Analysis and Synthesis of a Class of Neural Networks: Linear Systems Operating on a Closed Hypercube

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Abstract —In the present paper we investigate the qualitative properties of a class of neural networks described by a system of first-order linear ordinary differential equations which are defined on a closed hypercube of the state space with solutions extended to the boundary of the hypercube. When solutions are located on the boundary of the hypercube, the system is said to be in a saturated mode.

The class of systems considered herein retains the basic structure of the Hopfield model and is easier to analyze, synthesize and implement than the Hopfield model. An efficient analysis method is developed which can be used to completely determine the set of asymptotically stable equilibrium points and the set of unstable equilibrium points. The latter set can be used to estimate the domains of attraction for the elements of the former set. The synthesis procedure which we developed in [5] is modified and applied to the present class of neural networks. The class of systems considered herein can easily be implemented in analog integrated circuits.

The applicability of the present results is demonstrated by means of several specific examples.

I. Introduction

We WILL consider neural networks described by a system of first-order linear ordinary differential equations which are defined on a closed hypercube. We will refer to such systems as "linear systems in a saturated mode", (LSSM). Specifically, we will consider neural networks described by equations of the form

$$dx/dt = Tx + I \qquad \text{(M)}$$

with the constraints

$$-1 \leqslant x_i \leqslant 1, \qquad i = 1, \cdots, n$$

where $x = (x_1, \cdots, x_n)^T \in \boldsymbol{D}^n = \{x \in \boldsymbol{R}^n\colon -1 \leqslant x_i \leqslant 1,\ i = 1, \cdots, n\}$, $T = [T_{ij}]$ is an $n \times n$ constant matrix, and $I = (I_1, \cdots, I_n)^T$ is a constant vector. The main difference between LSSM systems (M) and usual linear systems is that the latter are defined on open subsets of $\boldsymbol{R}^n$ while the former are defined on the closed subset $\boldsymbol{D}^n$ of $\boldsymbol{R}^n$. For system (M), we will introduce a new kind of solutions, called solutions in the "saturated" mode. System (M) can easily be implemented by analog VLSI circuits.

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The present class of neural networks is closely related to the analog Hopfield model [1] and the results obtained in this paper can be applied directly to the Hopfield model.

The results which we establish for system (M) fall into one of two categories. One type of result addresses *analysis* while the other type pertains to *synthesis* procedures for system (M). In the following, we give a brief summary of the results developed herein:

Analysis: (These results are obtained under an appropriate set of assumptions, which we will give later.)

a) First, a definition of solutions of system (M) is developed. On the boundary $\partial\boldsymbol{D}^n$, solutions in the "saturated" mode are considered. Making use of this definition of solutions, we show that for any initial condition, there is a unique solution of (M), and all such solutions can be extended to the infinite time interval.

b) The concept of equilibrium point for (M) is made precise. An efficient algorithm is developed to determine the location of each equilibrium point of (M) and to determine whether it is asymptotically stable or unstable.

c) It is shown that system (M) has at most 3^n equilibrium points and at most 2^n asymptotically stable equilibrium points. The distribution of the equilibrium points is also discussed.

d) An energy function E for system (M) is specified. We show that there is a one-to-one correspondence between the set of local minimum points of E and the set of asymptotically stable equilibrium points of (M).

e) We also show that along each solution of (M), the energy function E decreases monotonically, and system (M) will not exhibit periodic solutions.

Synthesis:

We define the set $\boldsymbol{B}^n$ as $\boldsymbol{B}^n = \{x \in \boldsymbol{R}^n\colon x_i = 1 \text{ or } -1,\ i = 1, \cdots, n\}$. We will establish results along the following lines.

Given m vectors in $\boldsymbol{B}^n$, say $\alpha_1, \cdots, \alpha_m$, we wish to design a system (M) such that

1) $\alpha_1, \cdots, \alpha_m$ are asymptotically stable equilibrium points of system (M).

2) The system has no periodic solutions.

3) The total number of asymptotically stable equilibrium points of (M) in the set $\boldsymbol{B}^n - \{\alpha_1, \cdots, \alpha_m\}$ is as small as possible.

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4) The domain of attraction of each α_i is as large as possible.

The synthesis procedure developed in [5] is modified for system (M). In so doing, we consider a class of LSSM systems given by

$$dx/dt = T_\tau x + I_\tau \qquad (\mathrm{M}_\tau)$$

with the constraints

$$-1 \leqslant x_i \leqslant 1, \qquad 1 \leqslant i \leqslant n$$

where $x \in \boldsymbol{D}^n$ and $\tau \in \boldsymbol{R}$ is a parameter. The precise meaning of the parameter τ will be made clear later. Let L_a be the affine subspace of $\boldsymbol{R}^n$ generated by $\{\alpha_1, \cdots, \alpha_m\}$, which contains $\{\alpha_1, \cdots, \alpha_m\}$. When the parameter τ is large enough, we show that

a) system (M_τ) does not exhibit periodic solutions.

b) the set of asymptotically stable equilibrium points contained in $\boldsymbol{B}^n$ is approximately equal to $\boldsymbol{B}^n \cap L_a$.

The above synthesis method is also generalized to system (M) when the given data set (i.e., the desired set of asymptotically stable equilibrium points) is contained in $\Lambda = \{x \in \boldsymbol{D}^n\colon\ x_i = \pm 1 \text{ or } 0,\ i = 1, \cdots, n\}$ instead of $\boldsymbol{B}^n$.

For system (M_τ), we also discuss the effects of parameter variations in the system interconnecting matrix $T = [\boldsymbol{T}_{ij}]$.

The structure of the remainder of this paper is as follows. In Section II, we establish the notation used throughout. In Section III, we define the system (M) and we compare it with the analog Hopfield model. In this section, we also discuss aspects of the implementation and simulation of system (M). In Section IV, we state and prove the results summarized above under the category *Analysis* while in Section V, we establish the results and procedure summarized above under the category *Synthesis*. The applicability of the results of Section V is demonstrated by means of specific examples in Section VI. The paper is concluded with a few pertinent remarks in Section VII.

II. Notation

Let V and W be arbitrary sets. Then $V \cup W$, $V \cap W$, $V - W$ and $V \times W$ denote the union, intersection, difference, and Cartesian product of V and W, respectively. If V is a subset of W, we write $V \subset W$ and if x is an element of V, we write $x \in V$. If f is a function from V into W, we write $f\colon V \to W$ and we let $f(U) = \{f(x) \in W\colon x \in U\}$ for $U \subset V$, and $f^{-1}(y) = \{x \in V\colon f(x) = y\}$ for $y \in W$. Let $\varnothing$ denote the empty set, let $\boldsymbol{R}$ denote the set of real numbers, and let $\boldsymbol{R}^+ = [0, +\infty)$. If $V_1, \cdots, V_n$ are n arbitrary sets, their Cartesian product is denoted by $\Pi_{i=1}^n V_i = V_1 \times \cdots \times V_n$. If in particular, $V = V_1 = \cdots = V_n$, we write $\Pi_{i=1}^n V_i = V^n$. Let $\boldsymbol{R}^n$ be real n-space. If $x \in \boldsymbol{R}^n$, then $x^T = (x_1, \cdots, x_n)$ denotes the transpose of x. When using a norm for $x \in \boldsymbol{R}^n$, $|x|$, we will have in mind $|x| = \sqrt{\sum_{i=1}^n x_i^2}$. If $x \in \boldsymbol{R}^n$ and $Y \subset \boldsymbol{R}^n$, then $x \perp Y$ will mean that $x^T \cdot y = 0$ for all $y \in Y$. If $V \subset \boldsymbol{R}^n$, then $\overline{V}$, V^0 and ∂V represent the closure, interior and the boundary of V in $\boldsymbol{R}^n$, respectively. Also, we let $B(\tilde{x}, r) = \{x \in \boldsymbol{R}^n\colon\ |\tilde{x} - x| < r\}$ for $\tilde{x} \in \boldsymbol{R}^n$ and $r > 0$. Let $\boldsymbol{B}^n = \{x \in \boldsymbol{R}^n\colon\ x_i = 1 \text{ or } -1,\ i = 1, \cdots, n\}$ and $\boldsymbol{D}^n = \{x \in \boldsymbol{R}^n\colon\ -1 \leqslant x_i \leqslant 1,\ i = 1, \cdots, n\}$. If $x, y \in \boldsymbol{R}^n$, let $x * y = (x_1 y_1, \cdots, x_n y_n)^T$ and $\min(x) = \min\{x_i\colon\ 1 \leqslant i \leqslant n\}$. If $A = [A_{ij}]$ is an arbitrary matrix, then A^T denotes the transpose of A and the norm of A is defined as $|A| = \sup_{|x| \leqslant 1}\{|Ax|\}$. If A is a symmetric matrix, by $A > 0$ we mean that A is positive definite and by $A \geqslant 0$ we mean that A is positive semidefinite. $\mathrm{Sym}(n)$ denotes the symmetric group of order n. If $\{x_1, \cdots, x_m\} \subset \boldsymbol{R}^n$, then $\mathrm{Span}(x_1, \cdots, x_m)$ denotes the subspace of $\boldsymbol{R}^n$ generated by $x_1, \cdots, x_m$ and $A\,\mathrm{span}(x_1, \cdots, x_m)$ denotes the affine subspace of $\boldsymbol{R}^n$ generated by $x_1, \cdots, x_m$. If $x_0 \in \boldsymbol{R}^n$ and L is a subspace of $\boldsymbol{R}^n$, then $L + x_0$ denotes the affine subspace of $\boldsymbol{R}^n$ produced by shifting L by x_0, that is, $L + x_0 = \{y \in \boldsymbol{R}^n\colon\ y = x + x_0,\ x \in L\}$. Finally, if $x, y \in \boldsymbol{R}^n$, $\angle(x, y)$ denotes the angle between x and y, i.e., $\angle(x, y) = \arccos(x^T \cdot y/(|x||y|))$.

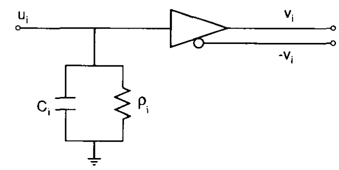


Fig. 1. The ith neural cell in the analog Hopfield model.

III. Neural Network Models and Implementations

We will consider neural networks described by LSSM systems given by

$$dx/dt = Tx + I \qquad (\mathrm{M})$$

with the constraints

$$-1 \leqslant x_i \leqslant 1, \qquad i = 1, \cdots, n \qquad (3.1)$$

where $x = (x_1, \cdots, x_n) \in \boldsymbol{D}^n$, $T = [T_{ij}]$ is an $n \times n$ constant matrix, and $I = (I_1, \cdots, I_n)^T$ is a constant vector.

Remark 3.1: System (M) is defined on the closed subset $\boldsymbol{D}^n$ of $\boldsymbol{R}^n$ instead of an open subset of $\boldsymbol{R}^n$ as is usually the case. The definition of solution of system (M) on the boundary of $\boldsymbol{D}^n$ will be precisely given in the next section. ■

One of the reasons for studying system (M) is to remedy several basic shortages of the analog Hopfield model [1].

In [1], Hopfield considers electric circuits $(\mathrm{C_H})$, given in Fig. 2, as neural systems. In such circuits, there are n identical nonlinear amplifiers. If we do not consider the input capacitance and input resistance, the input–output relation of the ith nonlinear amplifier is given by

$$v_i = g_i(\lambda u_i) = (2/\pi)\tan^{-1}((\pi/2)\lambda u_i)$$

where u_i is the input, v_i is the output and the parameter λ is the gain of the nonlinear amplifiers. It is assumed that the response time of each amplifier is negligible compared to the time constant determined by the input capacitance

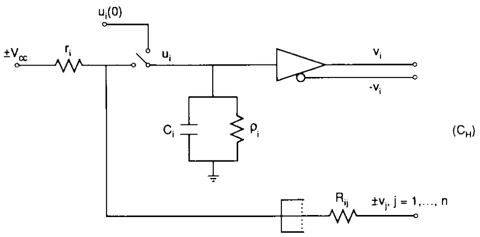


Fig. 2. Implementation of analog Hopfield model.

and input resistance. The ith nonlinear amplifier can be illustrated as in Fig. 1.

In circuit (C_H), neurons are connected to each other, as in Fig. 2, where $\pm V_{cc}/r_i$ is the input current to the ith neuron, $u_i(0)$ is the initial condition for the ith neuron, and R_{ij} denotes the resistor connecting the output of the jth neuron to the input of the ith neuron.

The circuit (C_H) can be described by the set of differential equations

$$\frac{du_i}{dt} = \frac{1}{C_i} \cdot \left\{ \sum_{j=1}^{n} \frac{1}{R_{ij}}(\pm v_j) - \left[\frac{1}{\rho_i} + \frac{1}{r_i} + \sum_{j=1}^{n} \frac{1}{R_{ij}}\right] u_i + \frac{\pm V_{cc}}{r_i} \right\}, \quad i = 1, \cdots, n. \quad (H_c)$$

If in (H_c), we let

$$T_{ij} = \begin{cases} +\dfrac{1}{R_{ij}}, & \text{if } R_{ij} \text{ is connected to } v_j \\ -\dfrac{1}{R_{ij}}, & \text{if } R_{ij} \text{ is connected to } -v_j \end{cases}$$

$$R_i = \frac{1}{\rho_i} + \frac{1}{r_i} + \sum_{j=1}^{n} \frac{1}{R_{ij}}$$

and

$$I_i = \frac{\pm V_{cc}}{r_i}$$

then (H_c) assumes the form

$$C_i(du_i/dt) = \sum_{j=1}^{n} T_{ij} v_j - u_i/R_i + I_i, \qquad i = 1, \cdots, n \quad (H)$$

which is given by Hopfield in [1].

