# Evolutionary Dynamics in Public Good Games

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**Abstract.** This paper explores the question whether boundedly rational agents learn to behave optimally when asked to voluntarily contribute to a public good. The dynamic game is described by an Evolutionary Algorithm, which is shown to extend the applicability of ordinary replicator dynamics of evolutionary game theory to problem sets characterized by finite populations and continuous strategy spaces. We analyze the learning process of purely and impurely altruistic agents and find in both cases the contribution level to converge towards the Nash equilibrium. The group size, the degree of initial heterogeneity and the propensity to experiment are key factors of the learning process.

## 1. Introduction

This paper explores the question whether boundedly rational individuals, who possess little information on the structure of the economy, learn to behave optimally when asked to voluntarily contribute to a public good. The agents engage in a dynamic non-cooperative game. The process of learning and strategy adaptation is modeled by an Evolutionary Algorithm (EA) which belongs to the general class of adaptive learning algorithms and closely relates our analysis to the working principles of evolutionary game theory.[1] In each round, a typical agent of the population reports his willingness to pay, which is his strategic variable. The provision level of the public good is determined, and the individual receives his utility. The player then revises his strategy in order to adapt to the changing environment. This is done by means of *learning by imitation* and *learning by experiments* (cf. Riechmann, 1999).

Strategic incentives in public goods games have at length been studied theoretically as well as in experiments (Isaac et al., 1985; Bergstrom et al., 1986; Andreoni, 1988b).[2] A widely accepted prediction on individual behavior is the Nash conjecture, that is, each person maximizes utility taking other people's behavior as given. The results from standard theory are clear. In incomplete information settings, there are no incentives to reveal individual preferences truthfully, the agents

try to free ride on the contributions of others, and the equilibrium provision level falls short of the Pareto-efficient one.

But do individuals actually *know* that free riding is the best response? If we consider the results from experimental economics, we find that (a) there is no significant evidence for free riding in one-shot games, and (b) there is convergence towards free riding in repeated games although there is no *exact* free riding (see Marwell and Ames, 1981; Isaac et al., 1984; Miller and Andreoni, 1991). So, there is good reason to conclude that standard theory neglects relevant aspects of individual behavior.

Andreoni (1988b) studies strategic interaction and learning in public goods experiments. He finds that the evidence from experimental data does not provide strong support neither for the prediction that agents act strategically nor do they successfully engage in learning processes. From this Andreoni (1988b) concludes that the individual behavior in laboratory experiments is driven by other factors such as altruism, social norms or bounded rationality.

Models of Bayesian learning (Bliss and Nalebuff, 1984; Fershtman and Nitzan, 1991; Gradstein, 1992) or approaches employing non-Nash conjectures (Sugden, 1985) – in our view – share the important shortcoming of endowing individuals with rather sophisticated information processing capacities. We drop the assumption of common knowledge, that is, every player in isolation has full knowledge of the relevant data and can costlessly figure out the equilibrium. Instead, our analysis relies on the assumption of *boundedly rational* agents who are involved in an adaptive learning process. Basically our model consists of a repeated game, where subjects report their individual valuation of the public good. The agents are neither informed on preferences nor on endowments of their co-players. We think that in general this layout corresponds to the standard design of experiments. It is a dynamic approach, because a single-shot game is not sufficient to allow subjects to learn the incentives, or following Andreoni (1988b, p. 292) "*Repetition appears to be necessary for subjects to approach free riding.*"

We draw from Miller and Andreoni (1991), who describe free riding as an outcome of an evolutionary game. The replicator dynamics of the adaptive learning process strengthen those strategies over time that perform well. Their findings support results from experiments on voluntary contribution to a public good. In their simulations the provision of the public good converges towards exact free riding. Furthermore, they demonstrate that convergence is delayed with an increase in group size, which also replicates results from experiments.

Unfortunately, an important shortcoming of replicator dynamics is that no strategy can be learned which differs from the ones already contained in the initial population. This, in general, turns out to be a severe problem in games with a continuous strategy space, for instance in the 'ultimatum game', the 'Cournot game', or 'rent-seeking games'. The purely imitative learning of replicator dynamics cannot develop any *new* strategies or recover those which have been wiped out. If free riding in the public good game is not part of the initial set of strategies, it cannot

be learned, and if the game starts with a homogeneous population, there will be no learning progress either. Miller and Andreoni (1991) cannot give a plausible explanation for an important result from free riding experiments, namely, that *exact* free riding is hardly observed.

For this reason we model the learning process by means of an Evolutionary Algorithm which displays two important features: first, the selection / reproduction mechanism reflecting an imitative strategy adoption generates learning dynamics which (in expectation) are identical to the replicator dynamics of evolutionary game theory, while simultaneously allowing for a continuous strategy space. Second, EAs also consist of a mutation operator, which – from a behavioral viewpoint – reflects the agent's propensity to (randomly) modify already existing strategies. By mutation, the agents can develop new strategies or recover those already lost at an earlier stage of the game. The EA we use provides results which in this respect are closer to the experimental findings.

We discuss a general model allowing for alternative preference specifications which vary from a pure public good to a purely private good, thereby also capturing the notion of impure altruism, where the agents receives additional utility, a so-called *warm glow*, from his donation to the public good (Andreoni, 1988a, 1990).

Our analysis will proceed as follows. Section 2 provides a short review of the underlying assumptions and corresponding results of the static game of voluntary contribution to a public good with incomplete information. Section 3 derives general results for the dynamic performance of strategies. Section 3 also describes the basics of the model of EA-learning, analyzes the learning dynamics, and establishes a link between evolutionary game theory and the theory of evolutionary algorithms. We examine two major principles of social learning models, i.e. learning by imitation and learning by experiments. Section 4 presents the simulations. Section 5 concludes and gives an outline for future research.

## 2. A Benchmark: Nash Equilibrium with Imperfect Information

Consider the standard model of voluntary contribution to a public good (cf. Varian, 1994; Cornes and Sandler, 1996, Ch. 6). Individual preferences and endowments are common knowledge, so there is no uncertainty regarding the strategies of other agents. Each individual's utility maximizing choice will depend on everyone else's. A Nash equilibrium is an allocation where each individual's chosen contribution is a *best response* to the other's. Typically, the Nash equilibrium is not Pareto-efficient. The equilibrium level of the public good falls short of the socially optimal one, which is referred to as *systemic free riding* (Cornes and Sandler, 1996, p. 159). In the particular case of quasi-linear preferences, the extent of free riding increases with the group size (Olson, 1965).

