Dynamics of Networking Agents Competing for High Centrality and Low Degree

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We model a system of networking agents that seek to optimize their centrality in the network while keeping their cost, the number of connections they are participating in, low. Unlike other game-theory based models for network evolution, the success of the agents is related only to their position in the network. The agents use strategies based on local information to improve their chance of success. Both the evolution of strategies and network structure are investigated. We find a dramatic time evolution with cascades of strategy change accompanied by a change in network structure. On average the network self-organizes to a state close to the transition between a fragmented state and a state with a giant component. Furthermore, with increasing system size both the average degree and the level of fragmentation decreases.

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Game theory conceptualizes many of the circumstances that drive the dynamics of social and economic systems. If such systems consist of many pairwise interacting agents, they can be modeled as networks. In such networks one can relate the function of a vertex to its position. For example, in business connections an agent would presumably like to be close, in network distance, to the average other agent [1]. This ensures the information received from other agents to be up-to-date [2] and will likely increase the agent's sphere of influence. At the same time the agent would seek to limit the work load by minimizing its degree (number of connections). In this Letter we define an iterative N-player game where agents try simultaneously to obtain high centrality and low degree. Agents remove and add edges by individual strategies. Furthermore, they update the strategies throughout the game by imitating successful agents. We assume the agents have only information about their immediate surroundings. As a result an agent can only relink to, or observe and mimic the strategies of, other agents a fixed distance away. Most recent studies of games on networks have considered a static underlying network defining the possible competitive encounters [3]. In other models where the network coevolves with the game [4], the agents are assigned additional variables which serve as the basis of the game. In our model, however, the score of an agent is determined by the network dynamics alone. This setting, apart from being conceptually simpler, makes the relation between the game and network dynamics more transparent. The rest of the Letter contains a precise definition of the model, an investigation of the time evolution of the strategies and network structure, and an investigation of the dependence on model parameters.

In our model N agents are synchronously updated over t_{tot} iterations. The number of edges M is allowed to vary. The initial configuration is an Erdős-Rényi network [5] of M_0 edges. Multiple edges or self-edges are not allowed. The score, in our game, is an effective score taking into account both the benefit of centrality and the inevitable cost of maintaining the network ties. We want the score of a vertex i to increase with centrality and decrease with its degree k_i. Of many centrality concepts [1] we choose to base our score on the simplest nonlocal centrality measure—closeness centrality (the reciprocal average path length from one vertex to the rest of the graph). Furthermore, if the network is disconnected we would like the score to increase with the number of vertices reachable from i. To incorporate this we use a slight modification of closeness:

$$c(i) = \sum_{j\in H(i)\setminus\{i\}} \frac{1}{d(i,j)}, \tag{1}$$

where $H(i)$ is the connected subgraph i belongs to and $d(i, j)$ is the graph distance between i and j. The purpose of the modification is to make the number of elements in the sum of Eq. (1) proportional to the number of vertices in the connected component i belongs to. This captures the idea that it is beneficial to belong to a large connected cluster. For clusters of equal size the modified closeness gives a higher weight on the count of nearer vertices, but otherwise captures similar features as the original closeness. We define a score function that incorporates the desired properties mentioned above:

$$s(i) = \begin{cases} c(i)/k_i & \text{if } k_i > 0 \\ 0 & \text{if } k_i = 0 \end{cases}. \tag{2}$$

In addition, we assume the accessible information is restricted to a close neighborhood of a vertex. To be precise, the moves allowed to a vertex is to delete or add edges to agents up to two steps away. Our assumption is motivated by the fact that in real world systems agents can be assumed to know more about the part of the graph that is closer to them.