Compared to other models, the Hopfield model is easily implemented by electric circuits and has been used in various applications (cf. [1]–[3]). However, as noted below, the Hopfield model suffers from several deficiencies.

Remark 3.2: 1) Concerning the implementation of system (H): Since the variable $u = (u_1, \cdots, u_n)^T$ of system (H) varies in $\boldsymbol{R}^n$, $|u|$ may assume very large values and scaling may pose problems in implementations. Furthermore, whenever the value of R_{ij} is altered to adjust the corresponding value of T_{ij} in (H), the value of R_i in (H) changes also.

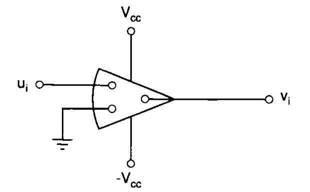


Fig. 3. The ith operational amplifier in (C_M).

2) Concerning the analysis of system (H): As discussed in [4], the set of equilibrium points for (H) corresponds to the set of solutions of the equations

$$\sum_{j=0}^{n} T_{ij} g(\lambda u_j) - u_i/R_i + I_i = 0, \qquad i = 1, \cdots, n.$$

Since it is difficult to solve this set of equations, it is hard to check the performance of system (H).

3) Concerning the synthesis of (H): In [2] and [4], the energy function for (H) given by

$$E(v) = -(1/2) v^T \cdot T \cdot v - v^T \cdot I + (1/\lambda) \sum_{i=1}^{n} \int_0^{v_i} (1/R_i) g_i^{-1}(\rho) \, d\rho$$

is employed. In the synthesis procedure for system (H), developed in [5], the third term in the energy function is neglected, with the consequence that the location of the equilibrium points can not be precisely synthesized.

The effects of the integrator terms in the energy function $E(x)$ can be reduced by increasing the gain λ of the nonlinear amplifiers. Thus λ is usually taken to be very large. As discussed in [5], large values of λ may result in periodic motions for system (H). In particular, this may happen when the diagonal elements of $T = [T_{ij}]$ have negative values. On the other hand, if we want to reduce the number of spurious equilibrium points of system (H) by the synthesis procedure introduced in [5], we need to choose large negative eigenvalues for T and the matrix T synthesized in this way will have negative diagonal elements. ■

Now let us consider electric circuits (C_M) as given in Fig. 5. In (C_M), there are n identical operational amplifiers as given in Fig. 3. In Fig. 3, u_i is the input voltage, v_i is the output voltage, and $\pm V_{cc}$ denotes power supply voltage. The input–output relation of the ith operational amplifier is given by

$$v_i = \begin{cases} V_{cc}, & \text{if } u_i > V_{cc}/\lambda \\ \lambda u_i, & \text{if } -V_{cc}/\lambda \leqslant u_i \leqslant V_{cc}/\lambda \\ -V_{cc}, & \text{if } u_i < -V_{cc}/\lambda \end{cases}$$

where λ is the gain of the operational amplifiers. By employing a feedback capacitor C_i, the ith operational amplifier becomes an integrator as illustrated in Fig. 4. In circuit (C_M), these n integrators are interconnected as

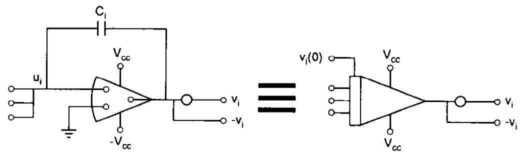


Fig. 4. The ith neural cell of system (M).

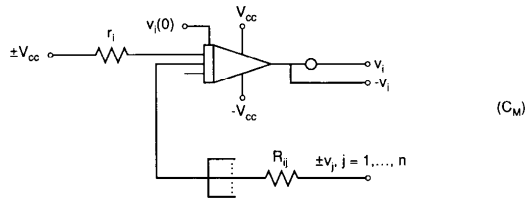


Fig. 5. Implementation of system (M).

shown in Fig. 5. In circuit (C_M), $\pm V_{cc}/r_i$ denotes the input current to the ith integrator, $v_i(0)$ represents the initial condition for the ith integrator, and each R_{ij} denotes the resistor connecting the output of the j-th integrator to the input of the ith integrator.

If in (C_M), we take the voltage V_{cc} as unit, then each component v_i of v can vary from -1 to 1. When $-1 < v_i < 1$ for $i = 1, \cdots, n$, the circuit (C_M) can be described by a set of ordinary linear differential equations as

$$\frac{dv_i}{dt} = \frac{1}{C_i}\left\{ \sum_{j=1}^{n} \frac{1}{R_{ij}}(\pm v_j) + \frac{\pm V_{cc}}{r_i} \right\}, \qquad i = 1, \cdots, n \tag{$\tilde{\mathrm{M}}_c$}$$

with the constraints

$$-1 < v_i < 1, \qquad i = 1, \cdots, n.$$

If in ($\tilde{\mathrm{M}}_c$), we let $C_i = 1$, $1 \leqslant i \leqslant n$, $x = (x_1, \cdots, x_n)^T = u$, $T = [T_{ij}]$, where

$$T_{ij} = \begin{cases} +\dfrac{1}{R_{ij}}, & \text{if } R_{ij} \text{ is connected to the } v_j \\ -\dfrac{1}{R_{ij}}, & \text{if } R_{ij} \text{ is connected to the } -v_j \end{cases}$$

and $I = (I_1, \cdots, I_n)^T$, where

$$I_i = \frac{\pm V_{cc}}{r_i}$$

then ($\tilde{\mathrm{M}}_c$) changes to the form

$$dx/dt = Tx + I \tag{$\tilde{\mathrm{M}}$}$$

with the constraints

$$-1 < x_i < 1, \qquad i = 1, \cdots, n.$$

Remark 3.3: In the usual applications, circuit (C_M) is operated in the unsaturated mode to represent (or simulate) linear system ($\tilde{\mathrm{M}}$). In the present paper, we will demonstrate that circuit (C_M), when operated in an extended mode, including saturation, exhibits properties associated with neural networks. In the next section, we will give a proper definition of solution for system (M) such that system (M) with the variable x defined on $\boldsymbol{D}^n$ indeed describes the behavior of the circuit (C_M) including the saturation. Thus (C_M) serves as an implementation of the neural network model (M). ■

In the next several sections, we will show that under certain conditions, the equilibrium points of (M) can be precisely determined. Also, by making use of the energy function

$$E(x) = -(1/2)x^T Tx - x^T I$$

the synthesis procedure developed in [5] can be modified and applied to system (M). Furthermore, it appears that operational amplifiers with large gains will stabilize circuit (C_M). Moreover, interconnecting matrices T with large negative eigenvalues can be used to synthesize (M) with a reduced number of spurious equilibrium points.

Remark 3.4: The above observations suggest that system (M) overcomes some of the disadvantages of system (H) while maintaining the basic structure of system (H). ■

Remark 3.5: A comparison of the models (H) and (M) can be described as follows. As pointed out in [4], in terms of the variables v_i, $i = 1, \cdots, n$, we can represent system (H) by

$$dv/dt = H(v)(-S(v) + Tv + I) \tag{$\mathrm{L_H}$}$$

where

$$v = (v_1, \cdots, v_n)^T \in (-1,1)^n$$
$$H(v) = \mathrm{diag}\left[\lambda/\left(C_1\left(dg_1^{-1}(v_1)/dv_1\right)\right), \cdots, \lambda/\left(C_n\left(dg_n^{-1}(v_n)/dv_n\right)\right)\right]$$
$$T = [T_{ij}],\ S(v) = \left(g^{-1}(v_1)/(\lambda R_1), \cdots, g^{-1}(v_n)/(\lambda R_n)\right)^T$$

and

$$I = (I_1, \cdots, I_n)^T.$$

System ($\mathrm{L_H}$) has the identical set of equilibrium points and the identical set of asymptotically stable equilibrium points as system ($\mathrm{L_M}$), given by

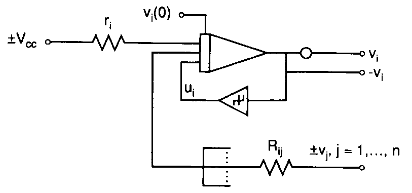
$$dv/dt = -S(v) + Tv + I. \tag{$\mathrm{L_M}$}$$

System ($\mathrm{L_M}$) can be implemented by the circuit (C_L) depicted in Fig. 6. In circuit (C_L), $u_i = (1/\lambda)g^{-1}(v_i)$, $i = 1, \cdots, n$ and each integrator is an ideal integrator in the sense that it can not be saturated. When the parameter λ tends towards infinity, the circuit (C_L) can be described by saturated integrators as in circuit (C_M). Thus, the two models (H) and (M) appear to be closely related. ■

Remark 3.6: System (M) is easily simulated by digital computers or by array processors using difference equations of the form (cf. [7])

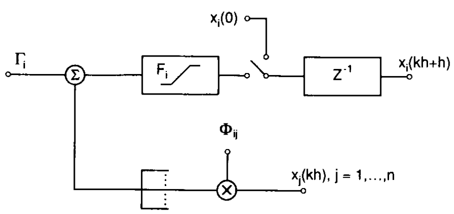
$$x((k+1)h) = F(\Phi x(kh) + \Gamma) \tag{$\mathrm{M_d}$}$$

where $x(kh) \in \boldsymbol{D}^n$, h is the sample period, F: $\boldsymbol{R}^n \to \boldsymbol{D}^n$,



(C_L)

Fig. 6. Implementation of system (L_M).



(D_M)

Fig. 7. Block diagram of (M_d).

$F(x) = (F_1(x_1), \cdots, F_n(x_n))^T$ with

$$F_i(\rho) = \begin{cases} 1, & \text{if } \rho > 1 \\ \rho, & \text{if } -1 \leqslant \rho \leqslant 1; \quad 1 \leqslant i \leqslant n \\ -1, & \text{if } \rho < -1 \end{cases}$$

and

$$\begin{cases} \Phi = \exp(hT) \\ \Gamma = \int_0^h \exp(\rho T)\, dp \cdot I. \end{cases}$$

The block diagram of (M_d) is shown in Fig. 7. In Remark 5.6, we will sow how to calculate the matrices Φ and Γ for the designed systems. ■

IV. Analysis

First, we give a proper definition of the solution for system (M) which describes the behavior of the electrical circuit (C_M) discussed in the previous section.

For each m, $0 \leqslant m \leqslant n$, let

$$\Lambda_m = \left\{ \xi = (\xi_1, \cdots, \xi_n)^T \in \Lambda \colon \xi_{\sigma(i)} = 0, 1 \leqslant i \leqslant m \text{ and} \right.$$
$$\left. \xi_{\sigma(i)} = \pm 1, m < i \leqslant n, \text{ for some } \sigma \in \mathrm{Sym}(n) \right\} \tag{4.1a}$$

where

$$\Lambda = \left\{ \xi = (\xi_1, \cdots, \xi_n)^T \colon \xi_i = \pm 1 \text{ or } 0, 1 \leqslant i \leqslant n \right\}. \tag{4.1b}$$

The numbers of elements in Λ_m is equal to $[n!/(m!(n-m)!)]2^{n-m}$, and the number of elements in $\Lambda = \cup \Lambda_m$, $0 \leqslant m \leqslant n$, is 3^n. For each $\xi \in \Lambda$, let

$$\boldsymbol{C}(\xi) = \left\{ x = (x_1, \cdots, x_n)^T \in \boldsymbol{R}^n \colon |x_i| < 1 \right.$$
$$\left. \text{if } \xi_i = 0, \text{ and } x_i = \xi_i \text{ if } \xi_i \neq 0 \right\}. \tag{4.1c}$$

From the notation given above, we have

Lemma 4.1: 1) $\Lambda_0 = \boldsymbol{B}^n$ and $\boldsymbol{C}(\xi) = \{\xi\}$, for any $\xi \in \Lambda_0$.

2) $\Lambda_n = \{0\}$ and $\boldsymbol{C}(0) = (\boldsymbol{D}^n)^0 = \{x \in \boldsymbol{R}^n \colon -1 < x_i < 1, i = 1, \cdots, n\}$.

3) $\partial(\boldsymbol{D}^n) = \cup \boldsymbol{C}(\xi)$), $\xi \in \cup \Lambda_m$, $\quad 1 \leqslant m \leqslant n$.

4) $\boldsymbol{D}^n = \cup \boldsymbol{C}(\xi)$, $\xi \in \cup \Lambda_m$, $\quad 0 \leqslant m \leqslant n$.

