Cooperation in Graph Structured Populations

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# ??

TBD

Allow agents to add their own links. Analyze the resulting network. Is it heterogeneous? Scale-free?

Regular graphs represent unrealistic representations of the real world.

Run simulations, gather statistics about resulting networks, compare to the characteristics for different network types: scale free, etc.

Two different forms up update dynamics (or evolutionary dynamics): synchronous and asynchronous. In synchronous updating, after all games in a generation have been played, the agents then proceed to update their strategies simultaneously. In asynchronous updating, a randomly chosen player plays the game and immediately updates its strategy followed by the next randomly shoes agent and so on. (Tanimoto – Fundamentals of Evolutionary Game Theory and its Applications).

# Social Dilemmas of Cooperation

(insert description of prisoner’s dilemma, snowdrift/chicken/hawk-dove, stag-hunt games here)

# Network Types

Random graphs

Small-world networks

Single-scale networks

Broad-scale networks

Scale-free networks

Explain heterogeneity of degree distributions…

# Related Work

Insert (Axelrod/Hamilton 1981) summary here.

(Nowak/May 1992) considered the scenario in which *n2* agents are arranged on a lattice and follow one of two fixed strategies: unconditional cooperation or unconditional defection. The edges of the lattice do not wrap around and therefore agents on the edges have fewer neighbors. In each generation, each agent plays the Prisoner’s Dilemma game with itself and its eight immediate neighbors achieving a fitness score equal to the sum of the payouts earned form each game. After all games for a generation have been played, the strategy of each agent is replaced with the strategy of the fittest agent among itself and its eight neighbors.

Using the following payouts: R=1, T=b>1, S=P=0; the authors run multiple simulations with varying values for *b* and various starting configurations. They find that when , cooperators and defectors coexist in continually shifting patterns.

## Impact of Network Topology on Evolution of Cooperation

In the following studies (list references), agents occupy the nodes of a graph with a fixed number of nodes *n* and a fixed average connectivity *z*. The fixed average connectivity implies that the number of edges in the graph is the same for the entire simulation. The agents follow one of two fixed strategies: unconditional cooperation or unconditional defection. The authors consider different social dilemma games and evaluate the impact of graph topology on the evolution of cooperation. In each generation, each agent plays the game with each of its neighbors achieving a fitness score equal to the sum of the payouts earned form each game. The number of games played by agent *i* is equal to the degree *ki* of the node it occupies and the total number of games played by all agents is equal to the number of edges in the graph. For non-homogeneous network structures, some agents may play more games than other agents.

The evolutionary dynamics are an extension of those used in pure strategy simulations in reference (Hauert/Doebeli 2004 Nature) to support graphs with both homogeneous and heterogeneous degree. After all games for a generation have been played, synchronous updating is used to evolve the agent strategies. For each agent *x* with payout *Px*, one of its neighbors *y* with payout *Py* is selected at random. If then agent *x* maintains its original strategy. Otherwise, agent *x*’s strategy is switched to the strategy of agent *y* with probability *p* defined as follows:

or

where *ki* is the degree of node *i*. For any agent *i*, and … need to get a better understanding of how this equation works.

The second term in the denominator may depend on the game being played. For example, it might be (T-P) for the snowdrift game.

Initially, an equal number of cooperators and defectors are randomly allocated to the nodes of the graph. After executing 10,000 generations to reach a stationary regime, the final 1000 generations are used to compute the equilibrium frequency of cooperators in the population[[1]](#footnote-1).

The results reported in each study are for *n* = 104 and *z* = 4.

(Pacheco Santos 2005 Network dependence of the dilemmas of cooperation)

Besides also reaching conclusions for non-regular and heterogeneous graph, this study provides results for the evolution of cooperation on regular graphs. The authors find that as the average connectivity *z* increases (thus moving closer to the well-mixed fully connected case), the ability of cooperators to thrive is decreased. However, for vales of *z* that are significantly smaller than the population size (thus the population is far removed from the well-mixed fully connected case), cooperators can out-perform defectors for small values of *b*.

The authors also analyze the impact of population size (while keeping a constant average connectivity z) on the success of cooperators. They find that the size of the population has little effect on the performance of cooperators. Leading to the insight that the structure of the network is more important than the size of the network. The authors note that for small populations, the graphs constructed using the growth and preferential attachment approach are not scale free. Leading to the insight that growing the network using preferential attachment is more important than the scale-free characeristics.

