CHAPTER 10

Random Numbers and a New Derivation

> Someone told me that each equation I included in the book would halve the sales.
>
> - Stephen Hawking, on "A Brief History of Time"

The size-density hypothesis led to the development of time-minimization theory, from which five other laws can be derived. But the size-density relation had never been tested for the topic where it was, in a sense, born: counties of the United States. The absence of county area data prior to 1930 made statistical tests impossible.

Toward the end of Fall term, 1978, an undergraduate majoring in Visual Communications, Doug McMullin, asked me to participate in his video-tape senior project (he had taken an introductory course from me several years earlier and needed some demographic information for the project). Doug entered the Sociology M.A. program the following June, and we worked on several projects together, one reported here and another in Chapter 12.

A Density-Density Equation

According to the size-density law, if a county lies in a region of density **D** its expected area would be

$$(1) \qquad E(a) = kD^{-2/3}$$

A more traditional meaning of "expectation", in statistics, is the average

$$\frac{A}{N} = kD^{-2/3}$$

where **A** is the area of the region and **N** is the number of counties in it. If the region's density D = P/A, we have

$$\frac{N}{A} = kD^{-2/3}$$

or, inverting the left side,

$$E(a) = \frac{\sum a_i}{N} = \frac{A}{N}$$

which may be thought of as a relationship between the two densities:

C = N/A, the number of county seats per square mile
D = P/A, the number of people per square mile

So we formulate the hypothesis

(2) $C = kD^{2/3}$

and test it with region-level P, A and N data.

The units of analysis for our study were all states and territories for which historical data were available.[1] In several cases we used units for which data were available even when they did not have state or territorial status, e.g., Maine (which was part of Massachusetts until 1820) and West Virginia (part of Virginia until 1863).

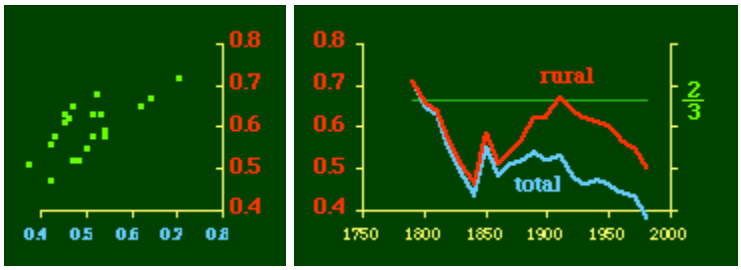
Previous research suggested that the independent variable should be made as specific as possible. We were able, with the data available, to calculate **rural** densities as well as total densities. We did so since the theory did not address urban clumps within territorial divisions; also, the 2/3 line developed for the world no doubt reflected primarily rural populations. The initial results of our analysis are shown in Table 10-1 and Figs. 10-1 to 10-3.

TABLE 10-1. U.S. COUNTIES, 1790-1980

census		TOTAL DENSITY			RURAL DENSITY		
year	n	b	r^2	$p\{\beta_{2/3}\}$	b	r^2	$p\{\beta_{2/3}\}$
1790	18	0.71	0.85	0.60035	0.71	0.84	0.54586
1800	22	0.65	0.92	0.67975	0.66	0.92	0.83769

1810	27	0.63	0.90	0.38676	0.64	0.90	0.51882
1820	27	0.55	0.86	0.01789	0.57	0.86	0.03509
1830	28	0.49	0.75	0.00350	0.51	0.75	0.00834
1840	30	0.43	0.71	0.00012	0.46	0.72	0.00053
1850	36	0.55	0.86	0.00470	0.58	0.87	0.02648
1860	40	0.48	0.80	0.00006	0.51	0.82	0.00047
1870	46	0.51	0.87	0.00001	0.54	0.90	0.00007
1880	47	0.52	0.84	0.00009	0.57	0.88	0.00345
1890	48	0.54	0.81	0.00205	0.62	0.86	0.19565
1900	48	0.52	0.82	0.00020	0.62	0.89	0.16043
1910	48	0.53	0.79	0.00200	0.67	0.89	0.89078
1920	48	0.48	0.74	0.00008	0.64	0.86	0.43018
1930	48	0.46	0.71	0.00005	0.62	0.82	0.28974
1940	48	0.47	0.69	0.00008	0.61	0.80	0.19727
1950	48	0.46	0.65	0.00012	0.60	0.80	0.12253
1960	48	0.44	0.59	0.00017	0.57	0.76	0.05561
1970	48	0.43	0.55	0.00013	0.55	0.74	0.01923
1980	48	0.38	0.37	0.00029	0.50	0.54	0.02180