5) For any $\xi, \eta \in \Lambda$, $\xi \neq \eta$, $\quad \boldsymbol{C}(\xi) \cap \boldsymbol{C}(\eta) = \varnothing$. ■

Suppose that $\xi \in \Lambda_m$ and $\sigma \in \mathrm{Sym}(n)$ such that

$$\xi_{\sigma(i)} = 0, \quad 1 \leqslant i \leqslant m \quad \text{and} \quad \xi_{\sigma(i)} = \pm 1, \quad m < i \leqslant n. \tag{4.2}$$

Subsequently, we will make use of the notation

$$\begin{aligned} T_{\mathrm{I,I}} &= \left[T_{\sigma(i)\sigma(j)} \right]_{1 \leqslant i, j \leqslant m,} \\ T_{\mathrm{I,II}} &= \left[T_{\sigma(i)\sigma(j)} \right]_{1 \leqslant i \leqslant m, m < j \leqslant n} \\ T_{\mathrm{II,I}} &= \left[T_{\sigma(i)\sigma(j)} \right]_{m < i \leqslant n, 1 \leqslant j \leqslant m}, \\ T_{\mathrm{II,II}} &= \left[T_{\sigma(i)\sigma(j)} \right]_{m < i, j \leqslant n} \end{aligned} \tag{4.3a}$$

$$\begin{aligned} I_I &= \left(I_{\sigma(1)}, \cdots, I_{\sigma(m)} \right)^T, \\ I_{\mathrm{II}} &= \left(I_{\sigma(m+1)}, \cdots, I_{\sigma(n)} \right)^T \end{aligned} \tag{4.3b}$$

and

$$\xi_{\mathrm{I}} = \left(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(m)} \right)^T, \qquad \xi_{\mathrm{II}} = \left(\xi_{\sigma(m+1)}, \cdots, \xi_{\sigma(n)} \right)^T. \tag{4.3c}$$

Remark 4.1: 1) For a given $\xi \in \Lambda_m$, there may exist different permutations in $\mathrm{Sym}(n)$ for which (4.2) is true. For these different permutations, the notation given above will be the same up to different orders in the components. Thus, the analysis and conclusions will be identical for any of the permutations used.

2) If $m = n$, we have $T_{\mathrm{I,I}} = T$, $I_{\mathrm{I}} = I$, $\xi_{\mathrm{I}} = \xi$ and the $T_{\mathrm{I,II}}, T_{\mathrm{II,I}}, T_{\mathrm{II,II}}, I_{\mathrm{II}}, \xi_{\mathrm{II}}$ do not exist. If $m = 0$, we have $T_{\mathrm{II,II}} = T$, $I_{\mathrm{II}} = I$, $\xi_{\mathrm{II}} = \xi$ and the $T_{\mathrm{I,I}}, T_{\mathrm{I,II}}, T_{\mathrm{II,I}}, I_{\mathrm{I}}, \xi_{\mathrm{I}}$ do not exist. ■

Definition 4.1: 1) Consider $\xi \in \Lambda_m$, $0 < m \leqslant n$, with $\sigma \in \mathrm{Sym}(n)$ such that $\xi_{\sigma(i)} = 0$, $1 \leqslant i \leqslant m$ and $\xi_{\sigma(i)} = \pm 1$, $m < i \leqslant n$. Also, consider the linear system defined by

$$dx_{\mathrm{I}}/dt = T_{\mathrm{I,I}} x_{\mathrm{I}} + T_{\mathrm{I,II}} \xi_{\mathrm{II}} + I_{\mathrm{I}} \tag{M_ξ}$$

where $x_{\mathrm{I}} = (x_{\sigma(1)}, \cdots, x_{\sigma(m)})^T$, $x_{\mathrm{II}} = (x_{\sigma(m+1)}, \cdots, x_{\sigma(n)})^T$ and $-1 < x_{\sigma(i)} < 1$ for $1 \leqslant i \leqslant m$.

(M_ξ) is said to be the *reduced linear system* of (M) over the region $\boldsymbol{C}(\xi)$.

2) For any $\xi \in \Lambda_m$, a C^1-function φ: $(0.\delta) \to \boldsymbol{C}(\xi)$ is said to be a (*local*) *solution* of LSSM system (M) if the vector function φ_{I} containing the $\sigma(i)$th component of φ, $1 \leqslant i \leqslant m$, is a solution of the linear system (M_ξ), i.e.,

$$d\varphi_{\mathrm{I}}(t)/dt = T_{\mathrm{I,I}} \varphi_{\mathrm{I}}(t) + T_{\mathrm{I,II}} \xi_{\mathrm{II}} + \mathrm{I}_{\mathrm{I}}, \qquad t \in (0, \delta) \tag{4.4a}$$

and

$$\min\left((T_{\mathrm{II,I}} \varphi_{\mathrm{II}}(t) + T_{\mathrm{II,II}} \xi_{\mathrm{II}} + \mathrm{I}_{\mathrm{II}}) * \xi_{\mathrm{II}} \right) \geqslant 0, \qquad t \in (0, \delta) \tag{4.4b}$$

where $\varphi_{\mathrm{I}} = (\varphi_{\sigma(1)}, \cdots, \varphi_{\sigma(m)})^T$ and $\varphi_{\mathrm{II}} = (\varphi_{\sigma(m+1)}, \cdots, \varphi_{\sigma(n)})^T$.

In particular, if $\xi \in \Lambda_m$, $m < n$, the solution φ is said to be in the *saturated mode*.

Remark 4.2: 1) $\varphi((0,\delta)) \subset \boldsymbol{C}(\xi)$ implies that

$$-1 < \varphi_{\sigma(i)}(t) < 1, \qquad i = 1, \cdots, m, \; t \in (0,\delta)$$

$$\varphi_{\sigma(i)}(t) = \xi_{\sigma(i)}, \qquad i = m+1, \cdots, n; \; t \in (0,\delta),$$

$$\text{i.e., } \varphi_{\mathrm{II}}(t) = \xi_{\mathrm{II}} \text{ on } (0,\delta).$$

2) When $m = n$, $\Lambda_n = \{0\}$. In this case, (4.4b) does not exist and we only need to consider (4.4a). Furthermore, the (local) solutions in $\boldsymbol{C}(0) = (\boldsymbol{D}^n)^0 = \{x \in \boldsymbol{R}^n: -1 < x_i < 1, i = 1, \cdots, n\}$ defined above for the LSSM system (M) will be identical to the usual solutions defined on the open subset $\boldsymbol{C}(0)$ for the linear system (M_0).

3)When $m = 0$, $\Lambda_0 = \boldsymbol{B}^n$ and for $\xi \in \Lambda_0$, $\boldsymbol{C}(\xi) = \{\xi\}$. In this case, (4.4a) does not exist and we only need to consider (4.4b). Furthermore, the only function with the range in $\boldsymbol{C}(\xi)$ is the constant function $\varphi(t) \equiv \xi$. ■

Definition 4.2: For $\tilde{x} \in \boldsymbol{D}^n$, a continuous function $\phi = \phi(\cdot, \tilde{x})$: $[0, \tilde{t}) \to \boldsymbol{D}^n$ is said to be a *solution of* (M) *starting at* $\tilde{x}$ if

a) $\phi(0, \tilde{x}) = \tilde{x}$, and

b) there are countably many non-interconnected open intervals $(t_i, t_i + \delta_i)$ such that $\overline{\cup(t_i, t_i + \delta_i)} = \overline{(0, \tilde{t})}$ and ϕ restricted to each $(t, t+\delta)$ is a (local) solution as defined in 1) above. ■

From the existence theory of ordinary differential equations (cf. [6]), we have

Theorem 4.1: 1) For any $x \in \boldsymbol{D}^n$, there is a unique solution $\phi(\cdot, x)$ for system (M).

2) Each solution of (M) can be uniquely extended to the infinite time interval $[0, +\infty)$. ■

In the following, we only consider solutions of (M) defined on the interval $[0, +\infty)$.

Definition 4.3: 1) A vector $\tilde{x} \in \boldsymbol{D}^n$ is said to be an *equilibrium (point) of* (M) if the function $\phi(\cdot, \tilde{x})$: $[0, \infty) \to \boldsymbol{D}^n$ defined by $\phi(t, \tilde{x}) \equiv \tilde{x}$, is a solution of (M).

2) Let $\tilde{x}$ be an equilibrium of (M).

i) $\tilde{x}$ is said to be *stable* if for any $\epsilon > 0$, there is a $\delta = \delta(\epsilon) > 0$ such that $|\phi(t, x) - \tilde{x}| < \epsilon$ for all $t \in [0, +\infty)$ whenever $|x - \tilde{x}| < \delta$.

ii) $\tilde{x}$ is said to be *asymptotically stable* if (i) it is stable and (ii) there is an $\eta > 0$ such that $\lim_{t \to +\infty} |\phi(t, x) - \tilde{x}| = 0$ whenever $|x - \tilde{x}| < \eta$.

iii) $\tilde{x}$ is said to be *unstable* if it is not stable. ■

We now consider the following assumptions for (M).

Assumption 4.1: 1) T is symmetric.

2) For any m, $0 \leqslant m \leqslant n$, and for any $\xi \in \Lambda_m$, the $m \times m$ matrix $T_{\mathrm{I,I}} = [T_{\sigma(i)\sigma(j)}]_{1 \leqslant i,j \leqslant m}$, is non-singular, where $\sigma \in \mathrm{Sym}(n)$ such that $\xi_{\sigma(i)} = 0$, $1 \leqslant i \leqslant m$ and $\xi_{\sigma(i)} = \pm 1$, $m < i \leqslant n$. ■

For system (M) satisfying Assumption 4.1, we define the following notation.

1) For $\xi = 0 \in \Lambda_n$, let

$$x_\xi = (x_{\xi 1}, \cdots, x_{\xi n})^T \in \boldsymbol{R}^n, \qquad x_\xi = -T^{-1}I. \quad (4.5a)$$

2) For $\xi \in \Lambda_m$, $0 < m < n$, with $T_{\mathrm{I,I}}, \cdots, T_{\mathrm{II,II}}, I_{\mathrm{I}}, I_{\mathrm{II}}$ defined in (4.4), let

$$x_\xi = (x_{\xi 1}, \cdots, x_{\xi n})^T \in \boldsymbol{R}^n \qquad (4.5b)$$

where

$$x_{\xi \mathrm{I}} = (x_{\xi\sigma(1)}, \cdots, x_{\xi\sigma(m)})^T = -(T_{\mathrm{I,I}})^{-1}(T_{\mathrm{I,II}}\xi_{\mathrm{II}} + I_{\mathrm{I}})$$

$$x_{\xi \mathrm{II}} = (x_{\xi\sigma(m+1)}, \cdots, x_{\xi\sigma(n)})^T = \xi_{\mathrm{II}}$$

and let

$$r_\xi \in \boldsymbol{R}, \qquad r_\xi = \min\left((T_{\mathrm{II,I}}x_{\xi\mathrm{I}} + T_{\mathrm{II,II}}\xi_{\mathrm{II}} + I_{\mathrm{II}}) * \xi_{\mathrm{II}}\right). \quad (4.5c)$$

3) For $\xi \in \Lambda_0 = \boldsymbol{B}^n$, $\boldsymbol{C}(\xi) = \{\xi\}$. Let

$$x_\xi = \xi \qquad (4.5d)$$

and

$$r_\xi \in \boldsymbol{R}, \qquad r_\xi = \min((T\xi + I) * \xi). \qquad (4.5e)$$

Assumption 4.2: Given Assumption 4.1 for system (M), we assume that for $\xi \in \Lambda_m$, $0 \leqslant m < n$, $x_\xi \notin \partial\boldsymbol{C}(\xi)$, i.e., $x_{\xi\sigma(i)} \neq \pm 1$, $i = 1, \cdots, m$. ■

Remark 4.3: For fixed T, Assumption 4.2 is true for almost all vectors I in $\boldsymbol{R}^n$. ■

Lemma 4.2: Suppose that system (M) satisfies Assumptions 4.1 and 4.2. Then for any $\xi \in \Lambda_m$, $0 \leqslant m < n$, $r_\xi \neq 0$.

Proof: Suppose that for some $\xi \in \Lambda_m$, $0 < m < n$, $r_\xi = 0$. Then from the notation given in (4.5), there is j, $1 \leqslant j \leqslant n - m$, such that the jth component of the vector $(T_{\mathrm{II,I}}x_{\xi\mathrm{I}} + T_{\mathrm{II,II}}\xi_{\mathrm{II}} + I_{\mathrm{II}})$ is equal to 0. Without loss of generality, assume that $j = 1$. Take $\zeta \in \Lambda_{m+1}$, where $\zeta_{\sigma(i)} = 0$ for $1 \leqslant i \leqslant m+1$ and $\zeta_{\sigma(i)} = \xi_{\sigma(i)}$ for $m+1 \leqslant i \leqslant n$. Then $r_\xi = r_\zeta \in \partial\boldsymbol{C}(\xi)$ and this contradicts Assumption 4.2. Suppose that for some $\xi \in \Lambda_0$, $r_\xi = 0$. Then we can arrive at a contradiction in the same manner. This proves the lemma. ■

Theorem 4.2: Suppose that system (M) satisfies Assumptions 4.1 and 4.2. Then for any m, $0 \leqslant m \leqslant n$, and for any $\xi \in \Lambda_m$, with the notation given above, we have the following results:

Case I: $m = n$, $\xi = 0 \in \Lambda_n$.

1) If $x_\xi \notin \boldsymbol{C}(\xi) = (\boldsymbol{D}^n)^0$, there is no equilibrium point of system (M) in $(\boldsymbol{D}^n)^0$.

2) If $x_\xi \in \boldsymbol{C}(\xi) = (\boldsymbol{D}^n)^0$, x_ξ is the unique equilibrium point of system (M) in $(\boldsymbol{D}^n)^0$. In particular,

i) if T is not negative definite, x_ξ is unstable, and

ii) if $T_{\mathrm{I,I}} = T$ is negative definite, x_ξ is asymptotically stable and there are no other equilibrium points in $\boldsymbol{C}(\xi) = \boldsymbol{D}^n$.

Case II: $0 < m < n$, $\xi \in \Lambda_m$.

A) If $r_\xi < 0$, there is no equilibrium point of system (M) in $\boldsymbol{C}(\xi)$.

B) If $r_\xi > 0$, we have that

1) If $x_\xi \notin \boldsymbol{C}(\xi)$, there is no equilibrium point of system (M) in $\boldsymbol{C}(\xi)$.

