CHAPTER 8

Size-Density from Time-Minimization

> Philosophy is written in this grand book — I mean the universe — which stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures, without which it is humanly impossible to understand a single word of it; without these, one is wandering about in a dark labyrinth.
>
> Galileo Galilei (1564-1642)

I had been fascinated by the -2/3 slope since 1971 when I first observed it, or one statistically indistinguishable from it, in the "world regression line" (Chapter 5).

The line is pictured in Fig 8-1. We may write its formula as

(1)

where A is area, D is density, and K is the intercept (the value of log A at the point where log D = 0, where D = 1).

FIG. 8-1. WORLD REGRESSION LINE
(N = 1,764 primary administrative subdivisions of 98 nations)

8-1.world.gif (3451bytes)

Between 1972 and 1975 a number of studies produced size-density slopes which did not depart significantly from this -2/3 value. Each time it happened I wondered Why? I tried to derive the slope theoretically (i.e., mathematically, from initial assumptions), but I failed each time until Spring term, 1975, when I was working with Jay Callen.

Dimensional Analysis: Time

Jay had been doing a lot of work with "central place theory", much of which involves explaining empirical regularities in terms of *least distance* models.[1] As it happened, I had been re-reading Zipf's principal work in which empirical regularities were derived in accordance with his principle of *least effort*.[2] At some point in all this I recalled a remark made by Professor Anderson, in a graduate school seminar in mathematical modeling, to the effect that well-developed theoretical fields tend to select *fundamental dimensions* and define other quantities in terms of these.

The classical example of such "dimensional analysis" is Newtonian Physics. Length is clearly a dimension, in a real sense the most basic one we have. With length, plus a few other dimensions defined in terms of length (e.g., time and mass — the distance a marker moves on a proper scale), other quantities can be defined as in the following table, the last column of which contains only s, m & t — length,

mass and time:

TABLE 8-1. DIMENSIONAL FORMULAS

Variable	Symbol	Formula	Dimensions
length, distance	s	s	s
area	A	sw	s^2
volume	V	swh	s^3
density	D	m/V	ms^{-3}
velocity	v	s/t	st^{-1}
acceleration	a	v/t	st^{-2}
momentum	p	mv	mst^{-1}
force	F	ma	mst^{-2}
work	W	Fs	ms^2t^{-2}
power	P	W/t	ms^2t^{-3}
gravity constant	g	g	st^{-2}
potential energy	PE	mgh	ms^2t^{-2}
kinetic energy	KE	1/2 mv^2	ms^2t^{-2}

Among other advantages, such reduction of all variables to one (or a few) dimensions makes it possible to rigorously "balance equations" dimensionally — have the same units on both sides of the equation, or specify the units for embedded "constants." Different sciences use different fundamental dimensions, e.g., Chemistry - energy; Economics - money (but see the "Labor Theory of Value" below). I recall Professor Anderson suggesting at some point in our conversations that one day, perhaps, *time* might prove to be a fundamental dimension in Sociology. I began to think about *least-time*.

Economics seems limited, as a *general* social science: what would its *least-cost* model mean in primitive societies which lack a money economy? In a similar vein, what could Geography's *least-distance* theories contribute to an understanding of future societies with "beam me up" technologies? I realize these are fictional, but our transportation systems make our ancestors view of distance hopelessly out of date also. Primtive societies where money is meaningless, future societies where distance is meaningless — how

do you theorize about both using the same social science model? Zipf's *least-effort* notion was a conscious attempt to combine monetary and distance (and other) costs, but it smacked of mixing apples and oranges (dollars + miles + sweat?), and it lacked rigor: how, exactly, do you measure (quantify) "effort"?

The solution, it seems to me, is to translate these disparate variables into the single dimension of *Time.* If I know the average velocity of a given transportation technology, it is easy enough to translate distance into time (distance is time, multiplied by velocitiy; time = distance/velocity). Depending upon whether I use a horse or an automobile, the county seat may be either a day or a half-hour away. Time is money, according to Franklin; money is time, multiplied by the rate at which it accumulates (time = money/rate): so much income, expenditure, rent, or interest per hour, day, month or year. Even effort, though poorly defined itself, might be translated into some expenditure of work-hours, given a level of technology (e.g., how long would it take someone to dig a given trench with no tools, with a shovel, with a backhoe?)

