Finite Mathematics

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Table of contents

Preface

This was made using Quarto.

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1 Introduction

This is a book created from markdown and executable code.

See Knuth (1984) for additional discussion of literate programming.

Part I

Linear Functions

2 Intro to Linear Functions

2.1 Definitions and Notation for Linear Functions

As you hop into a taxicab in Allentown, the meter will immediately read \$3.30; this is the "drop" charge made when the taximeter is activated. After that initial fee, the taximeter will add \$2.40 for each mile the taxi drives. In this scenario, the total taxi fare depends upon the number of miles ridden in the taxi, and we can ask whether it is possible to model this type of scenario with a function. Using descriptive variables, we choose m for miles and C for Cost in dollars as a function of miles: $C(m)$.

Here, $C(0)$ means the cost for travelling 0 miles (assuming you have entered the taxi). This cost is \$3.3. We can write this mathematically as

```
C(0)=3.3
```

Similarly, $C(2)$ is the cost of travelling 2 miles and can be computed as

```
C(2) = 3.3 + (2.4 x 2) = 8.1
```

Here, we take the base charge of \$3.3 and add it to the charge for riding 2 miles to get a grand total of 8.1.

In general, if you travel m miles, the cost, $C(m)$, can be computed as follows:

```
C(m)=3.3 + 2.4m
```

It is crucial to think carefully about the units of each component and how they relate. The expression below shows how this plays out in our taxi context:

$$C(m) = 3.3 \;\; dollars + 2.4 \; \frac{dollars}{mile} \times m \;\; miles$$

When dollars per mile are multiplied by a number of miles, the result is a number of dollars, matching the units on the 3.30, and matching the desired units for the cost, $C(m)$, of the ride (i.e., dollars).

We call a relationship such as this, a ***Function*** of m. The above function takes m (the miles traveled) as the ***Input*** and returns $C(m)$ (the cost of travelling m miles) as the ***Output***. As you will learn shortly, this is an example of a ***Linear Function***. There are many **types of functions** in mathematics and they are often named based on how the output values change in relation to changes in the input values. The function given above is called a *__linear function__ because the output values change proportionately to the input values.

2.2 Anatomy of a Linear Function

There are two parts to the function above; the first part (3.3) is FIXED while the second part, $2.4m$, VARIES depending on the value of m. While the fixed part of the function is important in determining the cost, it is the second part that plays an important role if we wanted to understand how "fast" the cost changes (in this case increases). As we will see later, the value 2.4 is known as the ***Rate of Change*** for the function $C(m)$. It tells us how fast, $C(m)$ changes as we change m. Furthermore, since this rate of change stays the same regardless of the value of m, we say that the rate is ***Constant*** which means that the cost changes at a constant rate.

In summary, a linear function has the following structure, where b is the fixed part, and mx is the variable part.

$$f(x) = b + mx$$

The higher the rate of change, the faster the output values change. For example, if we adjust the rate of change to 3.5 from 2.4 in the above scenario, you can expect the cost of riding to increase faster as you increase m.

2.3 Function Representations

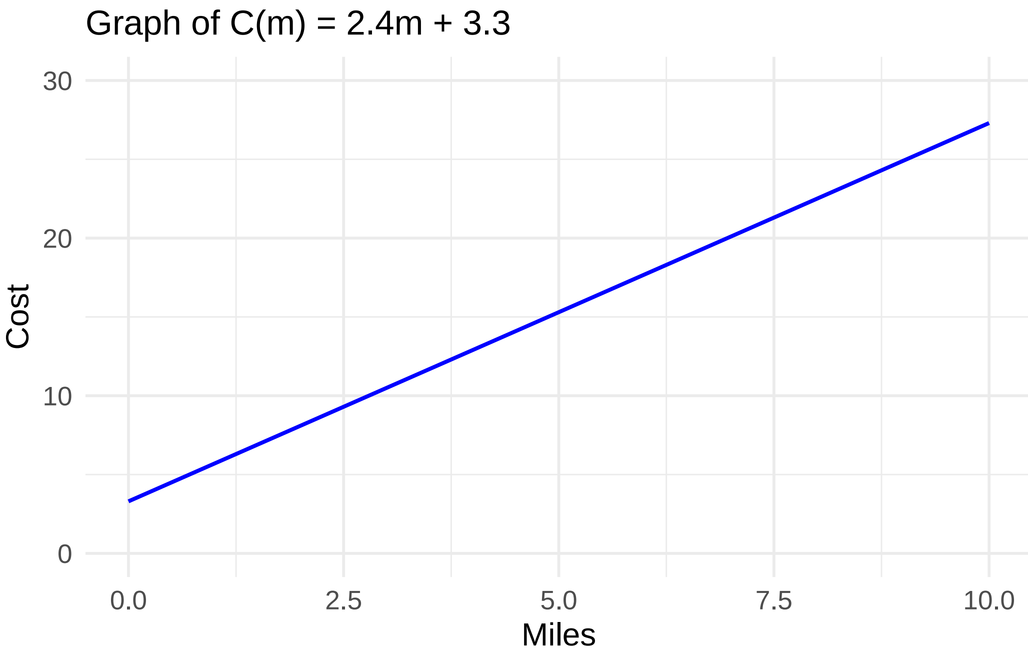
In the above section, we described the taxi cost function using words and represented it using a formula. Other tools for representing functions are tables and graphs.

Below is a table for the function above:

Miles (m)	0	1	2	3	4
Cost (in $)	3.3	5.7	8.1	10.5	12.9

Question: What are the advantages and disadvantages of using a table instead of a formula or verbal description?

We can also represent the function above using a graph. See below:



Since the cost is dependent on the miles traveled, we call it a ***Dependent Variable***. We call the miles (m) an ***Independent Variable***. By convention, we place the dependent variable on the horizontal axis (x-axis) and the dependent variable on the vertical axis (y-axis).

If you ride 0 miles, the cost is \$3.30, giving the coordinate $(m, C(m)) = (0, 3.30)$ on the graph. We call this point, the vertical or $C(m)-intercept$ (or $y-intercept$ in a general graph using x and y).

We call the above function, a ***Linear Function*** because its graph produces a straight line. This straight line results because the change in cost is consistent on any intervals of miles.

In a graph of any linear function, the rate of change is often referred to as ***Slope*** because it tells us how steep the line is. If the rate of change in the taxi scenario given above were, say, 5 dollars per mile, instead of 2.4, the line would be much steeper than it is. When a linear function is expressed in the form $f(x) = mx + b$, we call it ***slope-intercept form***. This form is the most common because it makes it easier to spot the *slope* and the $y-intercept$ which are important characteristics of linear functions.