2) If $x_\xi \in C(\xi)$, x_ξ is the unique equilibrium point of system (M) in $C(\xi)$. In particular,

i) if $T_{I,I}$ is not negative definite, x_ξ is unstable, and

ii) if $T_{I,I}$ is negative definite, x_ξ is asymptotically stable, an there are no other equilibrium points in $\overline{C(\xi)}$.

Case III: $\xi \in \Lambda_0 = \boldsymbol{B}^n$. (Note: $C(\xi) = \{\xi\}$.)

A) If $r_\xi < 0$, $x_\xi = \xi$ is not an equilibrium point of system (M).

B) If $r_\xi > 0$, $x_\xi = \xi$ is an asymptotically stable equilibrium point of system (M).

Proof:

Case I: In this case, $\xi = 0$ and $C(\xi) = (\boldsymbol{D}^n)^0$, and from Definition 4.3, solutions of (M) on $C(0)$ are identical to solutions of (M) on $C(0)$ in the usual sense. Then the conclusions of Case I of the theorem follow directly from the theory of linear differential equations (cf. [6]).

Case II: Consider $\xi \in \Lambda_m$, $0 < m < n$, and a constant function φ: $[0, +\infty) \to C(\xi)$, $\varphi(t) = \tilde{x}$. If $\tilde{x}$ is an equilibrium point of (M) in $C(\xi)$, then by Definition 4.1 we have that $T_{I,I}\tilde{x}_I + T_{I,II}\xi_{II} + I_I = 0$, i.e., $\tilde{x}_I = x_{\xi I}$ and $\min((T_{II,I}\tilde{x}_I + T_{II,II}\xi_{II} + I_{II}) * \xi) \geq 0$. Since $\tilde{x} \in C(\xi)$, $\tilde{x}_{II} = \xi_{II}$, and $x_\xi = \tilde{x} \in C(\xi)$. Also, $r_\xi \geq 0$ and by Lemma 4.2, $r_\xi > 0$. On the other hand, if $x_\xi \in C(\xi)$ and $r_\xi > 0$, let ρ: $[0, +\infty) \to C(\xi)$, $\varphi(t) = x_\xi$. Then φ is a C^1-function and satisfies (4.4). By Definition 4.1, φ is a solution of (M) and by Definition 4.3, $\varphi \equiv x_\xi$ is an equilibrium point of (M) in $C(\xi)$.

Furthermore, suppose that x_ξ is an equilibrium point of (M). Then there is an open neighborhood $\boldsymbol{U}$ of x_ξ in $\boldsymbol{R}^n$ such that $\min((T_{II,I}x_I + T_{II,II}\xi_{II} + I_{II}) * \xi_{II}) > 0$ for all $x \in \boldsymbol{U} \cap \boldsymbol{D}^n$. If $T_{I,I} < 0$, it can be proved by induction and by the theory of linear differential equations that all solutions of (M) starting in $\boldsymbol{U} \cap \boldsymbol{D}^n$ will monotonically converge to x_ξ as $t \to +\infty$. Thus x_ξ is an asymptotically stable equilibrium point of (M). On the other hand, if $T_{I,I}$ is not negative definite, then $T_{I,I}$ has a positive eigenvalue since $T_{I,I}$ is not singular. By the theory of linear differential equations, there is a $\delta_0 > 0$ such that $B(x_\xi, \delta_0) \cap \boldsymbol{D}^n \subset \boldsymbol{U} \cap \boldsymbol{D}^n$, and for any ϵ, $0 < \epsilon < \delta_0$, there is a function φ: $[0, t_0) \to C(\xi) \cap \boldsymbol{U}$ such that $\varphi(0) \in C(\xi) \cap B(x_\xi, \epsilon)$, $|\varphi(t_0) - x_\xi| > \delta_0$, and φ_I is a solution of linear system (M_ξ). Since $\varphi([0, t_0)) \subset \boldsymbol{U}$, φ_{II} satisfies (4.4b) and φ is a solution of system (M). Thus x_ξ is an unstable equilibrium point of (M) by Definition 4.3.

Finally, for purposes of contradiction, assume that x_ξ is an asymptotically stable equilibrium point of (M) and x_ζ is another equilibrium point of (M) in $\overline{C(\xi)}$. Since x_ξ is the unique equilibrium point of (M) in $C(\xi)$, $x_\zeta \in \overline{C(\xi)} - C(\xi)$. Then there is a k, $1 \leq k \leq m$, such that $\zeta_{\sigma(k)} = \pm 1$. By the theory of linear differential equations, the $\sigma(k)$th component of $Tx_\zeta + \mathrm{I}$ starting at x_ζ points to x_ξ. Thus $(Tx_\zeta + I)_{\sigma(k)} \cdot \zeta_{\sigma(k)} < 0$ and $r_\zeta < 0$. This contradiction shows that x_ξ is the only equilibrium point of (M) in $\overline{C(\xi)}$.

Case III: If $r_\xi < 0$, $\varphi \equiv x_\xi$ does not satisfy (4.4b) and x_ξ is not a solution of (M). If $r_\xi > 0$, then there is an open neighborhood $\boldsymbol{U}$ of x_ξ in $\boldsymbol{R}^n$ such that $\min((T_{II,I}x_I + T_{II,II}\xi_{II} + I_{II}) * \xi_{II}) > 0$, for all $x \in \boldsymbol{U} \cap \boldsymbol{D}^n$. It can be proved by induction and by the theory of linear differential equations that all solutions of (M) starting in $\boldsymbol{U} \cap \boldsymbol{D}^n$ will monotonically converge to x_ξ as $t \to +\infty$. Thus x_ξ is an asymptotically stable equilibrium point of (M) by Definition 4.3.

This concludes the proof of Theorem 4.2. ■

Remark 4.4: 1) Theorem 4.2 establishes an algorithm to locate all equilibrium points of (M) and to determine the stability properties of all equilibrium points.

2) In practice, due to noise, unstable equilibrium points can not be used as memory locations. However, by investigating the locations of the unstable equilibrium points of (M), the domain of attraction of each asymptotically stable equilibrium point of (M) can be approximately determined. We will illustrate this observation in Example 6.4.

3) Since system (M) is closely related to the analog Hopfield model, as discussed in Remark 3.5, Theorem 4.1 can be used to determine asymptotically stable equilibrium points for the Hopfield model. This will be illustrated in Example 6.3.

4) If for some $\xi \in \Lambda$, the corresponding $T_{I,I}$ is singular, Theorem 4.1 can still be applied to other vectors in Λ. If Lemma 4.1 is not true, we need to consider the case $r_\xi = 0$. In fact, we can study this case using the same method as was employed in Theorem 4.1. In this case, the conclusions might be less straightforward. ■

Theorem 4.3: If system (M) satisfies Assumptions 4.1 and 4.2, we have the following results:

1) There is at most one equilibrium point of (M) in each region $C(\xi)$, $\xi \in \Lambda$.

2) There are at most 3^n equilibrium points for system (M).

3) If for $\xi \in \Lambda$ the corresponding x_ξ is an asymptotically stable equilibrium point for system (M), then

i) x_ξ is the unique equilibrium point of system (M) in $\overline{C(\xi)}$, and

ii) x_ξ is the unique asymptotically stable equilibrium point of system (M) in $\overline{C(\xi)} \cup C(\zeta)$, where $\zeta \in \Lambda$ such that $x_\zeta \in \overline{C(\zeta)}$. In particular, if $\xi \in \Lambda_0 = \boldsymbol{B}^n$ and $x_\xi = \xi$ is an asymptotically stable equilibrium point of system (M), then $x_\xi = \xi$ is the unique asymptotically stable equilibrium point of system (M) in the region $\{x \in \boldsymbol{D}^n$: $-1 < x_i\xi_i \leq 1$, $1 \leq i \leq n\}$.

4) There are at most 2^n asymptotically stable equilibrium points for system (M).

Proof: Part 1) follows from Theorem 4.2. Part 2) follows from the fact that $\boldsymbol{D}^n$ is separated into 3^n regions by $C(\xi)$, $\xi \in \Lambda$. Part 3) has been proved in Theorem 4.2. Part 4) follows from Part 3). ■

Remark 4.5: Theorem 4.3 is not only very useful to simplify the algorithm given in Theorem 4.2 for checking the location of equilibrium points of system (M) but it is also useful to determine whether a given set of vectors can be synthesized as a set of asymptotically stable equilibrium points of system (M) or not (cf. Synthesis Procedure 5.2). ■

Definition 4.4: If T is symmetric, then

1) the *energy function* E: $\boldsymbol{D}^n \to \boldsymbol{R}$ of system (N) is defined by

$$E(x) = -(1/2)x^TTx - x^TI \tag{4.6}$$

and

2) $x \in \boldsymbol{D}^n$ is said to be a (*local*) *minimum* of the energy function E if there is an open neighborhood $\boldsymbol{U}$ of x in $\boldsymbol{R}^n$ such that $E(x) \leqslant E(y)$, for any $y \in \boldsymbol{D}^n \cap \boldsymbol{U}$. ∎

Theorem 4.4: Suppose that system (M) satisfies Assumptions 4.1 and 4.2. Then it is true that

1) along a non-equilibrium solution of (M), the energy function E given in (4.6) decreases monotonically; and

2) each non-equilibrium solution of (M) converges to an equilibrium of (M).

Proof: 1) It is sufficient to consider the proof for local solutions of (M) as defined in Definition 4.1. Consider a non-constant local solution of (M) given by φ: $(0,\delta) \to \boldsymbol{C}(\xi)$, $\xi \in \Lambda_m$. With the notation given in (4.3), we have on $(0,\delta)$,

$$\begin{aligned} E(\varphi(t)) &= -(1/2)\varphi(t)^T \cdot T \cdot \varphi(t) - \varphi(t)^T \cdot I \\ &= -(1/2)\left(\varphi_{\mathrm{I}}(t)^T, \varphi_{\mathrm{II}}(t)^T\right) \\ &\quad \cdot \begin{bmatrix} T_{\mathrm{I,I}} & T_{\mathrm{I,II}} \\ T_{\mathrm{II,I}} & T_{\mathrm{II,II}} \end{bmatrix} \cdot \begin{bmatrix} \varphi_{\mathrm{I}}(t) \\ \varphi_{\mathrm{II}}(t) \end{bmatrix} \\ &\quad - \left(\varphi_{\mathrm{I}}(t)^T, \varphi_{\mathrm{II}}(t)^T\right) \cdot \begin{bmatrix} I_{\mathrm{I}} \\ I_{\mathrm{II}} \end{bmatrix} \\ &= -(1/2)\big(\varphi_{\mathrm{I}}(t)^T \cdot T_{\mathrm{II}} \cdot \varphi_{\mathrm{I}}(t) \\ &\quad + 2\varphi_{\mathrm{I}}(t)^T \cdot T_{\mathrm{I,II}} \cdot \xi_{\mathrm{II}} + \xi_{\mathrm{II}}^T \cdot T_{\mathrm{II,II}} \cdot \xi_{\mathrm{II}}\big) \\ &\quad - \left(\varphi_{\mathrm{I}}(t)^T \cdot I_{\mathrm{I}} + \xi_{\mathrm{II}}^T \cdot I_{\mathrm{II}}\right) \end{aligned}$$

and by (4.4)

$$\begin{aligned} dE(\varphi(t))/dt &= -\left(T_{\mathrm{II}} \cdot \varphi_{\mathrm{I}}(t) + T_{\mathrm{I,II}} \cdot \xi_{\mathrm{II}} + I_{\mathrm{I}}\right) \cdot d\varphi(t)/dt \\ &= -\left(T_{\mathrm{II}} \cdot \varphi_{\mathrm{I}}(t) + T_{\mathrm{I,II}} \cdot \xi_{\mathrm{II}} + I_{\mathrm{I}}\right)^2 < 0. \end{aligned}$$

This proves Part 1).

2) Part 2) follows from the theory of ordinary differential equations (cf. [6]). ∎

Remark 4.6: 1) Theorem 4.4 shows that system (M) has no oscillatory solutions.

2) In practice, due to noise, all solutions of system (M) will only converge to asymptotically stable equilibrium points of system (M). ∎

Theorem 4.5: Suppose that system (M) satisfies Assumptions 4.1 and 4.2. Then there is a one-to-one correspondence between the set of local minima of the energy function E and the set of asymptotically stable equilibrium points of system (M).

Proof: Suppose that $\tilde{x}$ is an asymptotically stable equilibrium point of system (M). Then there is an open neighborhood $\boldsymbol{U}$ of $\tilde{x}$ in $\boldsymbol{R}^n$ such that all solutions of (M) starting in $\boldsymbol{U} \cap \boldsymbol{D}^n$ will converge to $\tilde{x}$ as $t \to +\infty$. Since along any solution φ, the value of the energy function E decreases monotonically and the energy function E is continuous, we have that $E(\tilde{x}) < E(x)$ for $x \in \boldsymbol{U} \cap \boldsymbol{D}^n$. Thus $\tilde{x}$ is a local minimum of E.

