# Problem Set #4 —— Runge-Kutta on Pendulum Simulation

University of Colorado - Boulder
CSCI4446 — Chaotic Dynamics

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## 1 RK4 Implementation

See Github Repository if interested in the implementation. The code simulates a pendulum with user-defined forcing and damping via the Fourth-Order Runge Kutta Methods. Important parameters are as follows:

1. $t_0$: Initial time at which the pendulum starts
2. $dt$: Discrete time step for each RK4 calculation
3. $N$: Number of time steps to simulate
4. $A$: Amplitude of forcing function
5. $m$: Mass of the pendulum
6. $l$: Length of the pendulum
7. $\alpha$: Frequency of forcing function
8. $\beta$: Damping coefficient
9. $g$: Acceleration due to gravity (9.80665 meters per second per second)
10. $\theta_0$: Initial position of the pendulum
11. $\omega_0$: Initial velocity of the pendulum

The differential equation being modeled is:

$$ml\frac{d^2\theta}{dt^2} + \beta l\frac{d\theta}{dt} + mg\sin\theta = A\cos(\alpha t) \tag{1}$$


Figure 1: Undamped, Unforced Pendulum State Space Diagram Beginning at $\{3, 0.1\}$

where $\theta$ is a function of $t$.

To model this with RK4, the second-order equation is reinterpreted as system of first-order equations using the dummy variable $\omega = \frac{d\theta}{dt}$:

$$\begin{aligned}\frac{d\theta}{dt} &= \omega \\ \frac{d\omega}{dt} &= \frac{A}{ml}\cos(\alpha t) - \frac{\beta}{m}\omega - \frac{g}{l}\sin\theta\end{aligned}$$

The vector $\vec{x} = \{\theta, \omega\}^T$ can then be passed through RK4, allowing RK4 to process each element on its own.

## 2 State Space Analyses

### 2.1 Initial condition near unstable fixed point

Letting $t_0 = 0$, $dt = 0.005$, $N = 400$, $m = 0.1$kg, $l = 0.1$m, $g = 9.80665$ m/s$^2$, and all other constants to 0, *i.e.* undamped and unforced, and setting the starting state to $\{\theta_0, \omega_0\}^T = \{3, 0.1\}^T$, we achieve the state space result in Figure 1 This set of initial conditions corresponds to the physical situation where the pendulum is just a few degrees shy of being pointed straight up while moving slowly counterclockwise. As one might imagine, the pendulum would not have enough velocity to reach the top, and would fall back down under the influence of gravity before swaying back and forth forever, each time nearly reaching the top.

