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Synthesis of Reverberating Neural Networks

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Abstract. In the first Part explicit methods are given, following the work of Refs. [1—3], for the design of networks whose reverberations cannot exceed prefixed periods no matter how coefficients are changed, as well as of networks obeying pre-assigned constants of motion. In the second Part the role of coupling strengths in determining cyclic behaviors is investigated and shown to lead to new methods for the design of reverberating networks.

The present paper is the direct continuation, after a rather long time lapse in which all our activities were absorbed by the growth of our Research Group to the dimension of a full size Laboratory, of earlier works [1—3] in which a deterministic mathematical theory of neural networks was formulated and several concrete problems were discussed and solved.

The relevant feature of our approach was the use of matrix algebra, rather than boolean logic; this permitted to prove, among other things, that the rank of the coupling matrix plays an important rôle in the study of networks whose cyclic sequences of states ("reverberations") are wanted to exceed in no case a pre-assigned maximal period.

The instantaneous activity of the network is described, in our model, by "Neuronic Equations"; learning, i.e. adaptation, by "Mnemonic Equations", and is of some order of magnitude slower than the processes described by Neuronic Equations: this assumption, which is in many ways essential to the study of the subject, was called the "Adiabatic Learning Hypothesis" [1].

The present work in part completes that expounded in Ref. [3] in part introduces a wider view of the reverberation problem, of which new solutions are given that do not necessarily depend upon the rank of the coupling matrix, so as to prepare the way for a systematic study of adaptive processes, on which we propose to report in the future. Our main concern here is to restrict the Neuronic Equations to types which secure that adaptation will not change the wanted basic reverberation properties.

Part I completes and extends the work of Ref. [3]; it begins with a short review of our notation, for convenience of the reader. Part II discusses a new approach to reverberations, based on the consideration of the relative magnitudes of coupling coefficients.

I. Remarks on Rank of Coupling Matrix

1. Notation. We record only the essential points of our formalism, referring for further details to the cited references.

Let:

$$1(x)=\begin{cases}1, & x>0\\ 0, & x\leqq 0\end{cases};\quad \operatorname{sgn} x=\sigma(x)=\begin{cases}1, & x>0\\ 0, & x=0;\\ -1, & x<0\end{cases}$$

Let τ be a constant delay time; a_{hk} coupling coefficients (elements of a matrix A: they describe facilitations and inhibitions); s_h the threshold of neuron h; write:

$$\boldsymbol{u}=1[\boldsymbol{v}] \quad \text{or} \quad \operatorname{sgn} x=\sigma[\boldsymbol{v}]$$

to denote that each component of $\boldsymbol{u}$ is the Heaviside or signum function of the corresponding component of $\boldsymbol{v}$. Let $u_h(t)$ $(h=1,2,\ldots,N)$ describe the state of neuron h at time t; quantize time so that $u_{h,m}=u_h(m\tau)$, m integer. Our Neuronic Equations take then the form, for a network whose state depends only upon the immediately preceding one (a restriction maintained throughout this work):

$$\boldsymbol{u}_{m+1}=1[A\,\boldsymbol{u}_m-\boldsymbol{s}] \tag{1}$$

or equivalently ($\boldsymbol{v}_m=A\boldsymbol{u}_m-\boldsymbol{s}$)

$$\boldsymbol{v}_{m+1}=A\,1[\boldsymbol{v}_m]-\boldsymbol{s} \tag{2}$$

which reduces to:

$$\boldsymbol{v}_{m+1}=\Gamma\sigma[\boldsymbol{v}_m] \quad (\Gamma=1/2A) \tag{3}$$

if

$$\sum_k a_{hk}=2s_h;$$

it is also assumed that $v_{hm}\neq 0$ always. The special form (3) describes *normal systems* (in our terminology) [2], which play a relevant rôle in the theory.

A *reverberation* of period R is a solution of (2) or (3) such that

$$\boldsymbol{v}_{m+R}=\boldsymbol{v}_m \quad (\text{all } m\geqq \text{some suitable } m_0\geqq 0).$$

2. Remarks on the Rank of the Coupling Matrix. In Ref. [3] special consideration was given to networks reverberating with period $R \ll 2^N$ ($N =$ no. of neurons $=$ order of A and Γ); the rank K of Γ was shown to be of central importance in the study of this problem, and a typical decomposition of the matrix Γ which exhibits neatly the wanted properties was used with advantage. Emphasis was put in Ref. [3] on conditions (realizable by means of simple additional controlling devices) which secure that a network will only reverberate with period $= R$ or $\leqq R$ (after a transient of duration $\leqq R$).