A natural extension to this basic model is to relax the requirements on individual knowledge and to allow for private information. It is easy to demonstrate that this adds to the problem of free riding such that *informational free riding*

(cf. Cornes and Sandier, 1996, Ch. 7.4) can be observed. We consider a simple model with $n \geq 2$ consumers, one private good $x$ and one public good $G$. Each agent is endowed with exogenous wealth $w_i$. The individual consumer participates in a Lindahl *tâtonnement* process and divides wealth between private consumption $x_i$, and a contribution to the public good $b_i = \theta_i G$, such that $w_i = x_i + b_i$. The individual cost share $\theta_i$ corresponds to the personalized Lindahl price. For simplicity, we normalize the price of the private good, the public good, and marginal costs for both goods to unity. In order to capture so-called *warm glow*-effects from imperfect altruism, we also assume that the agent receives utility from the pure 'act of giving'. The utility function of consumer $i$, $U(x_i, b_i, G)$, takes on the general quasi-linear form

$$U(x_i, b_i, G) = x_i + \beta_i \ln \left( b_i^\alpha G^{1-\alpha} \right), \tag{1}$$

$G$ represents the (aggregate) level of the public good, which is simply the sum of all individual contributions, $G = \sum_{i=1}^{n} b_i$. Denoting the aggregate contribution of all agents but agent $i$ by $G_{-i}$, $G_{-i} = \sum_{j=1, j \neq i}^{n} b_j$, the utility function can be rewritten as

$$U(x_i, b_i, G) = x_i + \beta_i \ln \left( b_i^\alpha (G_{-i} + b_i)^{1-\alpha} \right), \tag{2}$$

Note that the quasi-linear form of the utility function allows for the exclusion of income effects. The parameters $\alpha$, $1 - \alpha$, denote the weights the agent assigns to the private and the public part of his contribution. The model reduces to a private good model in case of $\alpha = 1$ and becomes a pure public good model for $\alpha = 0$. Due to the logarithmic part of the utility function, we have to restrict the agents' strategy spaces to strictly positive contributions: $b_i > 0 \forall i$.

The general form of $U(x_i, b_i, G)$ is known to all agents, while the individual valuation of the public good, $\beta_i > 0$, is private information, such that the model becomes a model of imperfect information. In what follows, we refer to $\beta_i$ as the *true* value of the preference parameter to distinguish it from the actually reported value $b_i$.

Substitution of the budget constraint enables us to write the consumer's maximization problem for an arbitrary report $b_i$ as

$$\max_{b_i} \quad w_i - b_i + \beta_i \ln \left( b_i^\alpha G^{1-\alpha} \right), \tag{3}$$

while taking the actions of others as given. From this, the individual reaction function is implicitly determined by the first-order condition, where $b_i^*$ denotes agent $i$'s best response

$$b_i^{*2} + (G_{-i} - \beta) b_i^* - \alpha \beta G_{-i} = 0. \tag{4}$$

Consequently, the only admissible symmetric Nash equilibrium follows as

$$b_i^* = \frac{\beta_i}{n}[1 + (n-1)\alpha] \,\forall i. \tag{5}$$

This result is straightforward: Equation (5) indicates the well-known result that, as long as the good has at least some properties of a public good, i.e. as long as $\alpha < 1$, it is utility maximizing for consumer $i$ to significantly underreport the true value $\beta_i$. An announcement that preference data are used for the purpose of estimating Lindahl prices creates incentives to decrease voluntary contributions so as to understate true demands. In short, the agent has an incentive to misrepresent preferences, in order to get a lower personalized Lindahl price $\theta_i$. Since this result applies to each of the $n$ agents, the level of public good provision will be too low, if it is determined in accordance with the reported valuations. So, besides the systemic type of free riding, we additionally observe informational free riding.

In this context, two benchmark cases are of special interest: In case of a pure public good ($\alpha = 0$), the single possible symmetric Nash equilibrium is characterized by a common individual contribution of $b_i^* = \beta_i/n$ for all $i$. In the opposite case of $\alpha = 1$, the good $G$ in focus is a private good, such that the respective symmetric individual Nash strategy is to truthfully reveal the own preferences, i.e. $b_i^* = \beta_i \,\forall i$.

The one-period game is easily extended to a repeated non-cooperative game. The Nash prediction continues to hold, and each round is an exact replication of the one-period game: Free riding is the best strategy for sufficiently large populations. Although we may drop the assumption of full information with respect to the (possibly heterogeneous) form of the opponents' utility functions and endowments, the agents of this model must at least be fully informed on the reports of the opponents $b_{-i}$ and the aggregation rule for $G$. Otherwise, agents were not able to compute the aggregate provision level of the public good and their corresponding cost shares $\theta_i$.

The repeated game will be the starting point for our learning model of the next chapter, but with one major difference. To capture the notion of *bounded rationality*, the agents of the evolutionary model are provided with as little *ex ante* information as possible regarding the structure of the game. They possess no knowledge with respect to other agents preferences, endowments, or decision rules. The only information they acquire is related to the *ex post* performance of the other agents' strategies compared to their own.

## 3. Dynamic Performance of Strategies

In this paper, we will use evolutionary algorithms (EAs) to model the dynamics of the repeated public good game described above. Interpreting the resulting dynamics as evolutionary processes of social learning, one standard approach to the analysis

of these phenomena is the application of replicator dynamics of evolutionary game theory (Vega-Redondo, 1996; Weibull, 1995). The reasons why we make use of evolutionary algorithms – instead of replicator dynamics – are easy to name. EAs display the typical evolutionary dynamics. In fact, their expected time paths are identical to the time paths of models applying ordinary replicator dynamics. But at the same time, EAs possess some advantages over replicator dynamics when it comes to the analysis of public goods problems.

First, ordinary replicator dynamics are restricted to the domain of infinitely large populations. These dynamics trace the development of population shares of strategies over time. Translating these population shares back into absolute numbers of agents playing the respective strategies will, most probably, not lead to integer values as long as the total number of agents is finite. Infinitely large populations are not of concern here, as we intend to analyze the effect of the population size on both, the dynamics and the outcome of the learning process.[3]

Second, ordinary replicator dynamics are restricted to the domain of discrete strategy spaces. Indeed, the leading principle of replicator dynamics is that they allow for the detailed tracing of the development of the population share of every single strategy over time. Consider, for instance, a discrete strategy space of $z$ many strategies. Then, a model relying on replicator dynamics is represented by a system of $z - 1$ simultaneous difference equations.[4] Here, the analysis reaches its natural limit for a continuous (i.e. infinitely large) strategy space. Even for a discrete strategy space, if it is sufficiently large, the solution is probably computationally more costly compared to the use of evolutionary algorithms.[5] However, the above described model of public good provision, allows for the choice of an arbitrary (positive) individual contribution, thereby making it a model with a continuous strategy space and excluding the use of ordinary replicator dynamics.