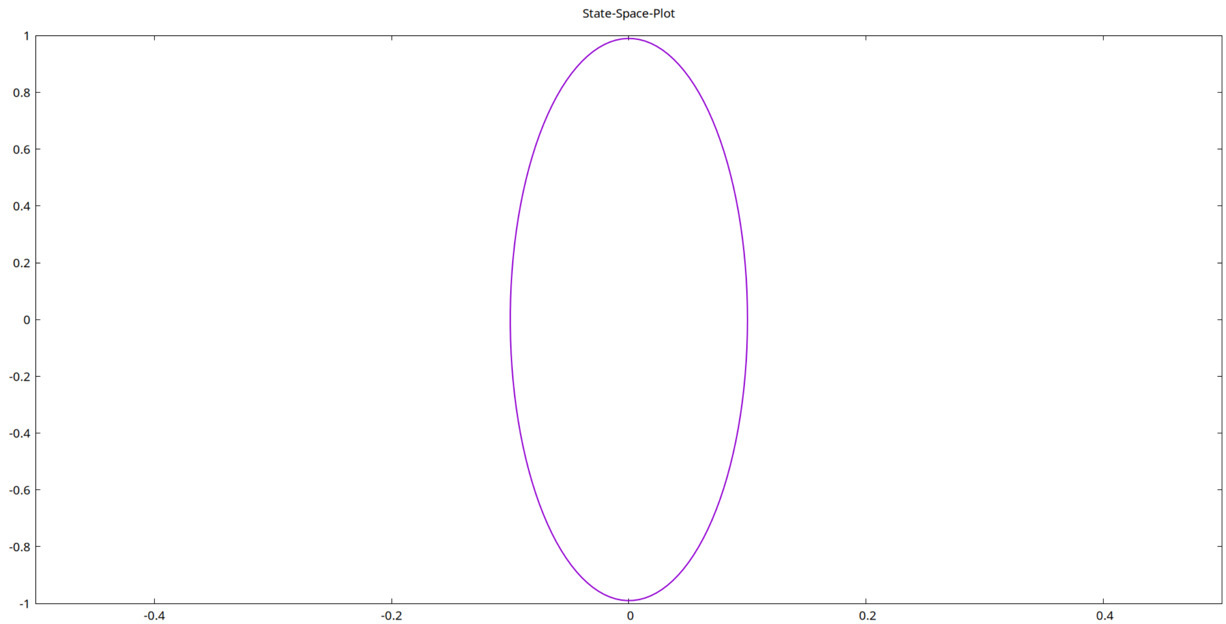



Figure 2: Undamped, Unforced Pendulum State Space Diagram Beginning at $\{0.1, 0\}$

The initial condition is very close to the unstable fixed point at $\{\pi, 0\}$, where the pendulum is perfectly balanced at the top.

### 2.2 Initial condition near stable fixed point

By setting the initial conditions near the stable fixed point of $\{0, 0\}$ with $\{0.1, 0\}$, *i.e.* the pendulum is set just to the right of the low point, we get a similar state space trajectory, as seen in Figure 2. Note the axes ranges are changed from 1. Where $|\omega|$ reached nearly 20 meters per second in 1, $|\omega|$ only did not even exceed one meter per second in 2. Additionally, 2 has a trajectory closer to that of a perfect an ellipse, as compared to 1.

## 3 Undamped State Space Portrait

In Figure 3, we can see a portrait of the state space of the system. The set of initial conditions contains the stable fixed point, small perturbations of the unstable fixed point, several points near the unstable fixed point, conditions that result in infinite rotations around the axis of the pendulum, and conditions that result in infinite swaying without going "over the top". With a higher value of $\beta$ the "swirls" would converge to the fixed point faster, and the reverse would occur with a lower $\beta$ value.

## 4 Damped State Space Portrait

By changing the damping parameter $\beta$ to 0.25, the same set of initial conditions produces the state space portrait in Figure 4.

