Impact of Population Structure on Cooperation

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# ??

TBD

Allow agents to add their own links. Analyze the resulting network. Is it heterogeneous? Scale-free?

Regular graphs represent unrealistic representations of the real world.

Run simulations, gather statistics about resulting networks, compare to the characteristics for different network types: scale free, etc.

Two different forms up update dynamics (or evolutionary dynamics): synchronous and asynchronous. In synchronous updating, after all games in a generation have been played, the agents then proceed to update their strategies simultaneously. In asynchronous updating, a randomly chosen player plays the game and immediately updates its strategy followed by the next randomly shoes agent and so on. (Tanimoto – Fundamentals of Evolutionary Game Theory and its Applications).

# Social Dilemmas of Cooperation

(insert description of prisoner’s dilemma, snowdrift/chicken/hawk-dove, stag-hunt games here)

prisoner’s dilemma – PD

snowdrift/chicken/hawk-dove game – SG

stag-hunt game - SH

# Network Types

Random graphs

Small-world networks

Single-scale networks

Broad-scale networks

Scale-free networks

Explain heterogeneity of degree distributions…

# Related Work

## Evolution of Cooperation in Well Mixed Populations

Insert (Axelrod/Hamilton 1981) summary here.

Others…?

## Evolution of Cooperation on a Square Lattice

In [2], the authors consider a scenario in which *n* agents are arranged on a square lattice with each agent connected to its nine immediate neighbors. The edges of the lattice do not wrap around giving agents on its edges fewer neighbors. Unlike other studies reviewed in this section, in each generation, each agent plays the game with itself as well as its nine immediate neighbors. After all games for a generation have been played, the strategy of each agent is replaced with the strategy of the fittest agent among itself and its eight neighbors.

The authors only consider the PD game and use a simplified payout structure for that game with R=1, T=b and S=P=0 where b>1. At the start of each simulation, a specified fraction of cooperators are allocated to nodes in the lattice with the remaining nodes allocated to defectors. Various starting configurations are considered consisting of different fractions of cooperators and different initial placement of those cooperators. The authors run multiple simulations with varying values for *b* and various starting configurations. They find that when , cooperators and defectors coexist in the population.

(insert summary of [3] here)

## Impact of Network Topology on Evolution of Cooperation

The well-mixed population and square lattice cases described above are special cases of agents allocated to a considered in the previous section corresponds to the case when agents occupy the nodes of a fully connected graph.

In this section, the impact of alternative graph topologies on the evolution of cooperation is considered. In each case, agents are allocated to the nodes of a graph with a fixed number of nodes equal to the number of agents *n*. The graph has a fixed number of edges giving the graph a fixed average connectivity *z*. The agents follow one of two strategies: unconditional cooperation or unconditional defection. The well-mixed case considered previously corresponds to the case of a fully connected graph while the square lattice case corresponds to a special case where the degree of each node is nine except for the nodes around the edges.

The graph is updated synchronously. For each generation, each agent plays the social dilemma game being investigated with each of its neighbors achieving a fitness score equal to the sum of the payouts earned from each game. The number of games played by agent *i* is equal to the degree *ki* of the node it occupies and the total number of games played by all agents is equal to the number of edges in the graph. For non-homogeneous network structures, some agents will play more games than other agents.

The evolutionary dynamics are the same as those used in [3] for simulations involving pure strategies except extended to handle graphs with heterogeneous degree. After all games for a generation have been played, the nodes are updated simultaneously. For each agent *x* with payout *Px*, one of its neighbors *y* with payout *Py* is selected at random. If then agent *x* maintains its original strategy. Otherwise, agent *x*’s strategy is switched to the strategy of agent *y* with probability *p* defined as follows:

where and *ki* is the degree of node *i*.

where For any agent *i*, and … need to get a better understanding of how this equation works.

Initially, an equal number of cooperators and defectors are randomly allocated to the nodes of the graph. After executing 10,000 generations to reach a stationary regime, the final 1000 generations are used to compute the equilibrium frequency of cooperators and defectors in the population.