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When a vertex i updates its position, it selects another vertex in a set X [the neighborhood $\Gamma(i)$ if an edge is to be removed, or the second neighborhood $\Gamma_2(i) = \{j: d(i, j) = 2\}$ if an edge is to be added]. This is done by successively applying six tie-breaking *actions*: (i),(ii) Choose vertices with maximal (minimal) degree (MAXD/MIND). (iii),(iv) Choose vertices with maximal (minimal) centrality in the sense of Eq. (1) (MAXC/MINC). (v) Pick a vertex at random (RND). (vi) Do not add (or remove) any edge (NO). The strategies of a vertex is encoded in two sextuples $\mathbf{s}_{\rm add} = (s_1^{\rm add}, \ldots, s_6^{\rm add})$ and $\mathbf{s}_{\rm del}$ representing a priority ordering of the addition and deletion actions, respectively. If $\mathbf{s}_{\rm add}(i) = (\text{MAXD, MINC, NO, RND, MIND, MAXC})$, then i tries at first to attach an edge to the vertex in $\Gamma_2(i)$ with highest degree. If more than one vertex has the highest degree, then one of these is selected by the MINC strategy. If still no unique vertex is found, nothing is done (by application of the NO strategy). Note that the selection procedure always terminated after strategies NO or RND are applied. If $X = \emptyset$, no edge is added (or deleted).

The strategy vectors are initialized to random permutations of the six actions. Every $t_{\rm strat}$th time step a vertex i updates its strategy vectors by identifying the vertex in $\Gamma_i \cup \{i\} = \{j: d(i, j) \leq 1\}$ with highest accumulated score since the last strategy update. Then i copies the parts of $\mathbf{s}_{\rm add}(j)$ and $\mathbf{s}_{\rm del}(j)$ that j used the last time step, and lets the remaining actions come in the same order as the strategy vectors prior to the update. For the purposes of making the set of strategy vectors ergodic, drive the strategy optimization [6] and model irrational moves by the agents [7] we swap, with probability p_s, two random elements of $\mathbf{s}_{\rm add}(j)$ and $\mathbf{s}_{\rm del}(j)$ every strategy vector update. Like the strategy space we also want the network space to be ergodic (i.e., that the game can generate all N-vertex graphs from all initial configurations). In order to ensure ergodicity disconnected clusters should be able to be reconnected. We obtain this by letting a vertex i attach to a random vertex (not just a second neighbor) with probability p_r every $t_{\rm rnd}$th time step. This is also plausible in real socioeconomic networks—even if agents are more influenced by their network surrounding, long-range links can form by other mechanisms (cf. Ref. [8]).

The outline of the algorithm is thus as follows: (1) Initialize the network to an Erdős-Rényi network with N vertices and M_0 edges. (2) Use random permutations of the six actions as $\mathbf{s}_{\rm add}$ and $\mathbf{s}_{\rm del}$ for all vertices. (3) Calculate the score for all vertices. (4) Update the vertices synchronously by adding and deleting edges as selected by the strategy vectors. With probability p_r an edge is added to a random vertex instead of a neighbor's neighbor. (5) Every $t_{\rm strat}$th time step, update the strategy vectors. For each vertex, with probability p_s, swap two elements in a vertex's strategy vector. (6) Increment the simulation time t and, if $t < t_{\rm tot}$, go to step 3. $n_{\rm avg}$ averages over different realizations of the algorithm are performed. We will use parameter values $M_0 = 3N/2$, $p_s = 0.005$, $t_{\rm strat} = 10$, $t_{\rm tot} = 10^5$, and $n_{\rm avg} = 100$ throughout the Letter (preliminary studies, and arguments presented below, give at hand that conclusions do not depend sensitively on these values).