The Labor Theory of Value

We see then that that which determines the magnitude of the value of any article is the amount of labour socially necessary, or the labour-time socially necessary for its production. Each individual commodity, in this connexion, is to be considered as an average sample of its class. Commodities, therefore, in which equal quantities of labour are embodied, or which can be produced in the same time, have the same value. The value of one commodity is to the value of any other, as the labour-time necessary for the production of the one is to that necessary for the production of the other. "As values, all commodities are only definite masses of congealed labour-time."

- Karl Marx, *Capital I:1:i*

These thoughts took me back further in my education, to an undergraduate course in the history of social thought, taught by Andrew P. Phillips at what was then San Francisco State College. Along with the ideas of Comte, Durkheim and Weber, it was there

Pete Phillips
San Francisco, June 1999

that I encountered the social thought of Karl Marx and, incidentally, his labor theory of value: the value of a commodity is a direct function of the average socially necessary labor ***time*** required for its production. When I first heard of this idea I appreciated the fact that, unlike most other social science theories, it aimed at being cross-cultural, pan-historical, i.e., general — presumably applicable to all societies, even non-monetized or hyper-technological ones.

At the time I learned of these ideas of Marx, I was taking a Social Psychology course in which I had occasion to review ideas of a number of philosophers about the "essential difference" between human and animal life. The text and teacher made much of so-called "symbolic interactionism", the doctrine which asserts that the fundamental difference between humans and animals is our reliance on "meaningful symbols" (vs. mere "signs" like barks and chirps). It didn't seem to me that this was much of an improvement over the ancient metaphysical assertion that we differ by virtue of our capacity for thought (in fact, Aristotle's notion of us as the *zoon politikon* seems a little more sophisticated). That was back before contemporary scientists began studying animal language. Against this notion that *thought* is what distinguishes us from other animals, or that "symbolic interactionism" should be the (dimensional?) basis of Sociology, let me quote one of my favorite lines from Shakespeare:

> But thought's the slave of life, and life time's fool;
> And time, that takes survey of all the world,
> must have a stop.[3]

Thought's the slave *of life* (all life), and life is *time's* fool — it could as easily have been written by Marx, or any biologist for that matter. I don't understand why Sociology (other than the classic intro text by the Lenskis) doesn't begin with Biology as Auguste Comte originally suggested. It would be a small step from there, in the study of human or non-human social systems, to see how relevant *time* really is as a fundamental dimension.

Just for fun, Table 9-2 contains a very small sample of Shakespeare's many other references to time. I once shut down an online literature search engine by trying to get it to list all his references to "time"; next time I tried it "time" had been removed from searches, treated as too common along with words like "the" and "is"):

TABLE 8-2. A FEW TIME REFERENCES FROM SHAKESPEARE

Time travels in divers paces with divers
person. I'll tell you who Time ambles
withal, who Time trots withal, who Time
Gallops withal, and who he stands still withal.
As You Like It, III,ii,328

Spite of cormorant devouring Time.
Love's Labors Lost, I,i,4

Come what may,
Time and the hour runs through the roughest day.
Macbeth, I,iii,146

If you can look into the seeds of time,
And say which grain will grow and which will
not.
Macbeth, I,iii,58

Time's the king of men.
He's both their parent, and he is their grave.
And gives them what he will, not what they
crave.
Pericles, II,iii,45

Yet do they worst, old Time: despite thy
wrong,
My love shall in my verse ever live young.
Sonnets, XIX

Like as the waves make towards the pebbled
shore
So do our minutes hasten to their end.
Sonnets, LX

What seest thou else
In the dark backward and abysm of time?