FIG. 10-1. DENSITY-DENSITY SLOPES

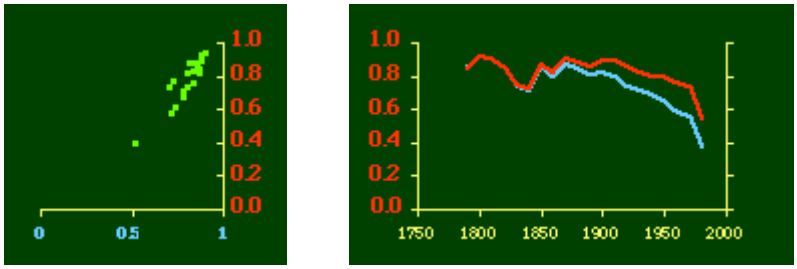


In the early years of this nation the population was almost entirely rural, therefore both independent variables produce virtually the same results. Increasingly, however, rural density diverges from total population density, and it shows results more in conformity with theory, as we thought it might. The left side of Fig. 10-1 shows a scatter diagram relating rural density values (Y-axis) to total density values (X-axis).

As Fig. 10-1 shows, the slope for rural density fits the theoretical 2/3 value at the birth of the nation — after 200 years of occupancy and just prior to westward expansion. With expansion it declines, presumably because population spread faster than county seats did. It then rises briefly in 1850, due I believe to the addition of Texas and California: each had long-established systems of local government under Mexican rule, and these were simply converted to counties on admission of the states to the Union. After that the slope generally rises again toward the theoretical 2/3

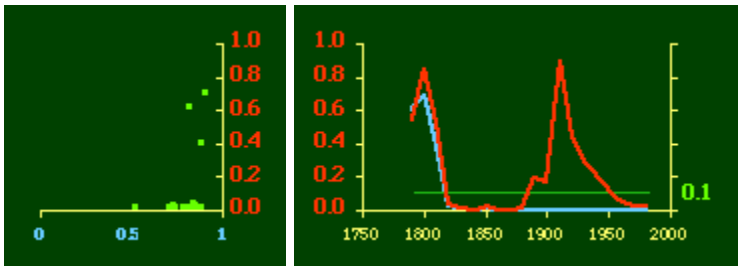
value With the closing of the frontier and the introduction of the automobile, it drifts toward zero again since subsequent population shifts no longer bring about the creation of new counties. Figs. 10-2 and 10-3 strengthen these conclusions.

FIG. 10-2. COEFFICIENT OF DETERMINATION



Obviously, both total and rural density correlated fairly highly with each other (left side of Fig. 2) and throughout the period, though each has been declining since the first part of the 20th century.

FIG. 10-3. PROBABILITY THAT β = 2/3



Here there is a dramatic difference between rural and total population density. The total density departs rapidly and permanently from the 2/3 value. Its high coefficient of determination and increasing number of cases guarantee that it must fail as the slope departs from 2/3. Rural density, however, falls away and then clearly returns to 2/3 (probabilities well above 0.10) with the closure of the frontier. Since then, due to the arrival of the automobile, the slope has been falling away toward zero, so the probability associated with 2/3 has likewise been on the decline.

The Result of Random Numbers?