2.4 Increasing and Decreasing Functions

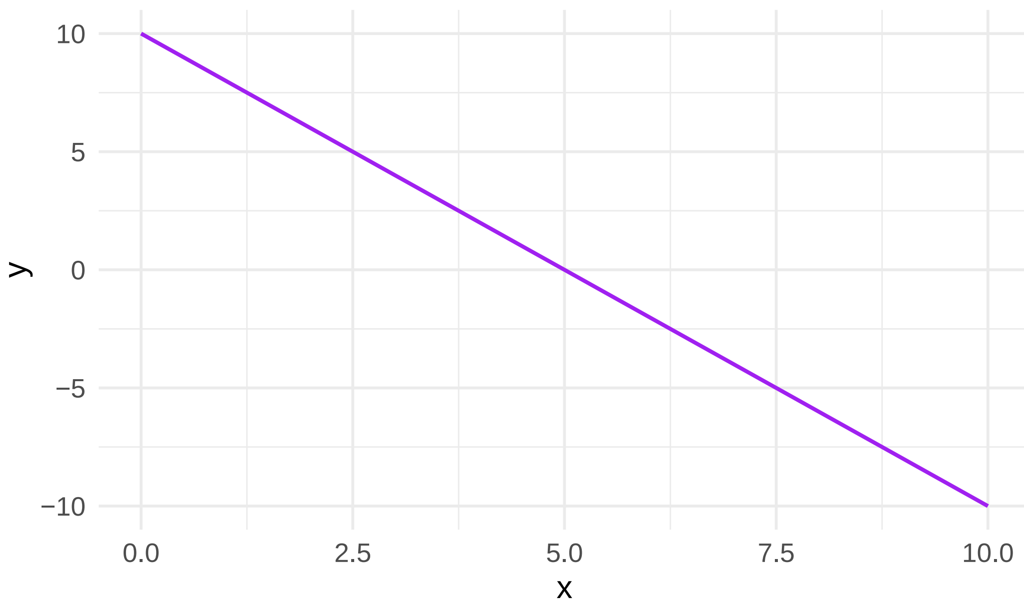
Notice in the above example that as you increase the number of miles, the cost of the ride goes up. This is because the rate of change (m) is positive.

Since as you increase the input value, the output value increases, we say that the function $C(m)$ is an increasing function. As can be seen on the graph, the line is rising from left to right. This is because the rate of change value is positive.

Generally, a linear function is said to be ***increasing*** if the slope m is positive and ***decreasing*** if the slope is negative.

2.5 Exercises

1. Create a real-life scenario that can be modeled by a decreasing linear function.
2. Write the formula for the function in exercise 1 above.
3. Describe how the graph of the function in exercise 2 would look like.
4. What would the graph of the function, $f(x) = 0x + 3$ look like?
5. Find the formula for the linear function, $y = f(x)$, graphed below:



6. Find the $y-intercept$ of a linear function whose rate of change is 2.5 and passes through the point $(3, 9)$.
7. Marcus currently owns 200 songs in his iTunes collection. Every month, he adds 15 new songs. Write a formula for the number of songs, N, in his iTunes collection as a function of the number of months, m. How many songs will he own in a year if the trend continues?

3 Rate of Change

3.1 Calculating Rate of Change

The rate of change (ROC) is perhaps the most important component of any linear function. As we have seen, it can tell you whether the function is increasing or decreasing and can be used to create the function formula (linear model) for a given real life situation. The question that arises is, "how can we compute the rate of change from given data (e.g., from a table?". In the earlier taxi example, suppose, we did not know the rate of change but only knew the cost of riding 4 miles and 7 miles. How can we use this information together with the fact that the function is linear to find the rate of change and even the formula for the function?

In the next few examples we explore ways of finding the rate of change and how to use it to create the linear function/model for given real-life situations. At the end, a formula for computing the rate of change is provided.

Example 1

The population of a city can be modeled approximately using a linear function. In 2002, the population was 23,400 and in 2006, it was 27,800.

a) Find the rate of change of the population for this city.

b) What is the formula of the linear function for the scenario? Let P be the population and t the time in years.

c) Assuming the model (function) holds true until 2024, what would be the population of the town in 2024?

Solution

a) Since we are told that the population grows linearly, we know that the growth between 2002 and 2003 is the same as the growth between 2003 and 2004, etc. Thus, to find the rate of change (i.e., population growth per year), we can divide the population change between 2002 and 2006 by the number of years as shown:

$$Rate\ of\ change = \frac{pop.\ in\ 2006\ - pop.\ in\ 2002}{2006 - 2002}$$

$$= 1100\ people\ per\ year$$

b) If we use 2002 as the base year(i.e., $t = 0$), then the FIXED $y - intercept$ value in the function is 23,400. We are left with writing down the function in the form $f(x) = mx + b$ where m is the rate of change and b is the constant (initial value). Our input variable is t so we use it to replace x.

$$f(t) = 1100t + 23,400$$

c) For 2024, $t = 22$ years. Thus,

$$f(22) = 1100 \times 22 + 23,400$$

$$= 47,600$$

Example 2

The summit of Africa's largest peak, Mt. Kilimanjaro, has two main ice fields and a glacier at its peak. Geologists measured the ice cover in the year 2000 ($t = 0$) to be approximately 1951 m^2; in the year 2007, the ice cover measured 1555 m^2.

a) Suppose that the amount of ice cover at the peak of Mt. Kilimanjaro is changing at a constant average rate from year to year. Find a linear model, $A = f(t)$ whose output is the area, A, in square meters in year t (where t is the number of years after 2000).

b) What do the slope and A-intercept mean in the model you found in (a)? In particular, what are the units on the slope?

C) Compute $f(17)$. What does this quantity measure? Write a complete sentence to explain.

d) If the model holds further into the future, when do we predict the ice cover will vanish?

Solution

a) We begin by finding the rate of change. Since we know that the rate of change is constant year after year, we can divide the difference in ice coverage between 2007 and 2000 by 7 to get the rate of change per year.

$$Rate\ of\ change = \frac{Coverage\ in\ 2007\ - Coverage\ in\ 2000}{2007 - 2000} \tag{3.1}$$

$$\tag{3.2}$$

$$= -56.57\ m^2\ per\ year \tag{3.3}$$

The general format of the function is $A(t) = mt + b$ where m is the rate of change and b is the $A(t)$-intercept (or the value of $A(0)$ which we know is 1951). Thus, the function is,

$$A(t) = -56.57t + 1951$$

b) The slope means that the ice for every additional year, the ice coverage decreases by $56.57m^2$. The units are square meters per year ($m^2/year$). The y intercept means that the initial coverage at year zero (when the measurement was first taken) is $1951m^2$.

c) $f(17) = (-55.57 \times 17) + 1951 = 1006.31$; This means that there were $1006.31mi^2$ of ice coverage on Mt. Kilimanjaro by 2017 (i.e., 17 years after 2000).

d) Remember that $A(t)$ is the function that gives the ice coverage after t years. Therefore, if the ice cover is zero, it means $A(t) = 0$. We compute t by solving the equation $-56.57t + 1951 = 0$ for t.

$$-56.57t + 1951 = 0$$

$$-56.57t = -1951$$

$$t = \frac{-1951}{-56.57}$$

$$= 34.49\ years$$

3.2 A Formula For ROC

From the foregoing examples, it should be readily clear that, given two input values x_1 and x_2 such that $x_1 \neq x_2$,

$$Rate\ of\ Change = \frac{Change\ in\ Output}{Change\ in\ Input}$$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example 3

If $f(x)$ is a linear function, $f(3) = -2$, and $f(8) = 1$, find an equation/formula for the function.

