LE/EECS 3172 3.0
Combinatorial Optimization

Final Exam

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Chapter 1

B-2 Construction of a stadium

A town council wishes to construct a small stadium in order to improve the services provided to the people living in the district. After the invitation to tender, a local construction company is awarded the contract and wishes to complete the task within the shortest possible time. All major tasks are listed in the following table. The durations are expressed in weeks. Some tasks can only start after the completion of certain other tasks. The last two columns of the table refer to question 2 which we shall see later.

Question 1: What is the earliest possible date of completing the construction?

Question 2: The town council would like the project to terminate earlier than the time announced by the builder (answer to question 1). To obtain this, the council is prepared to pay a bonus of €30K for every week the work finishes early. The builder needs to employ additional workers and rent more equipment to cut down on the total time. In the preceding table he has summarized the maximum number of weeks he can save per task (column "Max. reduct.") and the associated additional cost per week. When will the project be completed if the builder wishes to maximize his profit?

1.1 Question 1

1.1.1 Parameters

T enumerates $n = 18 + 1$ project tasks, ie. $T = \{1, \ldots, n+1\}$

A the matrix of arcs with $A \in \{0,1\}^{n+1 \times n+1}$; with elements $a_{ij} = \begin{cases} 1 & \texttt{task } j \text{ preceeds } \texttt{task } i \\ 0 & \text{otherwise} \end{cases}$

d_t the duration in weeks to complete task $t \in T$ with no additional labour hours allocated

Note: A ficticious task `Done` with $t = 19$ and $d_{19} = 0$ is also added which has all other terminal tasks as predecessors.

Data given for parameters

Table 1.1: Data for stadium construction

Task	**Description**	**Duration** d_t	**Predecessors** P_t	**Max. reduct.** r_t	**Add. cost** c_t
1	Installing the construction site	2	-	0	-
2	Terracing	16	1	3	30
3	Constructing the foundations	9	2	1	26
4	Access roads and other networks	8	2	2	12
5	Erecting the basement	10	3	2	17
6	Main floor	6	4, 5	1	15
7	Dividing up the changing rooms	2	4	1	8
8	Electrifying the terraces	2	6	0	-
9	Constructing the roof	9	4, 6	2	42
10	Lighting of the stadium	5	4	1	21
11	Installing the terraces	3	6	1	18
12	Sealing the roof	2	9	0	-
13	Finishing the changing rooms	1	7	0	-
14	Constructing the ticket office	7	2	2	22
15	Secondary access roads	4	4, 14	2	12
16	Means of signalling	3	8, 11, 14	1	6
17	Lawn and sport acessories	9	12	3	16
18	Handing over the building	1	17	0	-

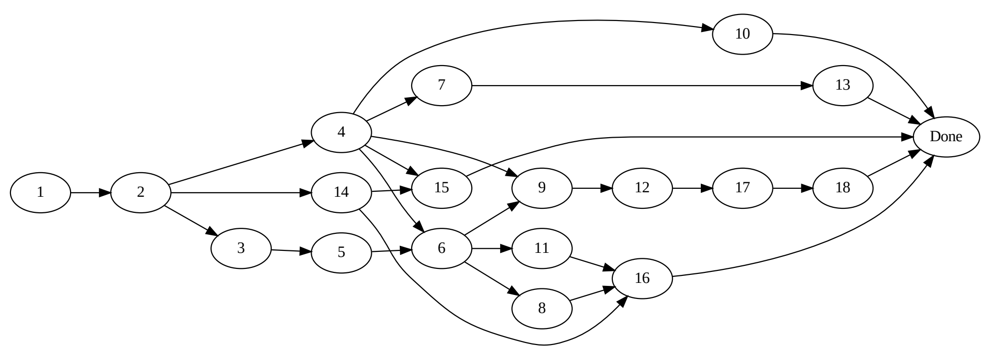
Note: Columns r_t and c_t is not used for question 1.