Third, ordinary replicator dynamics reflect the dynamics of pure imitation processes. They leave no room for the application of really *new* strategies, that is, strategies which have not been used before. In order to trace the development of any strategy of the strategy space, replicator dynamics require that each strategy is already represented in the initial population. This property can also be responsible for the common observation that, ultimately, the ordinary replicator process will lock-in in a state which does not necessarily represent a Nash-equilibrium of the underlying game.

Evolutionary algorithms possess none of the above mentioned shortcomings of ordinary replicator dynamics: they can (and, more than that, need to) deal with finitely large populations, they can cope with continuous strategy spaces, and they include a kind of mutation operator, preventing the process from locking-in into non-Nash states of the society.

In what follows, we give a short sketch of the evolutionary forces underlying the dynamic performance of strategies in an evolutionary algorithm. For simplicity, the $n$ agents are assumed to be identical with respect to their preferences and

endowments, such that $\beta_i = \beta$ and $w_i = w$, $\forall i = 1, \ldots, n$. We posit preferences, endowments and the population size to be unchanging over time.[6] In each period $t$, individual $i$ receives an endowment $w(t)$ and chooses a report $b_i(t)$ from the continuous set of feasible contributions $\mathcal{B} = \{b | 0 < b \leq w\}$.

Learning now involves that – as time goes by – the agent replaces poorly performing strategies by those performing well. But what makes one strategy perform better than another, and how does this information spread throughout the community? To answer this question, we will now focus on the dynamic evolution of an arbitrary strategy of time $t$, say the contribution level $b^m \in \mathcal{B}$. Consider a group of agents, each deciding to report the contribution $b^m$ in $t$. We identify these agents as 'type-$m$ agents'. The time-$t$ population share of type-$m$ agents will be denoted by $q^m(t)$. In $t$, the utility of a type-$m$ agent signaling a contribution $b^m$ is given by

$$U^m(t) = w(t) - b^m + \beta \ln((b^m)^\alpha G(t)^{1-\alpha}). \tag{6}$$

Equation (6) displays the well-known result that the single agent's utility is not only affected by his own action $b^m$. It is additionally determined by the strategic decisions of all members of the population which are reflected in the aggregate contribution level of the public good. Since $U^m(t)$ is a state-dependent variable, it changes over time as the agents revise their strategies.[7]

## 3.1. LEARNING CONCEPTS

Let us start with a short non-technical sketch of the two learning concepts *learning by imitation* and *learning by experiments* (cf. Riechmann, 1999), before going into the technical details of the learning dynamics. The first step in imitation learning is that a type-$m$ agent employs the available information regarding individual strategies and associated utilities in order to evaluate the success of his own strategy $b^m$. At the end of each round of the game, the individual compares the utility $U^m(t)$ he receives to the utility of other players, who possibly played different strategies. In a second step, he decides for period $t + 1$ as to whether or not to adopt an alternative strategy which just has proven to be more promising than the one he used previously. It is important to stress that no new strategies are generated within this process of imitative learning.[8]

The *new* strategies come into the population via experiments. The term *experiment* stands for a small-scale change in the strategy. The subject adds (subtracts) a small amount to (from) the chosen next-period contribution $b_i(t + 1)$ before entering the new round of the game. This increases the variety in the pool (distribution) of available strategies and is the only way leading out of strategic lock-ins.

The evolutionary algorithm we use to model the learning process deviates only little from the general design of genetic algorithms (Holland, 1975, 1992; Goldberg, 1989). In fact, the major difference is the real-valued coding. The algorithm consists

of two genetic operators: selection/reproduction and mutation, where the first one reflects *learning by imitation* and the second one reflects *learning by experiments*.[9] In what follows, we separately discuss the consequences of the two learning concepts for the dynamics of strategy choice and the resulting provision level of the public good. We focus on the similarities between evolutionary game theory and the theory of evolutionary algorithms (cf. Riechmann, 2001a).

## 3.2. LEARNING BY IMITATION

The basic operator of the EA employed here is the operator of selection and reproduction which is the same as the 'biased roulette wheel' operator from the canonical genetic algorithm (CGA) introduced by Holland (1975, 1992) and Goldberg (1989). At each time $t$, the population of the evolutionary algorithm consists of the strategies $b_i(t) \in \mathcal{B}$ played by the $n$ agents of the economy. Note that in the terminology of genetic algorithms, here the population is not the set of agents but the set of strategies.

Members of a time-$t$ population are transferred into the next-period one by means of selection and reproduction. The procedure is conducted a follows: A strategy is drawn randomly (with replacement) from the old population of strategies and copied into the new one. Each strategy has a certain probability of being selected and reproduced, which depends on its performance quality in the previous round of the game. This introduces state-dependency into our model. We use the *relative fitness* as a quality index for a strategy $b^m$, which is defined as the ratio of a type $m$ – agent's fitness $U^m(t)$ from (6) to the *aggregate fitness* of the whole population $U(t) = \int_{m|b^m \in B} n^m(t) U^m(t) \, dm$, where $n^m(t)$ denotes the number of agents currently using strategy $b^m$. We assume that the selection and reproduction probability equals the agent's *relative fitness*, which is a standard procedure in the theory of genetic algorithms. Thus, a higher relative fitness leads to a higher chance for a certain strategy of entering into the next population of strategies.

Selection/reproduction works in a way that the expected number $E[n^m(t+1)]$ of agents using strategy $b^m$ in period $t+1$, is determined by the following nonlinear stochastic difference equation[10]

$$E[n^m(t+1)] = \frac{U^m(t)}{U(t)} n^m(t) \, n. \tag{7}$$

The above equation (7) can be rewritten in a form explicitly showing the development of the *share* of agents who play the respective strategy. Obviously, the relative frequency $q^m(t)$ of agents playing strategy $b^m$ in period $t$ and the average utility $\bar{U}(t)$ are defined as

$$q^m(t) := \frac{n^m(t)}{n} \quad \text{and} \quad \bar{U}(t) := \frac{U(t)}{n}. \tag{8}$$

Using (8), it is now easy to demonstrate that the expected growth rate of a population share $q^m$ for a selection/reproduction scheme according to relative fitness can be written as

$$\mathrm{E}\left[\frac{q^m(t+1) - q^m(t)}{q^m(t)}\right] = \frac{U^m(t) - \bar{U}(t)}{\bar{U}(t)}. \tag{9}$$

The more successful a strategy was in the last period, the more likely it is to be imitated in the current one and its expected population share grows. Note that these *replication dynamics* or dynamics of imitation of (9) are structurally equivalent to the replicator dynamics of evolutionary game theory. By the *law of large numbers*, the EA-replication dynamics of selection/ reproduction converge towards the replicator dynamics of evolutionary game theory as the population size approaches infinity. For finite populations, EA-selection and reproduction provides an approximation for replicator dynamics. Note also that the result regarding the probability of strategies being imitated does not predict that the speed of learning increases, while superior strategies spread throughout the society, because the expected growth rate of the fraction $q^m(t)$ of people playing $b^m$ itself depends on the evolution of the entire population.