The studies reviewed in this section provide results for several different network topologies:

* Fully connected network: In this case, the network is a fully connected graph where the degree of each node is equal to the graph’s average connectivity . This case corresponds to the so-called *well-mixed* population case.
* Homogeneous structured network: In this case, the network is a regular graph where the degree of each node is equal to the graph’s average connectivity .
* Homogeneous unstructured network: In this case, the network is a random regular graph where the degree of each node is equal to the graph’s average connectivity .
* Watts-Strogatz network: In this case, the network is a small-world network generated using the Watts-Strogatz graph generation algorithm [13]. Starting from a regular ring graph with each node connected to its *k* nearest neighbors, edges in the graph are rewired randomly with probability p to produce a Watts-Strogatz graph. When *p*=0, the graph is unchanged while setting *p*=1 produces a random graph. For intermediate values of p, the graph displays small world characteristics of short path length and cliquishness as measured by the clustering coefficient[[1]](#footnote-1). For *p*<0.1 the path length decreases sharply while the clustering coefficient remains fairly constant while for *p*>0.1 the clustering coefficient decreases quickly.
* Single-scale network: In this case, the network is a type of small-world network with moderate heterogeneity where the degree of most nodes does not deviate significantly from the graph’s average connectivity [15].
* Scale-free network: In this case, the network is a graph generated using a process that involves preferential attachment. The resulting graph has strong heterogeneity with a degree distribution follows a power-law [14].
* Scale-free network with uniform attachment: In this case, the graph is generated using the same process as scale-free networks except that uniform attachment is instead of preferential attachment.
* Random scale free network: In this case, the network is a scale-free network in which the connections between nodes have been randomized to remove “age correlation” introduced by the process of generating the graph

In the following studies (list references), the authors consider different social dilemma games and evaluate the impact of graph topology on the evolution of cooperation.

The results reported in each study are for *n* = 104 ~~and~~ *~~z~~* ~~= 4~~.

**(Pacheco Santos 2005 Network dependence of the dilemmas of cooperation)**

In [4], the authors consider the cases of agents playing the PD and SG games on different topologies including homogeneous structured networks represented as regular ring-graphs, Watts-Strogatz networks and scale-free networks. The authors use the following payout structure for the PD and SG games. For PD the payout structure is 2>T=b>1, R=1 and P=S=0. For the SG game the payout structure is T=β>1, R=β-1/2, S=β-1 and P=0. For the SG game, the payouts are consolidated into a single parameter representing the cost-to-benefit ration of mutual cooperation.

### Homogeneous Regular Networks

The authors find that cooperation is difficult to maintain for both games when played on a regular ring. As expected, increasing *b* and *r* decreases the performance of cooperators. In addition, as the average connectivity *z* of the graph increases the population structure begins to mirror a well-mixed population and cooperation becomes more difficult.

### Watts-Strogatz - Small World and Random Networks

Considering graphs generated using the Watts-Strogatz algorithm with varying values for *p*, the authors find that the performance of cooperators improves as *p* increases. As with the case of a regular graph, cooperation suffers as the average connectivity *z* of the graph increases and the population approaches a well-mixed population. This leads to the result that cooperation receives the largest boost for a random graph (*p*=1) with small average connectivity.

### Scale Free Networks

For scale free graphs, the authors find that cooperation dominates for both games for almost the entire range of values for *b* and *r*. In addition, they find that, unlike for other graphs types considered, the performance of cooperators improves as the average connectivity *z* of the graph increases.

To evaluate the impact of preferential attachment, the authors consider a graph that is generated using the same process used to generate a scale-free graph except that preferential attachment is replaced with uniform attachment. While the uniform attachment network sustains cooperation better than a Watts-Strogatz network with *p*=1, a cooperators perform significantly better on a true scale-free network.

