1 1

Write out a proof that P is reversible with respect to μ. Conclude that μ is an invariant probability distribution for P.

2 2

Write out a proof that P is irreducible, and that if μ isn't perfectly uniform, then P is aperiodic. *[Hint: show that if $i \longrightarrow j$ under Q, then $i \longrightarrow j$ under P. For aperiodicity, consider a site where a transition could be rejected.]*

Therefore, if you pick any state to start in and run the chain for a long time n then the random variable X_n will be sampled from apximately the distribution μ. Going further, $X_n, X_{n+1}, ldots, X_{n+m}$ will all have distributions approximately μ. They won't be indepdendent, but if m is also large, then we can still show that

$$\sum_i f(i)\mu_i \approx \frac{1}{m+1}\sum_{k=0}^{m} f(X_k).$$

So expectations, with respect to the distribution μ can be worked out this way. This often works well in practice, event though it is sometimes hard to get rigorous bounds on how large n and m must be for answers to be reliable.

The Markov chain P is irreducible if $i \longrightarrow j$ $\forall i, j \in \mathcal{S}$. First, we must show that if $i \longrightarrow j$ under Q, then $i \longrightarrow j$ under P.

There are 2 cases to consider:

- If $u_j \geq u_i$ as $i \longrightarrow j$, then we always accept the move; the probability is always 1.

3 Results

Table 1: Results of 10 runs of Metropolis-Hastings algorithm

Run	$E\left[f(x)\right]$ estimate
1	151.7582975592603
2	151.78521559558445
3	151.71995644022766
4	151.72257839116807
5	151.7989403512341
6	151.77048793781816
7	151.73434402770206
8	151.7712540561242
9	151.723880973142
10	151.7255789619957

Figure 1: $l(i,j) \sim \text{Gamma}(k = 7.5)$

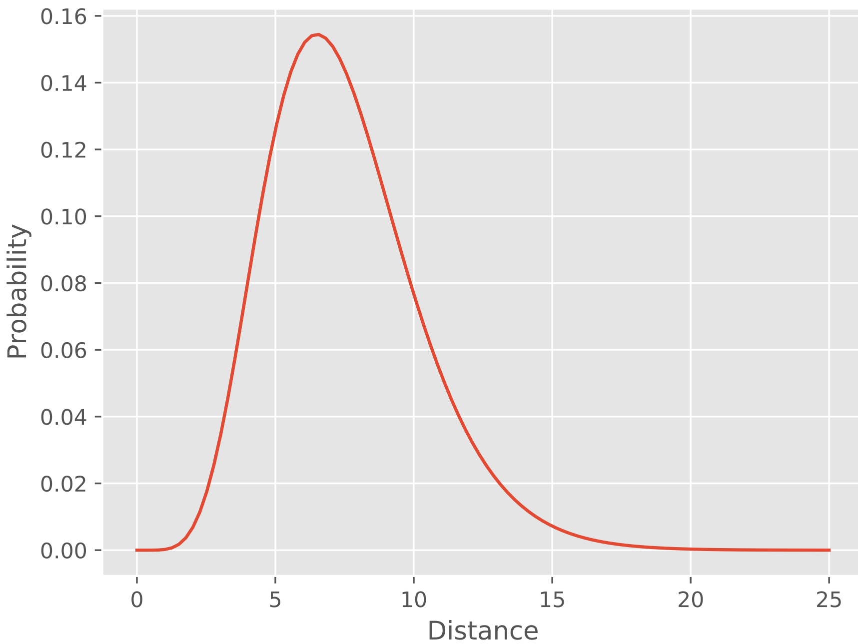

Figure 2: Several runs of Metropolis-Hastings algorithm $n = 10^6$ and $m = 10^4$