Task precedence

Some task $t \in T$ may have a predecessor $p \in T$. In other words task t can not commence until task p is finished. In this example, "Erecting the basement" can not be begun until after "Constructing the foundation" completes.

For some task $t \in T$, it has predecessors as the set $P_t = \{p_{t0}, \ldots, p_{tk} : p_{tk} \in T \text{ and } p_{tk} \text{ preceeds } p_{t0}\}$. Note that we could have $P_t = \emptyset$. One may visualize this graph as in fig. 1.1.

Figure 1.1: Graph of precedence dependencies of tasks



1.1.2 Decision variable

x_t indicates the week in which task $t \in T$ starts; $0 \leq x_t \in \mathbb{Z}$. For convience let $\boldsymbol{x} = (x_1, \ldots, x_t)^T \in \mathbb{Z}^n$.

1.1.3 Model

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & f(x_{n+1}) = x_{n+1} && \text{(1.1a)} \\ \text{subject to} \quad & x_i \geq a_{ij}(x_j + d_j), \forall (i,j) \in T \times T, && \text{(1.1b)} \\ & x_t \geq 0, \forall t \in T && \text{(1.1c)} \end{aligned}$$

The objective $f(x_{n+1})$ given by eq. (1.1a) is the start time of task with $t = 19$. Since $d_{19} = 0$, this also is the end time of the final task. In other words, we week to minimize the time of completion of the entire project.

Equation (1.1b) simply enforces that task i, which starts at x_i can not start until its predecessor j, which finishes at week $x_j + d_j$, is complete. When there is no arc from j to i, $a_{ij} = 0$ and the inequality constraint is satisfied. Equation (1.1b) is the canonical non-negativity constraint on $\boldsymbol{x}$.

1.1.4 Results

1.2 Question 2

1.2.1 Parameters

T enumerates $n = 18 + 1$ project tasks, ie. $T = \{1, \ldots, n+1\}$

A the matrix of arcs with $A \in \{0,1\}^{n+1 \times n+1}$; with elements $a_{ij} = \begin{cases} 1 & \texttt{task}\ j \text{ preceeds } \texttt{task}\ i \\ 0 & \text{otherwise} \end{cases}$

d_t the duration in weeks to complete task $t \in T$ with no additional labour hours allocated

r_t the maximum reduction in duration of task $t \in T$ in weeks that can be achieved by allocating additional labour

c_t cost in 1000€ / week when allocating additional labour to task $t \in T$.

b bonus paid in 1000€ per week the project finishes early; with $b = 30$

Refer to table 1.1 for values of the above parameters.

1.2.2 Decision variables

x_t indicates the week in which task $t \in T$ starts; $0 \leq x_t \in \mathbb{Z}$. For convience let $\boldsymbol{x} = (x_1, \ldots, x_{n+1})^T \in \mathbb{Z}^{n+1}$.

y_t indicates the number of weeks by which duration of task $t \in T$ is reduced; with $0 \leq y_t \in \mathbb{Z}$. For convience, let $\boldsymbol{y} = (y_1, \ldots, y_n)^T \in \mathbb{Z}^n$

1.2.3 Model

$$
\begin{aligned}
\underset{\boldsymbol{x},\boldsymbol{y}}{\text{maximize}} \quad & g(\boldsymbol{x},\boldsymbol{y}) = b\left(x_{n+1} - f_1^\star\right) && \text{(1.2a)}\\
\text{subject to} \quad & x_i \geq a_{ij}(x_j + d_j - y_j), \forall (i,j) \in T \times T, && \text{(1.2b)}\\
& y_t \leq r_t, \forall t \in T \setminus (n+1), && \text{(1.2c)}\\
& x_t \geq 0, \forall t \in T, && \text{(1.2d)}\\
& y_t \geq 0, \forall t \in T && \text{(1.2e)}
\end{aligned}
$$