On the other hand, suppose that $\tilde{x}$ is a local minimum of the energy function E. Then there is an open neighborhood $\boldsymbol{U}$ of $\tilde{x}$ in $\boldsymbol{R}^n$ such that $E(\tilde{x}) \leqslant E(x)$, $x \in \boldsymbol{U} \cap \boldsymbol{D}^n$. By Theorem 4.4.1, $\tilde{x}$ is an equilibrium point of system (M). If $\tilde{x}$ is not asymptotically stable, by Theorem 4.2, it is unstable. Thus there is a $\delta_0 > 0$ such that $B(\tilde{x}, \delta_0) \subset \boldsymbol{U}$ and for any ϵ, $0 < \epsilon < \delta_0$, there is a solution φ of (M) starting in $B(\tilde{x}, \epsilon) \cap \boldsymbol{D}^n$ which tends towards the exterior of $B(\tilde{x}, \delta_0) \cap \boldsymbol{D}^n$. Since along solution φ, the value of the energy function E decreases monotonically and E is continuous, we have that $E(\tilde{x}) > E(x_0)$ for some point $x_0 \in (\partial B(\tilde{x}, \delta_0)) \cap \boldsymbol{D}^n \subset \boldsymbol{U} \cap \boldsymbol{D}^n$. This contradiction shows that $\tilde{x}$ is an asymptotically stable equilibrium point of system (M).

This completes the proof of the theorem. ∎

Remark 4.7: 1) By Theorems 4.4 and 4.5, applications developed in [2] and [3] using the Hopfield model can also be realized by use of system (M).

2) As shown in [4], for system (L), which includes the Hopfield model as a special case, there is a one-to-one correspondence between the set of asymptotically stable equilibrium points of system (L) and the set of local minima of the energy function E_L: $(-1,1)^n \to \boldsymbol{R}$, given by

$$E_L(x) = -(1/2)x^T \cdot T \cdot x - x^T \cdot I + (1/\lambda) \sum_{i=1}^{n} \int_0^{x_i} (1/R_i) g_i^{-1}(\rho)\, d\rho.$$

When $\lambda \to +\infty$, $E_L(x) \to E(x) = (1/2)x^T \cdot T \cdot x - x^T \cdot I$. From this, we may conclude that near the location of each asymptotically stable equilibrium point of system (M), there is an asymptotically stable equilibrium point of system (L) when the gain λ is large. We believe (although we have not been able to prove this yet) that this relation establishes roughly a one-to-one correspondence between the sets of the asymptotically stable equilibrium points of the two systems (L) and (M). This is illustrated in Example 6.3. ∎

V. Synthesis

In the present section, we address a synthesis procedure for system (M) which is in the spirit of some of our earlier work [5].

Synthesis Problem: Given M vectors in $\boldsymbol{B}^n$, say $\alpha_1, \cdots, \alpha_m$, how can we properly choose a pair $\{T, I\}$ such that the resulting synthesized system (M) has the properties enumerated below?

1) $\alpha_1, \cdots, \alpha_m$ are asymptotically stable equilibrium points of system (M).

2) The system has no oscillatory solutions.

3) The total number of the spurious asymptotically stable equilibrium points (i.e., asymptotically stable equilibrium points of (M) contained in $\boldsymbol{B}^n - \{\alpha_1, \cdots, \alpha_m\}$) is as small as possible.

4) The domain of attraction of each α_i is as large as possible.

In view of the analysis results developed in the previous section, we may approach the preceding problem in the following manner.

Synthesis Strategy: Given m vectors in $\boldsymbol{B}^n$, say $\alpha_1,\cdots,\alpha_m$, find an $n\times n$ matrix $T=[T_{ij}]$ and a vector $I=(I_1,\cdots,I_n)^T$ such that:

1) T satisfies Assumption 4.1.

2) $x_{\alpha_i}=\alpha_i$, where $x_{\alpha_i}=T\alpha_i+I$. (5.1)

3) T has repeated eigenvalues equal to 1 and $-\tau$, $\tau>0$, and τ should be large.

Remark 5.1: In the following, we give the rationale for the above synthesis strategy. With Assumption 4.1 and by Remark 4.3, we may assume that Assumption 4.2 is also true for the synthesized system (M). Then all of the results in the previous section are applicable. We have:

1) By Theorem 4.4, the synthesized system (M) will have no oscillatory solutions.

2) By Theorem 4.2, for $1\leqslant i\leqslant m$, $x_{\alpha_i}=\alpha_i$ implies α_i is an asymptotically stable equilibrium point of the synthesized system (M).

3) As shown in Lemma 5.3 below, the smaller (i.e., the more negative) the eigenvalues of T are, the fewer spurious asymptotically stable equilibrium points for system (M) will exist. In general, this will result in larger domains of attraction for the remaining desired asymptotically stable equilibrium points. ■

Suppose that we are given m vectors in $\boldsymbol{B}^n$, say $\alpha_1,\cdots,\alpha_m$. Let

$$L=\operatorname{Span}(\alpha_1-\alpha_m,\cdots,\alpha_{m-1}-\alpha_m) \tag{5.2a}$$

and

$$L_a=\operatorname{Aspan}(\alpha_1,\cdots,\alpha_m). \tag{5.2b}$$

Then L is the linear subspace of $\boldsymbol{R}^n$ generated by the $(m-1)$ vectors $\alpha_1-\alpha_m,\cdots,\alpha_{m-1}-\alpha_m$, L_a is the affine subspace of $\boldsymbol{R}^n$ generated by the vectors $\alpha_1,\cdots,\alpha_m$, and $L_a=L+\alpha_m$. Assume that $k=\operatorname{rank}(L)$, $\{u_1,\cdots,u_k\}$ is an orthonormal basis of L, and $\{u_{k+1},\cdots,u_n\}$ is an orthonormal basis of $L^\perp$. Let $U^+=[u_1,\cdots,u_k]$, $U^-=[u_{k+1},\cdots,u_n]$,

$$T^+=\left[T_{ij}^+\right]=\sum_{i=1}^{k}u_iu_i^T=U^+(U^+)^T \tag{5.3a}$$

$$T^-=\left[T_{ij}^-\right]=\sum_{i=k+1}^{n}u_iu_i^T=U^-(U^-)^T \tag{5.3b}$$

and let

$$T_\tau=T^+-\tau T^- \quad\text{and}\quad I_\tau=\alpha_m-T_\tau\alpha_m \tag{5.4}$$

where $\tau\in\boldsymbol{R}$ is a parameter. For this class of synthesized LSSM systems

$$dx/dt=T_\tau x+I_\tau,\qquad \tau\in\boldsymbol{R} \tag{M_τ}$$

with the constraints

$$-1\leqslant x_i\leqslant 1,\qquad i=1,\cdots,n$$

we have the following result.

Lemma 5.1: 1) T^+ and T^- depend only on the given set $\{\alpha_1,\cdots,\alpha_m\}$ and they are independent of the choice of α_m and of the orthonormal basis $\{u_1,\cdots,u_n\}$ of $\boldsymbol{R}^n$. This is also true for T_τ and I_τ, for any $\tau\in\boldsymbol{R}$.

2) For any τ, T_τ is symmetric.

3) For any τ and for any $\alpha\in L_a$, $T_\tau\alpha+I_\tau=\alpha$. In particular, for each α_i, $i=1,\cdots,m$, $T_\tau\alpha_i+I_\tau=\alpha_i$.

Proof: 1) The proof of Part 1) can be found in linear algebra text books, e.g., [8].

2) Since $(T^+)^T=((U^+)(U^+)^T)^T=(U^+)(U^+)^T=T^+$, it follows that T^+ is symmetric. Also T^- is symmetric, and $T_\tau=T^+-\tau T^-$ is symmetric.

3) If $\alpha\in L_a$, there is a $\sigma=(\sigma_1,\cdots,\sigma_k)\in\boldsymbol{R}^k$ such that

$$\alpha=\sigma_1u_1+\cdots+\sigma_ku_k+\alpha_m.$$

Then

$$\begin{aligned}T_\tau\alpha+I_\tau&=T_\tau(\sigma_1u_1+\cdots+\sigma_ku_k)+(T_\tau\alpha_m+I_\tau)\\&=(\sigma_1u_1+\cdots+\sigma_ku_k)+\alpha_m=\alpha.\end{aligned}$$ ■

Lemma 5.2: For synthesized system (M_τ) with τ sufficiently large, Assumption 4.1 is true.

Proof: With the notation given in (4.1)–(4.3), it suffices to show that for each $\xi\in\Lambda_m$, $0<m\leqslant n$, there is a τ' such that for any $\tau>\tau'$, $T_{\tau\mathrm{I,I}}=[T_{\tau\sigma(i)\sigma(j)}]_{1\leqslant i,j\leqslant m}$, is nonsingular, where $\sigma\in\operatorname{Sym}(n)$ has the property that $\xi_{\sigma(i)}=0$, $1\leqslant i\leqslant m$ and $\xi_{\sigma(i)}=\pm1$, $m<i\leqslant n$. Suppose that there is a $\xi\in\Lambda_m$, $0<m\leqslant n$, such that for any $\tau>0$, $T_{\tau\mathrm{I,I}}$ is singular. Let $\boldsymbol{M}=\{x\in\boldsymbol{R}^n\colon x_i=0 \text{ if } \xi_i\neq0,\ 1\leqslant i\leqslant n\}$ and let P be the projection from $\boldsymbol{R}^n$ onto $\boldsymbol{M}$. Then, for $\tau>0$, there is $x_\tau\in\boldsymbol{M}$ such that $|x_\tau|=1$ and $P(T_\tau x_\tau)=0$. Then $T_\tau x_\tau\in\boldsymbol{M}^\perp$, $x_\tau^TT_\tau x_\tau=0\Rightarrow x_\tau^TT^+x_\tau-\tau x_\tau^TT^-x_\tau=0\Rightarrow\sin\theta-\tau\cos\theta=0$, where $\theta=\angle(x_\tau,T^+x_\tau)\Rightarrow\theta=\arctan(\tau)\Rightarrow\theta\to\pi/2$, as $\tau\to+\infty$. Let $\psi=\angle(T^+x_\tau,\boldsymbol{M})$. Then $T^+x_\tau\to x_\tau$, $\psi\to0$, as $\tau\to+\infty$. On the other hand, since $P(T^-x_\tau)=0$, we have that

$$\begin{aligned}0&=|P(T_\tau x_\tau)|=|P(T^+x_\tau-\tau T^-x_\tau)|\\&=|P(T^+x_\tau)-\tau P(T^-x_\tau)|=|P(T^+x_\tau)|=\cos\psi|T^+x_\tau|\\&=\cos\psi\sin\theta\Rightarrow\psi\equiv\pi/2.\end{aligned}$$

This contradicts that $\psi\to0$, at $\tau\to+\infty$. This proves the lemma. ■

From Remark 5.1 and Lemmas 5.1 and 5.2, we obtain the following:

Theorem 5.1: With the parameter τ sufficiently large, the synthesized LSSM system:

$$dx/dt=T_\tau x+I_\tau,\qquad \tau\in\boldsymbol{R} \tag{M_τ}$$

with constraints

$$-1\leqslant x_i\leqslant1,\qquad i=1,\cdots,n$$

has the following properties:

1) Each $\alpha\in\boldsymbol{B}^n\cap L_a$ is an asymptotically stable equilibrium point of (M_τ). In particular, this is true for $\alpha_1,\cdots,\alpha_m$.

2) There are no oscillatory solutions. ■

Remark 5.2: By Theorem 4.5, each $\alpha\in\boldsymbol{L}_a\cap\boldsymbol{B}^n$ is a local minimum of the energy function E: $\boldsymbol{D}^n\to\boldsymbol{R}$, $E(x)=-(1/2)x^TT_\tau x-x^TI_\tau$. In fact, it can be proved as in [5] that each $\alpha\in L_a\cap\boldsymbol{B}^n$ is a global minimum of the energy function E. ■

In the following, we address spurious asymptotically stable equilibrium points of (M_τ).

Lemma 5.3: For $-1<\tau_1<\tau_2$, if $\alpha\in \boldsymbol{B}^n$ is an asymptotically stable equilibrium point of system (N_{τ_2}), then α is also an asymptotically stable equilibrium point of system (N_{τ_1}).

Proof: Take $\lambda_1=(\tau_2-\tau_1)/(1+\tau_2)$ and $\lambda_2=(1+\tau_1)/(1+\tau_2)$. Then $\lambda_1,\lambda_2\geqslant 0$, $\lambda_1+\lambda_2=1$, and

$$\begin{aligned}T_{\tau_1}\alpha+I_{\tau_1}&=T^+(\alpha-\alpha_m)-\tau_1T^-(\alpha-\alpha_m)+\alpha_m\\&=\lambda_1\big(T^+(\alpha-\alpha_m)+T^-(\alpha-\alpha_m)+\alpha_m\big)\\&\quad+\lambda_2\big(T^+(\alpha-\alpha_m)-\tau_2T^-(\alpha-\alpha_m)+\alpha_m\big)\\&=\lambda_1\alpha+\lambda_2\big(T_{\tau_2}\alpha+I_{\tau_2}\big).\end{aligned}$$

If α is an asymptotically stable equilibrium point of system (N_{τ_2}), by Theorem 4.2, $\min((T_{\tau_2}\alpha+I_{\tau_2})*\alpha)>0$. It follows that

$$\begin{aligned}\min\big((T_{\tau_1}\alpha+I_{\tau_1})*\alpha\big)&=\min\Big(\big(\lambda_1\alpha+\lambda_2\big(T_{\tau_2}\alpha+I_{\tau_2}\big)\big)*\alpha\Big)\\&\geqslant\lambda_1\min(\alpha*\alpha)+\lambda_2\min\big((T_{\tau_2}\alpha+I_{\tau_2})*\alpha\big)>0.\end{aligned}$$

By Theorem 4.2, α is an asymptotically stable equilibrium point of system (N_{τ_1}). ■

Lemma 5.4: If in the synthesized system (M_τ), the parameter τ is sufficiently large and the constant vector $I_\tau=0$, then it follows that $\xi\in\boldsymbol{B}^n$ is an asymptotically stable equilibrium point of (M_τ) only if $\xi\in L_a$.