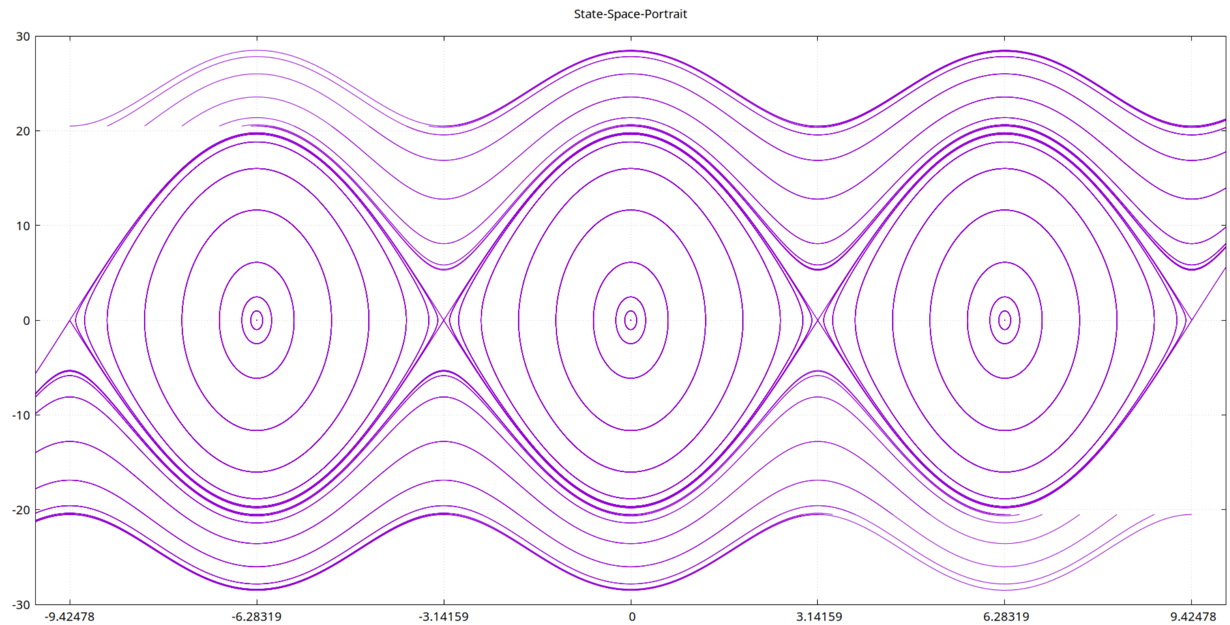



Figure 3: Undamped State Space Portrait

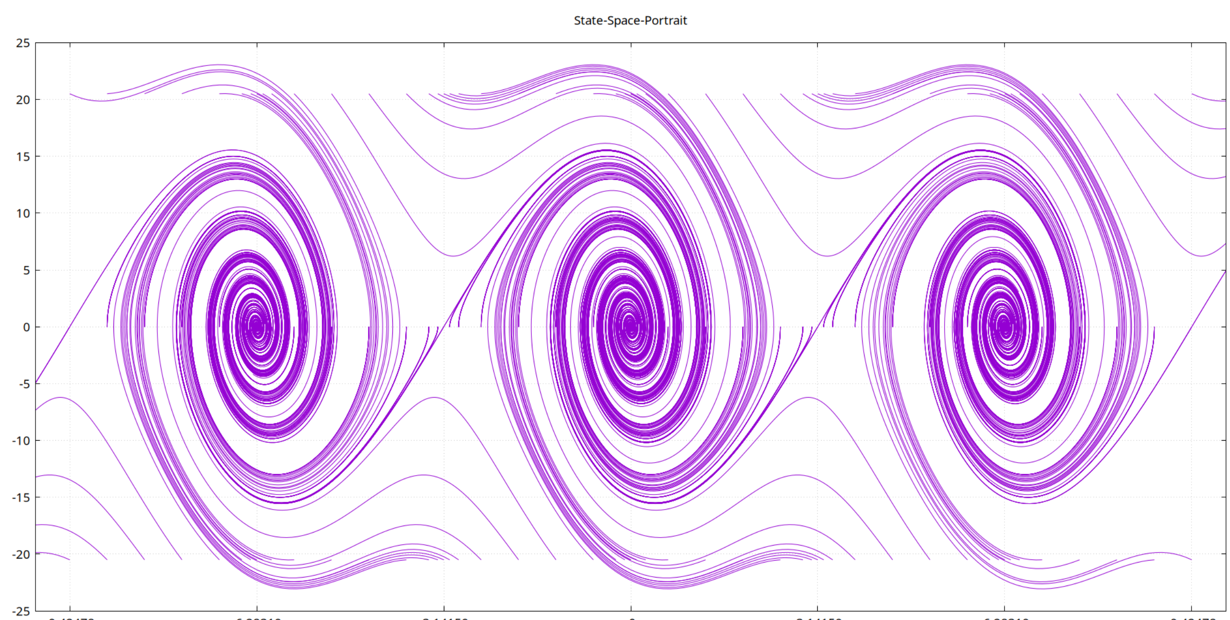



Figure 4: Damped State Space Portrait

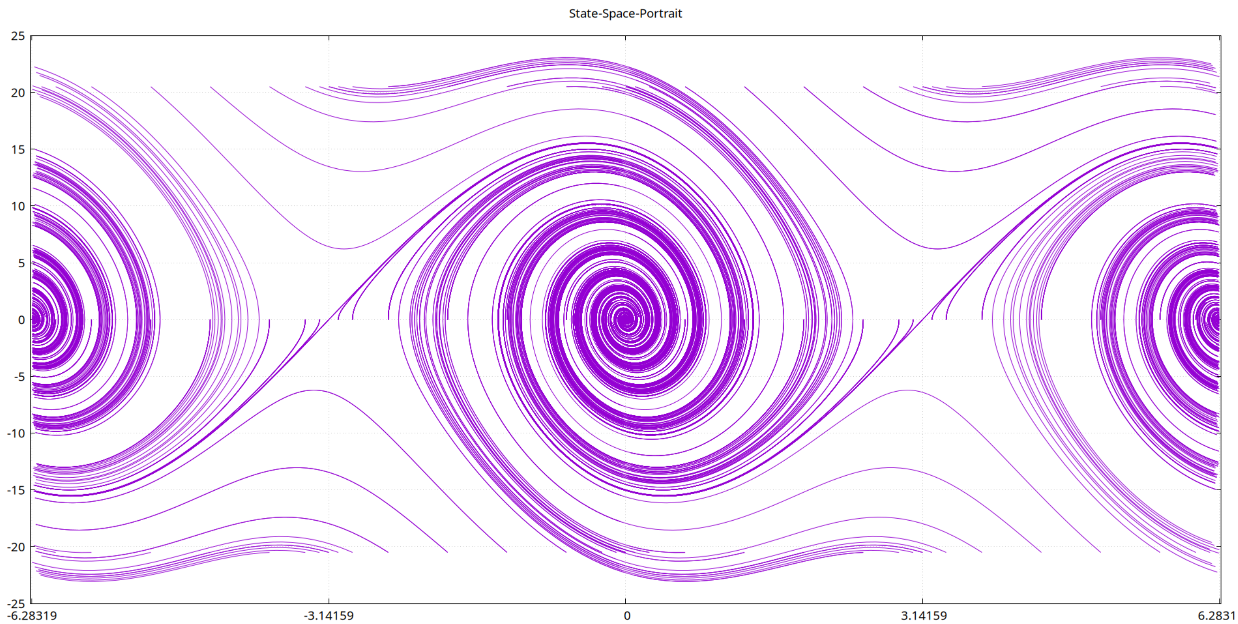



Figure 5: Damped State Space Portrait Modulo $2\pi$

# 5 Results Modulo $2\pi$

By changing the plotter so that it plots the results modulo $2\pi$, *i.e.* ignoring how many times the pendulum has rotated around its axis, we get the result in 5.

# 6 Driving the Pendulum

A pendulum's natural frequency is given by

$$f = \sqrt{\frac{g}{l}} \tag{2}$$

For the parameters we have used thus far, this comes out to $f \approx 9.902853124$ radians per second.

## 6.1 Parameter Exploration

For low values of $A$, such as $A = \frac{1}{2}$, there is a bifurcation from a fixed point to a periodic orbit at around $\alpha \neq 0$. When there is no forcing frequency but there is a forcing amplitude, the simulation converges to a fixed point at $\theta = A$, however, for any value of $\alpha$, the pendulum converges to a periodic orbit that is dependent on $\alpha$. Lower values of $\alpha$ result in slow, wide swings, while higher values produce fast, narrow swings.

For low $\alpha$ and $A = 1.2$, the pendulum switches between swinging back and forth and rotating around completely. The pattern does repeat, so it is not chaotic. Increasing $\alpha$ up to 40% of the natural frequency makes the pattern more complex, but it still repeats.