Our later experience has shown that most of the conditions of Ref. [3] are particularly effective in the design of networks which must give, as their response to an arbitrary initial stimulation, a reverberation which depends only upon the structure of the network, and *not* upon the form of the stimulation (after the transient is over, of course). The interest of this situation was described in Ref. [3]; we intend to emphasize in the present work, instead, the study of reverberations which depend upon both the structure of the network *and* upon the form of the initial stimulation (with a view to forthcoming studies on learning).

We begin by recalling specifically some results already proved in Ref. [3], which deserve a more complete treatment from our present point of view.

Divide, as in Ref. [3], N-space into a K-subspace, spanned by vectors $\boldsymbol{v}'$, and an $(N-K)$-subspace spanned by vectors $\boldsymbol{v}''$. Accordingly, Γ becomes

$$\Gamma = \begin{pmatrix} H & Y \\ X & Z \end{pmatrix}; \tag{4}$$

axes are of course taken so that the K-subspace comes in the first block. Eq. (3) yields:

$$\begin{aligned} \boldsymbol{v}'_{m+1} &= H\,\sigma(\boldsymbol{v}'_m) + Y\,\sigma(\boldsymbol{v}''_m) \\ \boldsymbol{v}''_{m+1} &= X\,\sigma(\boldsymbol{v}'_m) + Z\,\sigma(\boldsymbol{v}''_m). \end{aligned} \tag{5}$$

We impose now that (for $m > 1$):

$$\begin{aligned} \boldsymbol{v}'_{m+1} &= B\,\sigma(\boldsymbol{v}'_m) \\ \boldsymbol{v}''_{m+1} &= M\,\boldsymbol{v}'_{m+1}. \end{aligned} \tag{6}$$

(The same requirement was made in Ref. [3], but there we spoke only of sufficient conditions, here we study also necessity). In other words, we require that the whole nonlinear normal system (3) of order N behave in any case like a non-linear system of reduced order K (described by a square matrix B), i.e. having a reverberation period $R \leqq 2^K$; the state in the $(N-K)$-subspace is required to be linearly dependent on the *simultaneous* state of the K-subspace. Comparison of (5) and (6) shows, clearly, that it is *necessary* that

$$\begin{aligned} X &= M\,H \\ Z &= M\,Y \end{aligned}$$

in order for (5) and (6) to be *a priori* compatible for *all* conceivable states $\boldsymbol{v}'_m$ and $\boldsymbol{v}''_m$.

It was shown in Ref. [3] that a *sufficient* condition for (6) to be true obtains if the matrix M has at most one non-vanishing element in each row: then $\left([\sigma(M)]_{hk} = \sigma(M_{hk})\right)$:

$$\boldsymbol{v}'_{M+1} = B\,\sigma(\boldsymbol{v}'_m) = \{H + Y\,\sigma(M)\}\,\sigma(\boldsymbol{v}'_m), \tag{7}$$

so that

$$B = H + Y\,\sigma(M).$$

If such an M is assigned, Eq. (7) and

$$H = B - Y\,\sigma(M) \tag{8}$$

satisfy our request, B being an arbitrary matrix of order K.

Physically, we are discussing here the case of a network whose neurons in $(N-K)$-subspace repeat each the action of an individual neuron in K-space. This situation can be of interest, e.g., in error-correcting problems; although the mathematical solution is trivial, the problem is not trivial *per se*: because this type of result does not come easy from boolean algebra, and because all conditions deriving from rank (a concept which is quite critical and does not permit tolerances in the matrix elements) can for neural networks be extended to the whole "tolerance domain" of Γ (we define by this locution the set of all matrices, including Γ, which give always rise to exactly the same sequences $\boldsymbol{u}_m$ of states as Γ).

The considerations just made on formula (8) provide a concrete example; the relevant remark here is that, after Eq. (7), it is *necessary* that the rank of Γ be $\leqq K$ in order that (6) be always true.

3. Realization of Matrices with Learning-Invariant Rank. Whatever the mnemonic laws which may change the coupling coefficients of a learning (i.e. adaptive) network, an important demand to be made upon them, from our point of view, is that the rank K of the matrix Γ be not affected by their action. The reasons for this requirement are discussed in Refs. [1] and [3]; we wish to show here how simple algebraic decompositions of the matrix Γ lead naturally to constructions which guarantee the learning-invariance of its rank.

