Learning Objectives in MATH 3090

- Learn the fundamental thinking process of solving a real-world problem
- Master some basic numerical programming and analysis skills
- Train you the teamwork, presentation, and writing skills
- Make you realize by examples that the mathematics does connect to the real-world problems and have a broader understanding of the possible research directions and career paths

Problem: Record Insurance

- When organizing races, the committee has to pay a big bonus to the winners who break the world record. It is a major financial problem for the organizing committee, when no insurance is purchased
- Questions asked about what criteria an insurance company should use in estimating the premium and an organizing committee should use to determine whether to purchase such an insurance

Problem: Jet Lag

- Organizing international meetings is not easy in many ways, including the problem that some of the participants may experience the effects of jet lag. All these things may dramatically affect the productivity of the meeting

- Jet Lag asked the teams to create an algorithm that suggests the best places to hold a meeting given the number of participants, their home cities and other information that the meeting management company may request.

Undergraduate Math Modeling Contest

- *MCM: The Mathematical Contest in Modeling*
  ICM: The Interdisciplinary Contest in Modeling

MCM 2001 Problem B: [Escaping a Hurricane's Wrath (An Ill Wind...)]

Evacuating the coast of South Carolina ahead of the predicted landfall of Hurricane Floyd in 1999 led to a monumental traffic jam. Traffic slowed to a standstill on Interstate I-26. What is normally an easy two-hour drive took up to 18 hours to complete. Many cars simply ran out of gas along the way…

South Carolina

Cooling Coffee

Problem

Consider the cooling of a well stirred liquid so that the temperature does not depend on space. How to predict the temperature of the liquid based on some initial observations?

Case 1 Cooling Coffee

Discrete Time-Space Models: Newton Cooling Models

redefine THE POSSIBLE.

Cooling Coffee

Problem

Consider the cooling of a well stirred liquid so that the temperature does not depend on space. How to predict the temperature of the liquid based on some initial observations?

A Problem-Solving Framework

If you don’t know where you’re going, you will probably end up someplace else

---- Yogi Berra

Pólya's 4-Step Framework

1. **Understand** the problem: Determine where you are going?
2. **Plan** a strategy for solving the problem: Decide how to go about solving it
3. **Execute** your strategy and revise it if necessary: Carry it out
4. **Check** and interpret your result: Don't stop yet. This step may be the most important step

Step 1. Understand the problem

Be Sure You Understand the Problem:

- Read the problem carefully! If it helps, read it aloud
- Record the quantities and conditions that are given (often called the data of the problem)
- Identify the knowns and unknowns. Exactly what is to be determined?
- Draw a picture or diagram to help you organize the information and visualize the problem
- If possible, restate the problem in different ways to clarify or simplify it

Step 1. Understand the problem

Try to answer the following questions:

1. What is the unknown?

2. What factors affect the unknown quantity?

Step 1. Understand the problem

1. What is the unknown?
   - Temperature of the coffee
2. What factors affect the unknown quantity?
   - Initial temperature of the coffee
   - Number of times being stirred
   - Properties of air such as density, conductivity
   - Room temperature
   - Time
   - Surface or shape of the cup
   - Conductivity of the cup
   - Amount of sugar and crème added

Step 1. Understand the problem

3. Which do you think are the top 4 important factors?

4. How these factors affect the unknown?

5. Among these, which can be considered as a constant and which changes over time?

Step 2. Plan a strategy

This step is the most difficult; it requires creativity, organization, and experience

- Try to think of a similar or related problem
- Map out your strategy with a flow chart or diagram
- Identify the appropriate analytical or computational tools needed for the solution
- Approximation can be useful
- Consider special cases or specific examples
- Change perspective
- Take a break when necessary …

Step 2. Plan: Consider simple cases

Try to answer the following questions:

- What assumption can we make to simplify this problem?
- Notations will be used

Restate the problem

Problem

Consider the cooling of a well stirred cup of coffee in a room with constant temperature. Note that the temperature of the coffee is the same everywhere in the cup. How to predict the temperature of the coffee as a function of time?

Step 2. Plan: Name & List Variables

Notations:

t : time

t_k : time $k\Delta t$, where Δt is the size of the time steps.

u_k : temperature of a well stirred cup of coffee at time t_k

u_0 : initial temperature of the coffee (i.e., at time $t = 0$)

u_{sur} : temperature of the surrounding room temperature

c : insulation parameter of the cup, $c > 0$

Step 2. Plan: Modeling

Formulating the problem based on discrete form of Newton's law of cooling:

$$u_{k+1} - u_k = c\,\Delta t\left(u_{sur} - u_k\right)$$

$$\Rightarrow u_{k+1} = \left(1 - c\,\Delta t\right)u_k + c\,\Delta t\, u_{sur}$$

$$= a\,u_k + b$$

where $a = 1 - c\Delta t$ and $b = c\,\Delta t\, u_{sur}$

Step 2. Plan: Modeling

The mathematical model:

$$\begin{cases} u_{k+1} = a\,u_k + b & \qquad (1) \\ u_0 \text{ is known} \end{cases}$$

$$\text{where } a = 1 - c\Delta t \;\text{ and } b = c\,\Delta t\,u_{sur}$$

Eq. (1) is called the first order finite difference model

Step 2. Plan: Estimate Parameters in the Model

Which parameters in the model needs to be measured or estimated? How?