Solution

In this problem, we are looking at the input interval between 3 and 8. Thus, $x_1 = 3$ and $x_2 = 8$. To find the ROC for $f(x)$ we proceed as follows:

$$ROC = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{3.4}$$

$$\tag{3.5}$$

$$= \frac{f(3) - f(1)}{8 - 3} \tag{3.6}$$

$$\tag{3.7}$$

$$= \frac{1 - (-2)}{5} \tag{3.8}$$

$$\tag{3.9}$$

$$= \frac{3}{5} \tag{3.10}$$

Next, the general form of the linear function is $f(x) = mx + b$, where m is the ROC (aka slope. So, we can write, $f(x) = \frac{3}{5}x + b$. To find b, we can use one of the known values of $f(x)$, such as $f(8)$ and solve for b as follows:

$$f(8) = \frac{3}{5} \times (8) + b$$

$$1 = \frac{24}{5} + b$$

$$b = 1 - \frac{24}{5}$$

$$= -\frac{19}{5}$$

So, the equation becomes,

$$f(x) = \frac{3}{5}x - \frac{19}{5}$$

3.3 Point-Slope Equation Format

The equation $y = mx + b$ is called the slope-intercept form of a linear function (equation). In cases where you only know one of the points, say (x_1, y_1) and the slope m you can express the equation of the line as follows:

$$y - y_1 = m(x - x_1)$$

Where, (x_1, y_1) is the KNOWN point.

After this, you can then rearrange the equation into the slope-intercept format. You just need to be careful with your algebraic manipulation when doing this. See example below:

Example 4

A new house was sold for $296000 8 years after it was purchased. The original owners calculated that the house appreciated $2,500 per year while they owned it. Find a linear function that describes the above situation if x is the number of years since the building was purchased.

Solution

Let x be the number of years and $C(x)$ be the cost of the house after x years.

Note that, we do not know the initial price (i.e., b) but we know the *ROC* in cost to be 2,500 $ per year (i.e., a linear function). We also know the cost after 8 years (i.e, we know one point $(8, 296,000)$).

We can use this information and the concept of slope-point format to write the equation of the line as follows:

$$y - y_1 = m(x - x_1)$$

$$y - 296,000 = 2500(x - 8)$$

$$y - 296,000 = 2500x - 20,000$$

$$y = 2500x - 20000 + 296,000$$

$$y = 2500x + 276,000$$

Note that in the above equation, $y = C(x)$. So we are done. As a bonus, we know the cost of the house was $276,000 eight years ago.

3.4 Exercises

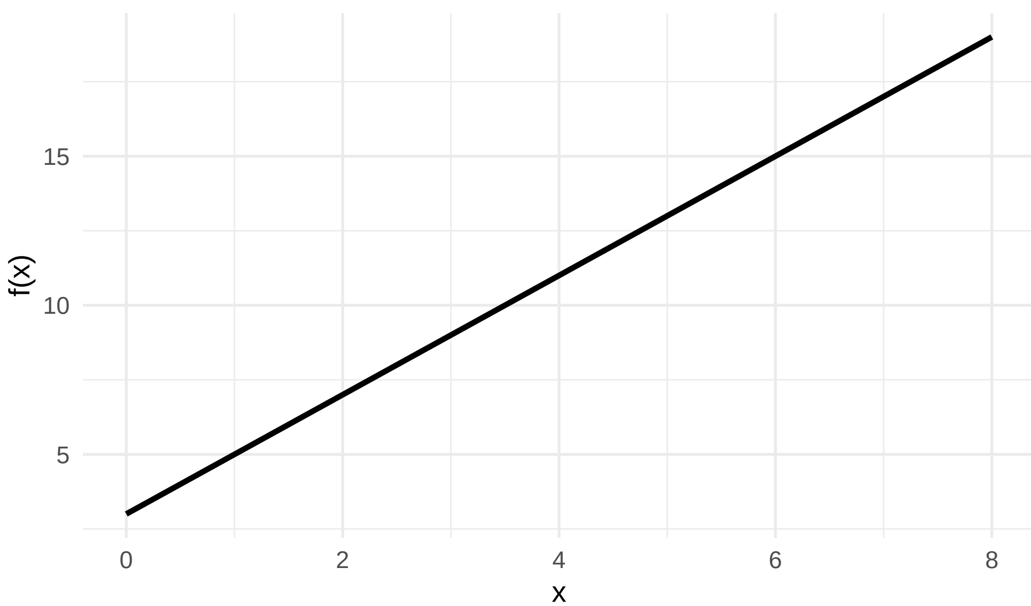
1. A town has a population of 2000 people at time $t = 0$. In each of the following cases, write a formula for the population P,of the town as a function of year t.

   a. The population increases by 90 people per year.
   b. The population increases by 1 percent per year.

2. Find the slope of the line connecting the points $(2, 7)$ and $(6, 11)$. Find the equation of this line.

3. Working as an insurance salesperson, Ilya earns a base salary and a commission on each new policy, so Ilya's weekly income, I, depends on the number of new policies, n, he sells during the week. Last week he sold 3 new policies, and earned \$760 for the week. The week before, he sold 5 new policies, and earned \$920. Find an equation for $I(n)$, and interpret the meaning of the components of the equation.

4. Find a formula for the linear equation graphed below. You can enlarge the graph by clicking on it.



5. An apartment manager keeps careful record of how the rent charged per unit corresponds to the number of occupied units in a large complex. See the table:

Monthly Rent (in \$)	650	700	750	800	850	900
Occupied apartments	203	196	189	182	175	168

a. Why is it reasonable to say that the number of occupied apartments is a linear function of monthly rent?
b. Let A be the number of occupied apartments and R the monthly rent charged (in dollars). If we let $A = f(R)$,what is the slope of the linear function $f(R)$? What is the meaning of the slope in the context of this question?
c. Determine a formula for $A = f(R)$.
d. If the rent were to be increased to \$1000, how many occupied apartments should the apartment manager expect? How much total revenue would the manager collect in a given month when rent is set at \$1000?

4 Applications

4.1 Intersecting Lines

As you saw earlier, graphs are one of the ways commonly used to represent linear functions. By examining graphs of linear functions, we learn a lot about the function. For example, we can quickly tell whether a function is increasing (positive slope), decreasing (negative slope) or neither (zero slope). We can also get a sense of how fast the output values are changing with change in the input values. In order to leverage this benefit of graphs to compare multiple linear functions, it is often helpful to graph the functions on the same grid.

Consider the claim below:

If two linear functions with **different** rates of change are graphed on the same grid, then, the lines must intersect at some point. Do you think this claim is true? Why or why not?

In the world of business and economics, **supply** and **demand** problems are sometimes modeled using linear functions. When the demand and supply meet (intersect) we have an equilibrium point. Consider the following example:

Example 1

The supply, in thousands of items, for custom phone cases can be modeled by the equation, $s(p) = 0.5 + 1.2p$ while the demand can be modeled by $d(p) = 8.7 - 0.7p$, where p is in the price in dollars. Find the equilibrium price and quantity, the intersection of the supply and demand curves.