It is, first of all, clear that a decomposition of type:

$$\Gamma = C\,\alpha\,D \tag{9}$$

where C has N rows and K columns, D has K rows and N columns and α is a square matrix of order K, secures that the rank of Γ does not exceed K, whatever the elements of C, α and D. Any construction scheme in which learning changes arbitrarily these elements, but not the basic structure (9), answers therefore satisfactorily our request; the matrix α may be used for exerting some specific controls on the network, otherwise we can suppose, without loss of generality, that it reduces to the unit matrix.

If we take:

$$\begin{aligned} C &= \begin{pmatrix} H^{(K,K)} \\ M^{(N-K,K)}\,H^{(K,K)} \end{pmatrix} \\ \alpha &= 1^{(K,K)} \\ D &= \left(1^{(K,K)} \middle| \Lambda^{(K,N-K)}\right) \end{aligned} \tag{10}$$

where the number of rows (columns) is denoted by the first (second) index, Eq. (9) gives immediately

$$\Gamma = \begin{pmatrix} H^{(K,K)} & [H\Lambda]^{(K,N-K)} \\ [M\,H]^{(N-K,K)} & [M\,H\,\Lambda]^{(N-K,N-K)} \end{pmatrix}$$

which is the special form of Eq. (4) used in Ref. [3] (our $H \equiv A$ of Ref. [3]).

We think it instructive to show in detail, as an example, how a network satisfying Eqs. (6) and (8) (and of course the requirement posed upon M for Eq. (8) to hold) can be easily realized with structures of type (9) and (10) (we put $\alpha = 1^{(K,K)}$). For greater generality we take the Neuronic Equations in the form (1), (2); then, starting with (9):

$$A = 2\Gamma = 2CD \tag{11}$$

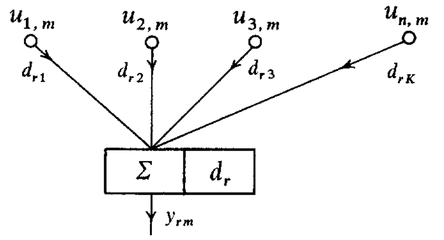
and setting $s'_h = \frac{1}{2} s_h$:

$$\boldsymbol{u}_{m+1} = 1[CD\boldsymbol{u}_m - \boldsymbol{s}']. \tag{12}$$

Call

$$y_m = D\boldsymbol{u}_m \tag{13}$$

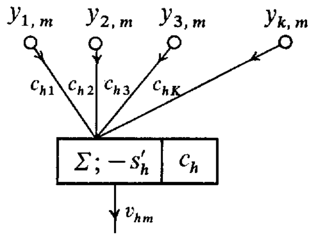
the K-vector produced by the action of $D^{(K,N)}$ on the N-vector $\boldsymbol{u}_m$. This action can be materialized by constructing K blocks d_r, one for each row of the matrix D, such that the output $\boldsymbol{u}_m$ at time m of N neurons enters the block d_r $(r = 1, 2, \ldots, K)$ and comes out as the $y_{r,m}$ given by Eq. (13):



To complete the transformation of the output at time m into the input of the network at the same time m we must now form the vector

$$\boldsymbol{v}_m = c\,\boldsymbol{y}_m - \boldsymbol{s}; \tag{14}$$

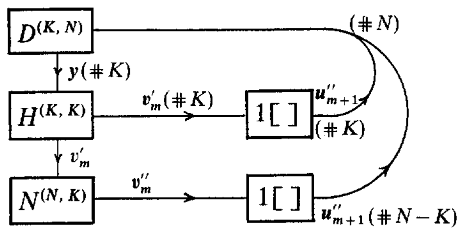
this can be done by means of N blocks c_h (one for each row of the matrix $C^{(N,K)}$):



The K blocks d_r multiply each incoming signal $u_{s,m}$ by a coefficient d_{rs} $(r = 1, 2, \ldots, N)$ and add; the N blocks c_h multiply each incoming signal $y_{r,m}$ by a coefficient c_{hr} $(r = 1, 2, \ldots, K)$, add and discriminate against the thresholds s'_h; they will give, after a delay τ, the response (12), which is the output $\boldsymbol{u}_{m+1}$ at time $m+1$ of the network.