### Impact of Population Size

The authors also analyze the impact of population size (while keeping a constant average connectivity z) on the success of cooperators. They find that the size of the population has little effect on the performance of cooperators. Leading to the insight that the structure of the network is more important than the size of the network. The authors note that for small populations, the graphs constructed using the growth and preferential attachment approach are not scale free. Leading to the insight that growing the network using preferential attachment is more important than the scale-free characeristics.

This study applies the same payout structure as (Nowak/May 1992).

**(Sanos, Pacheco 2005 A new route to the evolution of cooperation)**

**(Santos, Rodrigues, Pacheco 2005 Graph topology plays a determinant role in the…)**

The previous study considered agents occupying nodes of a homogeneous regular graph. In these two studies, the authors consider the scenario in which agents occupy the nodes of a heterogeneous graph with fixed number of nodes *n* and fixed average connectivity *z*. Two different types of graphs are considered: Watts-Strogatz[[2]](#footnote-2) and scale-free[[3]](#footnote-3). In each generation, each agent plays the Prisoner’s Dilemma game with each of its neighbors achieving a fitness score equal to the sum of the payouts earned form each game. Since the graph is heterogeneous, agents play different numbers of games resulting in different maximum possible scores for each agent.

Using the following payouts: R=1, T=b>1, S=P=0 (same as above); the authors run multiple simulations with varying values for *b* and various starting configurations. The authors find that graph topology has a significant impact on the performance of cooperators and defectors. Heterogeneous graphs have a significant positive impact on the ability of cooperators to survive and dominate a population of agents. In particular, graphs such as scale-free networks that are generated using the mechanisms of growth and preferential attachment have the largest positive impact on the performance of cooperators.

**(Santos, Rodrigues, Pacheco 2005 Epidemic spreading and cooperation dynamics…)**

The authors investigate the evolution of cooperation on a special class of graphs that exhibit small-world effects while still being homogeneous. This allows the impact of the small-world effect to be investigated independently of the effect of heterogeneity. Small-world networks are measured by two parameters: the average distance L between two nodes in the graph and the clustering coefficient C that measures the degree to which nodes cluster together.

Once again, the following payout structure is used for the game: R=1, T=b>1, S=P=0, the authors run simulations on both the special homogeneous small-world (HoSW) networks and the heterogeneous small-world (HeSW, aka scale-free) netwoks. A comparison of the results shows that for relatively high average distance L and clustering coefficient C, HoSW topology has a similar impact on the performance of cooperators to that of HeSW topology indicating that in this region small-world effects may be significant. However, in this region, cooperators still perform better on HeSW graphs showing that the effects of heterogeneity are also significant. As L and C decrease, the performace of cooperators on HeSW graphs becomes significantly better than on HoSW graphs showing that in this region heterogeneity is the driving force behind the enhanced performance of cooperators.

**(Santos, Pacheco, Lenaerts 2006 Evolutionary dynamics of social dilemmas in…)**

The authors consider a slightly different payout structure: R = 1, P = 0, and . The authors consider several different network topologies:

* Well-mixed populations: This corresponds to the case when the graph is fully connected resulting in a homogeneous network with average connectivity . For each node *i* in the network, . The authors reconfirm that cooperators are unable to compete with defectors when playing the Prisoner’s Dilemma in a well-mixed population.
* Homogeneous structured populations: In this case, the network is a regular graph with an average connectivity . The authors find that cooperators can outperform defectors in the Prisoner’s Dilemma when T is slightly larger than R and S is only slightly less than P. This small window of opportunity exists because the correlated spatial structure allows cooperators to form small clusters that resist invasion by defectors.
* Homogeneous unstructured populations: In this case, the network is a random graph where the degree *ki* of each node is equal to the average connectivity of the graph. The authors find that the ability of cooperators to outperform defectors is reduced in this case. The reduction occurs because the uncorrelated social structure no longer allows cooperators to form tight clusters that resist invasion by defectors. However, there is still a small window where cooperators can coexist with defectors showing that the reduction in connectedness provides some benefits to cooperation compared to the well-mixed fully-connected case.
* Heterogeneous structured populations: In this case, the network is a non-regular graph where the degree *ki* of each node is not necessarily equal to the average connectivity *z* of the graph. The authors consider two types of heterogeneous networks that fall into the class of small-world networks (Amaral 2000):
  + Single-scale network: a graph with moderate heterogeneity where the degree of most nodes does not deviate significantly from the graph’s average connectivity. The authors find that cooperators can outperform defectors in the Prisoner’s Dilemma when T is slightly larger than R and S is only slightly less than P. This window is slightly larger that the window provided by the homogeneous structured case. The single scale network has characteristics similar to the random graph in the homogeneous unstructured case that prevents compact clusters of cooperators from forming. However, the heterogeneity of the network offsets this effect and leads to an overall improvement in the conditions for cooperation.
  + Scale-free network[[4]](#footnote-4): a graph with strong heterogeneity whose degree distribution follows a power-law. The authors consider two types of scale free networks:
    - Random scale free network: In this scale-free network in which the connections between nodes remain random. The authors find that the introduction of scale-free characteristics into the network significantly improves the chances that cooperators can coexist with defectors in the Prisoner’s Dilemma game. Comparing this to the result obtained for single-scale networks shows that increasing heterogeneity appears to have a positive impact on cooperation. However, the randomness of the connections decreases the ability of cooperators to form tight clusters that resist invasion thus reducing the effectiveness of heterogeneity.
    - Barabási-Albert model: This scale-free network is grown using a process that involves preferential attachment. The process introduces “age correlation” in which older vertices have higher degree and are interconnected with each other. The authors find that the introduction of age correlation has a significant positive impact on the ability of cooperators to dominate defectors. The introduction of age correlation effectively eliminates the randomness that prevents clusters of cooperators from forming.

## Evolution of Network Topology

(Eguiluz Zimmrman 2005 Cooperation and the emergence of role differentiation…)

The authors consider the scenario in which *N* agents occupy the nodes of a graph. The graph contains edges that are initially inserted randomly between pairs of nodes. The constant *K* defines the average connectivity of the nodes in the graph (denoted *z* above). The agents follow one of two fixed strategies: unconditional cooperation or unconditional defection. In each generation, each agent plays the Prisoner’s Dilemma game with each of its neighbors achieving a fitness score equal to the sum of the payouts earned form each game. Following (Nowak/May 1992), the following payouts are used when playing the game: R=1, T=b>1, S=P=0.

After all games for a generation have been played, synchronous updating is used to evolve both agent strategies and the network topology. Following (Nowak/May 1992), after all games for a generation have been played, the strategy of each agent is replaced with the strategy of the fittest agent among itself and its neighbors. In addition, if an agent imitates a neighbor that is a defector then, with probability *p*, the link between the agent an the imitated defector is replaced with a link between the agent and an agent selected randomly from among all agents in the network.

Initially, the graph is populated with 60% cooperators randomly allocated to nodes in the graph. The remaining nodes are populated with defectors. After the simulation reaches a stationary state, the fraction of cooperative agents that exist in the population is computed. The authors report simulation results for various values of *p* and *b*. The values reported are averages over 100 simulation runs.

The authors collected results for the following range of values for *p* and *b*: and . The authors find that for , the fraction of cooperators in the population is kept above 90% for the range of values of *b* considered.

**(Santos, Pacheco, Lenaert PLoS 2006 Cooperation prevails when individuals adjust…)**

The authors consider the scenario in which *N* agents occupy the nodes of a graph with *NE* edges. Initially, each node is randomly connected to other nodes. Each agent follows one of two fixed strategies, unconditional cooperation or unconditional defection, and earns a fitness score that is equal to the sum of the payouts earned when playing social dilemma games against other agents. The payout matrix used for the games fixes R =1 and P = 0 and allows T and S to vary with and . As usual, the fitness score earned by the agents are used to update the strategies those agents follow. In addition, the fitness scores are also used to update the structure of the network. This leads to the co-evolution of the strategy composition of the population and the structure of the network that holds that population.