Proof: For $\xi\in\boldsymbol{B}^n-L_a$, let $\tau_0=\tan\theta$, where $\theta=\angle(T^+\xi,\xi)$. Then $\theta>0$ and for $\tau>\tau_0$,

$$\begin{aligned}\min\,(x_\xi*\xi)&=\min(\xi*(T^+\xi-\tau T^-\xi))\\&\leqslant\xi(T^+\xi-\tau T^-\xi)=\cos\theta-\tau\sin\theta\\&\leqslant\sin\theta\,(\tau_0-\tau)<0.\end{aligned}$$

By Theorem 4.2, ξ is not an equilibrium point. The lemma follows since there are only a finite number of elements in the set $\boldsymbol{B}^n-L_a$. ■

Lemma 5.5: If in the synthesized system (M_τ), the parameter τ is sufficiently large, then it follows that $\xi\in\boldsymbol{B}^n$ is an asymptotically stable equilibrium point of (M_τ) only if $\xi\in L_a\cup\boldsymbol{W}$, where

$$\boldsymbol{W}=\{x\in\boldsymbol{D}^n\colon |(1/2)r-T^-x|\leqslant|(1/2)r|\}\tag{5.5a}$$

where

$$r=T\alpha_m\tag{5.5b}$$

is the projection of 0 to L_a.

Proof: This lemma can be proved in the same manner as Lemma 5.4. ■

From Lemmas 5.3–5.5, we conclude the following:

Theorem 5.2: 1) When $\tau>-1$, the number of the spurious asymptotically stable equilibrium points of system (M_τ) contained in $\boldsymbol{B}^n$ decreases as τ increases.

2) When the parameter τ is sufficiently large, the set of the asymptotically stable equilibrium points of (M_τ) contained in $\boldsymbol{B}^n$ is $(L_a\cap\boldsymbol{B}^n)\cup\boldsymbol{W}$.

3) When the parameter τ is sufficiently large, the set of the asymptotically stable equilibrium points of (M_τ) contained in $\boldsymbol{B}^n$ is $L_a\cap\boldsymbol{B}^n$, if $I_\tau=0$. ■

Remark 5.3: Since the set $\boldsymbol{W}$ is considerably smaller than the set $\boldsymbol{D}^n$ it is unlikely that $\boldsymbol{W}\cap\boldsymbol{B}^n\neq\varnothing$. We may conclude that $(L_a\cap\boldsymbol{B}^n)\cup\boldsymbol{W}\approx L_a\cap\boldsymbol{B}^n$ when τ is large. This is confirmed in Examples 6.1 and 6.2 in the next section. ■

We conclude our discussion with the synthesis procedures given below.

Synthesis Procedure 5.1: Suppose we are given m vectors $\alpha_1,\cdots,\alpha_m$ in $\boldsymbol{B}^n$, which are to be stored as asymptotically stable equilibrium points for an n dimensional system (M). We proceed as follows:

1) Compute the $n\times(m-1)$ matrix:

$$Y=[\alpha_1-\alpha_m,\cdots,\alpha_{m-1}-\alpha_m].\tag{5.6}$$

2) Perform a singular value decomposition of Y and obtain the matrices U, V and Σ such that $Y=U\Sigma V^T$, where U and V are unitary matrices and where Σ is a diagonal matrix with the singular values of Y on its diagonal. (This can be accomplished by standard computer routines.) Let

$$Y=[y_1,\cdots,y_{m-1}]\tag{5.7a}$$

$$U=[u_1,\cdots,u_n]\tag{5.7b}$$

and

$$k=\text{dimension of Span}\,(y_1,\cdots,y_{m-1}).\tag{5.7c}$$

From the properties of singular value decomposition, we know that $k=$ rank of Σ, $\{u_1,\cdots,u_k\}$ is an orthonormal basis of $\mathrm{Span}(y_1,\cdots,y_{m-1})$ and $\{u_1,\cdots,u_n\}$ is an orthonormal basis of $\boldsymbol{R}^n$.

3) Compute

$$T^+=[T_{ij}^+]=\sum_{i=1}^{k}u_iu_i^T$$

$$T^-=[T_{ij}^-]=\sum_{i=k+1}^{n}u_iu_i^T.$$

4) Choose a large positive value for the parameter τ and compute

$$T_\tau=T^+-\tau T^-\quad\text{and}\quad I_\tau=\alpha_m-T_\tau\alpha_m.$$

Then all vectors in $L_a\cap\boldsymbol{B}^n$, where $L_a=\mathrm{Aspan}(\alpha_1,\cdots,\alpha_m)$, including $\alpha_1,\cdots,\alpha_m$, will be stored as asymptotically stable equilibrium points in the LSSM system

$$dx/dt=T_\tau x+I_\tau\tag{M_τ}$$

with constraints

$$-1\leqslant x_i\leqslant 1,\qquad i=1,\cdots,n.$$

5) If the parameter τ is sufficiently large, the set of the asymptotically stable equilibrium points contained in $\boldsymbol{B}^n$ is approximately equal to $L_a\cap\boldsymbol{B}^n$. ■

Remark 5.4: If we wish that in the synthesized system (M_τ) the constant vector $I_\tau=0$, we can modify Synthesis Procedure 5.1 as follows:

a) In step 1), take $Y=[\alpha_1,\cdots,\alpha_m]$.
b) In step 4), take $I_\tau=0$.

Then all conclusions will remain unchanged. In particular, each $-\alpha_i$, $i=1,\cdots,m$, will also be an asymptotically stable equilibrium point of the synthesized system (M_τ) and the number of elements in $L_a \cap \boldsymbol{B}^n$ will approximately be doubled. ■

By Theorems 4.2 and 4.3, Synthesis Procedure 5.1 can be generalized to store data given in $\Lambda = \{x \in \boldsymbol{D}^n\colon x_i = \pm 1 \text{ or } 0,\ 1 \leqslant i \leqslant n\}$. We have

Synthesis Procedure 5.2: Suppose that we are given m vectors $\alpha_1,\cdots,\alpha_m$ in Λ, which are to be stored as asymptotically stable equilibrium points for an n dimensional system (M). We proceed as follows:

1) For $\alpha, \alpha' \in \{\alpha_1,\cdots,\alpha_m\}$, $\alpha' \neq \alpha$, according to Theorem 4.3, the two vectors α and α' cannot be stored as asymptotically stable equilibrium points of system (M_τ), if either $\alpha \in \overline{C(\alpha')}$ or $\alpha' \in \overline{C(\alpha)}$, where

$$C(\alpha) = \left\{x = (x_1,\cdots,x_n)^T \in \boldsymbol{R}^n\colon |x_i| < 1 \text{ if } \alpha_i = 0 \text{ and } x_i = \alpha_i \text{ if } \alpha_i \neq 0\right\} \tag{5.8a}$$

$$C(\alpha') = \left\{x = (x_1,\cdots,x_n)^T \in \boldsymbol{R}^n\colon |x_i| < 1 \text{ if } \alpha'_i = 0 \text{ and } x_i = \alpha'_i \text{ if } \alpha'_i \neq 0\right\} \tag{5.8b}$$

as given in (4.1c).

2) Compute the $n \times (m-1)$ matrix

$$Y = [\alpha_1 - \alpha_m, \cdots, \alpha_{m-1} - \alpha_m].$$

3) Perform a singular value decomposition of Y and obtain the matrices U, V and Σ such that $Y = U\Sigma V^T$, where U and V are unitary matrices and where Σ is a diagonal matrix with the singular values of Y on its diagonal. Let $Y = [y_1,\cdots,y_{m-1}]$, $U = [u_1,\cdots,u_n]$, and $k =$ dimension of $\text{Span}(y_1,\cdots,y_{m-1})$. From the properties of singular value decomposition, we know that $k =$ rank of Σ, $\{u_1,\cdots,u_k\}$ is an orthonormal basis of $\text{Span}(y_1,\cdots,y_{m-1})$ and $\{u_1,\cdots,u_n\}$ is an orthonormal basis of $\boldsymbol{R}^n$.

4) Compute

$$T^+ = [T^+_{ij}] = \sum_{i=1}^{k} u_i u_i^T$$

$$T^- = [T^-_{ij}] = \sum_{i=k+1}^{n} u_i u_i^T.$$

5) Choose a large positive value for the parameter τ and compute

$$T_\tau = T^+ - \tau T^- \quad \text{and} \quad I_\tau = \alpha_m - T_\tau \alpha_m.$$

Then all vectors in $L_a \cap \Lambda$, where $L_a = \text{Aspan}(\alpha_1,\cdots,\alpha_m)$, including $\alpha_1,\cdots,\alpha_m$, will be stored as equilibrium points in the LSSM system

$$dx/dt = T_\tau x + I_\tau \tag{M_τ}$$

with constraints

$$-1 \leqslant x_i \leqslant 1, \qquad i = 1,\cdots,n.$$

6) For an $\alpha \in \{\alpha_1,\cdots,\alpha_m\}$, if $\alpha \in \boldsymbol{B}^n$, α is an asymptotically stable equilibrium point of (M_τ). If $\alpha \notin \boldsymbol{B}^n$, whether α is an asymptotically stable equilibrium point of (M_τ) or not can be determined using Theorem 4.2. ■

Remark 5.5: The synthesis procedures we developed in the foregoing for LSSM system (M) can be directly applied to the analog Hopfield model. This will be illustrated in Example 6.3 in the next section. ■

As discussed in Section III, system (M_τ) can be simulated by making use of the difference equation

$$x((k+1)h) = F(\Phi_\tau x(kh) + \Gamma_\tau) \tag{$M_{d\tau}$}$$

with the constraints

$$-1 \leqslant x_i \leqslant 1, \qquad i = 1,\cdots,n$$

where $x(kh) \in \boldsymbol{D}^n$, h is the sample period, $F\colon \boldsymbol{R}^n \to \boldsymbol{D}^n$, $F(x) = (F_1(x_1),\cdots,F_n(x_n))^T$, with

$$F_i(\rho) = \begin{cases} 1, & \text{if } \rho > 1 \\ \rho, & \text{if } -1 \leqslant \rho \leqslant 1, \qquad 1 \leqslant i \leqslant n \\ -1, & \text{if } \rho < -1 \end{cases}$$

and

$$\begin{cases} \Phi_\tau = \exp(hT_\tau) \\ \Gamma_\tau = \int_0^h \exp(\rho T_\tau)\, d\rho \cdot I_\tau. \end{cases}$$

For $1 \leqslant k \leqslant n$, let $\boldsymbol{I}_k$ be the $k \times k$ identity matrix. We can write

$$T_\tau = U\begin{bmatrix} \boldsymbol{I}_k & 0 \\ 0 & -\tau \boldsymbol{I}_{n-k} \end{bmatrix} U^T$$

where U is the $n \times n$ unitary matrix given in (5.7b) and $k =$ rank of Σ given in (5.7c). Then we have

Remark 5.6: In the corresponding difference equation $(M_{d\tau})$ of system (M_τ) given above, the parameter matrix Φ_τ and vector Γ_τ can be calculated as

$$\begin{aligned} \Phi_\tau = \exp(hT_\tau) &= \exp\left(hU\begin{bmatrix} \boldsymbol{I}_k & 0 \\ 0 & -\tau \boldsymbol{I}_{n-k} \end{bmatrix} U^T\right) \\ &= U \exp\left(h\begin{bmatrix} \boldsymbol{I}_k & 0 \\ 0 & -\tau \boldsymbol{I}_{n-k} \end{bmatrix}\right) U^T \\ &= U\begin{bmatrix} e^h \boldsymbol{I}_k & 0 \\ 0 & e^{-\tau h} \boldsymbol{I}_{n-k} \end{bmatrix} U^T \end{aligned} \tag{5.9a}$$

and

$$\begin{aligned} \Gamma_\tau &= U\begin{bmatrix} \int_0^h e^\rho \, d\rho\, \boldsymbol{I}_k & 0 \\ 0 & \int_0^h e^{-\tau\rho}\, d\rho\, \boldsymbol{I}_{n-k} \end{bmatrix} U^T \\ &= U\begin{bmatrix} (e^h - 1)\boldsymbol{I}_k & 0 \\ 0 & \dfrac{1}{-\tau}(e^{-\tau h} - 1)\boldsymbol{I}_{n-k} \end{bmatrix} U^T. \ ■ \end{aligned} \tag{5.9b}$$

Remark 5.7: Using the synthesis procedures introduced in this section, any vector in $L_a \cap \boldsymbol{B}^n$ will always be stored in the system (M_τ) as an asymptotically stable equilibrium point even if it is not in the given desired set of vectors $\{\alpha_1,\cdots,\alpha_m\}$. This can sometimes be used to simplify our synthesis procedure as will be demonstrated in Example 6.5 in the next section. ∎

VI. Examples and Applications

In the present section, we demonstrate the applicability of the results of the previous sections.

Example 6.1: In this example, we illustrate use of the Synthesis Procedure 5.1. The dimension of the system to be considered is $n=10$. Given are $m=5$ vectors specified by

$$\alpha_1=\begin{pmatrix}-1\\1\\-1\\1\\1\\1\\-1\\1\\1\\1\end{pmatrix},\quad \alpha_2=\begin{pmatrix}1\\1\\-1\\-1\\1\\-1\\1\\-1\\1\\1\end{pmatrix},\quad \alpha_3=\begin{pmatrix}-1\\1\\1\\1\\-1\\-1\\1\\-1\\1\\-1\end{pmatrix},$$

$$\alpha_4=\begin{pmatrix}1\\1\\-1\\1\\-1\\1\\-1\\1\\1\\1\end{pmatrix},\quad \alpha_5=\begin{pmatrix}1\\-1\\-1\\-1\\1\\1\\1\\-1\\-1\\-1\end{pmatrix}.$$

It is desired that these vectors be asymptotically stable equilibrium points of LSSM system (M).