The agent based simulation of the problem presented here will contain the operator of CGA selection/ reproduction as described above. We already mentioned in the beginning that the behavioral interpretation of this operator is the one of learning by imitation: Agents who recognize that their strategy was comparably less successful – or, alternatively was below average – discard their strategy and imitate another.

However, learning by pure imitation suffers from a severe drawback (cf. Riechmann, 1999): Strategies not contained in the very first population can never be learned, and strategies wiped out throughout the learning process can never be recovered. This implies that learning by pure imitation is a process which is highly path-dependent. In contrast to replicator dynamics, here, learning by imitation includes the case that even superior strategies may die out due to the random element in the selection/reproduction operator. From this follows immediately that the learning process might eventually *lock-in* at uniform contribution levels far away from the Nash-strategy as given in (5). This phenomenon is usually called *genetic drift*. We will come back to this point in the discussion of the simulations.

## 3.3. LEARNING BY EXPERIMENTS

An individual learning process purely relying on imitation is indeed not a very comprehensive notion of 'learning', since it completely ignores the creative part of the issue. For this reason we introduce the mutation operator as a means of *learning by experiments*.

Whereas the ordinary replicator dynamics do not possess elements of experimental learning, more elaborate variants, however, include forms of 'noise' in order to

prevent the dynamics from possible lock-in effects. Evolutionary, replicator-like dynamics including noise are described in two pioneering papers on Nash-refinements by Kandori et al. (1993) and Young (1993), who introduce the notion of stochastic stability.

The basic idea underlying the individual behavior during the step of learning by experiments is the following. The agent chooses a preliminary contribution level for the public good. We denote this preliminary level with $\tilde{b}_i(t)$. For an easier understanding, imagine that selection/ reproduction has already taken place and the preliminary level $\tilde{b}_i(t)$ is the starting point for the development of a new strategy $b_i(t)$. The value $\tilde{b}_i(t)$ is then subject to a stochastic refinement. A random variable is added to the individual contribution to the public good. The resulting strategy $b_i(t)$ is the one, the agent pretends to be his true demand for the public good in the next iteration of the game.

There are two ways to model the experimental process. We will call them *standard mutation* and *meta-mutation*. The first stands for a somewhat *naïve* view on experiments. The individual propensity to experiment remains unchanged over time. The second concept captures the notion of a *rule of thumb*, namely that experiments are intensified if they have proven to be successful and *vice versa*. In this case, the mutation rate is endogenized.

(a) *Standard mutation*: An agent's final strategy $b_i(t)$ is derived by adding a term $\varepsilon_i(t)$ to the agent's preliminary time-$t$ strategy, $\tilde{b}_i(t)$, where $\varepsilon_i(t)$ is a random number drawn from a Gaussian distribution with zero mean and a finite variance $\sigma^2$ [11]

$$b_i(t) = \tilde{b}_i(t) + \varepsilon_i(t), \quad \text{with} \quad \varepsilon_i(t) \sim \mathcal{N}(0, \sigma^2). \tag{10}$$

By using a Gaussian distribution for the *size of the experiment*, $\varepsilon_i(t)$, we sustain two plausible behavioral facts: Experiments are equally likely in any direction from the original strategy, and small experiments are more likely than big ones. We interpret the mutation variance $\sigma^2$ as the *propensity to experiment* (Riechmann, 2001b). The parameter $\sigma$ is a measure for the extent of experimentation an agent is willing to undertake. It is assumed to be identical for all agents and constant over the learning process. [12]

(b) *Meta-mutation*: Standard mutation implies by assumption that the experimental process has a constant diffusion. If we instead assume that the propensity to experiment is subject to a learning process itself, we allow the agent to decide on the variability of experiments he is willing to conduct. The main idea in this context is that the propensity to experiment should be smaller the closer the agent is to his optimal strategy. The notion of *learning to learn* by means of meta-mutation dates back to Bäck (1992a,b) and Bäck and Schütz (1996). It underlies the identical working principles as mutation itself. The propensity to experiment undergoes a random change which again is subject to a Gaussian distribution with zero mean and finite variance. In order to avoid infinite regressions in

mutation variances the stochastic process of meta-mutation is given by[13]

$$b_i(t) = \tilde{b}_i(t) + \varepsilon_i(t), \quad \text{with} \quad \varepsilon_i(t) \sim \mathcal{N}\left(0, \sigma_i^2(t)\right), \tag{11}$$

$$\sigma_i^2(t+1) = \sigma_i^2(t) + \eta_i(t), \quad \text{with} \quad \eta_i(t) \sim \mathcal{N}\left(0, \sigma_i^2(t)\right). \tag{12}$$

The process of meta-mutation is self-referential and thus highly path-dependent. Note that by the index $i$ the propensity to experiment now is a personalized variable of agent $i$. Endogenizing the mutation rate alters the coding of the genetic individual. This now contains two elements which may be inherited and modified by selection/ reproduction and mutation: $b_i, \sigma_i^2$.

With the introduction of mutation we simultaneously establish a way out of the local optimum or *lock in*-dilemma discussed above. With a positive propensity to experiment, there is always a positive probability that local optima will be left in finite time, and that already wiped-out strategies may be regained and possibly turn out to be superior when evaluated in an advanced stage of the game.

## 4. Simulations

So far, what do we know about the learning of optimal strategies in the case of voluntary contributions to a public good? The theoretical analysis of the preceding sections has shown that (a) superior strategies have better chances to survive and spread throughout the society, (b) non-astonishingly, a superior strategy is to behave as closely as possible to the Nash strategy, (c) purely imitative learning does not necessarily lead to the Nash strategy, and (d) the learning concepts act together in such a way, that an individual may learn to play Nash, even if the respective strategy was totally unknown to the players in the outset.