But what about randomness as the source of the 2/3 slope? Recall from Chapter 8 that the -2/3 size-density slope could be obtained from random numbers. What about the positive 2/3 slope of Eq. 2?

Beginning with the slope formula as before

$$(3) \qquad b_{DC} = \frac{COV(\log D, \log C)}{VAR(\log D)}$$

and substituting,

$$\log D = \log P - \log A$$
$$\log C = \log N - \log A$$

we arrive at the ungainly

$$\frac{COV(\log P,\log N) - COV(\log P,\log A) - COV(\log A,\log N) + VAR(\log A)}{VAR(\log P) + VAR(\log A) - 2COV(\log P,\log A)}$$

Ungainly or not, it produces the anticipated result: random numbers for P and N, together with multiplied pairs of random numbers for A as before, produce

$$(4) \qquad \begin{matrix} COV(\log P,\log N) \\ COV(\log P,\log A) \\ COV(\log A,\log N) \end{matrix} = 0 \quad \text{and} \quad VAR(\log A) = \frac{2\,VAR(\log P)}{2\,VAR(\log N)}$$

Substituting these in the "reduced" version of Eq. 3 produces

$$(5) \qquad b_{DC} = \frac{0 - 0 - 0 + 2}{1 + 2 - 0} = \frac{2}{3}$$

So is all this to be written off as the result of statistical computations on random numbers? Table 10-2 shows our computations for the relevant variances, covariances and ratios needed to test this idea.

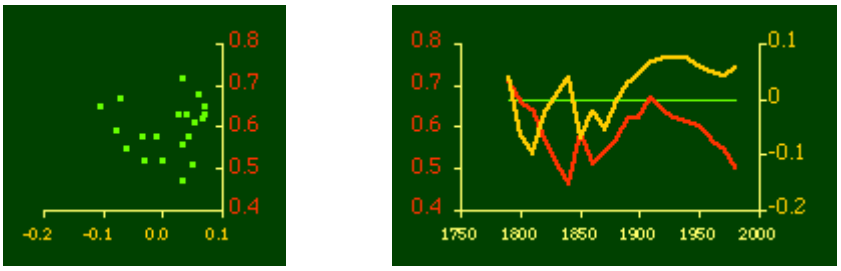
TABLE 10-2. A TEST OF THE "RANDOM NUMBERS" HYPOTHESIS [2]

		VARIANCES			COVARIANCES			RATIOS	
year	rural b	P	A	N	P,A	A,N	P,N	A:P	A:N
1790	0.71	0.13	0.29	0.12	0.04	0.10	0.10	2.13	2.42
1800	0.66	0.48	0.30	0.23	-0.07	0.05	0.28	0.62	1.31
1810	0.64	0.53	0.25	0.24	-0.10	0.02	0.30	0.48	1.05
1820	0.57	0.26	0.26	0.15	-0.03	0.10	0.14	0.98	1.69
1830	0.51	0.21	0.27	0.15	0.01	0.13	0.10	1.30	1.84
1840	0.46	0.21	0.24	0.16	0.04	0.15	0.11	1.15	1.51

1850	0.58	0.36	0.30	0.19	-0.07	0.09	0.18	0.83	1.61
1860	0.51	0.36	0.26	0.19	-0.02	0.12	0.18	0.73	1.37
1870	0.54	0.48	0.24	0.20	-0.06	0.08	0.24	0.49	1.17
1880	0.57	0.37	0.23	0.19	0.00	0.10	0.21	0.61	1.18
1890	0.62	0.26	0.22	0.18	0.03	0.11	0.18	0.85	1.21
1900	0.62	0.24	0.22	0.17	0.05	0.12	0.17	0.93	1.33
1910	0.67	0.20	0.22	0.16	0.07	0.12	0.16	1.12	1.35
1920	0.64	0.19	0.22	0.16	0.08	0.13	0.15	1.18	1.41
1930	0.62	0.19	0.22	0.16	0.08	0.13	0.14	1.20	1.40
1940	0.61	0.18	0.22	0.16	0.07	0.13	0.14	1.22	1.41
1950	0.60	0.16	0.22	0.16	0.06	0.13	0.13	1.37	1.41
1960	0.57	0.15	0.22	0.16	0.05	0.13	0.12	1.44	1.41
1970	0.55	0.15	0.22	0.16	0.04	0.13	0.11	1.47	1.41
1980	0.50	0.14	0.22	0.16	0.06	0.13	0.09	1.54	1.41