This study applies the same payout structure as (Nowak/May 1992).

(Sanos, Pacheco 2005 A new route to the evolution of cooperation)

(Santos, Rodrigues, Pacheco 2005 Graph topology plays a determinant role in the evolution of cooperation)

The previous study considered agents occupying nodes of a homogeneous regular graph. In these two studies, the authors consider the scenario in which agents occupy the nodes of a heterogeneous graph with fixed number of nodes *n* and fixed average connectivity *z*. Two different types of graphs are considered: Watts-Strogatz[[2]](#footnote-2) and scale-free[[3]](#footnote-3). In each generation, each agent plays the Prisoner’s Dilemma game with each of its neighbors achieving a fitness score equal to the sum of the payouts earned form each game. Since the graph is heterogeneous, agents play different numbers of games resulting in different maximum possible scores for each agent.

After all games for a generation have been played, the agent strategies are evolved as follows. For each agent *x* with payout *Px*, one of its neighbors *y* with payout *Py* is selected at random. If then agent *x* maintains its original strategy. Otherwise, agent *x*’s strategy is switched to the strategy of agent *y* with probability *p* defined as follows:

where *ki* is the degree of node *i*. For any agent *i*, and … need to get a better understanding of how this equation works.

Initially, an equal number of cooperators and defectors are randomly allocated to the nodes of the graph. After executing 10,000 generations to reach a stationary regime, the final 1000 generations are used to compute the equilibrium frequency of cooperators in the population[[4]](#footnote-4).

Using the following payouts: R=1, T=b>1, S=P=0 (same as above); the authors run multiple simulations with varying values for *b* and various starting configurations. The authors find that graph topology has a significant impact on the performance of cooperators and defectors. Heterogeneous graphs have a significant positive impact on the ability of cooperators to survive and dominate a population of agents. In particular, graphs such as scale-free networks that are generated using the mechanisms of growth and preferential attachment have the largest positive impact on the performance of cooperators.

(Santos, Rodrigues, Pacheco 2005 Epidemic spreading and cooperation dynamics…)

The authors investigate the evolution of cooperation on a special class of graphs that exhibit small-world effects while still being homogeneous. This allows the impact of the small-world effect to be investigated independently of the effect of heterogeneity. Small-world networks are measured by two parameters: the average distance L between two nodes in the graph and the clustering coefficient C that measures the degree to which nodes cluster together.

Once again, the following payout structure is used for the game: R=1, T=b>1, S=P=0. Using the same procedure for playing games during a generation and evolving agents strategies from one generation to the next, the authors run simulations on both the special homogeneous small-world (HoSW) networks and the heterogeneous small-world (HeSW, aka scale-free) netwoks. A comparison of the results shows that for relatively high average distance L and clustering coefficient C, HoSW topology has a similar impact on the performance of cooperators to that of HeSW topology indicating that in this region small-world effects may be significant. However, in this region, cooperators still perform better on HeSW graphs showing that the effects of heterogeneity are also significant. As L and C decrease, the performace of cooperators on HeSW graphs becomes significantly better than on HoSW graphs showing that in this region heterogeneity is the driving force behind the enhanced performance of cooperators.

(Santos, Pacheco, Lenaerts 2006 Evolutionary dynamics of social dilemmas in structured…)

Similar to the studies reviewed above, the authors consider the scenario in which agents occupy the nodes of a graph. The authors consider a slightly different payout structure: R = 1, P = 0, and . The evolutionary dynamics works in a similar fashion except that, due to the modified payout structure, the formula used to determine the probability that an agent’s trategy is replaced is given by the following:

The authors consider several different network topologies:

* Well-mixed populations: This corresponds to the case when the graph is fully connected resulting in a homogeneous network with average connectivity . For each node *i* in the network, . The authors reconfirm that cooperators are unable to compete with defectors when playing the Prisoner’s Dilemma in a well-mixed population.
* Homogeneous structured populations: In this case, the network is a regular graph with an average connectivity . The authors find that cooperators can outperform defectors in the Prisoner’s Dilemma when T is slightly larger than R and S is only slightly less than P. This small window of opportunity exists because the correlated spatial structure allows cooperators to form small clusters that resist invasion by defectors.
* Homogeneous unstructured populations: In this case, the network is a random graph where the degree *ki* of each node is equal to the average connectivity of the graph. The authors find that the ability of cooperators to outperform defectors is reduced in this case. The reduction occurs because the uncorrelated social structure no longer allows cooperators to form tight clusters that resist invasion by defectors. However, there is still a small window where cooperators can coexist with defectors showing that the reduction in connectedness provides some benefits to cooperation compared to the well-mixed fully-connected case.
* Heterogeneous structured populations: In this case, the network is a non-regular graph where the degree *ki* of each node is not necessarily equal to the average connectivity *z* of the graph. The authors consider two types of heterogeneous networks that fall into the class of small-world networks (Amaral 2000):
  + Single-scale network: a graph with moderate heterogeneity where the degree of most nodes does not deviate significantly from the graph’s average connectivity. The authors find that cooperators can outperform defectors in the Prisoner’s Dilemma when T is slightly larger than R and S is only slightly less than P. This window is slightly larger that the window provided by the homogeneous structured case. The single scale network has characteristics similar to the random graph in the homogeneous unstructured case that prevents compact clusters of cooperators from forming. However, the heterogeneity of the network offsets this effect and leads to an overall improvement in the conditions for cooperation.
  + Scale-free network[[5]](#footnote-5): a graph with strong heterogeneity whose degree distribution follows a power-law. The authors consider two types of scale free networks:
    - Random scale free network: In this scale-free network in which the connections between nodes remain random. The authors find that the introduction of scale-free characteristics into the network significantly improves the chances that cooperators can coexist with defectors in the Prisoner’s Dilemma game. Comparing this to the result obtained for single-scale networks shows that increasing heterogeneity appears to have a positive impact on cooperation. However, the randomness of the connections decreases the ability of cooperators to form tight clusters that resist invasion thus reducing the effectiveness of heterogeneity.
    - Barabási-Albert model: This scale-free network is grown using a process that involves preferential attachment. The process introduces “age correlation” in which older vertices have higher degree and are interconnected with each other. The authors find that the introduction of age correlation has a significant positive impact on the ability of cooperators to dominate defectors. The introduction of age correlation effectively eliminates the randomness that prevents clusters of cooperators from forming.
  + The authors consider graphs that acquire their scale-free characteristics due to fact that they are generated using a process that involves growth and preferential attachment.

## Evolution of Network Topology

(Eguiluz Zimmrman 2005 Cooperation and the emergence of role differentiation…)

The authors consider the scenario in which *N* agents occupy the nodes of a graph. The graph contains edges that are initially inserted randomly between pairs of nodes. The constant *K* defines the average connectivity of the nodes in the graph (denoted *z* above). The agents follow one of two fixed strategies: unconditional cooperation or unconditional defection. In each generation, each agent plays the Prisoner’s Dilemma game with each of its neighbors achieving a fitness score equal to the sum of the payouts earned form each game. Following (Nowak/May 1992), the following payouts are used when playing the game: R=1, T=b>1, S=P=0.

After all games for a generation have been played, synchronous updating is used to evolve both agent strategies and the network topology. Following (Nowak/May 1992), after all games for a generation have been played, the strategy of each agent is replaced with the strategy of the fittest agent among itself and its neighbors. In addition, if an agent imitates a neighbor that is a defector then, with probability *p*, the link between the agent an the imitated defector is replaced with a link between the agent and an agent selected randomly from among all agents in the network.

Initially, the graph is populated with 60% cooperators randomly allocated to nodes in the graph. The remaining nodes are populated with defectors. After the simulation reaches a stationary state, the fraction of cooperative agents that exist in the population is computed. The authors report simulation results for various values of *p* and *b*. The values reported are averages over 100 simulation runs.

The authors collected results for the following range of values for *p* and *b*: and . The authors find that for , the fraction of cooperators in the population is kept above 90% for the range of values of *b* considered.

**(Santos, Pacheco, Lenaert 2006 Cooperation prevails when individuals adjust their social ties)**

The authors consider the scenario in which *N* agents occupy the nodes of a graph with *NE* edges. Initially, each node is randomly connected to other nodes. Each agent follows one of two fixed strategies, unconditional cooperation or unconditional defection, and earns a fitness score that is equal to the sum of the payouts earned when playing social dilemma games against other agents. As usual, the fitness score earned by the agents are used to update the strategies those agents follow. In addition, the fitness scores are also used to update the structure of the network. This leads to the co-evolution of the strategy composition of the population and the structure of the network that holds that population.