In order to restrict C to the form (10) the blocks c_h can be conveniently split into two parts; the first K of them give as output the components of the K-vector $\boldsymbol{v}'_m$, which on the one hand is changed directly into the response (12), on the other is transmitted to $N-K$ blocks which perform the functions attributed to the matrix M of Eq. (8) and then, again, give the response (12).

We have therefore: a layer D of blocks d_r; a layer C of blocks c_h, which in turn reduces, in the cited example, to two layers H and M. Denoting these layers (which *are* the network: the N outputs of the layer H, M, or C, where the delay τ is effected, are those of the "neurons"; they act immediately upon the N inputs of the layer D, again those of the neurons) with a single, compact scheme, we find:



This scheme, of course, can be changed and implemented in many ways; a control matrix α can be inserted, as was said, between blocks D and H. We have obtained with it interesting results, both on the computer and in hardware, the reporting of which falls, however, beyond our present scope. The point which is relevant for us here is that, clearly, the rank of Γ can never exceed K no matter how the coupling values are changed within each block, and that, with M specified as in Eq. (8), the reverberation period R never exceeds 2^K.

4. *Remarks on Constants of Motion.* It was shown in Ref. [3] that, if the matrix A in Eqs. (1) or (2) has order N and rank K, there exist $N-K$ *constants of motion* of the network, namely the linear expressions

$$w_r = \boldsymbol{\gamma}_r \cdot \boldsymbol{v}(t) = -\boldsymbol{\gamma}_r \cdot \boldsymbol{s} \quad (r = 1, 2, \ldots, N-K) \tag{15}$$

where

$$A^T \boldsymbol{\gamma}_r = 0 \quad (A^T = \text{transpose of } A), \tag{16}$$

the vectors $\boldsymbol{\gamma}_r$ being the $N-K$ linearly independent solutions of Eq. (16).

We wish to show here how to construct a network of N neurons which admits $N-K$ constants of motion w_r fixed *a priori.*

Set

$$G \equiv (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \ldots, \boldsymbol{\gamma}_{N-K}) = \begin{pmatrix} G' \\ G'' \end{pmatrix} \tag{17}$$

where $\boldsymbol{\gamma}_h$ denotes the h-th column of the matrix G; G' projects onto K-subspace, etc. Then:

$$A^T G = 0, \tag{18}$$

that is, from Eqs. (4) and (10):

$$\begin{pmatrix} H^T & H^T M^T \\ \Lambda^T H^T & \Lambda^T H^T M^T \end{pmatrix} \begin{pmatrix} G' \\ G'' \end{pmatrix} = 0; \tag{19}$$

or

$$H^T G' + H^T M^T G'' = 0$$
$$\Lambda^T H^T G' + \Lambda^T H^T M^T G'' = 0;$$

these equations are both satisfied if we take:

$$G' = -M^T G''; \tag{20}$$

this shows that Eq. (18) is solved by taking G'' as an arbitrary square matrix of order $N-K$ and then determining G', and therefore all the vectors $\boldsymbol{\gamma}_h$ of def. (17), from Eq. (20).

Consider next the $N-K$ constants w_r of (15) as the components of an $(N-K)$-vector $\boldsymbol{w}$, which is given, after (17), by

$$\boldsymbol{w} = -G^T \boldsymbol{s}. \tag{21}$$

It is now clear that we can take in Eq. (19) H and Λ completely arbitrary, the only constraints on the matrix A of the network being Eqs. (20) and (21). The former determines M as

$$M^T = -G' G''^{-1} \tag{22}$$

if G'' has an inverse; the latter shows that this is possible in infinitely many ways. If the matrix G'' has rank $N-K$ we can in fact solve Eq. (21) $\left(\mathbf{s} \equiv \begin{pmatrix}\mathbf{s}'\\ \mathbf{s}''\end{pmatrix}\right)$ by setting:

$$\mathbf{s}'' = -[G''^T]^{-1}(\boldsymbol{w} + G'^T \mathbf{s}'). \tag{23}$$

We see thus that, once the constants of motion w_r, i.e. the matrix G, are assigned, all networks that admit those constants can be obtained by fixing H, Λ and $\mathbf{s}'$, arbitrarily and then determining M from (22) and $\mathbf{s}''$ from (23). The assignement of the w_r's alone would leave of course still greater arbitrariness, but it is hardly interesting.