A part of the time evolution of a run of the game is displayed in Fig. 1. Figure 1(a) and 1(b) show the fraction of the agents having a specific main addition ($s_1^{\rm add}$) and deletion action ($s_1^{\rm del}$), respectively. As we can see, the time evolution can be very complex, having sudden cascades of strategy changes. We do not display actions with lower priorities ($s_2, \ldots, s_6$), but we note that they are less clear-cut as they experience a lower selection pressure. Typically the time evolution shows rather lengthy quasistable periods punctuated by outbursts of strategy changing cascades (in both the addition and deletion strategies) as seen in Figs. 1(a) and 1(b). Not all strategies, as we will see later, invade the population. As illustrated in this example, MAXC is the most frequent main action for most parameter values, whereas MINC and MIND (and NO for addition) are rare. From the definition of the actions we anticipate differences in the network structure for time frames of different dominating strategies. This is indeed the case as evident from panels (c), (d), and (e) of Fig. 1 which display the average score $\langle s\rangle$, degree $\langle k\rangle$, and frac-

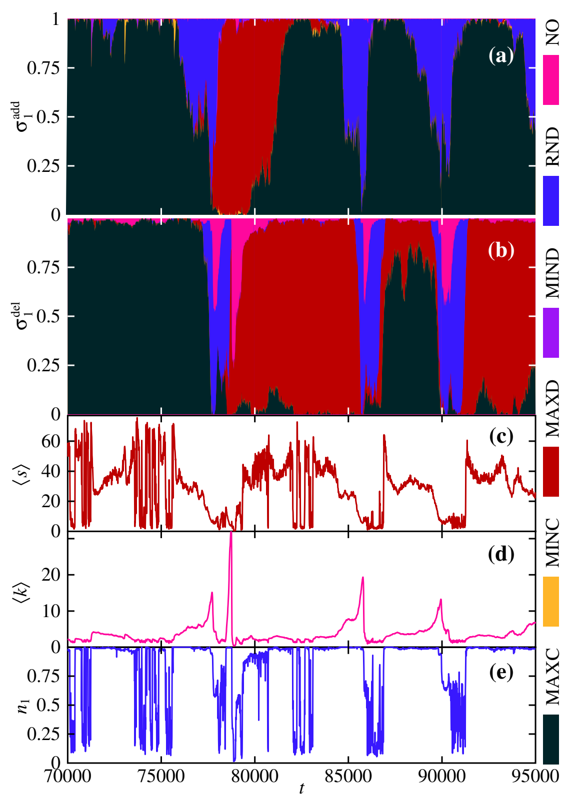

FIG. 1 (color online). An example run for a 200-agent system with $p_r = 0.012$. (a) and (b) show the fraction of vertices having a certain leading strategy for addition (a) and deletion (b), respectively. (c) shows the average score, (d) the average degree, and (e) the relative size of the largest connected component.

tion of vertices in the largest connected cluster n_1. The average score fluctuates wildly, suggesting that states of relative global prosperity are unstable. Likewise the degree has an intermittent time evolution with sudden high-degree spikes and periods of sparseness. Unsurprisingly, the high-degree spikes are located at the outbursts of the NO deletion strategy where edges are not deleted, but only added. The size of the largest connected cluster has an even more dramatic time evolution, fluctuating between fully connected and fragmented states. Note that the sudden drops in n_1 do not always have a corresponding decrease in degree—such fragmentation processes can thus result from rewiring a bridge to another cluster to a vertex in the own cluster. The large fluctuations in network parameters is also an indication that the systems is rather insensitive to initial parameter values.

Note that in Fig. 1(b) the strategies seem to differ in their ability to invade one another; e.g., MAXC is followed by a peak in RND. We investigate this qualitatively by calculating the transition matrix $\mathbf{T}$ with elements $T(s_1, s_1')$ giving the probability of a vertex with the leading action s_1 to have the leading action s_1' at the next time step. Note that the dynamics is not a Markov chain, and thus not fully determined by $\mathbf{T}$. If the current strategy was really independent of the strategy adopted in the previous time step, then we would have the relation $T_{ij} = \sqrt{T_{ii}T_{jj}}$. So we measure the deviation from a Markov-chain null model by assuming the diagonal (i.e., the frequencies of the strategies) and calculating $\mathbf{\Theta}$ defined by