The evolution of the strategies and network structure occur at different time scales. Let *τe* be the time scale at which strategy updates occur and *τa* be the time scale for network structure updates. In this case, the ratio specifies how frequently structure updates occur relative to strategy updates. For example, if and then indicating that structure updates occur three times as often as strategy updates. The probability *p* that a strategy update occurs during any time step is given by the following formula:

Given this, a structure update occurs with probability . In the example presented earlier, the probability that a strategy update occurs during any time step is while the probability that a structure update occurs is . Therefore, on average one strategy update occurs for every three structure updates.

For both types of updates, an agent *x* is chosen at random and another agent *y* is chosen at random from agent *x*’s neighbors. The two agents play the game with each of its neighbors and earn fitness scores *Px* and *Py* respectively.

In the case of a strategy update, after the games have been played, the strategy of *y* replaces the strategy of *x* with probability *pe* given by the following equation:

The parameter *βe* represents the selection strength and determines how strongly the fitness score impacts the decision to replace *x*’s strategy with *y*’s strategy. As , the dynamics approach neutral drift where each strategy has 50% chance of selection. As , the dynamics approach imitation dynamics in which the strategy of the fittest agent is always selected.

In the case of a structure update, agent *x* attempts to rewire the link between it and agent *y* if it is dissatisfied with the current link. An agent is satisfied with a link if the agent on the other end is a cooperator and dissatisfied otherwise. In the case that agent *x* is dissatisfied, the link is switched to a random neighbor of agent *y* with probability *pa* given by the following equation:

This equation is almost identical to the equation for *pe* except that the roles of *Px* and *Py* are switched and selection strength is controlled by the parameter *βa*.

Initially, the structure of the network is a homogeneous random graph in which the degree of each node is equal to the average connectivity *z* of the graph. An equal number of cooperators and defectors are randomly allocated to the nodes of the graph. Each simulation is run until the fraction of cooperators reaches 100% or the number of generations reaches 108. In the case that the fraction of cooperators does not reach 100%, the average fraction of cooperators over the last 1000 generations is used as the result. The authors run simulations with varying values for T, S and W. For each set of parameter values, 100 simulations are run, each with a different starting configuration, and the results are averaged to determine the fraction of cooperators that survive evolution.

The authors find that for *W* = 0 and moderate selection strength, the results reproduce the predictions for finite, well-mixed populations. As *W* increases, it becomes easier for cooperators to survive until *W* reaches a critical value (*Wcritical*) at which cooperators dominate for any value of T and S. The value of *Wcritical* increases as *z* increases as expected since there are more links to be rewired in order to reach a state where cooperators can dominate.

The authors find that the value of *W*, as well as the structure of the game payouts, determines the level of heterogeneity that evolves in the network. In general, as *W* increases, the heterogeneity of the evolved network increases. When the payout structure of the game favors defectors but allows cooperators to coexist at low levels, the few cooperators that remain accumulate large numbers of links leading to highly heterogeneous networks. The authors find that large T promotes heterogeneity more than large S.

Finally, the authors find that the value of *Wcritical* decreases as either of the selection strength parameters (*βe* and *βa*) increase. Smaller selection strength values allow less fit agents to survive. Prior to the network being rewired, cooperators are generally less fit than defectors. Smaller β values allow less fit cooperators to survive long enough to restructure the network into a state where cooperators dominate defectors.

Let *Nk* equal the number of nodes with degree *k* and *kmax* be the maximum degree possessed by any node in the graph. Then, the *degree of heterogeneity* of the graph *h* is given by the following equation:

And the *cumulative degree distribution* of the graph *D(k)* is given by the following equation:

The authors of [1] propose an alternative analysis of the evolution of cooperation in the two-person Iterated Prisoner’s Dilemma game.

In the one-shot Prisoner’s Dilemma game, rational choice theory[[6]](#footnote-6) stipulates that two rational (or utility maximizing) players will each choose to defect leading to the next-to-worst possible payout for each player. In order for the socially optimal outcome of mutual cooperation to be obtained, there must be a sufficiently high probability that the two players will meet again. In this case, the theory predicts that two rational players will choose to cooperate. The standard explanation given for this outcome is that the two players apply forward-looking logical deduction and come to the conclusion that mutual cooperation is the best course of action.

