let us describe the obtained results analytically. We start from the case of the developing network. One may use the simple continuous approach [8, 23] that gives exact results for the scaling exponents as was demonstrated in [22]. Since one site is added per unit of time, then the total number of sites is t and each site is labeled by the time of its birth $s \leq t$. Then the equation for the average connectivity of the site s at time t, $\overline{k}(s,t)$, in the continuous limit may be written in the following form:

$$\frac{\partial \overline{k}(s,t)}{\partial t} = \frac{\overline{k}(s,t)}{\int_0^t \mathrm{d}u \overline{k}(u,t)} + 2c \frac{\overline{k}(s,t)\left[\int_0^t \mathrm{d}u \overline{k}(u,t) - \overline{k}(s,t)\right]}{\left[\int_0^t \mathrm{d}u \overline{k}(u,t)\right]^2 - \int_0^t \mathrm{d}u \overline{k}^2(u,t)}, \qquad \overline{k}(t,t) = 1 \tag{1}$$

(we wrote also the boundary condition — one link connected with a new site is added at each time step). The first term is the same as in the BA model. The second one describes the increase of the connectivity due to the addition of new links between old sites with probability proportional to the product of connectivities of the connected sites.

What is the reason for such a form of this term? In the model under consideration, the multiple links are absent. Nevertheless, for a moment, let us allow new links to appear also between directly connected sites. Then, in the discrete version, the contribution of new links between old sites to the growth rate of k_i (connectivity of a site i) is $ck_i \sum_{j\neq i} k_j / \sum_{l>m} k_l k_m$. Now, we assume that the effect of the multiple links is not essential at long times (we checked the validity of this assumption by simulation). In continuous limit, from this expression, one obtains immediately the term under discussion.

This term is simplified at long times, if one omits relatively small contributions in both numerator and denominator in eq. (1). Then eq. (1) takes the form

$$\frac{\partial \overline{k}(s,t)}{\partial t} = (1+2c)\frac{\overline{k}(s,t)}{\int\limits_0^t \mathrm{d}u\overline{k}(u,t)}\,. \tag{2}$$

Applying $\int_0^t \mathrm{d}s$ to eq. (2) one gets

$$\int_0^t \mathrm{d}s\frac{\partial \overline{k}(s,t)}{\partial t} = \frac{\partial}{\partial t}\int_0^t \mathrm{d}s\overline{k}(s,t) - \overline{k}(t,t) = 1+2c\,, \tag{3}$$

so we obtain the obvious relation $\int_0^t \mathrm{d}s\overline{k}(s,t) = 2(1+c)t$. Now eq. (1) is of the following simple form:

$$\frac{\partial \overline{k}(s,t)}{\partial t} = \frac{1+2c}{2(1+c)}\frac{\overline{k}(s,t)}{t}\,. \tag{4}$$

Its solution is $\overline{k}(s,t) = (s/t)^{-\beta}$ with the exponent

$$\beta = \frac{1+2c}{2(1+c)}\,. \tag{5}$$

To obtain the exponent of the distribution of connectivities, γ, one uses the general relation between the scaling exponents of growing networks [22, 23]

$$\beta(\gamma-1) = 1 \tag{6}$$

that was obtained on the assumption that both $\overline{k}(s,t)$ and $P(k)$ show scaling behaviour, $\overline{k}(s,t) \propto (s/t)^{-\beta}$ at $s \ll t$, $P(k) \propto k^{-\gamma}$ for large k. From these assumptions, one gets [22] a general form of the probability $p(k,s,t)$ that the site s has the connectivity k at time t. The normalization condition for it leads immediately to eq. (6). From our simulation, we know that both $\overline{k}(s,t)$ and $P(k)$ show the power law behaviour for $c > 0$. Therefore,

$$\gamma = 2 + \frac{1}{1+2c}\,, \tag{7}$$

so we get both scaling exponents for the developing network.

Let us consider now the decaying network. Again we apply the continuous approach. In fact, the removal of old links with equal probability seems to be equivalent to the decrease of the connectivities of old sites with probability proportional to their particular values. Hence, both cases under consideration represent two sides of the same process, and eq. (2) may also be applicable to this case. Nevertheless, one should account for the fact that only existing links may be removed. Therefore, we prefer to make the calculations more thoroughly.

One introduces the average number of links between the sites s and s' at time t, $\overline{n}(s,s',t)$, where $0 \leq s \leq s' \leq t$. The average connectivity may be expressed in terms of this quantity:

$$\overline{k}(s,t) = \int\limits_0^s \mathrm{d}u\,\overline{n}(u,s,t) + \int\limits_s^t \mathrm{d}w\,\overline{n}(s,w,t)\,. \tag{8}$$

The set of equations for $\overline{n}(s,s',t)$ is

$$\overline{n}(s,t,t) = \frac{\int_0^s \mathrm{d}u\,\overline{n}(u,s,t) + \int_s^t \mathrm{d}w\,\overline{n}(s,w,t)}{\int_0^t \mathrm{d}s \left[\int_0^s \mathrm{d}u\,\overline{n}(u,s,t) + \int_s^t \mathrm{d}w\,\overline{n}(s,w,t)\right]},$$

$$\frac{\partial \overline{n}(s,s',t)}{\partial t} = c\,\frac{\overline{n}(s,s',t)}{\int_0^t \mathrm{d}s \int_s^t \mathrm{d}s'\,\overline{n}(s,s',t)}\,. \tag{9}$$