Table 10-2 shows that none of the covariances stabilize at the value of zero expected under the random numbers hypothesis. Nor do the ratios hover around the expected value of two. Let us look at each of these expectations in turn, in relation to changes in the slope relating center-density to rural-density.

FIG. 10-4. THE COVARIANCE OF LOG-P AND LOG-A IS UNRELATED TO VARIATION IN THE C-D SLOPE



The left side of Fig 10-4 shows that the covariance(P,A) is unrelated to the value of b. If we look at their values over time (right side) this becomes very clear. The expected values — b = 2/3 & cov(P,A) = 0 — are shown by the horizontal line. Both are near these values in 1790. During the 19th century they approach and depart from these values in **opposite** directions. In 1910, when the slope has returned to its time-minimizing value, the value of cov(P,A) is as far above its random-numbers value as it ever gets. As it returns toward zero, the slope departs from 2/3.

Figs 10-5 and 10-6 show that the covariances (A,N) and (P,N) are similarly unrelated to the slope, as raw values (left sides), or over time (right sides). This shows most clearly

after 1900 — as the slope decays cov(A,N) remains constant (no new states, very few new counties) and cov(P,N) approaches zero.

FIG. 10-5. THE COVARIANCE OF LOG-A AND LOG-N IS UNRELATED TO VARIATION IN THE C-D SLOPE

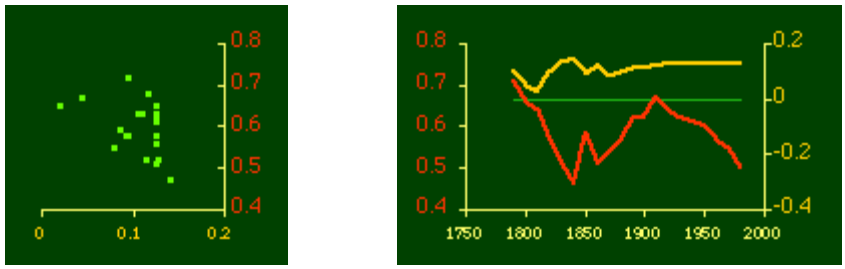
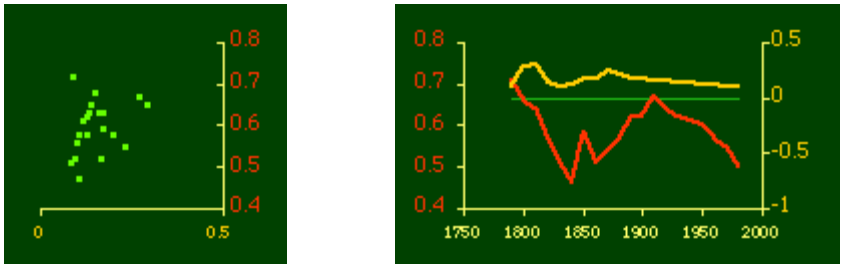


FIG. 10-6. THE COVARIANCE OF LOG-P AND LOG-N IS UNRELATED TO VARIATION IN THE C-D SLOPE



The same thing seems true of the variance ratios shown in Figs. 10-7 and 10-8. Each should equal 2, according to the random number argument. The only point at which they approach this value is in 1790. After that they don't behave as we would expect in connection with the 2/3 slope. State values for A and N are virtually unchanged through most of this century; the other ratio drifts toward a value of 2 while the density-density slope drifts toward zero.