To-morrow, and to-morrow, and to-morrow,
Creeps in this petty pace from day to day,
To the last syllable of recorded time;
And all our yesterdays have lighted fools
The way to dusty death. Out, out brief candle!
Life's but a walking shadow, a poor player
That struts and frets his hour upon the stage,
And then is heard no more: it is a tale
Told by an idiot, full of sound and fury,
Signifiying nothing.
Macbeth, V,v,17

A forted residence 'gainst the tooth of time,
And razure of oblivion.
Measure for Measure, V,i,12

There are many events in the womb of time which will be delivered.
Othello, I,ii,377

The Tempest, I,ii,49

Is there no respect of place, persons, nor time in you?
Twelfth Night, II,iii,100

Time is like a fashionable host
That slightly shakes his parting guest by the hand,
And with his arms outstretch'd as he would fly,
Grasps in the comer ....
Troilus and Cressida, III,iii,165

Envious and calumniating time.
Troilus and Cressida, III,iii,174

The end crowns all,
And that old common arbitrator, Time,
Will one day end it.
Troilus and Cressida, IV,v,223

Time Minimization Theory

Begin with this: Assume that ***social structures evolve in such a way as to minimize the time expended in their operation***. Under this constraint, determine how large a county (or other territorial division) should be (in order to enforce law, maintain order, provide services, or whatever).

One way of thinking about the value assumed by any variable is to assume it to be the result of contradictory forces: those tending to increase it, and those tending to decrease it. The larger a county is, other things equal, the more people there are to pay for its operations. On the other hand, the larger it is the harder it will be for distant citizens to reach, or be reached by, the county seat of government.

Originally, the "maintenance" time of a county was donated by people who volunteered their time as its officials. With the growth of county government, however, the operations of a county are carried out by paid employees. The money required to pay them comes from taxes and other sources. Ultimately, however, we may imagine the money as the result of time expenditures by those making up the county (e.g., as taxed income, itself a product of time).

Sharing the maintenance costs creates a *centrifugal* force, a tendency to increase the size the county, but as distance to the center increases the "interaction" time also increases, creating a *centripetal* force, a tendency to decrease county size. It seems reasonable to assume that actual county sizes represent a balance of these opposing tendencies, a minimizing of both "maintenance" and "interaction" times somehow taken together.

We don't know what the maintenance time actually is, only that it represents some expenditure of hours, **h**. We also don't know how the burden is distributed among the citizens. We can refer to counties as being more or less costly in terms of average or per-capita time expenditures,

where **P** is the population of the county. By a similar argument, ignoring the geographic distribution of the citizens within a county, we can specify an average interaction time

where **S** is the average distance[4] to the county seat and **v** is the average velocity under existing transportation technology.

The time expended in the operation of this social structure is the sum

total time = maintenance time + interaction time

(2)

Now, how do we get from this expression to Eq. 1?

We are trying to find an optimum value for **A**, the area served by the county, but **A** does not appear in Eq. 2, at least not directly. The definition of **density** is population divided by area

The average distance to the county seat must be proportional to the square root of county area (by dimensional analysis a length in, e.g., miles must be proportional to the square root of an area in square miles), so with the constant of proportionality, w, we obtain the equation

With these substitutions we can re-write Eq. 2

(3)

We can now write T as a function of A, treating D, h, w and v as (temporary) constants,

(4)

These centrifugal and centripetal forces appear in Fig. 8-2, and Fig. 8- 3 indicates a minimum, **least-time** expenditure.

FIG. 8-2. CONTRADICTORY EFFECTS OF AREA ON TIME EXPENDITURE

8-2.times.gif (2152bytes)

FIG. 8-3. TOTAL TIME EXPENDITURE AS A FUNCTION OF AREA

8-3.total.gif (2015bytes)

Taking the derivative[5] of T with respect to A gives us

(5)

Setting the derivative equal to zero and solving for A produces

(6)

We want to express Area as a function of Density, so we extract it from the right-hand term of Eq. 6 to obtain

(7) 

Reducing the constants to a single factor **k**

(8) 

The logarithmic transformation of Eq. 8, with K = log k, is

(9) 

Eq. 9 is identical to Eq. 1, the "world regression line" which we sought to derive theoretically. Thus, it follows directly from the assumption of time-minimization that the size-density slope should be negative two-thirds.