Let f^* be the optimal value of the objective function found in results for question 1 in section 1.1.4, ie. the earliest time which the project can be completed without allocating additional labour. The completion date which maximizes the bonus paid to the countractor is given by:

$$y_{n+1} = x_{n+1} - f_1^\star.$$

Then the total bonus for completing the job early is given by

$$g(\boldsymbol{x},\boldsymbol{y}) = b y_{n+1} = b\left(x_{n+1} - f_1^*\right). \tag{1.3}$$

The above eq. (1.3) is the quantity that the contractor seeks to maximize and becomes the objective function eq. (1.2a). Equation (1.2b) is similar to eq. (1.1b), in section 1.1.4, but takes into account the reduction in duration for the previous task. The maximum possible reduction in duration for task $t \in T$ is limited by r_t. This is represented by the family of constraints in eq. (1.2c). Equation (1.2d) and eq. (1.2e) are simply the family of canonical non-negativity constraints on decision variables $\boldsymbol{x}$ and $\boldsymbol{y}$.

1.2.4 Results

Chapter 2

D-1 Wagon load balancing

Three railyay wagons with a carrying capacity of 100 quintals (1 quintal = 100 kg) have been reserved to transport sixteen boxes. The weight of the boxes in quintals is given in the following table. How shall the boxes be assigned to the wagons in order to keep to the limits on the maximum carrying capacity and to minimize the heaviest wagon load?

2.1 Parameters

B enumerates $m = 16$ boxes; ie. $B = \{1, \ldots, m\}$

W enumerates $n = 3$ railway wagons; ie. $W = \{1, \ldots, n\}$

c denotes maximum capacity of a wagon in quintals, with $c = 100$

μ_b denotes the weight of box $b \in B$

Refer to table 2.1 for the values of the above parameters.

Table 2.1: Weight of boxes

Box	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Weight	34	6	8	17	16	5	13	21
Box	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
Weight	25	31	14	13	33	9	25	25

2.2 Decision variable

x_{bw} indicates box $b \in B$ is loaded in wagon $w \in W$; $x_{bw} \in \{0, 1\}$. For convience let $\boldsymbol{x} \in \{0, 1\}^{m \times n}$ with entries $x_{bw}, \forall (b, w) \in B \times W$.

m ancillary variable denoting upper bound of weights in quintals of all wagons; $0 \leq m \in \mathbb{Z}$

2.3 Model

$$\begin{aligned}
\underset{\boldsymbol{x}}{\text{minimize}} \quad & f(m) = m && (2.1a)\\
\text{subject to} \quad & \sum_{w \in W} x_{bw} = 1, \forall b \in B, && (2.1b)\\
& m \geq \sum_{b \in B} \mu_b x_{bw}, \forall w \in W, && (2.1c)\\
& \sum_{b \in B} \mu_b x_{bw} \leq c, \forall w \in W, && (2.1d)\\
& x_{bw} \in \{0, 1\}, \forall (b, w) \in B \times W, && (2.1e)\\
& m \geq 0 && (2.1f)
\end{aligned}$$

The objective we week to minimize eq. (2.1a) is simply m or rather, the upper bound on the weights of each wagon $w \in W$. Each box $b \in B$ must be loaded in exactly one wagon $w \in W$ as stated by eq. (2.1b).

For m to be the least upper bound of the weights of each wagon $w \in W$, m must be greater than all wagon weights as expressed in eq. (2.1c). Minimization will lower it until it is equal to the maximum of the weights of the loaded wagons. Note that this objective is wrong in the text.

Equation (2.1e) enforces that the decision variable $\boldsymbol{x}$ is a binary indicator, and eq. (2.1f) is the canonical non-negativity constraint on m.

2.4 Results

Chapter 3

D-3 Tank loading

Five tanker ships have arrived at a chemical factory. They are carrying loads of liquid products that must not be mixed. Refer to table 3.1 for these quantities. Nine tanks of different capacities are available on site. Some of them are already partially filled with a liquid. Table 3.2 lists the characteristics of the tanks (in tonnes). Into which tanks should the ships be unloaded (question 1) to maximize the capacity of the tanks that remain unused, or (question 2) to maximize the number of tanks that remain free?