The evolution of the strategies and network structure occur at different time scales. Let *τe* be the time scale at which strategy updates occur and *τa* be the time scale for network structure updates. In this case, the ratio specifies how frequently structure updates occur relative to strategy updates. For example, if and then indicating that structure updates occur three times as often as strategy updates. The probability *p* that a strategy update occurs during any time step is given by the following formula:

Given this, a structure update occurs with probability . In the example presented earlier, the probability that a strategy update occurs during any time step is while the probability that a structure update occurs is . Therefore, on average one strategy update occurs for every three structure updates.

For both types of updates, an agent *x* is chosen at random and another agent *y* is chosen at random from agent *x*’s neighbors. The two agents play the game with each of its neighbors and earn fitness scores *Px* and *Py* respectively.

In the case of a strategy update, after the games have been played, the strategy of *y* replaces the strategy of *x* with probability *pe* given by the following equation:

This is the *pairwise comparison* process introduced in Traulsen/Nowak/Pacheco Phys Rev E 2006). The parameter *βe* represents the selection strength and determines how strongly the fitness score impacts the decision to replace *x*’s strategy with *y*’s strategy. As , the dynamics approach neutral drift where each strategy has 50% chance of selection. As , the dynamics approach imitation dynamics in which the strategy of the fittest agent is always selected.

In the case of a structure update, agent *x* attempts to rewire the link between it and agent *y* if it is dissatisfied with the current link. An agent is satisfied with a link if the agent on the other end is a cooperator and dissatisfied otherwise. In the case that agent *x* is dissatisfied, the link is switched to a random neighbor of agent *y* with probability *pa* given by the following equation:

This equation is almost identical to the equation for *pe* except that the roles of *Px* and *Py* are switched and selection strength is controlled by the parameter *βa*. To ensure that the network remains connected, the link to agent *y* cannot be removed if this is agent *y*’s only link.

Initially, the structure of the network is a homogeneous random graph in which the degree of each node is equal to the average connectivity *z* of the graph. An equal number of cooperators and defectors are randomly allocated to the nodes of the graph. Each simulation is run until the fraction of cooperators reaches 100% or the number of generations reaches 108. In the case that the fraction of cooperators does not reach 100%, the average fraction of cooperators over the last 1000 generations is used as the result. The authors run simulations with varying values for T, S and W. For each set of parameter values, 100 simulations are run, each with a different starting configuration, and the results are averaged to determine the fraction of cooperators that survive evolution.

The authors find that for *W* = 0 and moderate selection strength, the results reproduce the predictions for finite, well-mixed populations. As *W* increases, it becomes easier for cooperators to survive until *W* reaches a critical value (*Wc*) at which cooperators dominate for any value of T and S. The value of *Wc* increases as *z* increases as expected since there are more links to be rewired in order to reach a state where cooperators can dominate.

The authors find that the value of *W*, as well as the structure of the game payouts, determines the level of heterogeneity that evolves in the network. In general, as *W* increases, the heterogeneity of the evolved network increases. When the payout structure of the game favors defectors but allows cooperators to coexist at low levels, the few cooperators that remain accumulate large numbers of links leading to highly heterogeneous networks. The authors find that large T promotes heterogeneity more than large S.

Finally, the authors find that the value of *Wc* decreases as either of the selection strength parameters (*βe* and *βa*) increase. Smaller selection strength values allow less fit agents to survive. Prior to the network being rewired, cooperators are generally less fit than defectors. Smaller β values allow less fit cooperators to survive long enough to restructure the network into a state where cooperators dominate defectors.