Those theoretical observations now have to pass the empirical test. We conduct numerical simulations to underline our findings and are especially interested in the following questions: First, how does the number of agents in the game affect the learning process? Second, what is the impact of initial heterogeneity? Third, how does the propensity to experiment affect the convergence properties of the learning process? Fourth, does the learning process display any regularities? Fifth, how do different degrees of impure altruism affect the learning process? Last, to what extent does convergence towards the Nash-behavior occur?

In order to analyze the more technical questions like the impact of the degree of initial heterogeneity in the population of agents, the group size and the propensity to experiment, we consider the special case of a pure public good, while later on, we return in our simulations to the more general case of different intensities in impure altruism.

The simulations are based on the evolutionary algorithm as described in the previous section. In order to derive results for the sensitivity of the learning process with respect to the group size, we performed simulations with $n = 50, n = 200$

and $n = 500$ agents respectively. Of course, the population size does not affect the true valuation $\beta$ or initial endowments w. Those parameters were assumed to be constant and identical across agents. The parameter settings are $w = 100$, and $\beta = 20$. The symmetric Nash equilibrium of the private-information economy is given by (5). In the case of a pure public good ($\alpha = 0$) this implies individually optimal contributions of $b^* = 0.4$ for a group of 50 agents, $b^* = 0.1$ for 200 agents, and $b^* = 0.04$ for 500 agents. The individually optimal contributions asymptotically converge to complete free riding for an increasing group size.

Each simulation consists of 50 000 periods, such that each single period corresponds to one instance of the play. The simulation results are rendered in the subsequent figures. We contrast the learning dynamics of a single simulation with the averaged learning dynamics of 50 simulations.

## 4.1. GROUP SIZE AND HETEROGENEITY

Figure 1 displays the society's average contribution to a pure public good for three settings. These differ with respect to the group size which we varied from $n = 50, n = 200$ to $n = 500$, while assuming standard mutation with a propensity to experiment set to $\sigma = 0.03$.

Regarding the initial population, we consider two different settings. The first is characterized by a homogeneous initial population. It is motivated by the idea that there is no significant free riding in the first period of a repeated game in public good experiments (Marwell and Ames, 1981). For this reason we assume our agents initially to be naïve or some kind of good-natured. They reveal their preferences truthfully in the initial period, so we give them a reason for learning to be selfish. With identical agents, our initial group composition is uniform with $b_i(0) = 20, \forall_i = 1, \ldots, n$.[14]

We contrast this setting with an identically specified model, except for assuming heterogeneous agents in the outset. Heterogeneity is obtained by randomizing the initial contributions. The $b_i(0)$ are i. i. d. and bounded in the interval $0 < b_i(0) \leq w$.

Figure 1 displays the simulation results. Figure 1(a) and (c) plot a single time-path of mean contributions in the economy while Figure 1(b) and (d) show the average of 50 simulations. The single time-path is more erratic due to the effects of mutation which cancel out by (10) to (12) when the number of simulations or the group size increases.

Independent of the population size, the contributions converge towards the respective Nash-equilibria in the long-run. Nevertheless, Figure 1 shows that convergence requires a considerable amount of time. Convergence is delayed with an initially homogeneous distribution of strategies in the population which does not give way to imitative learning. This becomes especially obvious in Figures 1 (a) and 1 (b) for the learning dynamics of the smallest group of 50 members. The learning progress can only be achieved by experiments. Once there is some heterogeneity in the population, imitation is effective and improves the learning process.
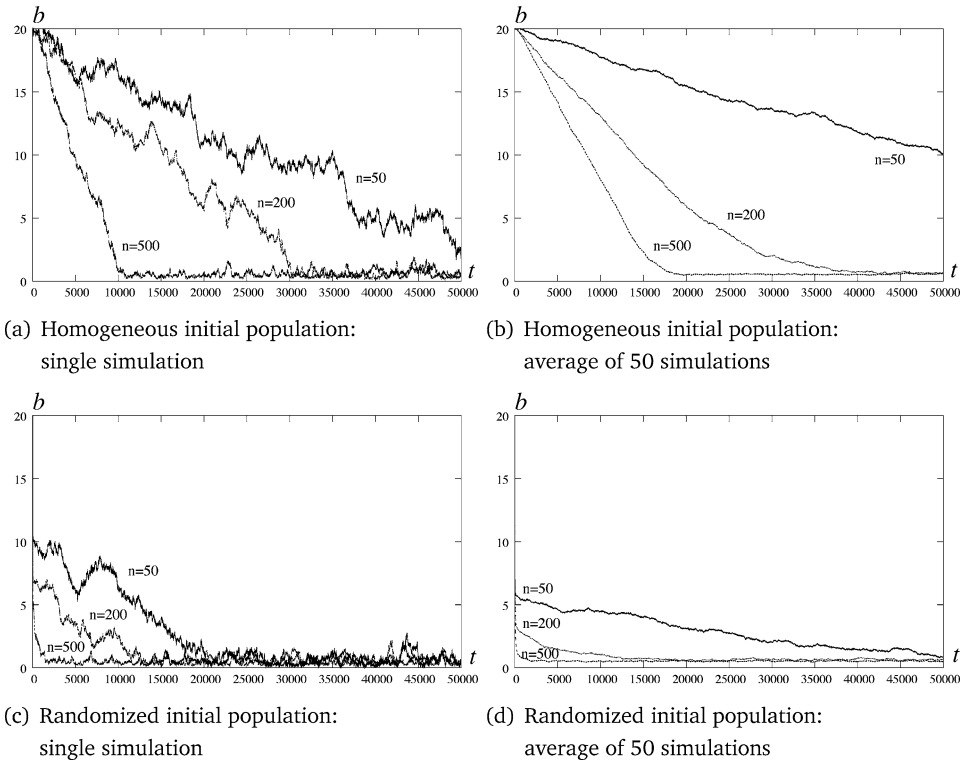
(a) Homogeneous initial population:
single simulation

(b) Homogeneous initial population:
average of 50 simulations

(c) Randomized initial population:
single simulation

(d) Randomized initial population:
average of 50 simulations

*Figure 1.* Convergence and population size, $n = 50$, $n = 200$, and $n = 500$.

This becomes obvious in the case of a randomized initialization. Convergence occurs within a fraction of time required by the uniform-population algorithm. Our results are a direct application of the so-called *Fisher principle* (Metcalfe, 1994; Birchenhall et al., 1997) which says that learning occurs faster the more heterogeneous a population is.