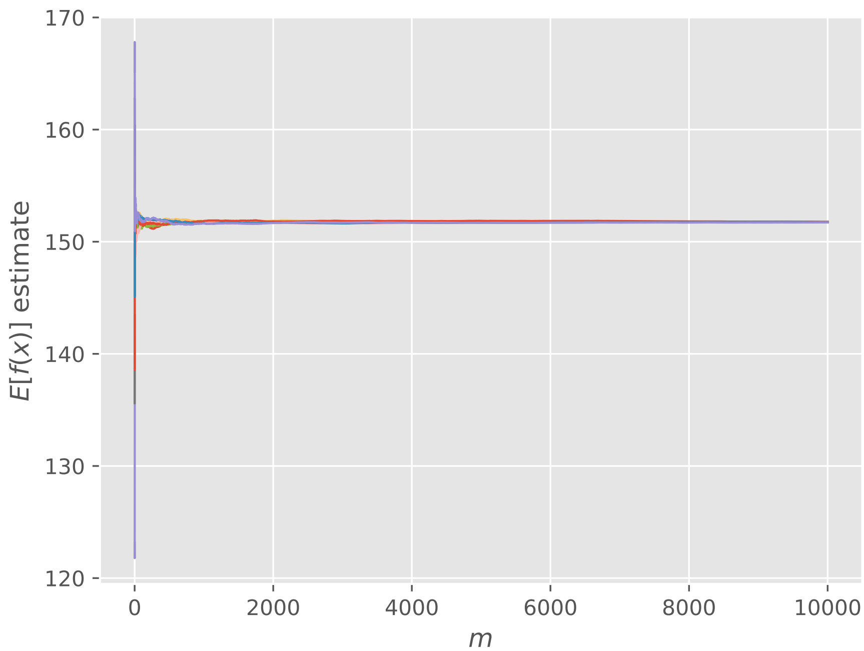

Table 1 shows the convergence of the MCMC. The burn-in phase is not depicted.

Appendix

Table 2: Distances between pairs of cities (i, j) for $i, j \in \{0, 9\}$ and $i \neq j$

i	j	$l(i,j)$	i	j	$l(i,j)$	i	j	$l(i,j)$	i	j	$l(i,j)$	i	j	$l(i,j)$
0	1	11.26	2	11	13.25	8	10	4.95	11	4	7.32	16	18	6.79
0	2	11.92	2	12	8.08	8	11	10.29	11	5	7.04	16	19	5.14
0	3	6.02	2	13	15.34	8	12	7.86	11	6	6.13	17	2	4.03
0	4	5.74	2	14	4.70	8	13	5.82	11	7	5.34	17	3	4.16
0	5	8.05	2	15	13.81	8	14	3.23	11	12	14.47	17	4	6.58
0	6	14.37	2	18	8.12	8	15	8.22	11	13	6.84	17	5	5.15
0	7	9.05	2	19	4.63	8	16	6.86	11	14	13.87	17	6	7.10
0	8	5.09	3	4	4.27	8	17	7.35	11	15	8.47	17	7	7.25
0	9	10.81	3	5	9.03	8	18	4.50	12	5	6.70	17	10	6.54
0	10	3.00	3	6	3.46	8	19	15.86	12	6	7.86	17	11	8.55
0	11	7.43	3	7	3.77	9	2	11.78	12	7	6.49	17	12	3.17
0	12	6.67	3	12	8.88	9	3	5.63	12	13	6.71	17	13	8.76
0	13	3.04	3	13	9.89	9	4	6.60	12	14	11.08	17	14	9.74
0	14	11.61	3	14	7.47	9	5	4.36	12	15	6.48	17	15	9.77
0	15	6.37	3	15	9.96	9	6	8.22	13	5	9.27	17	18	6.96
0	16	6.88	4	5	14.08	9	7	9.63	13	6	4.28	17	19	6.61
0	17	7.25	4	6	4.28	9	10	7.84	13	7	13.30	18	3	3.94
0	19	6.64	4	7	5.54	9	11	6.62	13	14	9.35	18	4	10.56
1	2	6.89	4	12	6.72	9	12	6.46	13	15	5.58	18	5	13.23
1	3	5.27	4	13	12.53	9	13	5.10	14	7	10.12	18	6	13.52
1	4	5.11	4	14	9.28	9	14	6.23	14	15	5.13	18	7	10.42
1	5	7.55	4	15	4.94	9	15	8.94	15	7	9.11	18	11	6.20
1	6	6.03	5	6	4.72	9	18	10.98	16	1	7.09	18	12	8.71
1	9	7.93	5	7	5.91	9	19	5.59	16	2	9.97	18	13	6.36
1	11	11.22	5	14	8.21	10	3	7.85	16	3	6.82	18	14	5.02
1	12	5.48	5	15	8.39	10	4	9.40	16	4	5.74	18	15	14.79
1	13	8.99	6	7	5.08	10	5	7.12	16	5	6.00	18	19	5.46
1	14	2.97	6	14	10.78	10	6	7.28	16	6	4.76	19	3	8.73
1	15	5.40	6	15	5.37	10	7	6.48	16	7	9.43	19	4	6.78
1	17	9.07	8	1	7.00	10	11	12.31	16	9	8.66	19	5	10.94
1	18	5.72	8	2	6.02	10	12	4.34	16	10	4.08	19	6	8.01
2	3	8.26	8	3	10.55	10	13	3.35	16	11	11.61	19	7	13.82
2	4	4.77	8	4	5.51	10	14	2.68	16	12	9.64	19	11	4.46
2	5	3.99	8	5	3.72	10	15	9.40	16	13	8.25	19	12	4.87
2	6	6.67	8	6	6.77	10	18	4.39	16	14	10.74	19	13	7.57
2	7	9.61	8	7	8.01	10	19	4.96	16	15	7.20	19	14	7.19
2	10	7.26	8	9	12.26	11	3	8.05	16	17	7.78	19	15	4.11