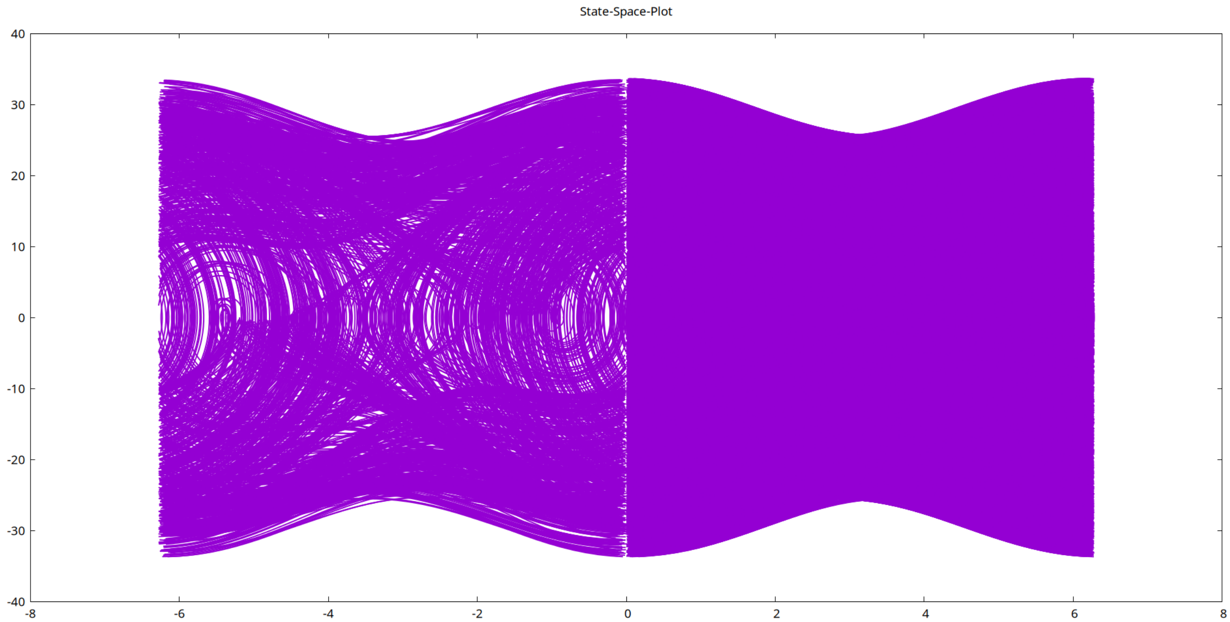


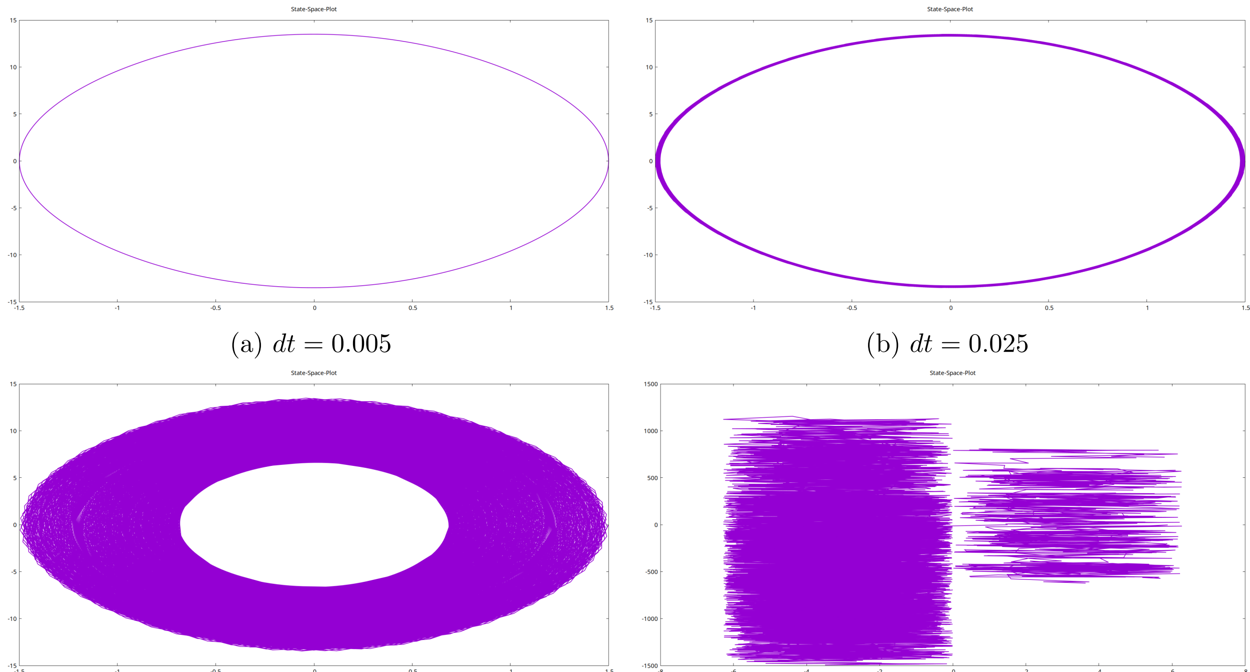
Figure 6: Damped State Space Plot with Driving—Chaotic

### 6.2 Chaotic Results

When $\alpha$ reaches 50% of the natural frequency, chaotic results appear, suggesting a bifurcation from an unstable periodic orbit to chaos somewhere between $\alpha = 4$ and $\alpha = 5$ when $A = 1.2$—other values of $A$ will have different associated values of $\alpha$ for the bifurcation. Figure 6 shows the results for $A = 1.2, \alpha = 0.5$ after 1 000 000 iterations at $dt = 0.005$. For positive values of $\theta$ (right side), the pendulum has appeared at every attainable location in the state space, and for negative $\theta$ (left side), it has almost reached every value. After more iterations, the left side would appear completely filled in. By increasing $A$ and $\alpha$, higher values of $\omega$ could be achieved as the pendulum would be driven to swing even faster.

## 7 Time-Step Exploration

The time step of 0.005 seconds was sufficiently small for reasonable accuracy in the simulation. However, by increasing the time step, the error drastically increases, to the point that the results are unusable. Figure 7 shows results after 10 000 iterations with increasing sizes of $dt$. 7a shows high accuracy at $dt = 0.005$, and by increasing $dt$ by a factor of 5, that accuracy diminishes (7b), though with still the "right idea". Multiplying by another factor of 2, we get to the point where the location and velocity of the pendulum is completely obscured, but the method is still able to determine that the pendulum is swinging back and forth, indicating it is working correctly. 7d shows useless data (note that $\omega$ reaches almost 1 500 meters per second.)




(a) $dt = 0.005$

(b) $dt = 0.025$

(c) $dt = 0.05$

(d) $dt = 0.5$

Figure 7: Pendulum with Initial Condition $\{1.5, 0\}$