1 Changing room temperature

Modify the coffee model to account for a room temperature that starts at 20 °C, and increases at a constant rate to a maximum 26 °C in 2 hours, and then stays at the maximum temperature. Assume that the initial temperature of the cofee is 100 °C and after 10 minutes, the temperature of the coffee is 90 °C. Modify the codes and draw the room temperature and coffee temperature in the same diagram for 6 hours.

1.1 Building model

The independent variable of the model is time t in hours. The observed temperature of the coffee $u_{obs}(t)$ is a function of time, with units °C.

Let u_{sur} be the surruounding temperature, then u_{sur} is given by

$$u_{sur}(t) = \begin{cases} 20 + 3t & 0 \le t \le 2 \\ 26 & t > 2 \end{cases}. \tag{1.1}$$

We use the model presented in class, but modify it to use $u_{sur}(t)$ to get

$$\frac{du}{dt} = c(u_{sur}(t) - u) \tag{1.2}$$

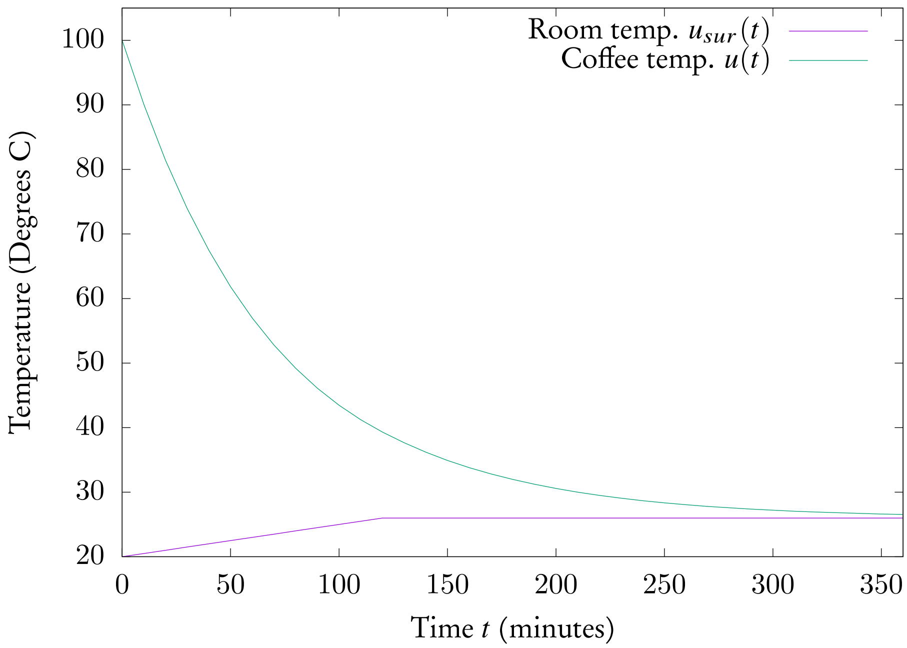
Now, we discretize the model with the timestep $h = \Delta t$ to get

$$\frac{u_{k+1} - u_k}{h} = hc(u_{sur}(t_k) - u_k) \tag{1.3}$$

where $t_k = k\Delta t$ is the discrete time, at the k^{th} step. This is the Newton Forward method.

1.2 Result

Figure 1.1: Coffee in room with changing temperature



Description: The analytic solution $U(t)$ plotted for $\rho = 0.1$, $u_m = 6$ °C, and $t^* = 50$ minutes. The disconuity is clearly visible when the milk is added to the coffee. The horizontal line shows u_d. The intersection of the lines is when $U(t) = u_d$.

2 When to add milk?

Do you add the milk to coffee straight away? People often ask wheter it is better to add the cold milk to a hot cup of coffee straight away or wait for a while to let it cool down and then add the cold milk.

Estimate the optimal time to add cold milk so that the coffee cools down to the drinkable temperature fastests. Let us assume that the drinable temperature is 50 °C. Assume that the initial temperature of the coffee is 100 °C and the insulation of the cup is 0.01282. Room temperature is fixed at 22 °C. Is your result reasonable? Explain it briefly. Based on your result, make suggestion to the people about when to add coffee.

You can use and modify the model and codes we developed in the class for the cooling of the coffee in a constant room temperature. Assume that the temperature of the coffee is independent on special variables. Make further assumptions if needed.