3.1 Question 1

3.1.1 Parameters

T enumerates the $n = 9$ tanks, ie. $T = \{1, \ldots, n\}$

L enumerates the $m = 5$ types of liquid arriving at the facility: $\{\texttt{Benzol}, \texttt{Butanol}, \texttt{Propanol}, \texttt{Styrene}, \texttt{THF}\}$ ie. $L = \{1, \ldots, m\}$.

F the indices of the prefilled tanks, ie. $\{\texttt{Benzol}, \texttt{THF}\}$ with $F = \{T \in L : \text{tank } t \text{ is prefilled}\} \subset T$

a_l denotes the amount (in tonnes) of liquid $l \in L$ arriving at the facility

c_t denotes the capacity (in tonnes) of tank $t \in T$

Q_t denotes the innitial amount (in tonnes) of liquid in tank $t \in F$

P_t denotes the type of liquid initially in tank $t \in F$

R_l remainder of liquid $l \in L$ after filling prefilled tanks $t \in F$ to capacity with liquid of same type, given by

$$R_l = \begin{cases} a_l - \displaystyle\sum_{t \in F, P_t = l} c_t - Q_t & \text{if } t \in F \\ a_l & \text{otherwise} \end{cases} \tag{3.1}$$

Refer to table 3.1 for the values of a_l and table 3.2 for the values of F, c_t, Q_t, and P_t.

Table 3.1: Various types of liquid arriving at facility

Type	**Benzol**	**Butanol**	**Propanol**	**Styrene**	**THF**
Qauntity (in tonnes) a_l	1200	700	1000	450	1200

Table 3.2: Facillity tank capacities and innitial quantities

Tank	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Capacity (tonnes) c_t	500	400	400	600	600	900	800	800	800
Initial type Q_t	-	`Benzol`	-	-	-	-	`THF`	-	-
Initial amount (tonnes) P_t	0	100	0	0	0	0	300	0	0

3.1.2 Decision variable

x_{lt} indicates that remaining liquid of type $l \in L$ is pumped into tank $t \in T$, after prefilling; for convience let $\boldsymbol{x} \in \{0,1\}^{m \times n}$ with entries x_{lt}

3.1.3 Model

$$
\begin{aligned}
\underset{\boldsymbol{x}}{\text{minimize}} \quad & f(\boldsymbol{x}) = \sum_{l \in L} \sum_{t \in T \setminus F} c_t x_{lt} && (3.2a)\\
\text{subject to} \quad & \sum_{t \in T \setminus F} c_t x_{lt} \geq R_l, \forall l \in L, && (3.2b)\\
& \sum_{l \in L} x_{lt} \leq 1, \forall t \in T \setminus F, && (3.2c)\\
& x_{lt} \in \{0,1\}, \forall (l,t) \in L \times T && (3.2d)
\end{aligned}
$$

Maximizing the remaining capacity is equivalent to minmizing the used capacity in a tank as expressed in eq. (3.2a). The family of constraints eq. (3.2b) ensures that all remaining liquid, after prefilling, is pumped into tanks for all liquids $l \in L$. Liquids of different types can not be allowed to mix in a tank. This is guaranteed by eq. (3.2c). Lastly, eq. (3.2d) states that $\boldsymbol{x}$ must be a binary variable.

3.1.4 Results

3.2 Question 2

We approach question 2 in almost an identical way as in question 1. The only thing that changes is the objective function. The model for question 2 is given below.