(Note that c is negative now!) The first equality of eq. (9) describes the links added to the network together with new sites as in the BA model. We again set the number of links connected with each new site to be unit. Applying $\int_0^t \mathrm{d}s$ to this equality we get

$$\int_0^{s'} \mathrm{d}s\,\overline{n}(s,s',t) = 1\,. \tag{10}$$

The second equality of eq. (9) shows how $\overline{n}(s,s',t)$ changes due to the removing of links between the old sites. Application of $\int_0^t \mathrm{d}s \int_s^t \mathrm{d}s'$ to this equality leads to the other obvious relation, $\int_0^t \mathrm{d}s \int_s^t \mathrm{d}s'\,\overline{n}(s,s',t) = (1+c)t$.

Let us search the solution of eq. (9) in the scaling form

$$\overline{n}(s,s',t) = \frac{1}{t}\mathcal{N}\left(\frac{s}{t},\frac{s'}{t}\right). \tag{11}$$

Then

$$\mathcal{N}(\xi,1) = \frac{\int_0^\xi \mathrm{d}\zeta\,\mathcal{N}(\zeta,\xi) + \int_\xi^1 \mathrm{d}\zeta'\,\mathcal{N}(\xi,\zeta')}{\int_0^1 \mathrm{d}\xi \left[\int_0^\xi \mathrm{d}\zeta\,\mathcal{N}(\zeta,\xi) + \int_\xi^1 \mathrm{d}\zeta'\,\mathcal{N}(\xi,\zeta')\right]},$$

$$\left[1 - \xi\frac{\partial}{\partial\xi} - \xi'\frac{\partial}{\partial\xi'}\right]\mathcal{N}(\xi,\xi') = c\,\frac{\mathcal{N}(\xi,\xi')}{\int_0^1 \mathrm{d}\xi \int_\xi^1 \mathrm{d}\xi'\,\mathcal{N}(\xi,\xi')}, \tag{12}$$

and

$$\int_0^1 \mathrm{d}\xi\,\mathcal{N}(\xi,1) = 1\,. \tag{13}$$

One sees that

$$\int_0^1 \mathrm{d}\xi \left[\int_0^\xi \mathrm{d}\zeta\,\mathcal{N}(\zeta,\xi) + \int_\xi^1 \mathrm{d}\zeta'\,\mathcal{N}(\xi,\zeta')\right] = 2(1+c) \tag{14}$$

and

$$\int_0^1 \mathrm{d}\xi \int_\xi^1 \mathrm{d}\xi'\,\mathcal{N}(\xi,\xi') = (1+c)\,. \tag{15}$$

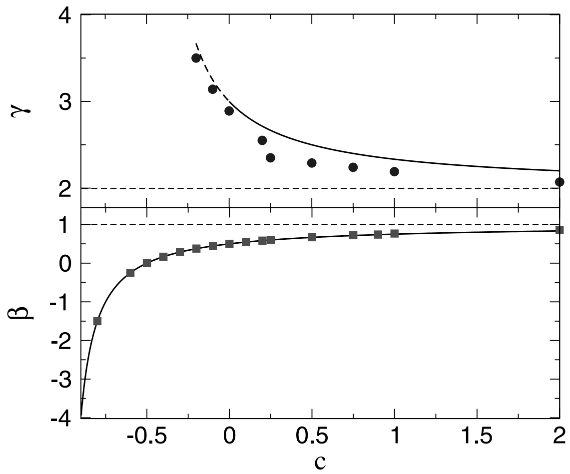

Fig. 2 – Exponents β of the average connectivity and γ of the distribution of connectivities *vs.* c, *i.e.* *vs.* the rate of removal ($c < 0$) or addition ($c > 0$) links between old sites. Points are obtained from the simulations. The lines are found analytically (see eqs. (5) and (7)). For the decaying network, the scaling behaviour of $P(k)$ is observed only in the narrow region of c close to zero.

The solution of eq. (12) may be found in the form

$$\mathcal{N}(\xi, \xi') = B\,\xi^{-a}\xi'^{\,-b}\,, \tag{16}$$

where B, a, b are constants. Inserting eq. (16) into eq. (12), accounting for eqs. (14) and (15), we obtain the exponents $a = b = 1-1/[2(1+c)]$. Equation (13) gives $B = 1-a = 1/[2(1+c)]$. Substitution of eq. (16), accounting for eq. (11), into eq. (8) leads to the expression $\beta = a = 1-1/[2(1+c)]$ that is exactly the same as in the previous case, see eq. (5). Note that now it is possible to use the relation between the scaling exponents, eq. (6), only in the region near $c = 0$, since only in this region we observed the scaling behaviour of $P(k)$. For these values of c we again get the old expression eq. (7) for the exponent γ.