FIG. 10-7. THE RATIO OF LOG-A TO LOG-P IS UNRELATED TO VARIATION IN THE C-D SLOPE

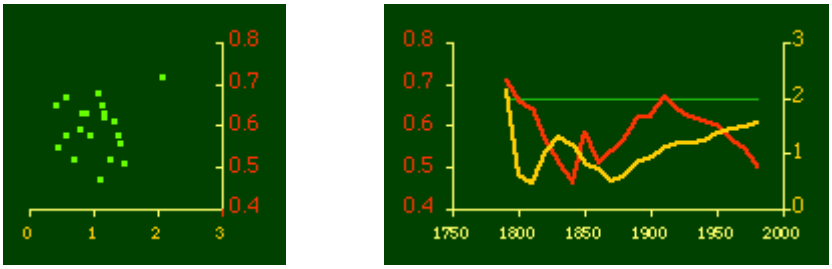
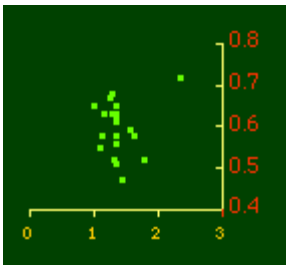
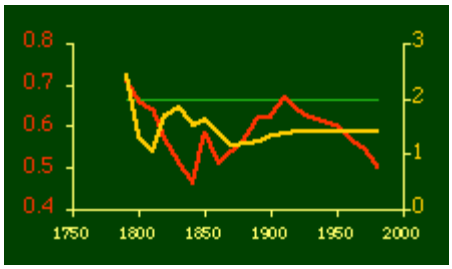


FIG. 10-8. THE RATIO OF LOG-A TO LOG-N IS UNRELATED TO VARIATION IN THE C-D SLOPE





I was on sabbatical leave in Greece, California and Washington, D.C. during 1980/1, but stayed in contact with Doug who was house-sitting for us and working on his M.A. thesis (see Chapter 12). We published the results reported here in 1981.[3]

A New Derivation from Time-Minimization

Sometime in 1981 I realized that the "density-density" equation, Eq. 2, could be derived directly from considerations of time minimization, without going through the size-density derivation as an intermediary step. The result is a set of four equations, one of which is the original size-density equation.

We begin as before with the assumption that social structures evolve in such a way as to minimize the time expended in their operation. We also retain the notion of maintenance time and interaction time. The only difference now is that we no longer focus on the size of a single county. Rather, we ask **How many counties** should a region have, given its population and area?

Maintenance time will refer to the total time cost required to provide the state with **N** counties. With **h** as the average cost per county, total

$$\text{maintenance time} = hN$$

Interaction between residences and county seats will require travel. The average distance **S** is traversed at the average velocity **v**. The **P** individuals making roundtrips will expend an

$$\text{interaction time} = \frac{2S}{v}P$$

Both these statements are true by definition: They state only that an amount may be expressed as an average (h or S/v) times the number of cases (N or P) .

The time expended in the operation of the N counties is given by the sum

$$(6) \qquad T = hN + \frac{2S}{v}P$$

In the earlier derivation we expressed the average distance as function of county area. This was a consequence of dimensional analysis. Here again we express average distance as a function of the square root of area, but it is now the **average** area for the set of N counties. The average area will be the total region's area **A** divided by N. Thus, with w as the constant of proportionality,

$$(7) \qquad T = hN + \frac{2w(A/N)^{1/2}}{v}P$$

We re-write the equation as

$$(8) \qquad T = hN + \frac{2w}{v}P\,A^{1/2}\,N^{-1/2}$$

differentiate with respect to N, set equal to zero and solve for N

$$(9) \qquad \frac{dT}{dN} = h - \frac{w}{v}P\,A^{1/2}\,N^{-3/2} = 0$$

$$h = \frac{w}{v}P\,A^{1/2}\,N^{-3/2}$$

$$N^{-3/2} = \frac{w}{hv}P\,A^{1/2}$$

$$(10) \qquad N = k\,P^{2/3}\,A^{1/3}$$

where $k = (w/hv)^{2/3}$.