3.2.1 Model

$$
\begin{aligned}
\underset{\boldsymbol{x}}{\text{minimize}} \quad & g(\boldsymbol{x}) = \sum_{l \in L} \sum_{t \in T \setminus F} x_{lt} && (3.3a)\\
\text{subject to} \quad & \sum_{t \in T \setminus F} c_t x_{lt} \geq R_l, \forall l \in L, && (3.3b)\\
& \sum_{l \in L} x_{lt} \leq 1, \forall t \in T \setminus F, && (3.3c)\\
& x_{lt} \in \{0,1\}, \forall (l,t) \in L \times T && (3.3d)
\end{aligned}
$$

Equation (3.3a) is similar to eq. (3.2a) in section 3.1.3. The only difference between $g(\boldsymbol{x})$ and $f(\boldsymbol{x})$ is that $g\boldsymbol{x})$ does not include the tank capacities c_t. The result is, we are only counting the number of tanks that used. Once again, minimizing this quantity is clearly equivalent to maximizing the number of unused tanks.

3.2.2 Results

Chapter 4

E-1 Car rental

A small car rental company has a fleet of 94 vechiles distributed among its 10 agencies. The location of every agency is given by its geographical coordinates X and Y in a grid based on kilometers. We assume that the road distance between agencies is approximately 1.3 times the Euclidean distance. Refer to table 4.1 for the geographical coordinates of all agencies, the inventory of cars required the next morning, and the inventory of cars in the evening preceeding this day. Refer to table 4.2 for the available and required inventories of the various agencies.

Suppose the cost for transporting a car is €0.50 per km, determine the movements of cars that allow the company to re-establish the required numbers of cars at all agencies, minimizing the total cost incurred for transport.

4.1 Parameters

- A — enumerates $n = 10$ rental agencies, ie. $A = \{1, \ldots, n\}$
- $\boldsymbol{X}_a$ — denotes the geographical coordinates of an agency $a \in A$, with $\boldsymbol{X}_a \in \mathbb{R}^2$, for convience let $\boldsymbol{X} \in \mathbb{R}^{2\times n}$ with columns $\boldsymbol{X}_a, \forall a \in A$
- r_a — number of cars required by agency $a \in A$ on next morning
- s_a — number of cars agency $a \in A$ has in the evening
- E — part of the partition of A with $E = \{a \in A : s_a > r_a\} \subset A$
- N — part of the parition of A with $N = \{a \in A : s_a < r_a\} \subset A$
- c — transporation cost in € / km, with $c = €0.50$ / km
- λ — Euclidean distance to road distance multiplier, with $\lambda = 1.3$

Table 4.1: Geographical coordinates of agencies

Agency a	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
X coordinate (km grid) $X_a^{(x)}$	0	20	18	30	35	33	5	5	11	2
Y coordinate (km grid) $X_a^{(y)}$	0	20	10	12	0	25	27	10	0	15

Table 4.2: Available and required inventory of agencies

Agency a	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Cars present s_a	8	13	4	8	12	2	14	11	15	7
Cars required r_a	10	6	8	11	9	7	15	7	9	12

4.2 Decision variable

x_{ij} denotes the number of cars transpored from an agency with excess $i \in E$ to an agency with shortage $j \in N$; $0 \le x_{ij} \in \mathbb{Z}$; for convience let $\boldsymbol{x} \in \mathbb{Z}^{|E| \times |N|}$ with entries x_{ij}

4.3 Model

$$\begin{aligned}
\underset{\boldsymbol{x}}{\text{minimize}} \quad & f(\boldsymbol{x}) = \sum_{i \in E} \sum_{j \in N} c\lambda x_{ij} \sqrt{\boldsymbol{X}_i - \boldsymbol{X}_j} && \text{(4.1a)} \\
\text{subject to} \quad & \sum_{i \in E} x_{ij} = s_j - r_j, \forall j \in N, && \text{(4.1b)} \\
& \sum_{j \in N} x_{ij} = s_i - r_i, \forall i \in N, && \text{(4.1c)} \\
& x_{ij} \ge 0, \forall (i,j) \in E \times N && \text{(4.1d)}
\end{aligned}$$

4.4 Results

Chapter 5

E-5 Combining different modes of transport

A load of 20 tonnes needs to be transported on a rouate passing through five cities, with a choice of three different modes of transport: rail, road, and air. In any of the three intermediate cities it is possible to change the mode of transport but the load uses a single mode of transport between two consecutive cities. Refer to table 5.1 for the cost of the various modes of transport between pairs of cities.