Lack of Intentionality in Time Minimization

I want to emphasize something about time-minimization as I have used the term: I make no assumption about people consciously or purposely minimizing time. Least-distance and least-cost theories are often put forward as criteria for *rational* decision making. Zipf presents his own theory of *least-effort* as similarly rational, conscious, purposive. We feel comfortable enough with models which describe the way a decision maker makes rational decisions employing one or more of these criteria. And rational decision makers certainly do try to manage time expenditure (think of all the time-and-motion studies done in industry throughout this century).

In contrast, I am explicitly *not* trying to specify rules to be used by a rational decision maker. Nor am I in any way asserting that people are inclined toward "rational" behavior. I am only trying to account for the relationship described by Eq. 1. The relationship between size and density results from, and describes, a pattern of territorial division which is the product of

countless numbers of decisions, more-or-less rational, by countless numbers of people, over very long periods of time and often huge expanses of space. The actors involved were effectively unconscious of one another and certainly unaware of the size-density relationship itself. I have no reason to assume any of them consciously tried to establish a negative two-thirds size-density relation or that they purposefully minimized time.

What I am putting forth here is pure theory, invention, fiction, imagination. I am only trying to state the simplest **assumption** which, **if** true, **would** account for the observed results. I make no assertion of its truth (in fact, I don't think most people do consciously minimize time, most of the time; they even "waste" it doing things they "like" - see below). It is as if I had observed the relation between temperature and pressure in a contained gas and began to speculate about perfectly elastic tiny billiard balls whizzing around in perfectly empty space, i.e., the kinetic theory of gases. In fact, the particles of a gas tend to be not-perfectly-elastic, double-atom molecules, whizzing around in not-perfectly-empty space. But it doesn't matter: the model is good enough to make sense of the results (at the level of a simple gas law). That is all I am doing with the assumption of time-minimization.

I belabor this point because I think Sociology tends to misunderstand the nature of scientific theory. Whereas most scientific explanation accounts for patterns in the recurrent behavior of very large numbers of events (atoms, molecules, cells, organisms, species, planets, galaxies), Sociology tends toward a highly individual-voluntaristic view of its "actors". When we think of crime, we usually picture a criminal rather than a crime rate; when we think of adolescence or adolescents, we seldom envision a population pyramid.

I illustrate this tendency in class lectures with

reference to an exhibit of 12th century Chinese scroll paintings I saw at the DeYoung Museum in San Francisco when I was an undergraduate. The right-hand side of Fig 8-4 is a badly beat-up poster I still have from that exhibit. The two pictures on the left are details from the center-right (houses) and lower-left (some people) of that poster. Some of paintings were enormous, as much as 50 feet when fully unrolled. Ordinarily they were scrolled, a slave standing on either side, slowly revealing portions, so that the viewers, lounging before the changing scene, had a sense of taking a journey through vast regions. The painting was supposed to inspire thoughtful commentary, sometimes written on the painting itself in the form of poetry.

FIG. 8-4. EMPHASIS ON THE EXTERNAL WORLD

I was fascinated by this view of the place of humans in nature. Here are the sky, mountains, valleys, river gorges. There may be a forest or an orchard, a bridge. Often people are shown amidst the buildings or natural surroundings. Who are they? How are they grouped? The people in such a picture are subject to all sorts of laws: meteorological, biological, physical, nutritional, sociological. You don't have to enter their minds to account for their behavior any more than you have to enter the minds of molecules or monkeys

to explain theirs. There is a **structure** in which these people, and thousands of others elsewhere and earlier, live out their lives.

The other rooms in the museum at that time were populated with European works, mostly portraits, suggested here in Fig 8-5. Though some of them had great artistic merit, their view of human nature was very different from you get in Fig. 8-4 The entire emphasis here is on the individual and his or her mood or character or claim-to-fame. About the only hint you get that they even live in a world outside themselves might be in their title (the 15th Earl implies that there were 14 before him) or the placement of a pillar in the background (**his** pillar in **his** palace in **his** world).