1) Compute the $n\times(m-1)=10\times 4$ matrix

$$Y=[y_1,y_2,y_3,y_4]=\begin{pmatrix}-2&0&-2&0\\2&2&2&2\\0&0&2&0\\2&0&2&2\\0&0&-2&-2\\0&-2&-2&0\\-2&0&0&-2\\2&0&0&2\\2&2&2&2\\2&2&0&2\end{pmatrix}$$

where $y_i=\alpha_i-\alpha_5$ for $1\leqslant i\leqslant 4$.

2) Perform a singular value decomposition of Y to obtain the matrices U, V, and Σ such that $Y=U\Sigma V^T$. We have, $k=\text{rank of }\Sigma=4$.

3) Compute

$$T^+=\left[T_{ij}^+\right]=u_1u_1^T+\cdots+u_4u_4^T$$

and

$$T^-=\left[T_{ij}^-\right]=u_5u_5^T+\cdots+u_{10}u_{10}^T.$$

4) Choose the parameter $\tau=10$ and compute

$$T_\tau=T^+-10T^-$$

$$=\left(\begin{matrix}-2.1314e+00&-6.423e-01&-2.1679e+00&-2.5693e+00&-3.1314e+00\\-6.4234e-01&-7.0292e+00&4.0146e-01&8.8321e-01&-6.4234e-01\\-2.1679e+00&4.0146e-01&-7.2701e+00&1.6058e+00&-2.1679e+00\\-2.5693e+00&8.8321e-01&1.6058e+00&6.4672e+00&-2.5693e+00\\-3.1314e+00&-6.4234e-01&-2.1679e+00&-2.5693e+00&-2.1314e+00\\2.4088e-01&-2.4891e+00&-1.5255e+00&1.0438e+00&2.4088e-01\\4.0146e-01&-4.8175e-01&1.1241e+00&-1.9270e+00&4.0146e-01\\-4.0146e-01&4.8175e-01&-1.1241e+00&1.9270e+00&-4.0146e-00\\-6.4234e-01&2.9708e+00&4.0146e-01&8.8321e-01&-6.4234e-01\\1.5244e+00&2.5693e+00&-2.3285e+00&7.2263e-01&1.5255e+00\end{matrix}\right.$$

$$\left.\begin{matrix}2.4088e-01&4.0146e-01&-4.0146e-01&-6.4234e-01&1.5255e+00\\-2.4891e+00&-4.8175e-01&4.8175e-01&2.9708e+00&2.5693e+00\\-1.5255e+00&1.1241e+00&-1.1241e+00&4.0146e-01&-2.3285e+00\\1.0438e+00&-1.9270e+00&1.9270e+00&8.8321e-01&-7.2263e-01\\2.4088e-01&4.0146e-01&-4.0146e-01&-6.4234e-01&1.5255e+00\\-4.9416e+00&-2.5693e+00&2.5693e+00&-2.4891e+00&-9.6350e-01\\-2.5693e+00&-6.9489e+00&-3.0511e+00&-4.8175e-01&-1.6058e+00\\2.5693e+00&-3.0511e+00&-6.9489e+00&4.8175e-01&1.6058e+00\\2.4891e+00&-4.8175e-01&4.8175e-01&-7.0292e+00&2.5693e+00\\-9.6350e-01&-1.6058e+00&1.6058e+00&2.5693e+00&-5.1022e+00\end{matrix}\right)$$

and

$$I_\tau = \alpha_m - T_\tau \alpha_m = \begin{pmatrix} 7.2263e-01 \\ 3.5328e+00 \\ -4.5766e+00 \\ 3.1314e+00 \\ 7.2263e-01 \\ 4.1752e+00 \\ 3.2920e+00 \\ -3.2920e+00 \\ 3.5328e+00 \\ -2.8905e+00 \end{pmatrix}.$$

By Theorem 4.2, we determine that $\alpha_1, \cdots, \alpha_5$ are asymptotically stable equilibrium points for the synthesized LSSM system

$$dx/dt = T_\tau S(x) + I_\tau, \qquad \tau = 10 \qquad (\mathrm{M}_\tau)$$

with the constraints

$$-1 \leqslant x_i \leqslant 1, \qquad 1 \leqslant i \leqslant n.$$

System (M_τ) has 8 additional asymptotically stable equilibrium points which are not in $\boldsymbol{B}^n$, and are given by

$$\alpha_1' = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \alpha_2' = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \alpha_3' = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ -1 \\ 0 \\ -1 \end{pmatrix},$$

$$\alpha_4' = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 1 \\ -1 \\ 0 \\ -1 \end{pmatrix}, \quad \alpha_5' = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad \alpha_6' = \begin{pmatrix} -1 \\ -.47406 \\ -.36636 \\ .24669 \\ 1 \\ 1 \\ .48695 \\ -.48695 \\ -.47406 \\ -1 \end{pmatrix},$$

$$\alpha_7' = \begin{pmatrix} 1 \\ -.47406 \\ -.36636 \\ .24669 \\ 1 \\ 1 \\ .48695 \\ -.48695 \\ -.47406 \\ -1 \end{pmatrix}, \quad \alpha_8' = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

Also by Theorem 4.2, we determine that (M_τ) has 70 unstable equilibrium points. ■

Remark 6.1: The data $\alpha_1, \cdots, \alpha_5$ used in Example 6.1 are identical to the data used in an example considered in [5]. For that example, using the same synthesis procedures as in the present paper, we obtained for the discrete Hopfield model a total of 15 spurious states. ■

TABLE I

average of total number of asymptotically stable equilibrium points of synthesized system	9.90
average of total number of asymptotically stable equilibrium points of synthesized system in $\mathbf{B}^n \cap L_a$	7.07
average of total number of asymptotically stable equilibrium points of synthesized system in $\mathbf{B}^n$-L_a	0
average of total number of asymptotically stable equilibrium points of synthesized system in $\mathbf{D}^n$-$\mathbf{B}^n$	2.83
average of total number of unstable equilibrium points of synthesized system	52.45
average of total number of vectors in the given data without being synthesized as asymptotically stable equilibrium points of synthesized system	0

Example 6.2: In order to ascertain how typical the results of Example 6.1 are, we repeated it 30 times using different sets of given vectors to be stored as asymptotically stable equilibrium points of system (M). Each set contains $m = 5$ vectors. For each given set of vectors, we synthesize system (M) as in Example 6.1. Table I summarizes these results. ■

Remark 6.2: The data used in Example 6.2 are identical to the data used in an example considered in [5]. For that example, we obtained for the discrete Hopfield model an average of 25.3 spurious states. Thus the LSSM model yields better results than the discrete Hopfield model. ■

Example 6.3: Our Synthesis Procedure with the parameter τ positive can be used to synthesize the analog Hopfield model (cf. [1] and [4]). Consider a Hopfield model with ten neurons described by

$$dx/dt = H(x)(-S(x) + Tx + I) \qquad (\mathrm{L})$$

where $x \in (-1,1)^{10}$, $S(x) = (s(x_1), \cdots, s(x_{10}))^T$, $s(\rho) = (1/\lambda)(2/\pi)\tan((\pi/2)\rho)$, $H(x) = \mathrm{diag}[1/s'(x_1), \cdots, 1/s'(x_{10})]$, and $\lambda = 100$. Suppose that we want to store $m = 5$ vectors, say, $\gamma_1, \cdots, \gamma_5$, in the system (L) and suppose that the exact locations of these five vectors are not important but they should be located in particular regions such that the sign of their components are the same as the given data in Example 6.1, that is, $\alpha_{i,j} = \mathrm{sgn}(\gamma_{i,j})$, $1 \leqslant i \leqslant 5$, $1 \leqslant j \leqslant 10$, where $\alpha_{i,j}$ and $\gamma_{i,j}$ are the jth components of vectors α_i and γ_i, respectively. Using the same T_τ and I_τ with $\tau = 10$ obtained in Example 6.1 and we obtain

$$dx/dt = H(x)(-S(x) + T_\tau x + I_\tau). \qquad (\mathrm{L}_\tau)$$

It seems that *efficient* methods of determining *all* of the asymptotically stable equilibrium points of *high dimensional* analog Hopfield models have not appeared in the literature. In the case of unstable equilibrium points, the situation is even worse. The close relationship between LSSM system (M_τ) and the analog Hopfield model (L_τ) suggests, as pointed out in Remarks 3.5 and 4.7, that the set of asymptotically stable equilibrium points of (M_τ) and the set of asymptotically stable equilibrium points of (L_τ) are approximately the same. To verify this observation, using the present example, we simulated system (L_τ) by Gill's method with step length = 0.01. In doing so, we used for Gill's method the *initial vectors* $\alpha_1,\cdots,\alpha_5$, respectively, to determine the five asymptotically stable equilibrium points of (L_τ) given by

$$\gamma_1 \approx \begin{pmatrix} -.99592 \\ .99599 \\ -.99600 \\ .99598 \\ .99594 \\ .99600 \\ -.99588 \\ .99588 \\ .99599 \\ .99588 \end{pmatrix}, \quad \gamma_2 \approx \begin{pmatrix} .99594 \\ .99599 \\ -.99600 \\ -.99588 \\ .99594 \\ -.99586 \\ .99598 \\ -.99598 \\ .99599 \\ .99588 \end{pmatrix}, \quad \gamma_3 \approx \begin{pmatrix} -.99592 \\ .99599 \\ .99586 \\ .99598 \\ -.99592 \\ -.99586 \\ .99598 \\ -.99598 \\ .99599 \\ -.99598 \end{pmatrix}$$

$$\gamma_4 \approx \begin{pmatrix} .99594 \\ .99599 \\ -.99600 \\ .99598 \\ -.99592 \\ .99600 \\ -.99588 \\ .99588 \\ .99599 \\ .99588 \end{pmatrix}, \quad \gamma_5 \approx \begin{pmatrix} .99594 \\ -.99587 \\ -.99600 \\ -.99588 \\ .99594 \\ .99600 \\ .99598 \\ -.99598 \\ -.99587 \\ -.99598 \end{pmatrix}.$$

Furthermore, with the initial vectors $\alpha_1',\cdots,\alpha_8'$ (determined in Example 6.1), we obtain, respectively, the eight asymptotically stable equilibrium points of (L_τ) given by

$$\gamma_1 \approx \begin{pmatrix} -9.9593e-1 \\ 9.9597e-1 \\ -1.5740e-3 \\ 1.7863e-3 \\ 9.9593e-1 \\ -9.9584e-1 \\ 9.9597e-1 \\ -9.9597e-1 \\ 9.9597e-1 \\ -1.8422e-3 \end{pmatrix}, \quad \gamma_2' \approx \begin{pmatrix} 9.9593e-1 \\ 9.9597e-1 \\ -1.5740e-3 \\ 1.7863e-3 \\ -9.9593e-1 \\ -9.9584e-1 \\ 9.9597e-1 \\ -9.9597e-1 \\ 9.9597e-1 \\ -1.8422e-3 \end{pmatrix}$$

$$\gamma_3' \approx \begin{pmatrix} -9.9594e-1 \\ 2.7891e-3 \\ -2.0755e-3 \\ 2.5102e-3 \\ 9.9592e-1 \\ 1.8154e-3 \\ 9.9593e-1 \\ -9.9593e-1 \\ 2.7891e-3 \\ -9.9592e-1 \end{pmatrix}, \quad \gamma_4' \approx \begin{pmatrix} 9.9592e-1 \\ 2.7809e-3 \\ -2.0777e-3 \\ 2.5124e-3 \\ -9.9594e-1 \\ 1.7904e-3 \\ 9.9593e-1 \\ -9.9593e-1 \\ 2.7809e-3 \\ -9.9592e-1 \end{pmatrix},$$

$$\gamma_5' \approx \begin{pmatrix} -9.9595e-1 \\ 2.0514e-3 \\ -2.7120e-3 \\ 9.9590e-1 \\ 3.9224e-3 \\ 9.9593e-1 \\ 2.5567e-3 \\ -2.5567e-3 \\ 2.0514e-3 \\ -9.9592e-1 \end{pmatrix}, \quad \gamma_6' \approx \begin{pmatrix} -9.8968e-1 \\ -4.7226e-1 \\ -3.6780e-1 \\ 2.4001e-1 \\ 9.9746e-1 \\ 9.9529e-1 \\ 4.9090e-1 \\ -4.9090e-1 \\ -4.7226e-1 \\ -9.9804e-1 \end{pmatrix},$$

$$\gamma_7' \approx \begin{pmatrix} 9.9746e-1 \\ -4.7226e-1 \\ -3.6780e-1 \\ 2.4002e-1 \\ -9.8968e-1 \\ 9.9529e-1 \\ 4.9090e-1 \\ -4.9090e-1 \\ -4.7226e-1 \\ -9.9804e-1 \end{pmatrix}, \quad \gamma_8' \approx \begin{pmatrix} 3.9078e-3 \\ 2.0472e-3 \\ -2.7143e-3 \\ 9.9590e-1 \\ -9.9595e-1 \\ 9.9593e-1 \\ 2.5570e-3 \\ -2.5570e-3 \\ 2.0472e-3 \\ -9.9592e-1 \end{pmatrix}.$$

Thus, in the present example, the set of asymptotically stable equilibrium points for system (M) and the set of asymptotically stable equilibrium points for the corresponding Hopfield model (H) (using the same matrix T and vector I) are indeed approximately identical, as discussed in Remark 4.7, and the former can be directly used to estimate the latter. ■