$$\Theta_{ij} = T_{ij}/\sqrt{T_{ii}T_{jj}}. \tag{3}$$

We find that the off-diagonal elements are much lower than 1 (the average off-diagonal Θ values are 0.014 for addition strategies and 0.010 for deletion). This reflects the contiguous periods of one dominating action. Some transitions, for example, between MAXC and RND (as observed above) are much above this average—$\Theta^{\text{del}}_{\text{MAXC,RND}} \approx \Theta^{\text{del}}_{\text{RND,MAXC}} \approx 0.027$. Futhermore, we note that $\mathbf{\Theta}$ is not completely symmetric as $\Theta^{\text{del}}_{\text{RND,NO}}$ is twice (about 3 standard deviations) as large as $\Theta^{\text{del}}_{\text{NO,RND}}$, meaning that it is easier for RND to invade NO as a leading deletion action than vice versa.

To get a more detailed view of the relation between the preferred actions and the structure of the network, we investigate the degree distribution $p(k)$ for different leading actions. We average $p(k)$ over all time steps that more than half of the agents adopt a certain strategy and the n_{avg} network realizations. In Fig. 2(a) we plot the degree distribution for the MAXC dominating addition action. It is conspicuously wide and seems to stay that way as N increases (though the precise functional form in the large-N limit is hard to assess). So despite that the network, at a given time, usually has one dominating strategy; the network structure is highly inhomogeneous. There are

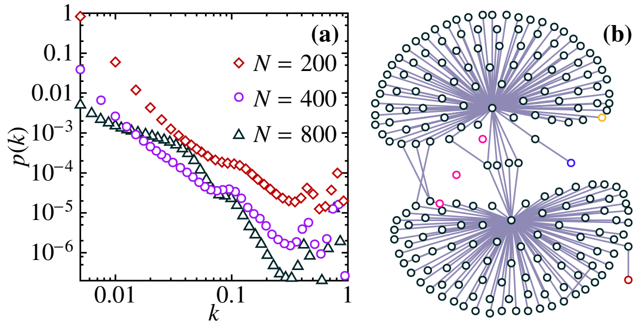

FIG. 2 (color online). The network structure with MAXC as the leading addition action. Parameter values are the same as in Fig. 1. Panel (a) shows the degree distribution for time steps where more than a half of the agents use MAXC as leading action. The curves are log-binned for large degrees. Errors are smaller than the symbol size. Panel (b) shows an example graph for $N = 200$. We emphasize that this is only one of a great variety on network topologies that emerge from the dynamics. The colors of the vertices represent the addition actions as in Fig. 1.

peaks in the degree distribution close to $k \approx 0.4N$ and $k \approx 0.8N$, meaning that the network has one or more hubs of extremely high degree. A snapshot of the network with two hubs, each with degree close to $N/2$ is seen in Fig. 2(b). Such a situation can indeed be rather stable: The most central vertices (the vertices between the hubs) have rather low degree, and thus have a very high score. Since these are in Γ_2 (but not in Γ) of most vertices, these will be the hubs of the next time step, and the old hub will likely be between these. Thus the property of being a hub will effectively oscillate between members of two sets of vertices.

Next we turn to the scaling of the strategy preferences and structural measures with respect to model parameters. In Fig. 3 we tune the fraction of random attachments p_r for three system sizes. In panels (a)–(c) we display the average fraction of leading addition actions among the agents $\langle\sigma_1^{\text{add}}\rangle$. As observed in Fig. 1(a) the dominant strategy is MAXC followed by MAXD and RND. The leading deleting actions, as seen in panels (d)–(f), are ranked similarly expect that MAXD has a larger presence. There are trends in the p_r dependences of $\langle\sigma_1^{\text{add}}\rangle$: As N increases, MAXC gains importance as an addition strategy and MAXD as a deletion strategy—the most intuitive strategies for obtaining high centrality and low degree. But there is apparently no incipient discontinuity in the given parameter range. The average degree, plotted in Fig. 3(g), is monotonically increasing with p_r and decreasing with N (if $p_r \gtrsim 0.1$). For all network models that we are aware of (allowing for fragmented networks) decreasing average degree implies a smaller giant component. In our model the picture is the opposite; as the system grows (for a fixed p_m) the giant component spans an increasing fraction of the network. This also means that the agents collectively reach the twin goals of keeping the degree low and the graph connected.