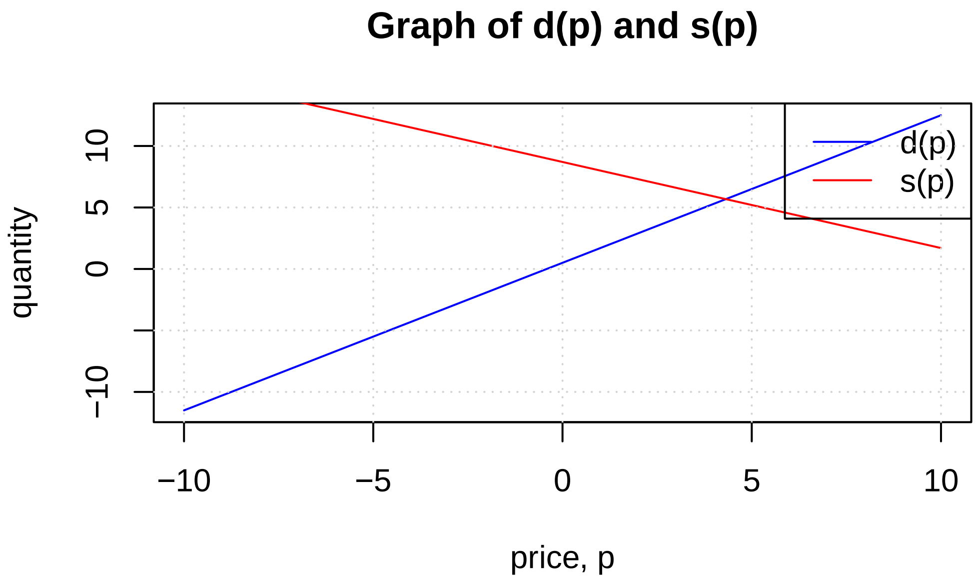
Solution

There are two ways to solve this problem. First, you can set up $s(p) = d(p)$ then solve algebraically for p or simply graph the two functions then look at the point of intersection.

Let us do both.

Graphical solution :

For the graphical solution, you simply graph the two functions (you can use tools such as desmos or geogebra) and read out the coordinates of the intersection point.



The approximate point of intersection for the graphs is $(4.3, 5.6)$. You can get more accurate values using technology. This intersection is a pair (x, y) where the first number is the price (input) and the second if the quantity (output).

Algebraic solution:

$$0.5 + 1.2p = 8.7 - 0.7p \tag{4.1}$$
$$\tag{4.2}$$
$$1.2p + 0.7p = 8.7 - 0.5 \tag{4.3}$$
$$\tag{4.4}$$
$$1.9p = 8.2 \tag{4.5}$$
$$\tag{4.6}$$
$$p = \frac{8.2}{1.9} \tag{4.7}$$
$$\tag{4.8}$$
$$= 4.32 \tag{4.9}$$

Thus, the equilibrium price is approximately \$4.32.

To find the quantity associated with this price, use any of the two functions to evaluate the output at $p = 4.32$. It does not matter which function you use because both of them have the same output value at the point of intersection (a shared point).

$$s(4.32) = 0.5 + (1.2 \times 4.32) \tag{4.10}$$

$$\tag{4.11}$$

$$= 5.68 \tag{4.12}$$

Thus, the two approaches give us the same solution.

4.2 Systems of Equations

Part II

Linear Programming

5 Geometric Method

5.1 Introduction to linear Programming

Managers are often called upon to make complicated decisions. For example, production managers often make decisions on what products to manufacture and in what quantities. In making such decisions, the manager must consider the available resources and how to utilize them for maximum profit. Note that resources are not limited to raw materials. They can include labor (human hours), farmland, machinery, etc. Resources, in general, are always limited and management must decide how to allocate them in order to get the maximum possible profit.

Linear programming (LP) is one of the most important methods used in management science to solve problems of the kind describe above. LP involves maximizing or minimizing a quantity, usually profit or cost, under some given constraints.

5.2 Mixture Problems & Charts

A mixture problem is a problem which includes combining limited resources to manufacture products that will generate maximum profit for the company.

These problems are common because most products that we use involve combining multiple resources in their production. Although there are other considerations in making production decisions, availability of resources is one of the most important constraints.

An ***optimal production policy (OPP)*** is a policy that,

> `(i) does not violate the constraints under which the company operates and,`
> `(ii) yields maximum profit.`

Example 1

A toy manufacturer can manufacture only skateboards and, only dolls, or some kind of skateboards and dolls. Skateboards require 5 units of plastic and can be sold for a profit of $ 1, while dolls require 2 units of plastic and can be sold for $0.55 profit. Only 60 units of plastic are available.

(a) Make a mixture chart to model this situation.
(b) What numbers of skateboards and/or dolls should the company make to maximize profit?

Before we solve this problem, note that it is a mixture problem because:

- Definite resources are available in limited quantities. The resource here is container units of plastic.
- Definite products can be made by combining (mixing) the resources. The products here are skateboards and dolls.

Solution

a) A mixture is a simple table that shows the resources, products, and profit. The chart displays the "verbal" information into a format that makes it easier to convert the problem to mathematical form (equations) that we can then solve. The rows of the mixture chart contain the products while the columns contain the resources and the profit margin. Below is a mixture chart for the problem above:

Table 5.1: **Mixture Chart for dolls and skateboards problem**

Products	Resource(s): Containers of plastic: 60	Profit (per unit)
Skateboard (x units)	5	\$ 1.00
Dolls(y units)	2	\$ 0.55

b) There are several methods of solving linear programming problems such as the one provided above. Many of the methods follow the following general steps:

- Translating the problem into a mathematical form,
- Identifying a set of possible solutions (feasible region) and,
- Identifying a solution that would give us maximum profit, i.e., the optimal b production policy.

We will consider the geometric approach first and discuss its advantages as well as limitations then present a more general method in the next chapter. After you have understood how these methods work, you will have an opportunity to use technology (e.g., Excel spreadsheets, and web applications) to solve LP problems. Note, however, that technology may not help you in translating the problem into mathematical terms. That part is done by humans (YOU).

5.3 The Geometric Method

The geometric method of solving linear programming problems involves creating a graph to visualize the **feasible region** (the set of likely solutions) and then identifying the correct solution from the feasible region. Since the inequalities involve linear functions, the feasible region is polygonal in shape. The type of polygon formed (triangle, quadrilateral, pentagon, etc) depends on the number of constraints you have.

> `A feasible set (region) for an LP problem is the collection of all physical possible solution choices that can be made.`

Let us proceed with the solution to Example 1. We will start by converting the problem into mathematical terms (inequalities and equations), creating a feasible region, and then use a technique called the **corner point principle** to choose the best solution from the feasible region.