In fig. 2, we plot the analytically obtained dependences β and γ *vs.* c together with the results obtained from the simulation. One may see that the correspondence between the simulation and the theory is quantitative. When c changes from -1 to 0, β increases from $-\infty$ to $1/2$ passing zero at $c = -1/2$. Subsequent increase of c to ∞ leads to growth of β up to 1 while γ decreases from 3 to 2. The particular case $c \to \infty$, $\gamma = 2$, corresponds to the situation when the network evolves only due to the addition of new links by the above defined rules.

Our results show that the permanent removing of links leads to a more essential change of the structure of a network than the addition of them. What is the reason for that? One may see that the decaying structure under consideration is, in fact, a changing set of disconnected clusters. Because of finite-size effects, we failed to find the position of the percolation threshold that may be defined for networks [12]. Nevertheless, we see that, at high enough rates of link removal, large clusters are certainly absent, and the appearing structures indeed have to demonstrate quite different properties than the networks with $c \geq 0$. We failed also to find any peculiarity in the distribution of clusters in the point of the scaling break, $c = -1/2$.

One may note that many real networks show power law distributions of connectivities with exponents between 2 and 3 [8, 20]. Such values of γ correspond to the case of developing networks (see eq. (5), $c > 0$). Nevertheless, that is only one of the possible reasons.

In summary, we have introduced a new parallel component of the evolution of growing networks. In addition to new links connecting new sites and old ones, links between old sites may appear or break with the relative rate c. In that way, we have accounted for the processes which certainly occur in many real growing networks. We have demonstrated that addition of this component to a scale-free network does not break the scaling behaviour in a wide range of the rate c. The values of the scaling exponents have turned to be dependent on this parameter.

* * *

SND thanks PRAXIS XXI (Portugal) for a research grant PRAXIS XXI/BCC/16418/98. JFFM was partially supported by the project PRAXIS/2/2.1/FIS/299/94. We also thank M. C. MARQUES for reading the manuscript and A. V. GOLTSEV, YU. G. POGORELOV and A. N. SAMUKHIN for many useful discussions. After this manuscript was submitted we learnt of a work of Albert and Barabási [24] which overlaps some of our results. We thank A.-L. BARABÁSI for informing us of this work before publication.

REFERENCES

[1] LAHERERRE J. and SORNETTE D., *Eur. Phys. J. B*, **2** (1998) 525.
[2] REDNER S., *Eur. Phys. J. B*, **4** (1998) 131.
[3] WATTS D. J. and STROGATZ S. H., *Nature*, **393** (1998) 440.
[4] ALBERT R., JEONG H. and BARABÁSI A.-L., *Nature*, **401** (1999) 130.
[5] HUBERMAN B. A. and ADAMIC L. A., *Nature*, **401** (1999) 131.
[6] HUBERMAN B. A., PIROLLI P. L. T., PITKOW J. E. and LUKOSE R. J., *Science*, **280** (1998) 95.
[7] BARABÁSI A.-L. and ALBERT R., *Science*, **286** (1999) 509.
[8] BARABÁSI A.-L., ALBERT R. and JEONG H., *Physica A*, **272** (1999) 173.
[9] WATTS D. J., *Small Worlds* (Princeton University Press, Princeton, New Jersey) 1999.
[10] BARTHÉLÉMY M. and AMARAL L. A. N., *Phys. Rev. Lett.*, **82** (1999) 3180; 5180(E).
[11] NEWMAN M. E. J. and WATTS D. J., *Phys. Lett. A*, **263** (1999) 341.
[12] NEWMAN M. E. J. and WATTS D. J., *Phys. Rev. E*, **60** (1999) 7332; MOORE C. and NEWMAN M. E. J., cond-mat/0001393, to be published.
[13] BARRAT A. and WEIGT M., *Eur. Phys. J. B*, **13** (2000) 547.
[14] TSALLIS C. and DE ALBUQUERQUE M. P., *Eur. Phys. J. B*, **13** (2000) 777.
[15] ERDÖS P. and RENÝI A., *Publ. Math. Inst. Hung. Acad. Sci.*, **5** (1960) 17.
[16] BOLLOBÁS B., *Random Graphs* (Academic Press, London) 1985.
[17] DERRIDA B. and POMEAU Y., *Europhys. Lett.*, **1** (1986) 45.
[18] DERRIDA B. and STAUFFER D., *Europhys. Lett.*, **2** (1987) 739.
[19] MONASSON R. and ZECCHINA R., *Phys. Rev. Lett.*, **75** (1995) 2432.
[20] AMARAL L. A. N., SCALA A., BARTHÉLÉMY M. and STANLEY H. E., cond-mat/0001458.
[21] DOROGOVTSEV S. N. and MENDES J. F. F., *Europhys. Lett.*, **50** (2000) 1.
[22] DOROGOVTSEV S. N., MENDES J. F. F. and SAMUKHIN A. N., cond-mat/0004434, submitted for publication.
[23] DOROGOVTSEV S. N. and MENDES J. F. F., *Phys. Rev. E*, **62** (2000) 1842.
[24] ALBERT R. and BARABÁSI A.-L., cond-mat/0005085.