2.1 Preliminary discussion and assumptions

A realistic physical treatment of the problem of mixing disimilar fluids at different temperatures is an enormously complicated problem. Simulating such a scenario requires both accurate $3D$ spacial fluid simulation coupled with a thermal dynamic simulation.

Instead of dealing with the complexity of mixing fluids we will confine ourselves to some simplifications. All physical properties of the liquids (asside from temperature) are taken to be identical. Mixing the milk into the coffee causes an instantaneous change temperature. The only factors we consider are the temperature of the milk $u_{milk}^{(i)}$ before mixing, and the ratio of the mass of the milk divided by the mass of the coffee ρ given by

2.2 Model

$$\frac{dU}{dt} = c(u_{sur} - U) \quad \text{for} \quad t \neq t^*. \tag{2.1}$$

$$U(t) = \begin{cases} u(t) & t < t^* \\ u_{mix} & t = t^* \\ v(t) & t > t^* \end{cases}. \tag{2.2}$$

$$\frac{du}{dt} = c(u_{sur} - u) \quad \text{for} \quad t < t^*, \quad \text{with} \quad u(0) = U(0) = u_0 \tag{2.3}$$

$$\frac{dv}{dt} = c(u_{sur} - v) \quad \text{for} \quad t > t^* \quad \text{with} \quad v(t^*) = v_0 = u_{mix} \tag{2.4}$$

$$T = \frac{m_1T_1 + m_2T_2}{m_1 + m_2} = \frac{m_1(T_1 + \rho T_2)}{m_1(1 + \rho)} = \frac{T_1 + \rho T_2}{1 + \rho}, \quad \text{where} \quad \rho = \frac{m_2}{m_1} \tag{2.5}$$

Taking *(2.5)*, with $T_1 = u(t^*)$, and the milk temperature u_m we can get u_{mix},

$$u_{mix} = v_0 = v(t^*) = \frac{u(t^*) + \rho u_m}{1 + \rho} \tag{2.6}$$

2.3 Analytical model

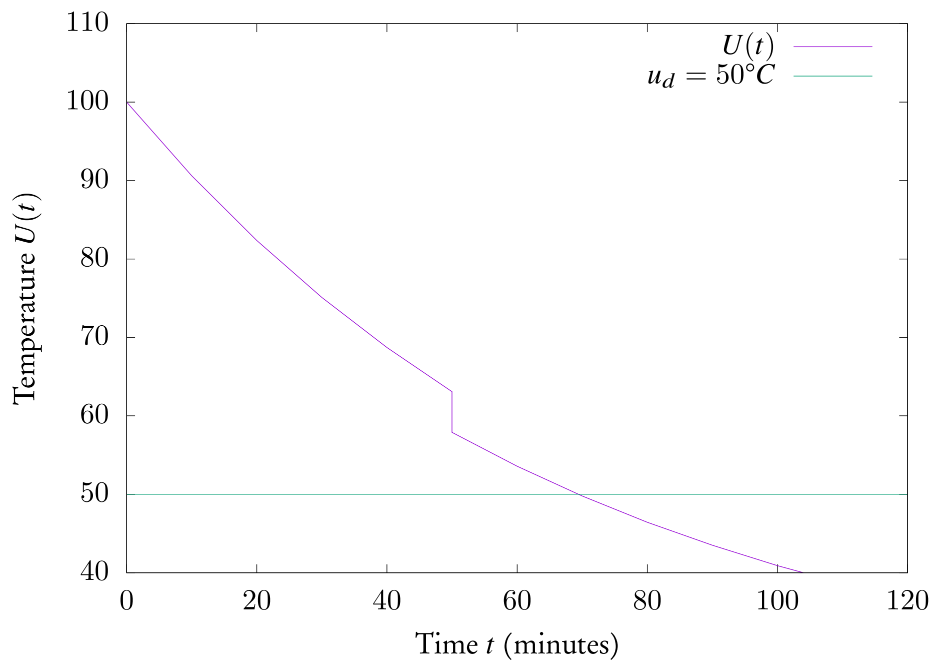
We can easily find a closed form solution for $U(t)$ by solving the constrained differential equation *(2.2)*.

$$\alpha(t) = \alpha_0 e^{-ct} \quad \text{and} \quad \beta(t) = \beta_0 e^{-c(t-t^*)} \tag{2.7}$$

where $\alpha(t) = u(t) - u_{sur}$ and $\beta(t) = v(t) - u_{sur}$, with intial conditions $\alpha_0 = \alpha(0) = u_0 - u_{sur}$ and $\beta_0 = \beta(t^*) = v_0 - u_{sur}$. The closed form solution is then

$$U(t) = \begin{cases} \alpha_0 e^{-ct} + u_{sur} & t < t^* \\ \beta_0 e^{-c(t-t^*)} + u_{sur} & t \geq t^* \end{cases} \tag{2.8}$$

Figure 2.1: Analytical solution



Description: The analytic solution $U(t)$ plotted for $\rho = 0.1$, $u_m = 6$ °C, and $t^* = 50$ minutes. The disconuity is clearly visible when the milk is added to the coffee. The horizontal line shows u_d. The intersection of the lines is when $U(t) = u_d$.