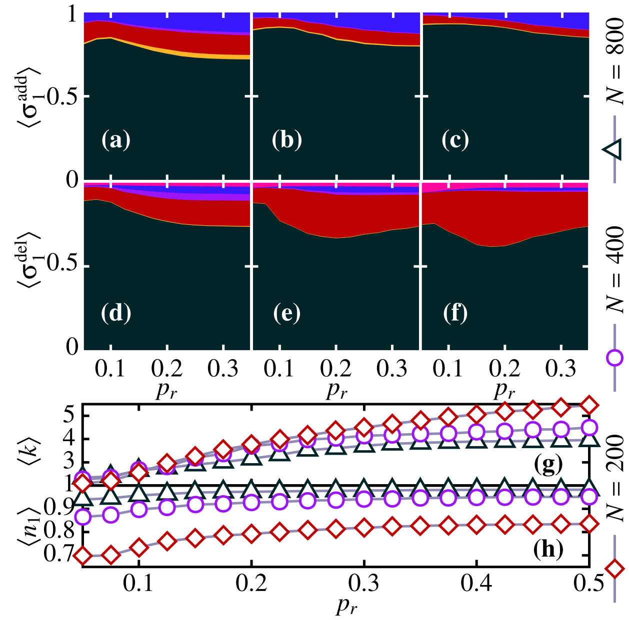

FIG. 3 (color online). The system's dependence on the fraction of random rewirings p_r and system size N. Panels (a), (b), and (c) show the fraction of preferred addition actions $\langle\sigma_1^{\mathrm{add}}\rangle$ for systems of 200, 400, and 800 agents, respectively. Panels (d), (e), and (f) show the fraction of preferred deletion actions for the same three system sizes, while (g) shows the average degree and (h) the average relative size of the largest connected component.

We also note [in Figs. 3(a)–3(f)] that although the system has the opportunity to be passive (i.e., agents having $s_1^{\mathrm{add}} = s_1^{\mathrm{del}} = \mathrm{NO}$), it does not. This is reminiscent of the red-queen hypothesis of evolution [9]—organisms need to keep evolving to maintain their fitness.

To summarize, we have investigated an N-player game of networking agents. The success of an agent i increases with the closeness centrality and the size of the connected component i belongs to, while it decreases with i's degree. Such a situation may occur in diplomacy, lobbying, or business networks, where an agent wants to be central in the network but not at the expense of having too many direct contacts. The dynamics proceed by the agents deleting edges and attaching new edges to their second neighbors according to strategies based on local information. Once every tenth time step the agents evaluate the strategies of the neighborhood and imitate the best performing neighbor to optimize their strategy. As the vertices of our model have no additional traits—their competitive situation is completely determined by their network position—the time evolution of strategies is immediately tied to the evolution of network structure. The evolutionary trajectories (of both strategies and the network structure) are strikingly complex, having long periods of relative stability followed by sudden transitions, spikes, or chaotic periods. One such instability is manifested in a transient fragmentation of the network; this occurs more rarely as the network size increases. In fact, the network gets more connected as size is increased; interestingly this is accompanied with a decreasing average degree—thus, with a growing number of actors the system gets better at achieving the common goal of being connected and keeping the degree low. We also observe that the network dynamics never reaches a fixed point of passivity (where the network is largely static); this suggests a situation similar to the red-queen hypothesis—agents have to keep on networking to maintain their success. We believe network positional games will prove to be a useful framework for modeling dynamical networks, and anticipate much future work in this direction.

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