Learning is improved with an increase in the group size. Larger populations converge considerably faster than smaller ones. This effect can primarily be attributed to the propensity to experiment. The chance (i.e. probability) of finding an agent, who carried out a comparably large and successful experiment is greater in a large population than in a small one. Thus, we expect the effects of experimentation to be more significant in larger groups. This outcome contradicts the results from laboratory experiments on voluntary contributions to a public good (see Miller and Andreoni, 1991, and references therein), where the speed of convergence was found to be inversely related to group size.

How can these contradictory results be explained? We believe that additional factors have to be taken into account. The group size ranged from 4 to 100 agents in the laboratory experiments cited above. In such small groups, an agent receives a comparably stronger feedback to his contribution than in larger ones. The effects of

his own actions on the aggregate provision level of the public good $G$ are no longer negligible. For this reason, we think that the agent might be inclined to respond stronger to payoff signals by undertaking more or bigger experiments in order to find out the optimal strategy. We suggest that in small groups the propensity to experiment, $\sigma^2$, should be higher than in large ones.

## 4.2. PROPENSITY TO EXPERIMENT

In order to analyze the effects of experimentation we perform simulations with the two types of mutation as described above. For standard mutation, the population is assumed to be initially homogeneous and comparably small. We set $b_i(0) = 20$ and $n = 200$, and let the mutation variance vary according to the following values $\sigma^2 \in \{0.01, 0.02, 0.03, 0.05, 0.1\}$.

Figure 2 (a) displays single runs of the society's mean contribution level while Figure 2(b) shows the average of 50 simulation runs. Both Figures demonstrate the expected result, that a high propensity to experiment accelerates the convergence process. For instance, a comparably large propensity to experiment of 10% has the advantage of rapidly leading to a heterogeneous population which promotes the process of imitative learning. But it also has the disadvantage as to that it hinders convergence once the individual strategies have come close to their optimal level. The agents do not stop undertaking large experiments. Especially from Figures 2(b) it becomes obvious that the time-path is more erratic than with lower propensities to experiment, and that the average contribution stays significantly above the symmetric Nash equilibrium of $b_i^* = 0.4$. On the contrary, a small propensity to experiment like 1% hardly leads to any learning progress since it does not carry enough heterogeneity into the population.

The simulation results from meta-mutation are more convincing. Figure 2(c) and (d) contrast the evolution of strategies in the case of standard mutation ($\sigma^2 = 0.03$) with meta-mutation for initially homogeneous agents. Meta-mutation takes account of the need to intensify experiments when strategies are uniform and far away from their optimal level on the one hand, and, on the other hand, to experiment less when the optimum is almost reached. This in general improves the learning process, which converges faster than in the case of standard mutation. Figure 2(c) displays the typical jumps in learning progress in the presence of meta-mutation. It thus reflects the process of information *exploration* and *exploitation*, which is a prominent feature of evolutionary algorithms, in almost an ideal manner. The big jumps as well as the long-lasting deadlocks of the learning process express the self-enforcing characteristics of meta-mutation, where periods of intense experimentation take turns with primarily imitative phases.

Figure 2(e) and (f) emphasize this interaction of imitative and experimental learning and show characteristic regularities of the learning process. By starting from a randomized initial population of $n = 200$, we contrast pure replication dynamics, that is $\sigma^2 = 0$, with standard-mutation at the comparably low level
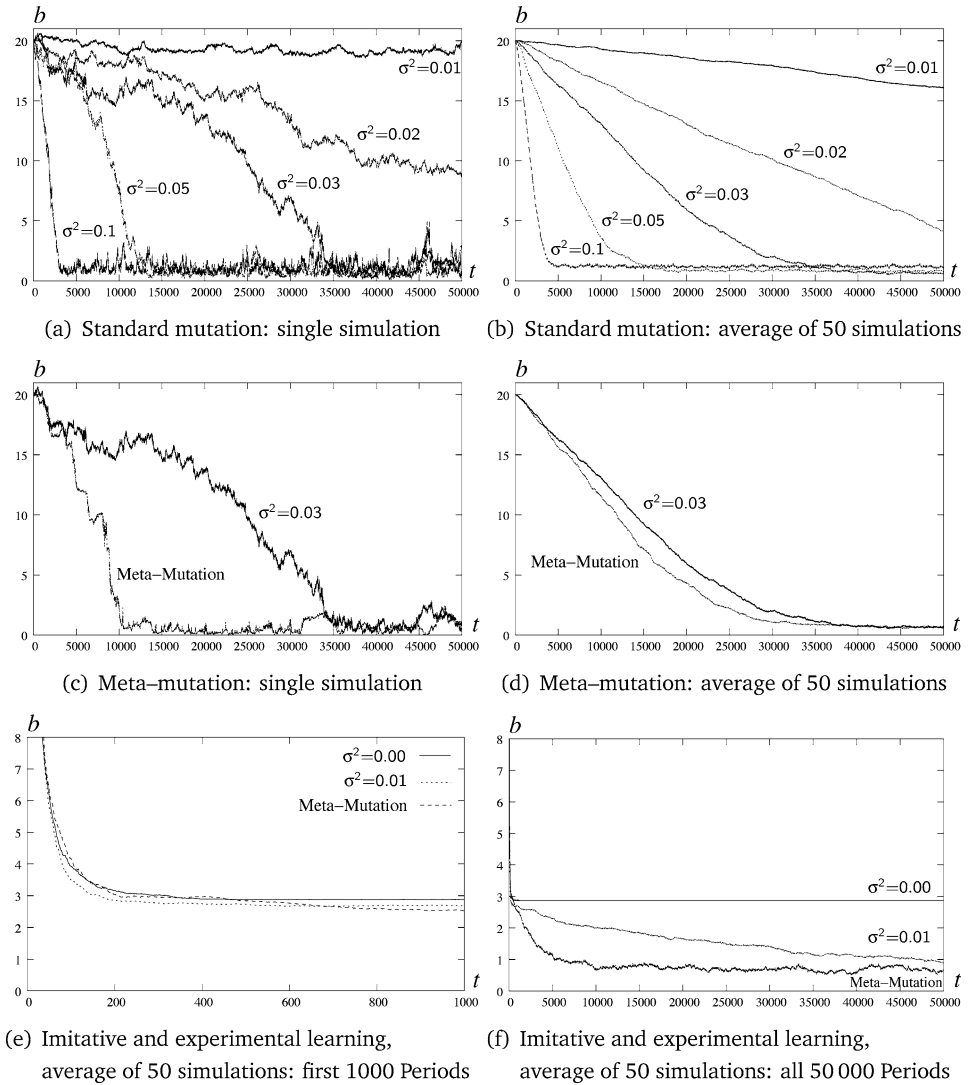
(a) Standard mutation: single simulation

(b) Standard mutation: average of 50 simulations

(c) Meta–mutation: single simulation

(d) Meta–mutation: average of 50 simulations

(e) Imitative and experimental learning, average of 50 simulations: first 1000 Periods

(f) Imitative and experimental learning, average of 50 simulations: all 50 000 Periods

*Figure 2.* Convergence and the propensity to experiment.