This explanation for cooperation in the Iterated Prisoner’s Dilemma game assumes that the participants have adequate cognitive abilities and the desire to do the following:

1. Estimate the probability of meeting their opponent in the future
2. Estimate the probability of being recognized by their opponent in a future interaction
3. Logically deduce that cooperation is the best course of action
4. Have confidence that their opponent also has adequate cognitive abilities to come to the same conclusion

In addition, the rational choice explanation assumes that cooperation emerges as a by-product of players seeking private gain. However, in human society, statements of group solidarity often justify acts of cooperation.

As an alternative to the rational choice model, the authors in [1] suggest a stochastic learning based model. In this model, the players are adaptive, backward-looking and reactive rather than purposive, forward-looking and preemptive. In this model, the game payoffs act as positive or negative rewards that reinforce or attenuate a player’s tendency to cooperate. Modeling players as having a tendency to cooperate means that players occasionally choose the unexpected action. The authors find that these random unexpected actions are critical to the ability of players to escape social traps.

The authors find that highly cooperative contestants can be draw into a stable non-cooperative social trap. However, the stochastic nature of the model allows for the chance that a fortuitous sequence of unexpected actions can lead the two actors out of the trap and into a regime of mutual cooperation. The authors find that, within the stochastic learning model, the evolution of cooperation is dependent on the length of the sequence of steps that must be coordinated in order to escape social traps.

The authors propose a *stochastic learning model of social exchange*. The model is based on the Bush-Mosteller stochastic learning model for binary choice [2]. In that model, the probability that an action will be selected at time *t* is given by the following:

where *O* is a positive constant less than one.

This model can be adapted to the Prisoner’s Dilemma game as follows. Assume the following standard payout matrix for the game:

|  |  |  |
| --- | --- | --- |
|  | **Cooperate (C)** | **Defect (D)** |
| **Cooperate (C)** | R, R | S, T |
| **Defect (D)** | T, S | P, P |

where the payouts conform to the following constraints: and . This payout matrix can be reduced to two parameters: magnitude (σ) and severity (γ) defined as follows:

For their analysis, the authors further simplify the payout matrix by instituting the following additional constraints on the payouts:

Given this simplification, and and the payout matrix can be reformulated as follows:

|  |  |  |
| --- | --- | --- |
|  | **Cooperate (C)** | **Defect (D)** |
| **Cooperate (C)** | σ, σ | -σγ, σγ |
| **Defect (D)** | σγ, -σγ | -σ, -σ |

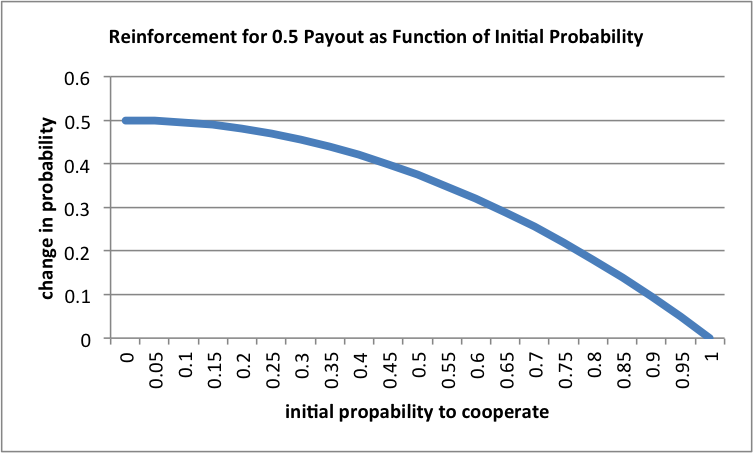
Let be the payout received by player *i* at time *t*. Given the constraints imposed on the payouts, when there is mutual cooperation (*R*) or the player unilaterally defects (*T*) and when there is mutual defection (*P*) or the player unilaterally cooperates (*S*).