5.3.1 Convering Mixture Chart into Mathematical Form

First, we know that we cannot manufacture negative number of objects (skateboards of dolls). So, negative numbers are not permitted in this context. Note however, that 0 is a possible number. Thus, we have two inequalities;

$$x \geq 0 \text{ and } y \geq 0$$

The symbol $>$ means grater than while $\geq$ means greater than or equal to.

We call the above two inequalities ***minimum constraints*** because they tell us the minimum that we can have for each product.

Since we have a limited supply of resources (in this case 60 units of plastic) we must also have inequalities for ***resource constraints***. Since we need 5 units of plastic to manufacture ONE skateboard, we will need $5x$ units to manufacture x units of skateboards. Similarly, we will need $2y$ units of plastic to manufacture y dolls. In total, we need $5x + 2y$ units of plastic to manufacture the dolls and skateboard. This value must not exceed 60. Thus, we have the inequality,

$$5x + 2y \leq 60$$

Notice that this time, we use the symbol for less than or equal to.

In this problem, we only have 3 inequalities but in a realistic problem, there would be hundreds or even thousands of them.

The last step in formulating the mathematical model is to make the **objective function**. This is the function that connects the profit to the resources. Since we know that each skateboard results in \$1.00 profit, we know that x dolls will result in a profit of \$1x and y dolls will result in a profit of \$0.55x. We do not know what the profit is but we know it is a function of both dolls and skateboards. We can denote the profit as P. So, we have the equation,

$$P = 1x + 0.55y$$

Notice that the objective function is an **equation** (not inequality) that gives a **specific** amount of profit as we vary the number of skateboards and dolls. In other words, P changes as we change x and y. So we can determine the value of x and$y$ that would produce maximum P.

5.3.2 Representing the Feasible Region

After creating the inequalities from the mixture chart, we can draw a picture to help us visualize the feasible region geometrically. Graphs are the most commonly used tools for visualizing the feasible region.

Notice that all the three inequalities (i.e., the minimum and resource constraints) are linear in the sense that when you graph them (assuming an equal sign) you will get a straight line. To take care of the fact that the inequalities admit a broad range of values, we,

a) Use a dotted line if the inequality is strictly less ($<$) or greater ($>$) and shade the region representing the constraints. For example, if $y > 2$, we draw a dotted line for $y = 2$ and shade the region **above** the line on the graph (this is the region that obeys the constraint).
b) Use a bold line if the inequality allows equality.

Below is a graph of the feasible region for our problem above. We have shaded the region where x values and y values are greater than 0, as well as the region where $5x + 2y$ is less than or equal to 60.

All points within the region labelled "feasible region" are possible solutions to our problem in the sense that they do not violate the constraints. For example, the feasible point $(8, 4)$ requires the company to manufacture 8 skateboards and 4 dolls. This "solution" does not violate any of the constraints given in the problem. To find the profit associated with this point, we use the objective (profit) function to compute the profit as follows:

\$\$

$$\begin{aligned} P &= x + 0.55y && (5.1)\\ &= 8 + (0.55 \times 4) && (5.2)\\ &= \$10.20 && (5.3) \end{aligned}$$

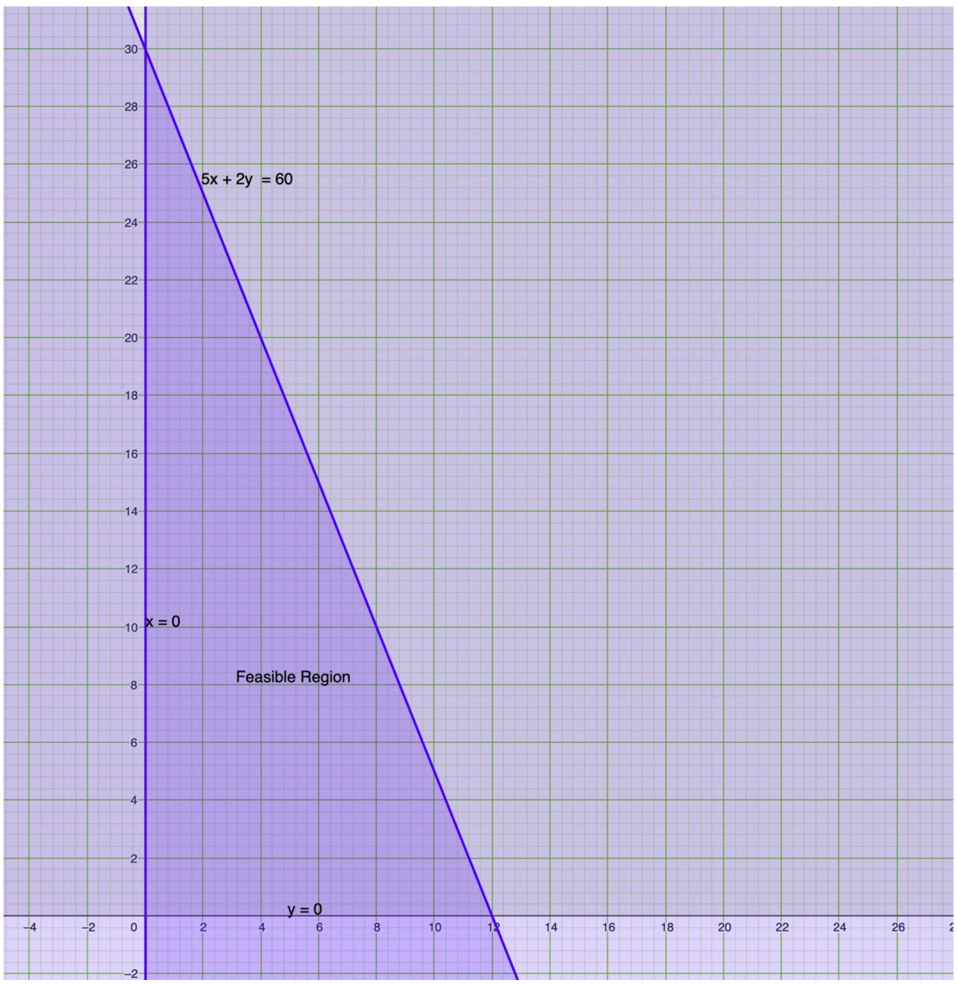


Figure 5.1: Graph of the Feasible Region

$$

Now, it is easy to show that there is a different point within the feasible region that would yield a higher profit while still obeying the constraints. Take, for example, the point $(2, 20)$ which means 2 skateboards and 20 dolls. The profit for this choice would be higher. See below:

$$

$$
\begin{aligned}
P &= 1.00x + 0.55y && (5.4)\\
&= 2 + (0.55 \times 20) && (5.5)\\
&= \$13.0 && (5.6)
\end{aligned}
$$

$$

Choosing a point in the feasible region that would result in maximum profit (optimal production policy) is not a trivial task. However, there is an genius technique known as ***The Corner Point Principle*** which we discuss next.