$\sigma^2 = 0.01$ and meta-mutation.[15] The Figure 2(e) and (f) display the average of 50 simulation runs.

The society's mean contributions for replicator dynamics inevitably lock-in. The lock-in levels of the simulations depend on the respective initial distribution of strategies within the population and are the results of genetic drift. There are two possible reasons for this phenomenon. Either, the optimal strategy is not contained in the initial set of strategies. Since there is no mutation, better strategies cannot be found. Or, a superior strategy has been eliminated in the very beginning

of the learning process due to the random element in the selection operator. At this point it is important to remember that replication dynamics by means of selection/reproduction follow a stochastic process and equal replicator dynamics only in expectation. Again, since there is no mutation, those strategies cannot be recovered. Learning then reduces to pure imitation. The learning rate is declining and the process comes to a standstill after approximately 500 iterations of the game.

The jump from imitative to experimental learning can be illustrated very nicely in the case of a random initial population combined with a low propensity to experiment. Imitation drives the learning progress and leads to an almost homogeneous population within the first 200 periods (see Figure 2(e), where the learning progress rapidly slows down after approximately 200 periods). Afterwards, progress is only achieved by experiments which are conducted at a comparably small level. This leads to the kinked time-path of average contributions in Figure 2(f). We find the low propensity to experiment to delay convergence if compared to meta-mutation. The learning process under meta-mutation is characterized by a continuous progress, although at a declining learning rate. We will come back to this point later, when we discuss the extent of convergence.

### 4.3. IMPURE ALTRUISM

The qualitative result we expect from the assumption of exclusive utility received from individual donations is: A Nash equilibrium is characterized by a situation where each agent significantly offers a positive contribution to the public good. The amount crucially depends on the preference weight $\alpha$. For instance, truthful revelation of $\beta$ is the dominant strategy, if the contribution is a private good. We simulated the learning process with a group size of $n = 200$, $\sigma^2 = 0.03$, and let the preference parameter vary according to the values $\alpha \in \{0.25, 0.50, 0.75, 1\}$ with the corresponding symmetric Nash equilibria of the private-information economy $b^* \in \{5.075, 10.05, 15.025, 20\}$.

The simulation results are given in Figure 3. Figure 3 (a) and (b) depict the respective time-paths for a random distribution of strategies $b_i(0)$.

We see the same mechanisms at work as in the pure public good case discussed before. Especially in Figure 3(b) it becomes obvious that the agents learn optimal behavior by means of imitation and experimentation. The average contributions to the public good in the economy converge towards the respective symmetric Nash equilibrium level. Table I gives a summary of our simulation results. The table confronts the theoretical results $b^*$ with the respective simulation results. The simulation results are the mean and variance of the population average contribution $b$ after 50 000 rounds of a simulation run, taken over 50 different simulations. From the data in the table, the hypothesis that the mean simulation results are equal to the respective theoretical results cannot be rejected at the standard levels of significance (by application of the Wilcox-test).

*Table I.* Summary of simulation results.

| Parameter $\alpha$ | Theoretical $b^*$ | Simulations mean b | Simulations var(mean b) | Number of simulation runs |
|---|---|---|---|---|
| 0.25 | 5.075 | 5.35 | 3.02 | 50 |
| 0.50 | 10.050 | 10.95 | 5.45 | 50 |
| 0.75 | 15.025 | 15.12 | 9.68 | 50 |
| 1.00 | 20.000 | 20.21 | 5.83 | 50 |



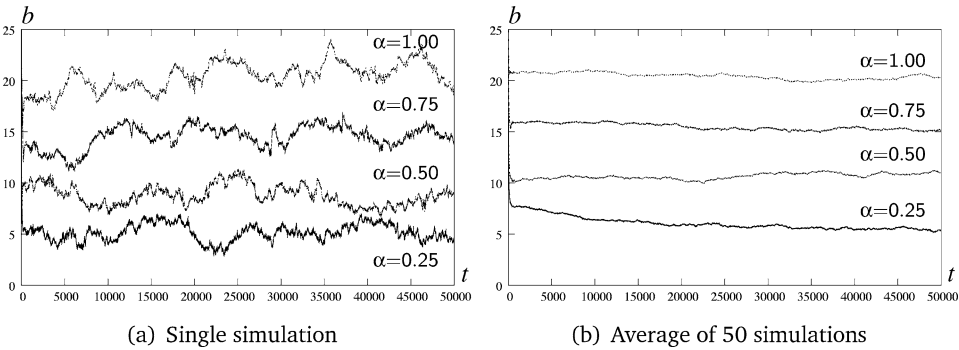(a) Single simulation          (b) Average of 50 simulations

*Figure 3.* Impure altruism, $n = 200$, $\sigma^2 = 0.03$.

## 4.4. EXTENT OF CONVERGENCE

We finally turn to the last question, namely, to what extent convergence of the learning process occurs. From the theoretical solution of the pure public good model with private information, we know that a symmetric Nash equilibrium as given in (5) is characterized by understatement of the individual preferences for the public good. The extent of free riding increases with the group size and the optimal individual contribution level approaches $b^* = 0$ as the number of agents goes to infinity. From laboratory experiments we know that *exact* free riding is hardly ever observed. Our simulations replicate the results from experimental economics and thus differ from the evolutionary learning dynamics discussed by Miller and Andreoni (1991).

One reason for this is, of course, that with standard mutation the agents never stop to experiment. There will always be random disturbances forcing the individual strategy away from its optimal value. This effect can be seen in Figure 1(a), (c), 2(a), (c) where single runs of the simulations are presented. It becomes most distinct in the case of a very high level of the propensity to experiment $\sigma^2 = 0.10$ in Figure 2(a). This finding parallels the wide-spread idea of some experimental economists that deviations from the Nash equilibrium occur due to 'noise' in agents' behavior (cf. Camerer, 2003).