$u_{sur},\ u_0$: direct measure

c : estimate.

Set the small observation time h. Approximate c :

$$u_{obser} - u_0 = c\,h\left(u_{sur} - u_0\right) \Rightarrow c \approx \frac{u_{obser} - u_0}{h\left(u_{sur} - u_0\right)}$$

Note: Measuring *c* for *n* times and taking an average give a more accurate approximation for *c*.

Step 2. Plan: Revise Name & List Variables

Notations:

t : time

t_k : time $k\Delta t$, where Δt is the size of the time steps

h : small length of observation time from $t = 0$

u_k : temperature of a well stirred cup of coffee at time t_k

u_0 : initial temperature of the coffee (i.e., at time $t = 0$)

u_{sur} : temperature of the surrounding room temperature

u_{obser} : temperature of the coffee at time h

c : insulation parameter of the cup, $c > 0$

Step 3. Execute and Revise

- Keep an organized written record of your work, which will be helpful if revisions are needed
- Double-check each step so that you don't propagate errors to the end of the solution
- Assess your strategy as you work; if you find a flaw, return to Step 2 and revise your strategy

Step 3. Execute: Try an example

Example :

Let $a = \frac{1}{2}$, $b = 2$, and $u_0 = 10$. Compute u_1 , u_2, and u_3

$$u_1 = \frac{1}{2}10 + 2 = 7.0$$

$$u_2 = \frac{1}{2}7 + 2 = 5.5$$

$$u_3 = \frac{1}{2}5.5 + 2 = 4.75$$

$$u_{200} = ?$$

Step 3. Execute: Solve it analytically?

- Can we compute the solution directly and how?

$$
\begin{aligned}
u_{k+1} &= au_k + b \\
&= a(au_{k-1} + b) + b \\
&= a^2 u_{k-1} + ab + b \\
&= a^2(au_{k-2} + b) + ab + b \\
&\ \vdots \\
&= a^{k+1}u_0 + b\left(a^k + \cdots + a^2 + a + 1\right) \\
&= a^{k+1}u_0 + b\frac{1-a^{k+1}}{1-a} \\
&= a^{k+1}\left(u_0 - \frac{b}{1-a}\right) + \frac{b}{1-a} = a^{k+1}(u_0 - u_{sur}) + u_{sur}
\end{aligned}
$$

Step 4. Check and interpret your result = Assessment

- Be sure that your result makes sense; for example, check that it has the expected units and that numerical values are sensible
- Recheck your calculations, or find an independent way of checking the result
- Check the consistency of the result by considering special or limiting cases
- Write the solution clearly and concisely

Step 4. Assessment

- Is it right? What would be a sensible solution?

  Analytic Solution

$$u_{k+1} = a^{k+1}\left(u_0 - u_{sur}\right) + u_{sur} \quad \text{i}$$

- Can you make a prediction of the temperature of the coffee using some common sense?

Step 4. Assessment

Steady State Theorem:

If $|a| < 1$, then

$$u_{k+1} \to u, \text{ where } u = au + b \ (\text{i.e., } u = u_{sur})$$

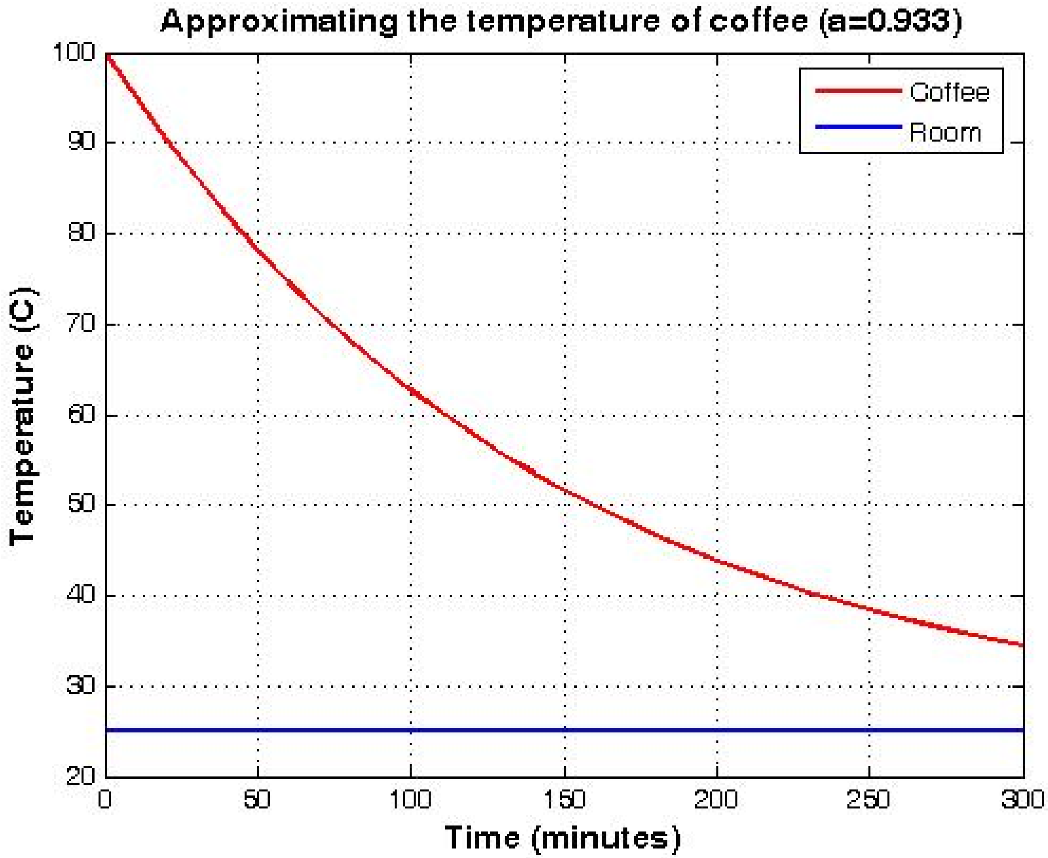
In addition, the difference between the approximation at $(k+1)$-th step and steady state solution is

$$u_{k+1} - u = a^{k+1}\left(u_0 - \frac{b}{1-a}\right).$$

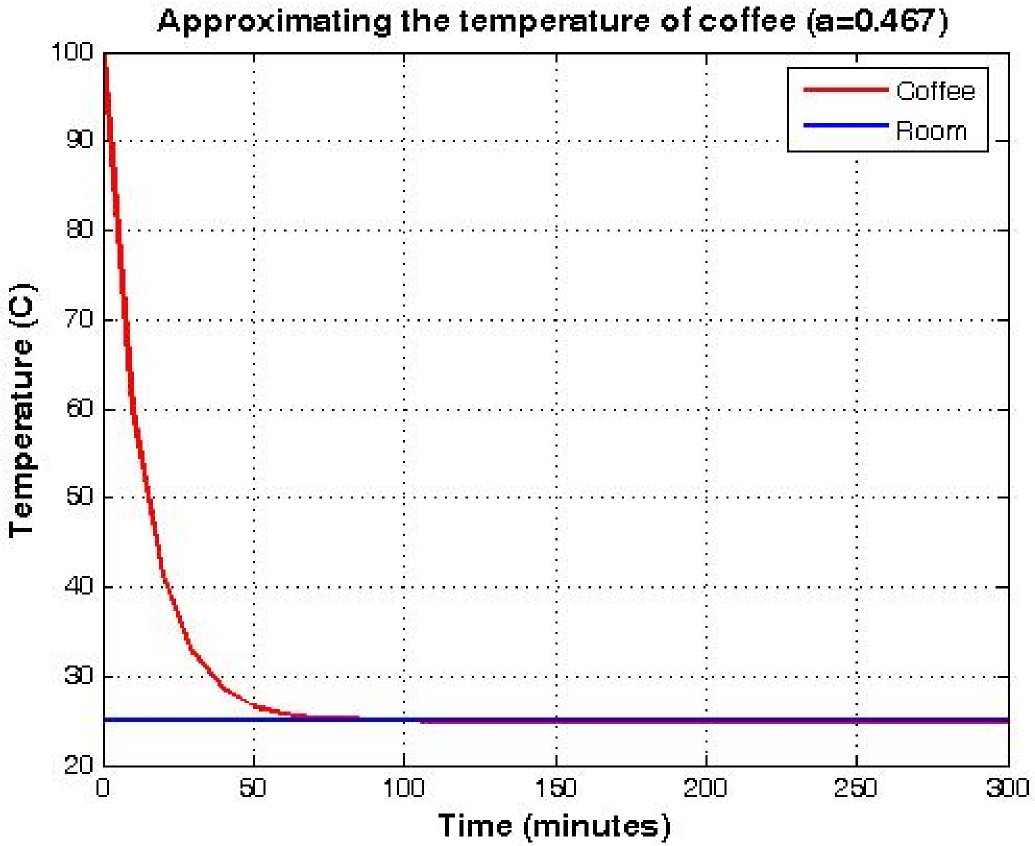
Step 3. Execute: Solve it differently?