Let *Nk* equal the number of nodes with degree *k* and *kmax* be the maximum degree possessed by any node in the graph. Then, the *degree of heterogeneity* of the graph *h* is given by the following equation:

And the *cumulative degree distribution* of the graph *D(k)* is given by the following equation:

**(Fu Hauert Nowak Wang Phy Rev E 78 2008 Reputation-based partner choice…)**

The authors extend the framework introduced in (Santos/Pacheco/Lenaert PLoS 2006) to allow reputation to influence the link rewiring process. The authors use the *image score* metric introduced in (Nowak/Sigmund 1998) to measure an agent’s reputation.

As in (Santos, Pacheco, Lenaert PLoS 2006), the evolution of strategies and network structure occurs asynchronously. A strategy update proceeds as defined in (Santos, Pacheco, Lenaert PLoS 2006). However, in addition, agent *x*’s reputation is also updated using the following formula:

where is the reputation of agent *x* at time *t*, equals 1 if agent *x* cooperated at time *t* and zero otherwise and for any agent *i*.

The structure update procedure is modified to allow reputation to guide the relinking process. An agent *x* is selected at random to replace the link that exists between itself and its lowest reputation neighbor. With probability *p*, the link that exists between agent *x* and its lowest reputation neighbor is replaced with a link to the agent with the highest reputation among agent *x*’s neighbors’ neighbors while with probability that link is replaced with a link to an agent selected at random from among all agents except agent *x*’s neighbors. To ensure that the network remains connected, the link to agent *x*’s lowest reputation neighbor cannot be removed if this is that agent’s only link.

In their simulations, the authors focus on a payout structure that is consistent with the Prisoner’s Dilemma game. Specifically, they use the payout structure introduced by (May/Nowak 1992): R=1, T=b>1, S=P=0. As in (Santos, Pacheco, Lenaert PLoS 2006), each simulation begins with an equal number of cooperators and defectors randomly allocated to the nodes in a homogeneous random network. For each simulation, a total of 106 generations are executed and the fraction of cooperators that survive evolution is determined by averaging over the last 1000 generation. The data presented in the paper results from averaging over 103 simulation runs.

For moderate parameter settings (N=104, z=10, b=1.2, W=1, p=0.5 and β=0.01), the authors find that cooperation steadily increases with cooperators coming to dominate the population after 25,000 generations. A growth in the frequency of links between cooperators (CC) mirrors the growth in cooperation with a corresponding decrease in the links to defectors. After reaching a steady state dominated by cooperators, as expected based on earlier studies (cite here), the network is highly heterogeneous with a degree distribution that follows a power law indicating that the network has properties similar to a scale free network. As expected, due to introduction of high reputation partner seeking behavior, the authors find a positive correlation between the reputation score of an agent and the degree of the node occupied by that agent.

Consistent with (Santos, Pacheco, Lenaert PLoS 2006), the authors observe a existence of a critical value *Wc* for *W* above which cooperators eliminate defectors from the population and that *Wc* increases with increasing *b* (T). However, the introduction of reputation-based partner switching lowers critical point at which cooperators are able to dominate. While (Santos, Pacheco, Lenaert PLoS 2006) report that for S=0 and T=1.8, *W* must be equal to approximately 2 before cooperators can dominate, the current study finds that, for the same values of S and T, cooperators can dominate for values of *W* as low as approximately 0.3.

The authors also consider a reputation model variation that includes reputation discounting:

Where is the discount rate that diminishes the value or earlier good deeds over time. When the agent’s last action alone determines it reputation while when the original model is restored. The authors find that if the frequency of partner switching is high enough (approximately ), smaller values of δ make it easier for cooperators to dominate. IN this case, when agents react immediately to defection, it makes it harder for defectors to maintain links that allow them to exploit cooperators.

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1. https://en.wikipedia.org/wiki/Clustering\_coefficient [↑](#footnote-ref-1)
2. https://en.wikipedia.org/wiki/Watts\_and\_Strogatz\_model [↑](#footnote-ref-2)
3. https://en.wikipedia.org/wiki/Scale-free\_network [↑](#footnote-ref-3)
4. https://en.wikipedia.org/wiki/Scale-free\_network [↑](#footnote-ref-4)