Astonishingly, experimentation does not even stop if the mutation rate is endogenized. The results on meta-mutation indicate that the propensity to experiment

does not converge to zero in the course of the learning process. This observation can be ascribed to two factors: The first reason is the prevailing strategic interaction of the agents in the game. If only a single agent changes his strategy, this change induces a variation in all the other agents' strategies as well. While (5) denotes the symmetric Nash equilibrium, there are many other, asymmetric equilibria as well. Accordingly, if one agent leaves the pattern of common play in the symmetric equilibrium, the others try to adapt their strategy in order to attain a new, asymmetric equilibrium. This process of ongoing adaptation, together with the disturbances by experimentation, leads to ongoing movements.

The second reason can also give an answer to the questions why convergence takes so long and why we observe a decrease in the learning rate. Even if we take account of the factors which accelerate learning like heterogeneity and meta-mutation it still takes hundreds of iterations of the game until the optimal strategy is learned. This should not be viewed as a general result for learning processes. In fact, our results can be ascribed to the special functional form of the underlying 'fitness function' as given by (1). Evolutionary algorithms are less capable of finding optima if the objective function is relatively flat in the neighborhood of the optimal value. This is the case with the logarithmic utility function we applied to our problem. The individual utility gains in the course of the iterations become smaller and smaller as the optimal value is approached such that differences in the relative fitness of strategies become negligible. In addition, small disturbances from experimentation may reset the learning process. This, of course, can be seen as a disadvantage of discussing models with memoryless bounded rational agents.

## 5. Conclusions

In this paper, we explored the question whether or not boundedly rational agents learn optimal strategies over time, when requested to voluntarily contribute to a public good. Our analysis relied on the standard Nash–Cournot approach of public economic theory that predicts free riding behavior, which – in the special case discussed in this paper – even increases with the group size.

The agents of our model played a repeated game and were endowed with only a minimum of information regarding the structure of the economy. The learning process was modeled by means of an evolutionary algorithm and analytically decomposed into two learning mechanisms: *learning by imitation* and *learning by experiments*. We demonstrated that the first concept equals the replicator dynamics of evolutionary game theory in expectation.

Simulations support our major theoretical result, that boundedly rational individuals actually learn to play Nash. Better strategies are adopted over time and the provision level of the public good converges towards the corresponding symmetric Nash equilibrium.

We were especially interested in the key factors that influence the learning dynamics. The convergence speed is affected by the size of the population, its degree

of heterogeneity, and by the propensity to experiment, all of them positively correlated with the learning progress. The learning process never comes to a rest in the sense of a homogeneous final population, where each agent has learned the symmetric Nash-strategy of the private information economy. Our results reproduce outcomes from laboratory experiments on voluntary contributions to public goods. This finding is due to the fact, that the agents of our dynamic model never stop to experiment in order to find a better strategy. This result even holds when the propensity to experiment is endogenized and for itself subject to a learning process. In our view, this is an intuitively plausible and realistic description of individual behavior. Learning is considerably slowed down, if the initial population is homogeneous. In this case, imitative learning is ineffective and the learning progress is mainly achieved by means of experiments.

Our approach is a rather elementary dynamic learning model which by intention was close to the standard game-theoretic approach. We wanted to illustrate the key determinants, the basic working principles and the relation of learning by means of evolutionary algorithms to other approaches of learning in games.

Straightforward extensions to this model would be to allow for more heterogeneity or to introduce some kind of (possibly finite) memory, which was beyond the scope of this paper as we assumed rather simple-minded agents. Another noteworthy extension is to discuss the effects of discrete public goods or thresholds on the learning process. In our model, a positive amount of the public good was always provided as long as an agent decided to offer a positive contribution. The strategic environment changes substantially if we consider a situation in the spirit of Gradstein (1992) or Dixit and Olson (2000), where at least a number $k$ of $n$ agents must announce a positive contribution to the public good, which otherwise would not be provided.

## Acknowledgements

## Notes

[1] Evolutionary algorithms (EAs) are a family of simulation methods resembling the basic working principles of biological evolution. The probably best known type of EAs are Genetic Algorithms (Clemens and Haslinger, 2001; Riechmann, 2001b).

[2] For a survey see Ledyard (1995) and Cornes and Sandler (1996).

[3] At this point it is interesting to note that Miller and Andreoni (1991) derive their results on the impact of the population size on the learning dynamics by implicitly relying on the assumption of an infinitely large population.

[4] This is probably one of the reasons why replicator dynamics are very rarely used for the analysis of models with more than a handful of different strategies. Very notable exceptions are the papers by Binmore and Samuelson (1994) and Binmore et al. (1995), which model the evolution of strategies in an ultimatum game.

[5]One may argue that the strategy space is limited by the memory of the computer system in use. For Genetic Algorithms (GAs), which are based on bit strings coding the strategies, the length of the bit strings limit the size of the strategy space, such that GAs can only handle discrete strategy spaces.

[6]Learning in general does not occur in isolation but always includes learning from others. By assuming identical agents, no individual adopts strategies from other agents with preferences different to his own.

[7]Miller and Andreoni (1991) used a quite similar setup to run simulations based on replicator dynamics.

[8]Moreover, this concept may even lead to absurd learning dynamics, if we assumed agents who are heterogeneous with respect to their true preferences $\beta_i$. A strategy which, in this context, is successful for one person might as well perform poorly for another and hence is not worth to be copied.

[9]There is only one value coded in each round of the algorithm, namely the value of the strategy $b_i(t)$. Consequently, there is no need for crossover.

[10]The stochastic element in the equation stems from the fact that in order to get integer numbers of agents of each given type, the actual results from the equation must be rounded.

[11]The equation reflects the standard way of setting up this mutation operator. For our model, which requires strictly positive contributions, we used a refined operator. If the result of a mutation is non-positive, we force the result to $\varepsilon$, a very small positive number, in our simulations usually set to $10^{-8}$. Thus, the complete version of (10) reads as

$$b_i(t) = \max\{b_i(t) + \varepsilon_i(t), \varepsilon\}, \quad \text{with} \quad \varepsilon_i(t) \sim \mathcal{N}(0, \sigma^2).$$

We have chosen this formulation to keep the analysis close to Bäck and Schütz (1996). Assuming instead the noise term $\varepsilon$ to be lognormally distributed, brings the advantage of eliminating the problem of possibly negative strategies and is a worthwhile extension of the paper, but beyond the scope of the present analysis.

[12]Due to the real-valued coding of the EA, mutation differs substantially from the bit-flipping-procedure of the binary-coded CGA.

[13]The same argument as given for Equation (10) in Footnote 11 applies.

[14]At this point recall our critique on replicator dynamics, where an initially homogeneous population would not learn anything.

[15]Note that it is necessary to randomize the initial population, because otherwise the learning process of replication dynamics gets stuck at the initial level, as does the replicator dynamics of evolutionary game theory.

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