5.3.3 The Corner Point Principle

The corner point principle has been touted as one of the most important insights into the theory of linear programming. The principle states that,

> `In a linear programming problem, the maximum value for the profit formula always corresponds to a corner point of the feasible region.`

For our feasible region above, there are **three corners**. These corners have coordinates $(0, 0)$, $(0, 30$, and $(12, 0)$. So, we use the profit function to compute the profit associated with each of these 3 points and choose the highest as our optimal production policy. See the calculations below:

For $(0, 0)$, the profit would be $P = 1.00(0)x + 0.55(0) = \$0.00$

For $(0, 30)$, the profit would be, $P = 1.00(0)x + 0.55(30) = \$16.50$

For $(12, 0)$, the profit would be, $P = 1.00(12)x + 0.55(0) = \$12.00$

Therefore, the ***optimal production policy*** would be to manufacture 0 skateboards and 30 dolls.

NOTE: In the real world, there would be a lot more corners which would make this process cumbersome. However, as mentioned earlier, there are computer programs that can do the job faster and more efficiently than humans.

5.3.4 Summary of the Geometric Method

1. Read the problem carefully to identify resources and products.
2. Make a mixture chart for the problem.
3. Assign an unknown quantities (often x, and y) to each product and use the mixture chart to write the resource and minimum constraints.
4. Write the profit formula as well.
5. Create a feasible region by graphing ther inequalities (you can use a program such as Geogebra or Desmos).
6. Find the coordinates of the corner points and evaluate the profit for each. The corner that gives maximum profit is the optimal production policy.

In the next example, we extend the toy problem above to include one more resource (person minutes). Read below:

Example 2

A toy manufacturer can manufacture only skateboards and, only dolls, or some kind of skateboards and dolls. Skateboards require 5 units of plastic and can be sold for a profit of $ 1, while dolls require 2 units of plastic and can be sold for $0.55 profit. Only 60 units of plastic are available. Furthermore, making one skateboard requires 15 person-minutes while making one doll requires 18-person minutes. There are only 360person person-minutes available.

(a) Make a mixture chart to model this situation.
(b) What numbers of skateboards and/or dolls should the company make to maximize profit?

Solution

a) Below is the new mixture chart,

Products	Resource 1: Plastic: 60	Resource 2: person-minutes 360	Profit
Skateboard (x units)	5	15	$ 1.00
Dolls(y units)	2	18	$ 0.55

b) We start by writing down the inequalities (constraints) and the profit function. We still have the same minimum constraints as from example 1:

$$x \geq 0 \text{ and } y \geq 0$$

For the resource constraints, we have two inequalities because we have two resources: For the plastic, we have

$$5x + 2y \leq 60$$

For the person hours, we have,

$$15x + 18y \leq 360$$

The profit function stays the same:

$$P = 1x + 0.55y$$

Next, we create a feasible region by graphing the inequalities. Notice that this new feasible region is smaller than the first and it has four corner points. The fourth point is as a result of the new inequality created by the additional resource constraints. As indicated earlier, the more resources you have, the more the corner points you expect.

The last step is to use the corner principle to find the optimal production policy: We check the profits for each of the corner:

For $(0, 0)$, the profit would be \$0.

For $(0, 20$, the profit would be, $P = 1.00(0)x + 0.55(20) = \$11.00$

For $(6, 15$, the profit would be, $P = 1.00(6)x + 0.55(15) = \$14.50$

For $(12, 0$, the profit would be, $P = 1.00(12)x + 0.55(0) = \$12.00$

Thus, in this new problem, the optimal production policy is 6 skateboards and 15 dolls for a maximum profit of \$ 14.50.

5.4 Exercises

For each description in exercises 1-4, create a mixture table and write one or more resource constraint inequalities. The unknown to use for each product is given in parenthesis:

1. Manufacturing one package of hot dogs(x) requires 6 oz of beef, and manufacturing one package of bologna (y) requires 4 oz of beef. There are 300 oz of beef available.
2. It takes 30 ft of 12-in. board to make one bookcase (x); it takes 72 ft of 12-in. board to make one table(y). There are 420 ft of 12-in. board available.
3. Manufacturing one salami(x) requires 12 oz of beef and 4 oz of pork. Manufacturing one bologna (y) requires 10 oz of beef and 3 oz of pork. There are 40 lb of beef and 480 oz of pork available.

For each of the following exercises, graph the feasible region, label each line segment bounding it with appropriate equation, and give the coordinates of every corner point.

4. $x \geq 0$; $y \geq 0$; $2x + y \leq 10$
5. $x \geq 0$; $y \geq 0$; $x + 2y \leq 12$; $x + 2y \leq 8$
6. $x \geq 2$; $y \geq 6$; $3x + 2y \leq 30$