The number of counties which will minimize time is thus given as a function of the region's population and area. Algebraic manipulation produces four more equations.

Division of Eq. 10 by A produces

$$(11) \quad \frac{N}{A} = k D^{2/3}$$

Division of Eq. 10 by P produces

$$(12) \quad \frac{N}{P} = k D^{-1/3}$$

Inversion of the left side of Eq. 11 results in

$$(13) \quad \frac{A}{N} = k D^{-2/3}$$

Inversion of the left side of Eq. 12 results in

$$(14) \quad \frac{P}{N} = k D^{1/3}$$

Eq. 11 is identical to Eq. 2 derived from expectation algebra.

Eq. 12 is the parallel to Eq. 11.; the latter gives the density of county seats, the "center density" (C = N/A), while the former gives county seats per capita, the **rate** (R = N/P).

Eq. 13 is the original size-density equation, except here the term on the left refers to the **average** county size in a region rather than the size of a given county.

Eq. 14 is a parallel to Eq. 13; it gives the average **population** per county. (I use lower case "a" and "p" to distinguish these from regional areas and populations.)

The relation among the four equations is shown in Table 10-3 (with D=P/A, C=N/A, R=N/P, a=A/N and p=P/N):

TABLE 10-3. NEWLY DERIVED EQUATIONS

(15)	$C = k D^{2/3}$	**RATE**	per unit area
(16)	$R = k D^{-1/3}$		per capita
(17)	$a = k D^{-2/3}$	**MEAN**	area
(18)	$p = k D^{1/3}$		population

The algebra is tedious (you can see it here), but it can be established that these four equations are statistically equivalent. That is, testing one of them is, in effect, to test all. It would thus appear to be a matter of convenience (which kind of data are available) — whatever equation is tested, the other results can be calculated algebraically. None of this was easy to get published.[4]

Appendix — Exact Probabilities

It is traditional in the social sciences to report findings at various significance levels, e.g., $p < .01$ or $p < .05$, just as I have done in earlier chapters. I switched, with the work reported in this chapter, to reporting exact probabilities. The change was partly due to the fact that it became technically possible for me to do so around this time (see below). I also did it because I agree with Lehmann's argument: [5]

> It is ... good practice to determine not only whether the hypothesis is accepted or rejected at the given significance level, but also to determine the smallest significance level ... at which the hypothesis would be rejected for the given observation. This number gives an idea of how strongly the data contradict (or support) the hypothesis, and enables others to reach a verdict based on the significance level of their choice.

Until the advent of modern computers we had to rely on tables of the t- distribution to report significance levels. Around the time of the research reported here, programmable calculators and personal computers made it easy to evaluate the integral describing the distribution. You can see the BASIC program I used here

. Next Chapter

NOTES:

[1] The source for historical population and area data was Bureau of the Census, *Historical Statistics of the United*

States, Colonial Times to 1970. Series A:195 (Population) and A:211-61, Washington, D.C., 1975. The number of counties per state was from Table 2-1.

[2] The caption symbols refer to logged values, e.g., *D* stands for *log D.*

[3] G. Edward Stephan and Douglas R. McMullin. "The Historical Distribution of County Seats in the United States: a Review, Critique and Test of Time-Minimization Theory", *American Sociological Review*, 46:907-17, (1981).

[4] The new derivation and the skeleton the argument of Eqs. 19-32 was published as G. Edward Stephan, "Territorial Division", *Social Forces*, 63:145-158, 1984. As in most Sociology journals, the typesetting made the equations virtually unreadable.

[5] E. L. Lehmann, *Testing Statistical Hypotheses*, :61-2, New York: Wiley, 1959.