MATH 3172 3.0
Combinatorial Optimization

Midterm I

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1 Hill climb

1.1 `grid1` and `grid2`

$$\texttt{grid1} = \begin{bmatrix} 3 & 7 & 2 & 8 \\ 5 & 2 & 9 & 1 \\ 5 & 3 & 3 & 1 \end{bmatrix}, \qquad \text{and} \qquad \texttt{grid2} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 8 & 10 \\ 0 & 2 & 4 & 8 & 16 \\ 1 & 4 & 8 & 16 & 32 \end{bmatrix}.$$

The global maximum of `grid1` and `grid2` was trivial to find using a naive implementation of the hill-climb. Both adjacent and diagonal state transitions were allowed to reduce the number of iterations.

Table 1: Hill-climb for three given discrete functions

Function $f(x)$	iterations	time (μs)	$x^\star$	$f(x^\star)$	success
`grid1`	3	112	$(1, 2)$	9	yes
`grid2`	2	39	$(3, 4)$	32	yes
`grid3`	8	115	$(7, 98)$	-7.4	no

1.2 `grid3` is problematic

Observe in the table, in the previous section, the naive hill climbing algorithm fails to find the global maximum of `grid3` around the point $x^\star = (1, 1)$.

Figure 1: 3D plot of $f_3(\boldsymbol{x})$

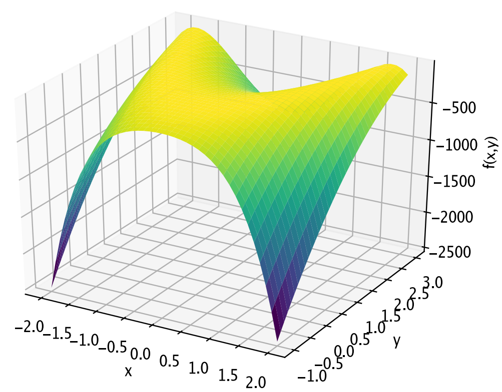
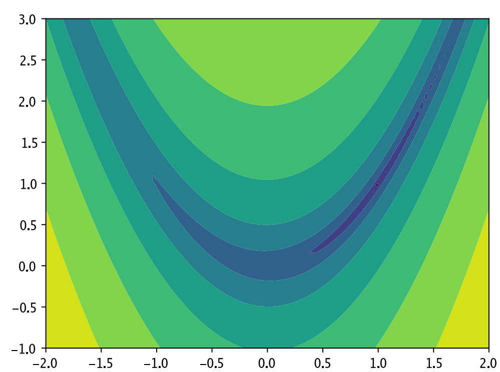


Figure 2: $(-f_3(\boldsymbol{x}))^{1/8}$ to emphasize narrow global maximum band



Especially when discretized, `grid3` has a ridge on which $\boldsymbol{x}^\star = (1, 1)$ lies. Hill climbing tends to get stuck along the sides this ridge, causing the algorithm to fail to find the global maximum. Aliasing, as a result of discretization, also results in several small isolated maxima near this ridge in the vicinity of $\boldsymbol{x}^\star$.

2 Simulated annealing

2.1 `grid1` and `grid2`

2.2 `grid3`

2.2.1 Cooling schedule

An adaptive additive exponential cooling schedule was used to find the global maximum of `grid3`.

$$T_k = T_n + (T_0 - T_n)\left(\frac{1}{1 + e^{\frac{2\ln(T_0 - T_n)}{n}\left(k - \frac{1}{2}n\right)}}\right) \tag{1}$$

Furthermore, we multiply T_k by an adaptive term $1 \le \mu \le 2$ which is calculated using the distance between the value of the current state $f(s_i)$ and the best value encountered so far $f^\star$.

$$T = \mu T_k = \left(1 + \frac{f(s_i) - f^\star}{f(s_i)}\right) T_k \tag{2}$$

In practice, we take the `np.abs` and use `np.clip(x, 1, 2)` to ensure that these assumptions are maintained.

The following parameters, initial temperature and number of cycles, denoted T_n and n respectivly, resulted in consistent convergence to the global maximum near $\boldsymbol{x}^\star = (0, 0)$:

$$T_n = 10 \qquad \text{and} \qquad n = 5000$$

2.2.2 Other cooling schedules considered

The following monotonic additive and multiplicative cooling schedules were also tried, but failed to produce good results:

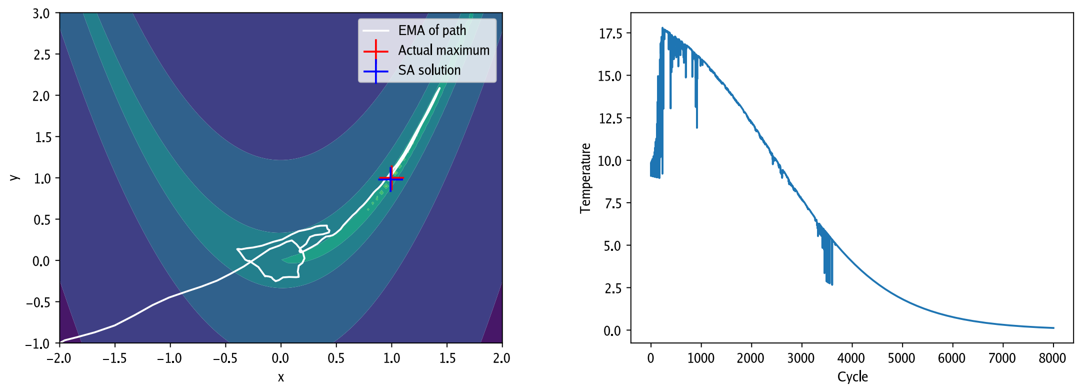
1. Linear cooling,
2. (a) exponential multiplicative cooling,
   (b) logarithmic multiplicative cooling,
   (c) quadratic multiplicative cooling,
3. (a) linear additive cooling,
   (b) quadratic aditive cooling,
   (c) exponential additive cooling.

2.2.3 Results

With the above parameters, our implementation of adaptive simulated annealing consistently converged very close to $\boldsymbol{x}^\star$ after $k = 35754$ cycles, taking on average 1.31 seconds.

Figure 3: Simulated annealing with adaptive exponential cooling schedule

(a) State trajectory (b) Adaptive-AE cooling schedule



Due to the large number of steps k, instead of showing the whole state trajectory, the white line denotes an exponential moving average or EMA of the trajectory with $\gamma = 0.002$. Given the k^{th} state s_k, the EMA denoted Y_k is calculated recursively using

$$Y_k = \begin{cases} s_1 & \text{for } k = 0 \\ \gamma s_k + (1-\gamma) s_{k-1} & \text{for } k > 1 \end{cases} \tag{3}$$

The blue cross denotes the solution found by our simulated annealing algorithm, and the red cross denotes $\boldsymbol{x}^\star = (0,0)^T$ which is the analytical solution of $\boldsymbol{x}^\star = \operatorname{argmax} f_3(\boldsymbol{x})$.

The figure on the right shows the exponential additive cooling schedule T_k being multiplied by a stochastic varrying term $1 \leq \mu \leq 2$.

2.2.4 Discussion

In case of `grid3` our implementation of simulated annealing is able to overcome the shortcommings of the hill climbing algorithm in *section 1*. Given an appropriate cooling schedule, the state trajectory is able to 'jump' around in the isolated maxima contained inside the narrow ridge containing $\boldsymbol{x}^\star$. This is an execellent practical demonstration of simulated annealing's abilitiy to converge to to a global maximum, when other methods fail and/or get stuck in local maxima.

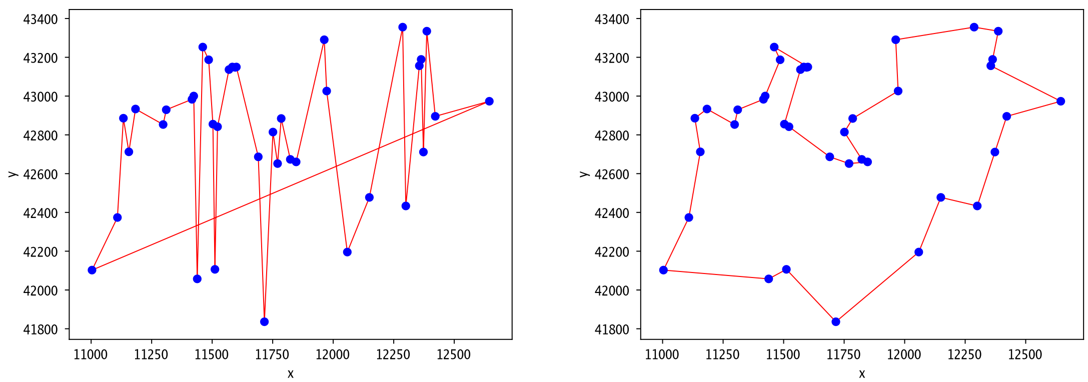
3 Traveling salesman problem

3.1 Results

Figure 4: Using 2-opt to for `dj38`

(a) Initial tour

(b) Optimum tour



Our implementation of `2-opt` finds the exact optimum tour after 20 iterations, in 318ms. The optimum tour length was found to be $f(\boldsymbol{t}^{\star}) = 6950$. Our results are identical with the results on the University of Waterloo's website[1].

[1] http://www.math.uwaterloo.ca/tsp/world/djtour.html DJ38 - Djibouti