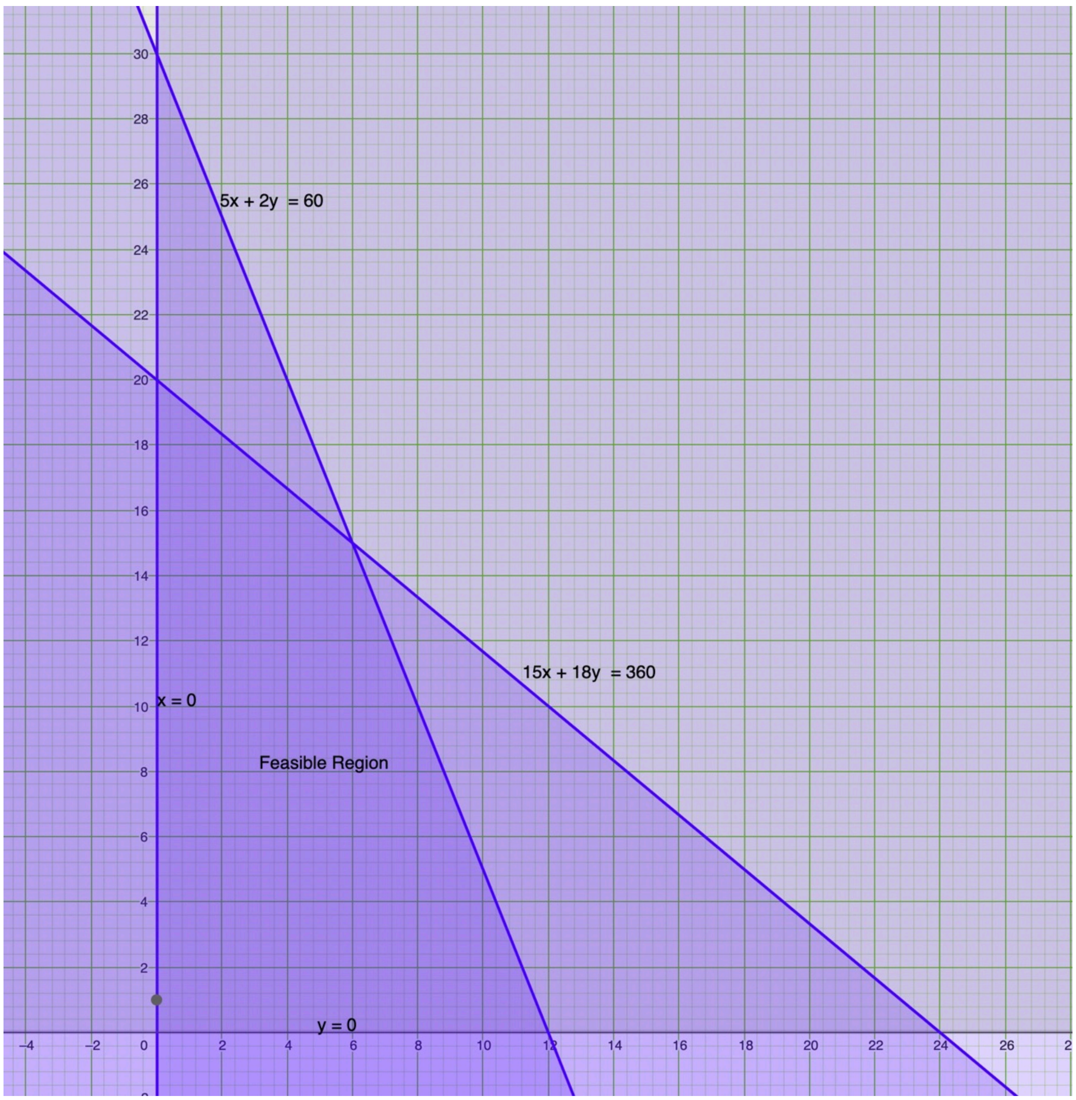


Figure 5.2: Updated Feasible Region

7. Determine whether the points $(2, 4)$ and/or $(10, 6)$ are points of the feasible region in exercises 4, 5, and 6.

8. Determine the maximum value of P given by $P = 3x + 2y$ subject to the constraints $x \geq 0$, $y \geq 0$, $x \leq 7$, and $y \leq 5$.

9. A linear programming problem has the following constraints: $x \geq 0$, $y \geq 0$, $5x - y \leq 15$, and $4y + x \leq 24$.

   a) Without graphing, determine the corner points of the feasible region for the LP problem?
   b) Sketch a graph of the feasible region.

10. Nuts Galore sells two spiced nut mixtures: Grade A and Grade B. Grade A requires 7 oz of peanuts and for every 8 oz of almonds. Grade B requires 9 oz of peanuts for every 8 oz of almonds. There are 630 oz of peanuts 640 oz of almonds available. Grade A makes Nuts Galore a profit of \$1.70, and Grade B makes a profit of \$2.40 per unit assembled. How many units of Grade A and Grade B nut mixtures should be made to maximize the company's profit, assuming that all the units can be sold?

11. Find the maximum value of P where $P = 3x + 2y$ subject to the constraints $x \geq 3$, $y \geq 2$, $x + y \leq 10$, and $2x + 3y \leq 24$.

12. A clothing manufacturer has 600 yd of cloth available to make shirts and decorated vests. Each shirt requires 3 yd of material and provides a profit of \$5. Each vest requires 2 yd of material and provides a profit of \$2. The manufacturer wants to guarantee that under all circumstances, there are minimums of 100 shirts and 30 vests produced. How many of each garment should be made to maximize the profit? If there are no minimum quantities, how, if at all, does the optimal production policy change?

13. A paper recycling company uses scrap cloth and scrap paper to make two different grades of recycled paper. A single batch of grade A recycled paper is made from 25 lb of scrap cloth and 10 lb of scrap paper, whereas one batch of grade B recycled paper is made from 10 lb of scrap cloth and 20 lb of scrap paper. The company has 100 lb of scrap cloth and 120 lb of scrap paper on hand. A batch of grade A paper brings a profit of \$500, whereas a batch of grade B paper brings a profit of \$250. What amounts of each grade should be made? How, if at all, do the maximum profit and optimal production policy change if the company is required to produce at least one batch of each type?

14. Courtesy Calls makes telephone calls for businesses and charities. A profit of \$0.50 is made for each business calls and \$0.40 for each charity call.It takes 4 min (on average) to make a business call and 6 min (on average) to make a charity call. If there are 240 min of calling time to be distributed each day, how should that time be spent so that Courtesy Calls makes a maximum profit? What changes, if any, occur in the maximum profit and optimal production policy if they must make at least 12 business calls and 10 charity call every day?

15. A factory manufactures chairs and tables, each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 39 hours; the second at most 42 hours; and the third at most 23 hours. A chair requires 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; a table needs 2 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is \$20 per unit for a chair and \$30 for a table, how many units of each should be manufactured to maximize profit?

6 The Simplex Method

6.1 Introduction to Simplex Method

As stated earlier, Linear Programming is one of the most commonly used methods for solving practical problems in business and even in government. However, the problems we have considered so far had a maximum of two products and relatively small feasible regions and few corner points. Many real life problems often involve much more products and result in very big feasible regions with hundreds or even thousands of corner points.

The simplex method provides a more general method of solving LP programs without relying on a **physical** feasible region. Consider the following LP problem through which we will discuss the details of the simplex method:

Example 1

A factory manufactures chairs, tables and bookcases each requiring the use of three operations: Cutting, Assembly, and Finishing. The first operation can be used at most 600 hours; the second at most 500 hours; and the third at most 300 hours. A chair requires 1 hour of cutting, 1 hour of assembly, and 1 hour of finishing; a table needs 1 hour of cutting, 2 hours of assembly, and 1 hour of finishing; and a bookcase requires 3 hours of cutting, 1 hour of assembly, and 1 hour of finishing. If the profit is \$20 per unit for a chair, \$30 for a table, and \$25 for a bookcase, how many units of each should be manufactured to maximize profit?

Let us start by creating a mixture chart for the problem:

Products	Resource: Cutting(600hrs)	Resource: Assembly(500hrs)	Resource: Finishing(300hrs)	Profit (\$)
Chairs	1	1	1	20
Tables	1	2	1	30
Bookcases	3	1	1	25

Next, we make the constraint inequalities assuming that the company makes x chairs, y tables, and z bookcases. Note that using x, y, and z is not advisable when you have more than 3 variables. It is recommended you use $x_1, x_2, x_3, ...$

Minimum constraints: Chairs: $x \geq 0$ Tables: $y \geq 0$ Bookcases: $z \geq 0$.

Resource constraints: For cutting: $x + y + 3z \leq 600$ For assembly: $x + 2y + z \leq 500$ For finishing: $x + y + z \leq 300$

Things to Note

- Unlike the problems encountered in the previous chapter, this problem includes 3 variables (x, y, z). This means that, we would have to create a three dimensional graph to visualize the feasible region- a task that is impossible for humans to execute by hand.
- Most application problems include way more than 3 variables, which means we would need a multidimensional graph to visualize the feasible region.
- The simplex method provides an algorithmic method that solves this type of problems without needing to create a visual feasible region. The method can also be used to solve problems in the previous chapter.
- The technical details of the simplex method are beyond the scope of the course but the algorithm itself is relatively easy to execute once you learn it.

You should try this problem on your own (see exercises) once you have understood the simplex method. For now, the step-by-step procedure of the simplex algorithm is presented.

6.2 Step-by-Step Simplex Method

We want to walk through the simplex method step-by-step until we find an optimal production policy and the maximum profit.