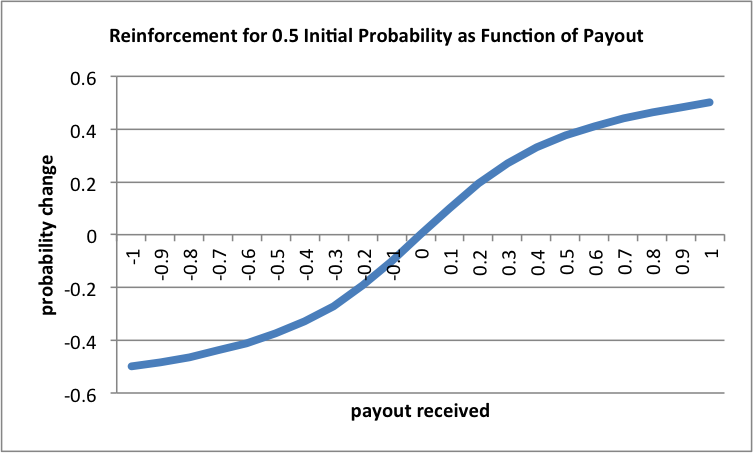
Let be the action taken by player *i* at time *t* defined as follows:

Then the probability that player *i* will cooperate at time *t* is defined as follows:

This adapted form of the model includes two reinforcement terms. The first of these applies when the player chooses to cooperate at time *t* and increases (decreases) the probability that the player will cooperate at time *t+1* if a positive (negative) payout was received at time *t*. The second term applies when the player chooses to defect at time *t* and decreases (increases) the probability that the player will cooperate at time t+1 if a positive (negative) payout was received at time *t*.

As seen in the following two charts, the reinforcement generated by the model decays with both the initial probability to cooperate and the size of the payout received. The first chart shows how the reinforcement provided given a constant payout of 0.5 decays as the initial probability to cooperate increases while the second chart shows how the reinforcement provided given a constant initial probability to cooperate of 0.5 decays as the absolute value of the payout received approaches one.





Given this model, the symmetric moves CC and DD provide positive cooperation reinforcements while the asymmetric moves CD and DC provide negative cooperation reinforcements. Let *pe* be the probability that that a player cooperates at equilibrium and let be the probability that a player defects at equilibrium. The reinforcements received by a player for each possible move and the probabilities of each move are given in the following table:

|  |  |  |
| --- | --- | --- |
| **Move** | **P(Move)** | **Reinforcement** |
| CC |  |  |
| DD |  |  |
| CD |  |  |
| DC |  |  |

At equilibrium, the positive cooperation reinforcements received by a player are balanced by the negative cooperation reinforcements received leading to no net change in the player’s probability of cooperation. This is captured in the following equation:

In computer simulations involving two agents using the model described above and playing a repeated Prisoner’s Dilemma, the authors find that two equilibriums exist along with a threshold that determines which equilibrium the agents will obtain. A non-cooperative equilibrium exists where the agents are caught in a “self defeating rut” in which asymmetric moves are too common for the agents to develop the trust required for mutual cooperation to prevail. A cooperative equilibrium exists where the agents have built up enough trust to withstand the occasional asymmetric move. A threshold exists such that if both agents’ probability to cooperate is above the threshold then the agents will be pushed into the cooperate equilibrium. Below this threshold, the agents are pushed into the non-cooperative equilibrium.

Agents can escape from the non-cooperative equilibrium through a process the authors call *stochastic collusion*. Stochastic collusion occurs when the agents are able to string together a sequence of synchronized symmetric moves that allow them increase their probability of cooperation to the point where they cross the threshold and get pulled into the cooperative equilibrium. Although, the agents select their actions at random, it appears as if the agents “are clever strategists who have finally engineered a tacit collusion”. Reducing the number of coordinated moves required to reach the threshold increases the chances of stochastic collusion occurring. Increasing the size of probability change that occurs with each move reduces the required number of moves. Increasing the magnitude (σ) of the payouts increases the size of the probability change and therefore increases the chances that the agents will escape the non-cooperative social trap.

# References

1. Macy, M. W., “Learning to Cooperate: Stochastic and Tacit Collusion in Social Exchange,” *American Journal of Sociology*, vol. 97, no. 3, pp. 808-843, 1991.
2. Bush, R. R., and F. Mosteller, *Stochastic Models for Learning*, New York: Wiley, 1955.

1. Calculated as the average frequency over the last 1000 generations [↑](#footnote-ref-1)
2. https://en.wikipedia.org/wiki/Watts\_and\_Strogatz\_model [↑](#footnote-ref-2)
3. https://en.wikipedia.org/wiki/Scale-free\_network [↑](#footnote-ref-3)
4. Calculated as the average frequency over the last 1000 generations [↑](#footnote-ref-4)
5. https://en.wikipedia.org/wiki/Scale-free\_network [↑](#footnote-ref-5)
6. https://en.wikipedia.org/wiki/Rational\_choice\_theory [↑](#footnote-ref-6)