FIG. 8-5 EMPHASIS ON THE INDIVIDUAL

8-5.vel (7k) 8-5.vir (6k) 8-5.gauguin.t (13k) 8-5.go (8k)

8-5.n (7k) 8-5.rembrandt. (11k) 8-5.fa (4041 8-5.mu (8k)

There is a similar contrast in point-of-view in much literature. Moby Dick begins with the highly personal

> Call me Ishmael. Some years ago—never mind how long precisely—having little or no money in my purse, and nothing particular to interest me on shore, I thought I would sail about a little and see the watery part of the world.

and ends after much struggle with

> Now small fowls flew screaming over the yet yawning gulf; a sullen white surf beat

> against its steep sides; then all collapsed,
> and the great shroud of the sea rolled on as
> it rolled five thousand years ago.

Huckleberry Finn, though less cosmic, moves similarly from the unique to the recurrent. It begins with "You don't know about me...." and concludes: "... Aunt Sally she's going to adopt me and civilize me, and I can't stand it. I been there before."

I don't see how anyone could theorize about human behavior when humans are viewed solely as willing, "motivated" individuals. (Isn't "motivation" or "motive force" what got all the pre-Galilean physicists back to Aristotle confused with regard to rest *versus* motion?) Each individual is just that: a genetically unique entity which, even if it were cloned a thousand times, would still be unique in terms of subsequent individual experience and personal "motivation". Predicting behavior would be a matter of getting to know that individual well enough to guess the next move correctly. The knowledge would not generalize beyond that individual psyche. Your intense love of football probably has little to say about mine for opera; your need to golf has little to do with my need to solve puzzles. If science is the study of **recurrent** events, then it has nothing to say about individuals and their "likes" or "wills" or "wants" or "motives". It should rather focus on the "common denominator" which by assumption underlies all recurrent actions and their patterns, something such as time-minimization.

A theory of human behavior which "makes sense" at an individual level (rational choice or libidinal repression) ought to be suspect. Theories should account for *patterns which appear in macro-level data*. A theory like time-minimization should not be judged in terms of intuitive "reality"; it need only account for the empirical finding better than any competitor.

Publishing the Derivation

I submitted the derivation to the *American Sociological Review* on April 8, 1975. The rejection arrived on June 9. One reviewer had no opinion regarding publication, but did offer some hints for tightening the presentation. The other wrote that

> The basic flaw in the analysis is the reliance on the characteristics of an imaginary average region. A truly operational model would use the real average distance to regional centers...,

But theory is imaginary. When a theory is confirmed by direct observation (e.g., cell theory through the microscope, the presence of Neptune or Pluto with a telescope) it is no longer theory but fact. A "truly operational model", as this reviewer seems to mean the words, would be a description of an actual county. Is that even science? It certainly isn't theory. The same reviewer didn't add to his credibility when he pointed out that I hadn't "defined little d in the fraction dT/dA" (the left side of Eq. 5 in the derivation above), and, even if I had, it should be canceled from the fraction.[6]

After fruitless argument over these points with the *ASR*, I submitted the paper to the *American Journal of Sociology*, January 16, 1976. It was rejected April 15. One referee described it as a "useful fragment that ties together your previous work quite well" (you could make the same remark about Newton's laws of motion). A second found "work on area-density less than inspiring" (an interesting criterion for judging scientific work). I asked for reconsideration and received some discussion about the determinants of density (my independent variable) and the statement

> Who cares about county seats — more civilly, what's the sociological payoff to a general theory with such limited scope?

I mentioned the attention Durkheim had paid to precisely this topic (Chapter 4) and got nowhere, the final rejection coming October 5, 1976.

A former undergraduate student, then studying for his doctorate in Social Psychology at the University of Nevada, Phil Knowles, suggested I submit the paper to *Science* which I did on November 1. There was no hope of getting a positive review, but I knew I would get a quick and competent one. Receipt was acknowledged on the 24th, and the review arrived January 3, 1977. The referee wrote that it was "a very simple and elegant derivation, which I think should be published." The editor, however, said that "We receive each week about three times as many good reports as we can publish...." and rejected it.

I wrote back on January 10 commenting only that, for a purportedly interdisciplinary scientific journal, I saw very little social science in *Science*. On January 18 I received the reply

> We are glad to accept for publication as a report in Science your paper on "Territorial Division: the Least-Time Constraint Behind the Formation of Sub-National Boundaries."