Consider the example below:

Example 2

Maximize the function $P = 3x + 4y + z$ subject to the conditions

$$3x + 10y + 5z \leq 120$$
$$5x + 2y + 8z \leq 6$$
$$8x + 10y + 3z \leq 105$$

Problem Posing: Note that this problem does not have a real world context, before you proceed, please write a real life LP problem that might suit the given profit function and constraints.

Step 1: Convert the inequalities to equations by adding **slack variables** and rewrite the profit function so that the constant value is on the right hand side of the equation:

$$3x + 10y + 5z + s_1 = 120$$
$$5x + 2y + 8z + s_2 = 6$$

$$8x + 10y + 3z + s_3 = 105$$

$$-3x - 4y - z + P$$

Slacks allow us to convert inequalities to equations by filling up the amount by which a quantity falls short of another. For example, s_1 is the amount by which the quantity $3x+10y+5z$ falls short of 120. So, by adding s_1 to $3x+10y+5z$ we achieve the equality. sh

Step 2: Create the initial **simplex tableau**. In the tableu, each equation appears in its own row with the profit function appearing as the last row. See table below:

x	y	z	s_1	s_2	s_3	P	
3	10	5	1	0	0	0	120
5	2	8	0	1	0	0	6
8	10	3	0	0	1	0	105
−3	−4	−1	0	0	0	1	0

Figure 6.1: Initial Tableau

Step 3: Choose the pivot column. The most negative entry in the last row is -4 which is in column 2. So, column 2 is our pivot column. See below:

x	y	z	s_1	s_2	s_3	P	
3	10	5	1	0	0	0	120
5	2	8	0	1	0	0	6
8	10	3	0	0	1	0	105
−3	−4	−1	0	0	0	1	0

Figure 6.2: Tableau showing Pivot column

Why do we choose the most negative element in the last row?

The most negative entry in the bottom row represents the largest coefficient in the objective function - the coefficient whose entry will increase the value of the objective function the quickest. Remember this is a maximization problem.

Step 4: Find the pivot row. To find the pivot row, divide the values on the far right by values of the pivot column. The row with the smallest quotient will be your pivot row. In this case,

$$120/10 = 12$$

$$6/2 = 3$$

$$105/110 = 10.5$$

The smallest quotient here is 3, which means the pivot row is row 2. The matrix below highlight the picot row:

$$\left[\begin{array}{cccccccc} x & y & z & s_1 & s_2 & s_3 & P & \\ \hline 3 & 10 & 5 & 1 & 0 & 0 & 0 & 120 \\ 5 & 2 & 8 & 0 & 1 & 0 & 0 & 6 \\ 8 & 10 & 3 & 0 & 0 & 1 & 0 & 105 \\ \hline -3 & -4 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Figure 6.3: Pivot row and column

Why does the smallest quotient identify a row? Using the quotients to identify the pivot element guarantees that we do not violate the constraints as we proceed with the algorithm.

Step 5: The 2 at the intersection of row 2 and column 2 is called a pivot element. We want to perform row operations to make it a 1. To do this, we simply divide the whole of row 2 by 2. This can be represented as $\frac{1}{2}R_2 \mapsto R_2$. This means that the new row 2 will be half of the previous row 2.

$\frac{1}{2} \times (5, 2, 8, 0, 1, 0, 0, 6) = (2.5, 1, 4, 0, .5, 0, 0, 3)$

The new tableau becomes,

$$\left[\begin{array}{cccccccc} x & y & z & s_1 & s_2 & s_3 & P & \\ \hline 3 & 10 & 5 & 1 & 0 & 0 & 0 & 120 \\ 2.5 & 1 & 4 & 0 & .5 & 0 & 0 & 3 \\ 8 & 10 & 3 & 0 & 0 & 1 & 0 & 105 \\ \hline -3 & -4 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Figure 6.4: Unitize pivot element

Step 6: We perform row operations to convert every entry above and below the new pivot element (1) into a 0. The following operations will achieve this:

- The new row 1 will be the difference between row and and 10 times row 2 i.e., $R_1 - 10R_2 \mapsto R_1$.
- The new row 3 will be the difference between the current row 3 and 10 times row 2.e., $R_3 - 10R_2 \mapsto R_3$.
- The new row 4 will be the current row 4 plus 4 times row 2 i.e., $R_4 + 4R_2 \mapsto R_4$.

We perform the actual computations below:

$R_1 - 10R_2 \mapsto R_1$: $(3, 10, 5, 1, 0, 0, 0, 120) - 10(2.5, 1, 4, 0, 0.5, 0, 0, 3) = (-22, 0, -35, 1, -5, 0, 0, 90)$.

$R_3 - 10R_2 \mapsto R_3$: $(8, 10, 3, 0, 0, 1, 0, 105) - 10(2.5, 1, 4, 0, 0.5, 0, 0, 3) = (17, 0, -37, 0, -5, 1, 0, 75)$.

$R_4 + 4R_2 \mapsto R_4$: $(-3, -4, -1, 0, 0, 0, 1, 0) + 4(2.5, 1, 4, 0, 0.5, 0, 0, 3) = (7, 0, 15, 0, 2, 0, 1, 12)$.

Now we put these new rows into our tableau. See below:

$$\left[\begin{array}{cccccccc}
x & y & z & s_1 & s_2 & s_3 & P & \\
\hline
-22 & 0 & -35 & 1 & -5 & 0 & 0 & 90 \\
2.5 & 1 & 4 & 0 & .5 & 0 & 0 & 3 \\
-17 & 0 & -37 & 0 & -5 & 1 & 0 & 75 \\
\hline
7 & 0 & 15 & 0 & 2 & 0 & 1 & 12
\end{array}\right]$$

Figure 6.5: One more Tableau

Notice that the last row has no negative numbers. This means we are done and we can directly read our solution. If there were any negative numbers left on the last row, you would have to do the process one more time (i.e., find new pivot column and row then perform subsequent row operations). This process continues until there are no negative numbers on the last row.

Step 7: Read the solution from the final tableau. Every column with a "1's" and "0's" would give us a value for a variable. In our case above,

$y = 3, s_1 = 90, s_3 = 75, P = 12$

All the others (i.e., (x, z, s_2)) will be zero.

This solution basically means, make 3 units of product y, and zero product x and z.

6.2.1 Summary of the Simplex Method

Below is a summary of the simplex method for maximization problems in LP:

a) Set up the problem. That is, convert the problem into mathematical terms. This involves creating the constraint inequalities and the objective function.
b) Convert the inequalities into equations. This is done by adding one slack variable for each inequality.
c) Construct the initial simplex tableau with the objective function as the bottom row. d)Identify the pivot column. The most negative entry in the bottom row identifies the pivot column.
d) Calculate the quotients. The smallest quotient identifies a row. The element in the intersection of the column identified in step 4 and the row identified in this step is identified as the pivot element. The quotients are computed by dividing the far right column by the identified column in step 4. A quotient that is a zero, or a negative number, or that has a zero in the denominator, is ignored.
e) Perform pivoting to make all other entries in this column zero.
f) When there are no more negative entries in the bottom row, we are finished; otherwise, we start again from step (d).