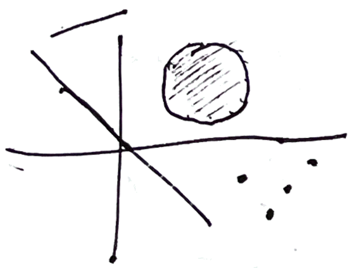
Hausdorff Measure

0. Motivation



Points, Lines, and Planes all have zero 3-dimensional Lebesgue measure. So no way to compare sizes of lower-dimensional subsets of $\mathbb{R}^3$.

A related Question: Can we define a purely metric notion of length, area, etc.?

~~Naive approach?~~

1. Definition:

$$H^p_\delta(A) = \inf \left\{ \sum_{j=1}^{\infty} (\text{diam } E_j)^p \;\middle|\; A \subseteq \bigcup_{j=1}^{\infty} E_j,\ \text{diam } E_j < \delta \right\}$$

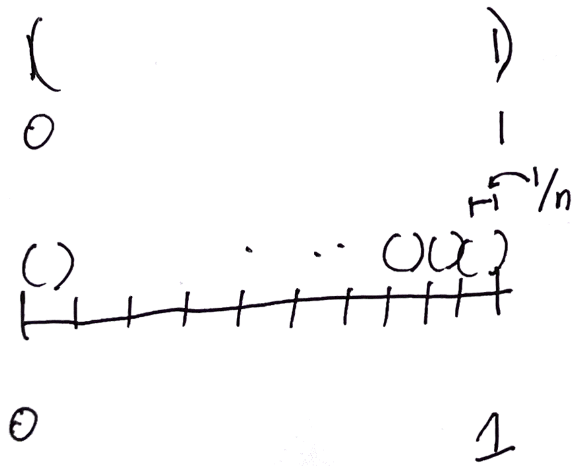
Then

$$H^p(A) = \lim_{\delta \to 0} H^p_\delta(A).$$

Note that as $\delta \to 0$, sets only are removed from the infemum, so $H^p_\delta(A)$ can only increase. Thus the limit exists, but might be ∞.

Hopefully this looks like an outer measure to you.

1-dimensional measure of a line:



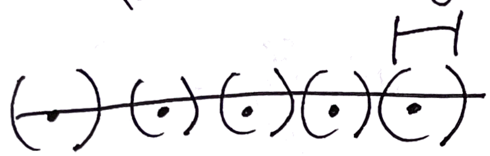
$\delta \geq 1$

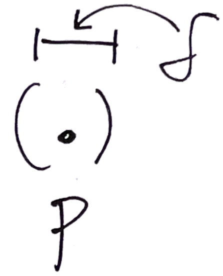
$\delta \geq 1/n$

So $$\sum_{j=1}^{n} \operatorname{diam}(E_j) = \sum^{n} 1/n = 1$$

So as $n \to \infty$, $H^1(L) = 1$.

H^0:





$$\sum (\mathrm{diam}\, E_j)^0 = \sum 1 \le$$

$$H^0(\mathbb{Q}) \le \infty$$

But for $H^\varepsilon(\{p\})$:

, $\varepsilon > 0$

$$\sum (\mathrm{diam}\, E_j)^\varepsilon = \sum \delta^\varepsilon = 0$$

So points (and countable sets) have zero Hausdorff measure for any dimension $p > 0$.

~~Def: A set A,~~

<u>Theorem:</u> If $H^p(A) < \infty$, then $H^q(A) = 0$ for all $q > p$.

Proof: $H^P(A) < \infty$, so for any $\delta > 0$

(Leave on Board for reference later) $\exists \{B_j\}$, $A \subseteq \bigcup B_j$, $\operatorname{diam} B_j < \delta$,

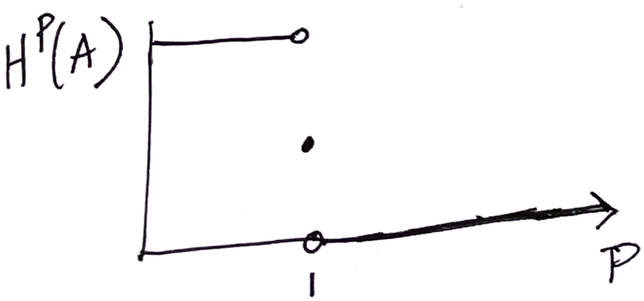
and $\sum (\operatorname{diam} B_j)^p \leq H^P(A) + 1$. $q > p$.