- Can we approximate the solution and how?

Step 4. Results: $u_{obser} = 95^o \quad dt = 10\,\text{min}$



Step 4. Results: $u_{obser} = 60^o \quad dt = 10\,\text{min}$



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Step 4. Results: $u_{obser} = -15^o \quad dt = 10\,\mathrm{min}$

Step 4. Results: $u_{obser} = -73^o \quad dt = 10\,\mathrm{min}$

Step 4. Results: $u_{obser} = 25^o \quad dt = 20\,\text{min}$

Step 4. Check and interpret your result

- Do your numerical solutions consistent with the analytical solution?

- Why?

Floating Point Number & Error

Computers use a finite subset of the rational numbers to approximate any real number

Let x be any real number.

Infinite decimal expansion : $x = \pm .x_1x_2\cdots x_d\cdots 10^e$

Truncated floating point number : $x \approx fl(x) = \pm .x_1x_2\cdots x_d 10^e$

where $x_1 \neq 0, 0 \leq x_i \leq 9,$

d : an integer, precision of the floating point system

e : an bounded integer

Floating point or **roundoff error** : $fl(x) - x$

Error Propagation

When additional calculations are done, there is an accumulation of these floating point errors.

Example: Let $x = -0.6667$ and $fl(x) = -0.667\ 10^0$ where $d = 3$.

$$\text{Floating point error}: \ fl(x) - x = ?$$

$$\text{Error propagation}: \ fl(x)^2 - x^2 = ?$$

Error Propagation

When additional calculations are done, there is an accumulation of these floating point errors.

Example: Let $x = -0.6667$ and $fl(x) = -0.667\ 10^0$ where $d = 3$.

$$\text{Floating point error}: \quad fl(x) - x = -0.0003$$

$$\text{Error propagation}: \quad fl(x)^2 - x^2 = 0.00040011$$

Accumulation Error

Let $U_0 = fl(u_0),\ A = fl(a)$, and $B = fl(b)$ so that

$$U_k = AU_{k-1} + B + \overline{R}_k = aU_{k-1} + b + R_k$$

where $\overline{R}_k$ is the roundoff error at k-th step

and R_k includes $\overline{R}_k$ and round errors assoicated with a and b.

Accumulation Error : $U_k - u_k = fl(u_k) - u_k$

Accumulation Error

- Can we guarantee that the accumulation error can keep reasonably small as the number of time steps increases?
- Can we provide a rough estimate on the accumulation error?

Accumulation Error

Accumulation Error Theorem

Consider the first order finite difference algorithm.

If $r = |a| < 1$ and the roundoff errors are uniformly bounded, *i.e.*,

$$|\bar{R}_k| \le R < \infty,$$

then the accumulated error is uniformly bounded, *i.e.*,

$$|U_k - u_k| \le r^k |U_0 - u_0| + R\frac{1-r^k}{1-r} \le \left(1 + \frac{1}{1-r}\right) R.$$

Summary on Cooling of Coffee

Model

Based on the discrete Newton cooling Law,

$$u_{k+1} = au_k + b$$

Steady State Theorem:

if $|a| < 1$, then

$$u_{k+1} \to u, \text{ where } u = au + b \ (\text{i.e., } u = u_{sur})$$

Accumulation Error Theorem:

if $|a| < 1 \& |\overline{R}_{k+1}| \leq R < \infty$, then

$$|U_{k+1} - u_{k+1}| \leq M < \infty, \text{where M independent of } k$$

Finally, What-if thinking

For example, what if the coffee is not well-stirred?

Finally, What-if thinking

What if $\Delta t \to 0$?

$$\begin{aligned} u_{k+1} - u_k &= c\Delta t\left(u_{sur} - u_k\right) \\ \frac{u_{k+1} - u_k}{\Delta t} &= c\left(u_{sur} - u_k\right) \end{aligned}$$

Finally, What-if thinking

Discrete Model:

$$u_{k+1} - u_k = c\Delta t\left(u_{sur} - u_k\right)$$

$$\frac{u_{k+1} - u_k}{\Delta t} = c\left(u_{sur} - u_k\right)$$

$$\Rightarrow$$

Continuous Model:

$$\text{As } \Delta t \to 0, \quad \frac{u_{k+1} - u_k}{\Delta t} \to \frac{du}{dt}$$

$$\therefore \quad \frac{du}{dt} = c\left(u_{sur} - u\right)$$