Example 6.4: To illustrate how to estimate the domain of attraction of each asymptotically stable equilibrium point of system (M), we consider an example given by

$$dx/dt = Tx + I, \qquad -1 \leqslant x_i \leqslant 1, \qquad i = 1,2,3 \quad (M_3)$$

where $x \in \boldsymbol{D}^3$,

$$T = \begin{pmatrix} -2.6667 & -3.6667 & -3.6667 \\ -3.6667 & -2.6667 & -3.6667 \\ -3.6667 & -3.6667 & -2.6667 \end{pmatrix}$$

and

$$I = \begin{pmatrix} -3.6667 \\ -3.6667 \\ -3.6667 \end{pmatrix}.$$

This system is synthesized by Synthesis Procedure 5.1 with the given data set

$$\alpha_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

This system has three asymptotically stable equilibrium points α_1, α_2 and α_3, and four unstable equilibrium points given by

$$\beta_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \beta_4 = 3.667 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

The reason for choosing this example is that the domain of attraction of each α_i can be determined by inspection of the symmetry of the set $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\mathrm{Span}(\alpha_1, \alpha_2, \alpha_3)$ relative to the axis passing through the point $x_0 = (1,1,1)^T$, as illustrated in Fig. 8. For instance, the domain of attraction of α_1 is the polygon with vertices given by $\alpha_1, \gamma_1, \gamma_2, \gamma_3, \gamma_4$, where

$$\gamma_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \gamma_4 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

In Fig. 8, we see that all unstable equilibrium points are indeed located on the boundaries of the domain of attrac-

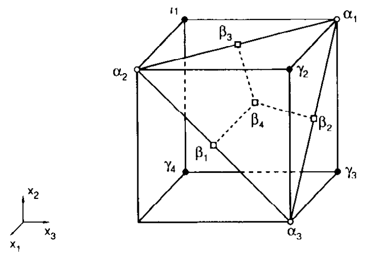


Fig. 8. Location of Aspan$(\alpha_1, \alpha_2, \alpha_3)$.

Fig. 9. Gray levels.

tion of α_1, α_2, and α_3. In general, the domain of attraction of an asymptotically stable equilibrium point of system (M) may approximately be determined by the set of unstable equilibrium points of (M) in its vicinity. ■

Remark 6.3: As discussed in Example 6.3, we can also use the domain of attraction of an asymptotically stable equilibrium point of system (M) (e.g., as obtained in Example 6.4) to estimate the domain of attraction of the corresponding asymptotically stable equilibrium point of the corresponding Hopfield model. ■

Example 6.5: Let us consider patterns made up of 9×9 small boxes. Each pattern corresponds to a vector in $\boldsymbol{R}^{81}$ with each component value varying from -1 to 1 determined by the gray level (cf. Fig. 9) in the corresponding box. If the gray level in a box is white (black) the value of the corresponding component is -1 (1).

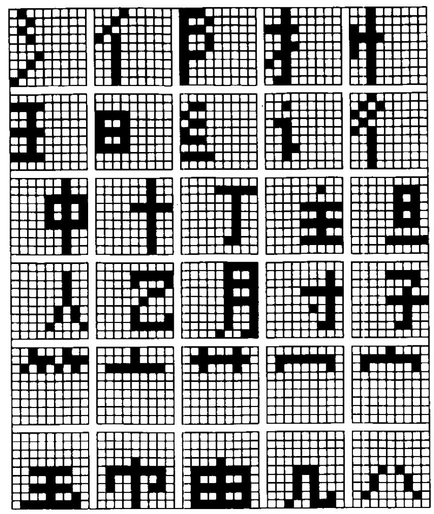


Fig. 10. 30 Chinese characters.

Suppose that we wish to synthesize a system (M), with dimension $n = 81$, which will "remember" certain Chinese characters. Since many Chinese characters can be separated into two basic components, using the present synthesis procedure, we only need to synthesize a system (M) by employing these basic components. The resulting system will automatically "remember" all possible combinations of the above components. For instance, let us consider a data set $\{\alpha_1, \cdots, \alpha_{31}\}$, where each α_i, $1 \leqslant i \leqslant 30$, corresponds to a Chinese character in Fig. 10, respectively. In particular,

$$\alpha_7 = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}$$

and

$$\alpha_{18} = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \end{bmatrix}.$$

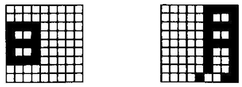
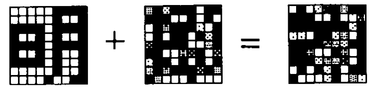


Fig. 11. Two basic Chinese characters: "sun" and "moon".



(a) (b) (c)

Fig. 12. The input pattern. (a) Pattern. (b) Noise. (c) Initial input.

The Chinese character corresponding to α_7 means "sun" and the Chinese character corresponding to α_{18} means "moon" (see Fig. 11). Also, in the given data set, α_{31} corresponds to "nothing," i.e.,

$$\alpha_{31}=\begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}.$$

To simplify our calculation, we will synthesize system (M) as discussed in Remark 5.4, resulting in $I_\tau = 0$. In this case, we have $L_a = \text{Aspan}(\alpha_1,\cdots,\alpha_{31}) = \text{Span}(\alpha_1,\cdots,\alpha_{31})$. Also, since we will check our results using a digital computer instead of constructing an analog circuit, we will synthesize directly the corresponding discrete system (M_d), as discussed in Remarks 3.6 and 5.6. The exact procedure is as follows:

1) Let $Y = [\alpha_1,\cdots,\alpha_{31}]$.

2) Perform a singular value decomposition of Y as described in Synthesis Procedure 5.1 to obtain the unitary matrix U and the constant $k = \text{rank of Span}(\alpha_1,\cdots,\alpha_{31})$.

3) Let $\tau = 100$ and let the sample period $h = 0.01$. Next, calculate the $n \times n$ matrix

$$\Phi_\tau = U\begin{bmatrix} e^h \boldsymbol{I}_k & 0 \\ 0 & e^{-\tau h}\boldsymbol{I}_{n-k}\end{bmatrix}U^T$$

as given in (5.10), where $\boldsymbol{I}_k$ and $\boldsymbol{I}_{n-k}$ denote the $k\times k$ identity matrix and $(n-k)\times(n-k)$ identity matrix, respectively.

4) The synthesized system is given as

$$x((k+1)h) = F(\Phi_\tau x(kh)) \qquad (\text{M}'_\text{d})$$

where

$$F(\rho) = \begin{cases} 1, & \text{if } \rho > 1 \\ \rho, & \text{if } -1 \leqslant \rho \leqslant 1 \\ -1, & \text{if } \rho < -1. \end{cases}$$

By Theorem 5.1, in addition to $\{\alpha_1,\cdots,\alpha_{30}\}$, system ($\text{M}'_\text{d}$) can store at least 125 other data given by

$$\beta_{ij} = (1/2)\alpha_i + (1/2)\alpha_j, \qquad 1 \leqslant i \leqslant 10,\ 11 \leqslant j \leqslant 20 \tag{6.1a}$$

and

$$\beta_{ij} = (1/2)\alpha_i + (1/2)\alpha_j, \qquad 21 \leqslant i \leqslant 25,\ 26 \leqslant j \leqslant 30. \tag{6.1b}$$

Thus by this method, we can easily synthesize a system (M) with a total number of data stored which is greatly in excess of the dimension of system (M).

We tested the synthesized system (M'_d) on Micro VAX II computer, using different initial conditions as inputs. The results show that vectors β_{ij} given in (6.1) are indeed stored as asymptotically stable equilibrium points of system (M'_d). Below, we illustrate one of our tests with a specific input. Let

$$\alpha = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 \\ -1 & -1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \end{bmatrix}$$

where α corresponds to the Chinese character which means "bright," shown in Fig. 12(a). Since $\alpha = (1/2)\alpha_7 = (1/2)\alpha_{18}$, by Theorem 5.1, α will be an asymptotically stable equilibrium point of system (M'_d). To illustrate this, we use the randomly generated noise pattern given in Fig.

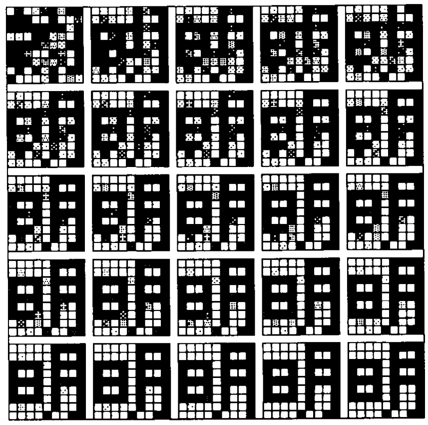


Fig. 13. Patterns corresponding to 24 states of system (M_d').

12(b). The input vector β is chosen as

$$\beta = \begin{pmatrix} -1.5883e+00 & 8.5641e-01 & -1.7455e+00 & -7.8640e-01 & -6.0213e-01 & 2.6491e+00 & 6.7701e-01 & 1.7214e+00 & 1.6857e+00 \\ 3.8494e-01 & 3.1822e-01 & 9.3525e-01 & -2.6039e+00 & -2.0815e+00 & 1.8863e+00 & -1.9788e+00 & -9.1153e-01 & -3.4685e-01 \\ 2.4785e+00 & 1.5546e+00 & 5.5772e-01 & 2.5630e+00 & 7.0001e-01 & 2.1686e+00 & 1.2835e-01 & -8.3534e-01 & 2.7454e+00 \\ -9.3104e-01 & -9.6184e-01 & -2.4556e+00 & 2.2943e+00 & 2.9161e-01 & -8.0473e-01 & -6.8852e-01 & -1.7563e+00 & 2.5769e+00 \\ 7.2871e-01 & 2.3746e+00 & 1.6503e+00 & 2.7366e+00 & -3.2764e-01 & -6.4740e-01 & -7.4018e-01 & 3.8742e-01 & 2.1784e+00 \\ 2.6002e+00 & 9.9647e-01 & -2.0726e+00 & -3.7206e-01 & -1.3817e+00 & 7.8871e-01 & -3.7080e-01 & -2.9843e+00 & 7.8695e-01 \\ 1.0585e-01 & 1.7251e+00 & 9.8233e-01 & -7.2520e-01 & -4.1936e-01 & -6.3976e-01 & -1.2922e-01 & -2.9962e+00 & 2.0927e+00 \\ -1.7130e+00 & 1.8079e-01 & -1.4569e+00 & 6.8815e-01 & -8.2293e-01 & 1.7970e+00 & -1.1923e+00 & 7.0962e-01 & 4.2602e-01 \\ 3.6531e-01 & -1.5630e+00 & -2.4871e+00 & -2.6758e+00 & -8.0707e-01 & 7.2772e-01 & -2.8560e+00 & -5.2942e-01 & -9.5342e-01 \end{pmatrix}.$$

where β is depicted in Fig. 12(c) and is obtained from adding the patterns in Figs. 12(a) and (b). β has at least 21 bits which differ from α.

With β given as the initial vector for system (M_d'), α was determined in 24 steps, shown in Fig. 13, as the desired asymptotically stable state.

In the present example, the speed of convergence of the initial state to the desired state of system (M_d') was very fast. This suggests that in certain applications, such as associative memories, it may be practical to implement the LSSM system (M_d) on digital array processors (rather than analog circuits). ■

VII. Concluding Remarks

In the present paper we demonstrated that a class of linear analog circuits, when allowed to operate in saturation, exhibits properties usually attributed to neural networks. We call this class of networks *LSSM systems* (linear systems operating in a saturated mode).

For such LSSM systems, we developed efficient analysis and synthesis procedures. Also, we compared the properties of LSSM systems with those of the analog Hopfield model. There are indications that LSSM systems do not possess some of the shortcomings of the Hopfield model. Also, we demonstrated that our synthesis procedure for LSSM systems can be used as an aid in the synthesis of Hopfield neural networks.

To demonstrate the applicability of our results, we considered five specific examples.

References

[1] J. J. Hopfield, "Neurons with graded response have collective computational properties like those of two-state neurons," in *Proc. Nat. Acad. Sci. USA*, vol. 81, pp. 3088–3092, May 1984.

[2] J. J. Hopfield and D. W. Tank, "'Neural' computation of decisions in optimization problems," *Biol. Cybern.*, vol. 52, pp. 141–152, 1985.
[3] D. W. Tank and J. J. Hopfield, "Simple 'neural' optimization networks: An A/D converter, signal decision circuit, and a linear programming circuit," *IEEE Trans. Circuits Syst.* vol. CAS-33, pp. 533–541, May 1986.
[4] J. Li, A. N. Michel, and W. Porod, "Qualitative analysis and synthesis of a class of neural networks," *IEEE Trans. Circuits Syst.*, vol. 35, pp. 976–986, Aug. 1988.
[5] ____, "Qualitative analysis and synthesis of a class of neural networks: Variable structure systems with infinite gains," *IEEE Trans. Circuits Syst.*, vol. 36, pp. 713–731, May 1989.
[6] R. K. Miller and A. N. Michel, *Ordinary Differential Equations*. New York: Academic, 1982.
[7] K. J. Astrom and B. Wittenmark, *Computer Controlled Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1984.
[8] A. N. Michel and C. J. Herget, *Mathematical Foundations in Engineering and Science*. Englewood Cliffs, NJ: Prentice-Hall, 1981.

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Wolfgamg Porod (M'85), for a photograph and biography please see page 243 of the February 1989 issue of this TRANSACTIONS.