Furthermore, changing the mode of tranport incurs an associated cost. This cost is indepdependent of location. Refer to table 5.2 for these costs.

How should we organize the transport of the load at the least cost?

5.1 Parameters

M enumerates the $n = 3$ modes of transport: $\{\texttt{rail}, \texttt{road}, \texttt{air}\}$, with $M = \{1, \ldots, n\}$

L enumerates the $k = 5$ legs of the route between the cities, ie. $L = \{1, \ldots, k\}$

c_{ml} denotes the transportation cost per tonne on leg $l \in L$ of the route using mode $m \in M$

t_{ij} denotes the cost per tonne of transfering the cargo from mode $i \in M$ to mode $j \in M$ at a given city

q denotes the weight of the cargo in tonnes with $q = 20$ tonnes

Refer to table 5.1 for values of c_{ml} and to table 5.2 for values of t_{ij}.

Table 5.1: Transportation costs between pairs of cities using various modes of transport

	1-2	**2-3**	**3-4**	**4-5**
Rail	30	25	40	60
Road	25	40	45	50
Air	40	20	50	45

Table 5.2: Costs of changing mode of transport

from/to	**Rail**	**Road**	**Air**
Rail	0	5	12
Road	8	0	10
Air	15	10	0

5.2 Decision variables

x_{ml} indicates that mode $m \in M$ is used on leg $l \in L$ of the route; $x_{ml} \in \{0,1\}$; for convenience, let $\boldsymbol{x} \in \{0,1\}^{n\times k}$ with entries x_{ml}

y_{ijl} indicates whether cargo is transfered from mode $i \in M$ to mode $j \in M$ after leg $l \in L$ of the journey; $y_{ijl} \in \{0,1\}$; for convenience, we define the tensor $\boldsymbol{y} \in \{0,1\}^{n\times n\times k}$

5.3 Model

The total tranportation cost is given by

$$g(\boldsymbol{x}) = \sum_{m\in M} \sum_{l\in l} c_{ml} x_{ml}. \tag{5.1}$$

The total cost of transfering the cargo between modes of transportation is

$$h(\boldsymbol{y}) = \sum_{i\in M} \sum_{j\in M} \sum_{l\in L\setminus\{k\}} t_{ij} y_{ijl} \tag{5.2}$$

$$\begin{aligned}
\underset{\boldsymbol{x}}{\text{minimize}} \quad & f(\boldsymbol{x},\boldsymbol{y}) = g(\boldsymbol{x}) + h(\boldsymbol{y}) && \text{(5.3a)}\\
\text{subject to} \quad & \sum_{m\in M} x_{ml} = 1, \forall l \in L, && \text{(5.3b)}\\
& \sum_{i,j\in M} y_{ijl} = 1, \forall l \in L \setminus \{k\}, && \text{(5.3c)}\\
& x_{il} \geq y_{ijl}, \forall i,j \in M, l \in L \setminus \{k\}, && \text{(5.3d)}\\
& x_{ml} \in \{0,1\}, \forall m \in M, l \in L, && \text{(5.3e)}\\
& y_{ijl} \in \{0,1\}, \forall i,j \in M, l \in L && \text{(5.3f)}
\end{aligned}$$

5.4 Results

Chapter 6

F-4 Airline hub location

The airline FAL (French Air Lines) sepcializes in freight transport. The company links the major French cities with cities in the United States, nameley Atlanta, Boston, Chicago, Marseille, Nice, and Paris. The average quantities in tonnes transported every day by this company between these cities are given in table 6.1.

We shall assume that the transport cost between two cities i and j is proportional to the distance that seperates them. The distances in miles are given in table 6.2.