The conditions just stated show what kind of adaptation can take place in such a network without altering its constants of motion.

II. Rôle of Coupling Strengths

1. The concept of rank of a matrix Γ has proved [3] a corner-stone in our approach to reverberating networks; it should be clear, however, from the considerations made at the very inception of this theoretical approach ("first stability criterion" of sect. 4, B.2 of Ref. [1]) of which the definition of the tolerance domain of Γ is only a convenient re-wording, that the results thereby obtained for a matrix Γ of rank K can be profitably extended to its whole tolerance domain, i.e. to whole classes of matrices of unrestricted rank. This aspect will be exhaustively explored in a forthcoming paper; the considerations that follow are intended to provide a first example of structural properties of coupling matrices, not directly connected to rank, which offer other solutions of the reverberation problem and permit useful insights into the rôle played in this context by the relative strengths of individual coupling coefficients.

We use the form (1) of the Neuronic Equations, and define as *equivalent to* A, or belonging to the *equivalence class of* A, all matrices contained in the tolerance domain of A before specified. Let, consistently with the previous notation,

$$V_h(\boldsymbol{u}) = \sum_{j=1}^{N} a_{hj} u_j - s_h; \tag{24}$$

all quantities u_j, v_h are taken at the same time, the time label is omitted because unimportant here. We are interested in situations in which some of the coefficients a_{hj} in (24) are dominant with respect to others $a_{hj'}$, in the sense that the firing of neurons j' either does not affect, or affects only exceptionally, the firing of neuron h. It is convenient, for this purpose, to define the *class* of the coupling of a neuron h with a set I $(I \leqq N)$ of input neurons $k_1, k_2, \ldots, k_I$, as follows.

Each state of the set I is characterized by a vector $\boldsymbol{z}$ having I components $(=0, 1)$; consider a class Z_q containing q such states $(q < 2^I)$:

$$Z_q \equiv \{\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_q\}; \tag{25}$$

we shall say that the coupling of neuron h with the neurons of I is *of class* Z_q if the coefficients $a_{hk_1}, \ldots, a_{hk_I}$ in (24) can be always annihilated without altering the sign of $V_h(\boldsymbol{u})$ when, and only when, the state of I does *not* belong to Z_q.

If, in particular, I contains only one neuron k, only two classes are possible:

$$Z_1 \equiv \{0\} \quad \text{and} \quad Z_1 \equiv \{1\};$$

in the first case putting $a_{hk} = 0$ when $u_k = 1$ does not alter $v_h(\boldsymbol{u})$, so that one obtains an equivalent matrix by annihilating a_{hk} altogether; the second case describes the situation in which the annihilation of a_{hk} would lead instead out of the equivalence class: changes of a_{hk} can certainly affect the rank. We shall say, for short, that the *coupling* a_{hk} *is of class* 0 in the first case, *of class* 1, *or strong,* in the second.

We can thus rephrase, if we wish, the former definition of class of the coupling of h with I by saying that it is of class z_q when the coefficients $a_{hk_1}, \ldots, a_{hk_I}$ behave as strong when the state of I is contained in z_q, as of class 0 otherwise.

2. Conditions for couplings to be of a given class are easily obtained. Call p_h $(p_h > 0)$ the smallest positive value, n_h $(n_h \leqq 0)$ the largest negative value that $v_h(\boldsymbol{u})$ can assume when the components 0, 1 of $\boldsymbol{u}$ range among all 2^N *a priori* available states.

The necessary and sufficient condition for a_{hk} to be of class 0 is that

$$n_h \leqq a_{hk} < p_h. \tag{26}$$

To see this, it suffices to write the expression (24) as

$$v_h(\boldsymbol{u}) = a_{hk} u_k + \sum_{j \neq k} a_{hj} - s_h \tag{27}$$

and to remaind that it can never assume a value between n_h and p_h. Annihilation of a_{hk} will not change the sign of (27) if, and only if, (26) is satisfied.

Likewise, the necessary and sufficient condition for the coupling of h with I, as described in the previous section, to be of the class Z_q defined by (25) is that the relations:

$$n_h \leqq \sum_{k=k_1}^{k_I} a_{hk} z_k < p_h \tag{26'}$$

be verified by all, and only, the states $\boldsymbol{z}_i$ which do *not* belong to Z_q. The proof is identical to the one just mentioned.