2.4 Discrete Newton Forward model

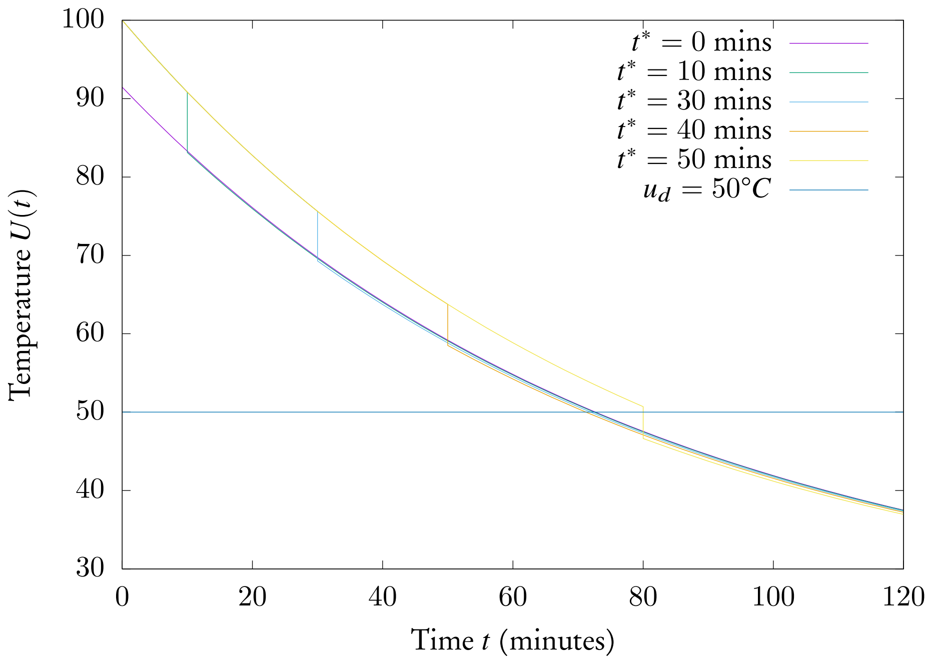
It is straightforward to create a discrete model. Let $h = \Delta t$ denote the iteration step size, and let $k \in \{0, 1, \ldots\}$ denote the iteration step. The initial step is to set

$$U_0 = u_0.$$

$$U_k = \begin{cases} U_{k-1} - hc(U_{k-1} - u_{sur}) & \text{for } hk < t^* \\ \frac{U_{k-1}+\rho u_m}{1+\rho} & \text{for } hk \leq t^* \leq h(k+1) \\ U_{k-1} - hc(U_{k-1} - u_{sur}) & \text{for } hk > t^* \end{cases}, \tag{2.9}$$