$$\begin{aligned}
\sum (\operatorname{diam} B_j)^q &= \sum (\operatorname{diam} B_j)^{q-p} (\operatorname{diam} B_j)^p \\
&\leq \delta^{q-p} \sum (\operatorname{diam} B_j)^p \\
&\leq \delta^{q-p} (H^P(A) + 1)
\end{aligned}$$

So as $\delta \to 0$, $\sum (\operatorname{diam} B_j)^q \leq 0$.

Contrapositive says: If $H^P(A) > 0$, then $H^q(A) = \infty$ for $q < p$.

Together, graph of Hausdorff measure over p:



s ← special value of Hausdorff measure.

At most 1 such

Cor: $\cancel{\exists!}$ $p \geq 0$ s.t. $\cancel{H^P(A)}$ $0 < H^P(A) < \infty$

(Maybe), it might not exist.

Def: Hausdorff Dimension.

$$\dim_H(A) = \inf\{p \geq 0 \mid H^p(A) = 0\}$$

$$= \sup\{p \geq 0 \mid H^p(A) = \infty\}$$

Line : dimension 1

Plane : dimension 2

$+$ H^n agrees w/ ℓ^n Lebesgue on $\mathbb{R}^n$

So agrees w/ our intuition about Manifolds.

What about the Cantor set ?

$$H^1(C) = 0$$

$$H^0(C) = \infty$$

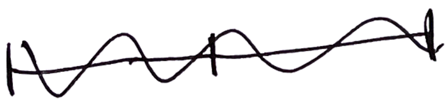
$$0 \leq \dim_H(C) \leq 1$$

~~Cantor set is actually two copies of itself, scaled down by 1/3.~~

~~Breif Detour:~~

Intuition of ~~#~~ Dimension in metric/ measure spaces.

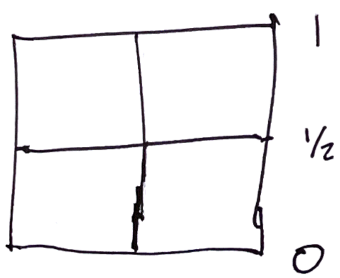
$= L$ ~~$M(L) = 1$~~



$$M\left(\tfrac{1}{2}L\right) = \left(\tfrac{1}{2}\right)^{1} M(L)$$

← measure scaling

$= \frac{1}{2}L$ ← A Metric scaling



S

$$M\left(\tfrac{1}{2}S\right) = \left(\tfrac{1}{2}\right)^{2} M(S)$$

What if Not?

Suppose $M\left(\frac{1}{2}L\right) = \left(\frac{1}{2}\right)^{1-\varepsilon} M(L)$, $\varepsilon > 0$.

If μ is really a measure:

$$\mu(\tfrac{1}{2}L) + \mu(\tfrac{1}{2}L) = \mu(L)$$

But then

$$\mu(L) = 2\mu(\tfrac{1}{2}L) = 2\left(\tfrac{1}{2}\right)^{1-\varepsilon}\mu(L)$$

$$= 2^{\varepsilon}\mu(L)$$

so $\mu(L) = 0$ or $\mu(L) = \infty$.

Looks like the proof on the board.
Shows the tension/relationship between the metric structure of the set, and the measure structure of the set.

Back to the cantor set:
The cantor set contains two copies of itself, scaled by $\frac{1}{3}$. so

$$\cancel{2\mu(\tfrac{1}{3}C) = \mu(C). \quad \text{so}}$$

$$\mu(\tfrac{1}{3}C) = \cancel{\left(\tfrac{1}{3}\right)^{\log_3(2)}}\ \tfrac{1}{2}\mu(C) = \left(\tfrac{1}{3}\right)^{\log_3(2)}\mu(C).$$

So $\dim_H(C) = \log_3(2) \approx 0.6309$

Matches our expectations:

Not quite a line, Not quite a point.

Sufficently dense that it is almost a line, but not quite, still lots of holes.