Nothing further by way explanation or commentary. I was ecstatic. It was published the following April.[7] I have received more, and more interesting, communication as a result of this very brief publication in *Science* than I have from all my other publications in Sociology journals combined. I have reproduced the article here, retaining as much of the original typography as possible (I couldn't create fully justified paragraphs however).

Earlier Derivations

Jay Callen has, over the years, brought to my attention three earlier studies from other parts of the

world which derive relations identical or related to Eq. 1:

* In England, using a least-distance assumption, Palmer[8] derived

where **v** is the density of service centers and **u** is the density of population. This is identical to an equation derived differently in chapter 10 from Eq. 1.

* Earlier, in Greece, also using a least-distance model (which he equated with least-effort) Virirakis[9] produced

where P is the population and R is the radius of imaginary circular areas in the i-th density category. Substituting DA = P (from the definition for density) and $A = \pi R^2$ (the area of a circle), produces

which is virtually identical with Eq. 6.

* Earlier still, in Poland, Mycielski and Trzeciakowski,[10] in a derivation too long to be described here, start from cost considerations and propose a method for locating gasoline service stations efficiently (at least by their criteria). The authors pass **en route** through two equations (on p. 68) compatible with Eqs. 8 and 9.

Size-Density from Random Numbers

Around the time of the publication in **Science**, I developed, at the suggestion of my good friend and colleague Chuck Gossman (and initially to my horror), a derivation of the -2/3 slope from random numbers.

Consider the size-density regression coefficient, the covariance of log D and log A, divided by the variance of log D:

(10)

Re-writing the denominator and the numerator [11] enables us to "reduce" the size-density slope, expressing it as a function of the variances of log P and log A and the covariance of log P and log A:

Charles Gossman
Lake Padden WA, March 2001

(11)

Now draw three sets of random numbers. Label the first set **P**. Multiply the second two sets (area is two dimensional) and label the product set **A**. It follows that the covariance is

(12)

and the variances must be related by

(13)

Substituting these results in the "reduced" formula

(14)

Aaack! I will return to this disturbing result in Chapter 10.

Next Chapter

NOTES:

[1] e.g., Ronald Abler, John S. Adams and Peter Gould, *Spatial Organization: The Geographer's View of the World*, Englewood Cliffs, NJ: Prentice-Hall, 1971.

[2] George K. Zipf, *Human Behavior and the Principle of Least Effort*. Cambridge, MA: Addison-Wesley, 1949.

[3] *Henry IV, Part I*, V, iv, 81-82. In all of Shakespeare's gloomy references to time's ruination (see Table 8-2) he seems to exempt only love:

> Love's not Time's fool, though rosy lips and cheeks
> Within his bending sickle's compass come.
> *Sonnet* 116

[4] I follow the practice in physics of letting *S* represent distance since *d* has special mathematical significance and *l* ("ell" for "length") is easily confused with the numeral representing unity.

[5] *Minimization*: this will be in the methodological appendix.

[6] As any first-quarter calculus student knows *dy/dx* is just a symbolic way of saying *the derivative of y with respect to x*. it could have been written *y'* or *f'(x)* rather than *dy/dx*. In particular, "little d" is not a factor which can be "canceled". By the logic of my reviewer, the letter H should be written as 1, since it looks like eleven divided by eleven. After mentioning this review in a presentation to an audience of scientists, one suggested that I should ask the reviewer to evaluate

If you cancel *n* the answer is "six".

[7] G. Edward Stephan, "Territorial Division: the Least-Time Constraint Behind the Formation of Subnational Boundaries", *Science*, 196:523-4, April 27, 1977.

[8] D. S. Palmer, "The Placing of Service Points to Minimize Travel", *Operational Research Quarterly*,

24:121-123, 1973.

[9] John Virirakis, "Minimum Effort as a Determinant of the Area, Population, and Density of Residential Communities", *Ekistics*, 27:362-371, 1969.

[10] Jerzy Mycielski and Witold Trzeciakowski, "Optimization of the Size and Location of Service Stations", *Journal of Regional Science*, 5:59-68, 1963.

[11] The algebraic details of this argument will be presented more fully in Chapter 10