The airline is planning to use two cities as connection paltforms (**hubs**) to reduce the transport costs. Every city is then assigned a single hub. The traffic between cities assigned to a given hub H_1 to the cities assigned to the other hub H_2 is all routed through the single connection form H_1 to H_2 which allows the airline to reduce the transport cost by 20%. Determine the two cities to be chosen as hubs in order to minimize the transport cost.

6.1 Parameters

C enumerates $m = 5$ cities: $\{\texttt{Atlanta}, \texttt{Boston}, \texttt{Chicago}, \texttt{Marseille}, \texttt{Nice}\}$, ie. $C = \{1, \ldots, n\}$

n denotes the number of hubs to choose, with $n = 2$

q_{ij} denotes the average quantity of goods in tonnes transported between city $i \in C$ and $j \in C$ in a given day

d_{ij} denotes the distance from city $i \in C$ to city $j \in C$

c_{ijkl} is proportional to the transportation cost from city $i \in C$ to city $j \in C$ through hubs $k \in C$ and $l \in C$

r is the reduced hub to hub transport cost factor, with $r = 0.8$

c_{ijkl} is calculated from d_{ij} and r as follows:

$$c_{ijkl} = d_{ik} + r d_{kl} + d_{lj}, \forall i, j, k, l \in C \tag{6.1}$$

Note: When any one of the two suscripts are equal, we have $d_{ii} = 0$.

Table 6.1: Average daily quantities of freight in tonnes

	Atlanta	**Boston**	**Chicago**	**Marseille**	**Nice**	**Paris**
Atlanta	0	500	1000	300	400	1500
Boston	1500	0	250	630	360	1140
Chicago	400	510	0	460	320	490
Marseille	300	600	810	0	820	310
Nice	400	100	420	730	0	970
Paris	350	1020	260	580	380	0

Table 6.2: Distances between pairs of cities in miles

	Boston	**Chicago**	**Marseille**	**Nice**	**Paris**
Atlanta	945	605	4667	4749	4394
Boston		866	3726	3806	3448
Chicago			4471	4541	4152
Marseille				109	415
Nice					431

6.2 Decision variables

x_{ijkl} indicates that the cargo flows from city $i \in C$ to city $j \in C$ via hubs $k \in C$ and $l \in C$, in other words

$$x_{ijkl} = \begin{cases} 1 & \text{cargo flows from city } i \text{ to } j \text{ via hub } k \text{ to } l \\ 0 & \text{otherwise} \end{cases},$$

for convenience, define the tensor $\boldsymbol{x} \in \{0,1\}^{n \times n \times n \times n}$ with entries x_{ijkl}

y_i indicates whether city $i \in C$ is a hub, in other words

$$y_i = \begin{cases} 1 & \text{city } i \text{ is a hub} \\ 0 & \text{otherwise} \end{cases},$$

for convenience, define the tuple $\boldsymbol{y} = (y_1, \ldots, y_n) \in \{0,1\}^n$

6.3 Model

$$\begin{aligned}
\underset{\boldsymbol{x}}{\text{minimize}} \quad & f(\boldsymbol{x}) = \sum\sum_{i,j,k,l \in C}\sum\sum c_{ijkl} c_{ijkl} q_{ij} x_{ijkl} && (6.2a)\\
\text{subject to} \quad & \sum_{i \in C} y_i = n, && (6.2b)\\
& \sum_{k,l \in C}\sum x_{ijkl} = 1, \forall i,j \in C, && (6.2c)\\
& x_{ijkl} \leq y_k, \forall i,j,k,l \in C, && (6.2d)\\
& x_{ijkl} \leq y_l, \forall i,j,k,l \in C, && (6.2e)\\
& x_{ijkl} \in \{0,1\}, \forall i,j,k,l \in C, && (6.2f)\\
& y_i \in \{0,1\}, \forall i \in C && (6.2g)
\end{aligned}$$

6.4 Results