3. Consider now a neural network in which each of the N neurons has a strong coupling with the first K neurons, and a coupling of class Z_q with the remaining $N-K$ (here, $I = N-K$); we may call the first K *master neurons*, the remaining $N-K$ *neurons of class* Z_q. The states of the network are described by vectors

$$\begin{pmatrix}\boldsymbol{y}_i\\ \boldsymbol{z}_j\end{pmatrix}, \quad \text{with} \quad i = 1, \ldots, 2^k; \quad j = 1, \ldots, 2^{N-K},$$

$\boldsymbol{y}_i$ describing the master neurons and $\boldsymbol{z}_j$ the neurons of class Z_q.

In order to compute the maximum permissible reverberation period in this situation, we have simply to count how many states are allowed by the network at time $m+1$ following an arbitrary state at time m. Out of the 2^{N-K} *a priori* possible states z_j, we must distinguish those which correspond to the class Z_q

from the remaining ones. The latter are, by definition, handled by the network as if they were identical with:

$$\begin{pmatrix} y_i \\ 0 \end{pmatrix}; \tag{28}$$

the former can be listed as

$$\begin{pmatrix} y_i \\ z_1 \end{pmatrix}; \dots, \begin{pmatrix} y_i \\ z_q \end{pmatrix}. \tag{29}$$

The subsequent state at time $m+1$ is uniquely determined by each of the states (28), (29): the network, however excited at time 0, from time 1 on can only therefore assume

$$(1+q)\,2^K \tag{30}$$

different states; this number is thus the upper bound on the reverberation period we were looking for.

Consider, as a check, the class $Z_{2^{N-K}-1}$ formed by *all* possible states of the last $N-K$ neurons with the exception of the state 0. (30) gives as upper bound on the reverberation period

$$(1+2^{N-K}-1)\cdot 2^K = 2^N,$$

as expected. If, on the other hand, all couplings of the N neurons to the last $N-K$ neurons are of order 0, then clearly $q=0$ and the maximum period is 2^K.

4. In conclusion, it is instructive to consider some particular cases which correspond to appropriate choices of the class Z_q. Call $Z_{(r)}$ the class formed by all the states whose number of vanishing components is $\leqq r$. We propose to study a network whose first K neurons are master neurons and the remaining $N-K$ are of class $Z_{(r)}$.

To do so, we may begin by fixing in some suitable but arbitrary way the first K coupling coefficients and the thresholds for each of the N neurons of the network. In order that the last $N-K$ neurons have a coupling of class $Z_{(r)}$ with h, we must have, from (27):

$$n_h \leqq \sum_{i=1}^{q} a_{hk_i} < p_h \tag{31}$$

for any arbitrary choice $k_1, \dots, k_q$ of $q < N-K-r$ from among the last $N-K$ neurons. All conditions (31) are easily satisfied if we take, for instance, all coefficients a_{hk} for all $k > K$ equal to one another and such that

$$n_h \leqq q\cdot a_{hk} < p_h$$

for all

$$q < N-K-r$$

and

$$a_{hk} \geqq \frac{p_h}{q}$$

or

$$a_{hk} < \frac{n_h}{q}$$

for every $q \geqq N-K-r$.

These requirements are fulfilled by taking e.g.,

$$a_{hk} = \frac{p_h}{N-K-r}$$

or

$$a_{hk} = \frac{n_h}{N-K-r} - \varepsilon,$$

with a suitable $\varepsilon > 0$.

The number of states belonging to the class $z_{(r)}$ is, clearly:

$$\sum_{j=1}^{r} \binom{N-K}{j},$$

so that the upper bound on the reverberation period for such a network is given by (30) as:

$$R = 2^K\left(1 + \sum_{j=0}^{r} \binom{N-K}{j}\right),$$

with $r \leqq N-K-1$. For $R = N-K-1$ we find again, of course:

$$R = 2^N.$$

For $r=0$ we find

$$R = 2^{K+1}.$$

In conclusion, we remark that this method of realizing reverberating networks with *a priori* bounded periods should be of interest in the study of learning or adaptation.

Admissible changes of the network can take place both by having master neurons shift there rôles with other neurons, or by allowing, through learning, only some pre-assigned of states in (30) to assume a significant role.

It is also of interest to note that in some solid-state devices now under investigation master neurons appear both naturally and necessarily, and that some biological neurons may be thought of as playing a similar role in living neuronal systems.

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