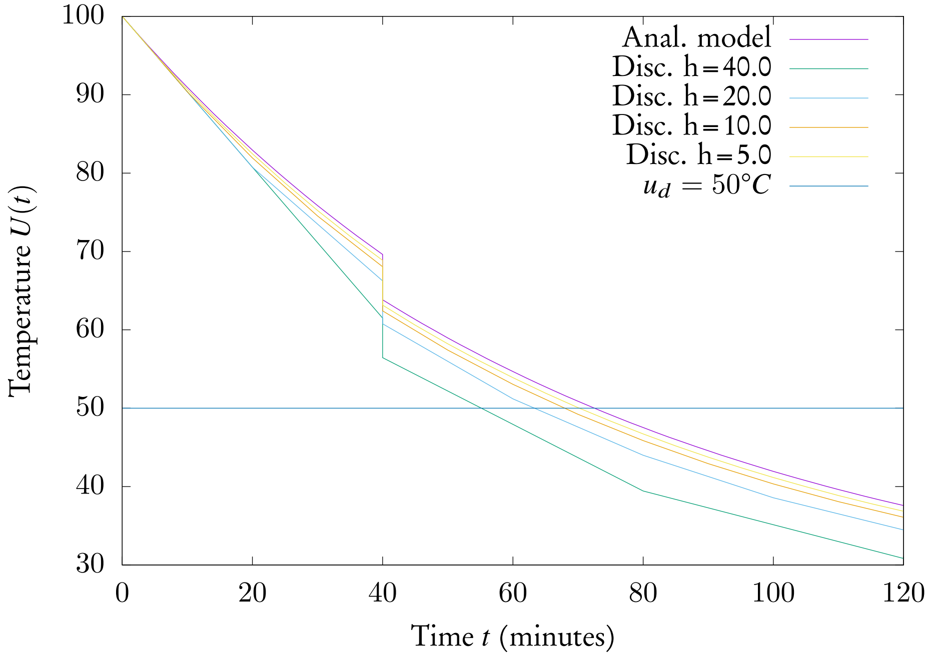
In practice an additional point is added at t^* to better plot the disconuity with a vertical line. This restricts the implementation to values of t^* which satisfy $\texttt{fmod}(t^*, h) = 0$. `fmod` is the floating point modulus function in the C++ *standard library* as defined in the `<cmath>` header.

Figure 2.2: Discrete solution with varying t^*



Discussion The analytic solution $U(t)$ plotted for $\rho = 0.1$, $u_m = 6$ °C, and $t^* = 50$ minutes. The disconuity is clearly visible when the milk is added to the coffee. The horizontal line shows u_d. The intersection of the lines is when $U(t) = u_d$.

Figure 2.3: Comparing analytical model with discrete model for various time steps h



Discussion Observe that the discrete model approaches the analytical solution as the time step h decreases. The discretization step introduces a negative error into the solution. This example uses the following parameters values: $u_0 = 100.0$ °C, $u_{sur} = 22.0$ °C, $t^* = 40$ mins., $\rho = 0.1$, and $u_m = 6.0$ °C.

2.5 Conclusions and analysis

Observe that the magnitude of the first derivative of $U(t)$ decreases as $t \rightarrow \infty$. In order for $U(t)$ to reach u_d in the minimum amount of time, it is best to add the milk as late as possible. This maximizes rate of the first portion of Newton cooling. Infact, the model intersects $y = u_d$, slightly earlier as t^* increases. Keep in mind that there is a limit to this.

There exists some $t_{ideal} > 0$ such that if $t^* > t_{ideal}$, adding milk to the coffee instantly cools it below $U(t) < u_d$.

We can derive the optimum time t_{ideal} by starting from

$$u_{mix} = \frac{u(t^*) + \rho u_m}{1+\rho} = u_d = 50.0 \text{ °C}. \tag{2.10}$$

$$u(t^*) = u_d(1+\rho) - \rho u_m$$

$$(u_0 - u_{sur})e^{-ct^*} + u_{sur} = u_d(1+\rho) - \rho u_m$$

$$-ct^* = \ln\left[\frac{u_d(1-\rho) - \rho u_m}{u_0 - u_{sur}}\right]$$

$$t_{ideal} = t^* = -\frac{1}{c}\ln\left[\frac{u_d(1-\rho) - \rho u_m}{u_0 - u_{sur}}\right] \tag{2.11}$$

Here, t_{ideal} is the latest we can add milk without cooling the coffee below u_d. This minimizes the cooling time.

3 Compound interest

You invest \$100 in a savings account paying 6% per year. Let $y(t)$ be the amount in your account after t years. If the interest rate is compounded continuously, then $y(t)$ solves the ODE initial value problem

$$\frac{dy}{dt} = ry$$

where $r = 6\% = 0.06$ and $y(0) = \$100$.

a. What is the analytic solution $y(t)$ to this initial value problem?

   Start with the ensatz $y(t) = Ae^{rt}$, and plug into equation above.

$$\frac{dy}{dt}(t) = rAe^{rt} = ry(t)$$

   Take ensatz at $t \to 0$, to determine constant A,

$$y(0) = Ae^{r\cdot 0} = A \implies A = y(0) = \$100.$$

   The analytic solution is therefore,

$$y(t) = 100e^{rt}. \tag{3.1}$$

b. What is the balance in your account after 10 years with each of the following methods of compounding interest, yearly, monthly, quarterly, daily, and continuous compounding?

   Compound interest can be solved analytically using *(3.1)*, as follows:

$$y(10) = 100e^{0.06\cdot 10} = \$182.21.$$