Potentiels à 1 dimension en escalier

Cohen H1 p68

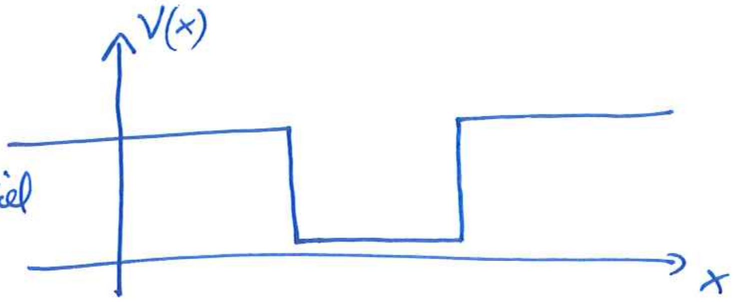
II.11

$$i\hbar\,\partial_t \psi(t,x) = -\frac{\hbar^2}{2m}\partial_x^2 \psi(t,x) + V(x)\,\psi(t,x)$$

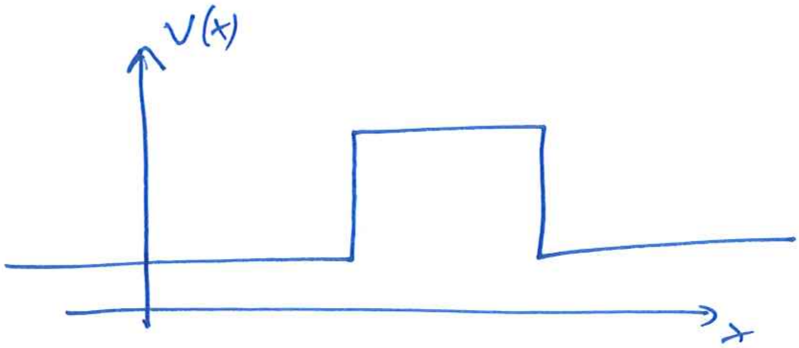
avec $V(x)$ constant par morceau.

exemples

puit de potentiel



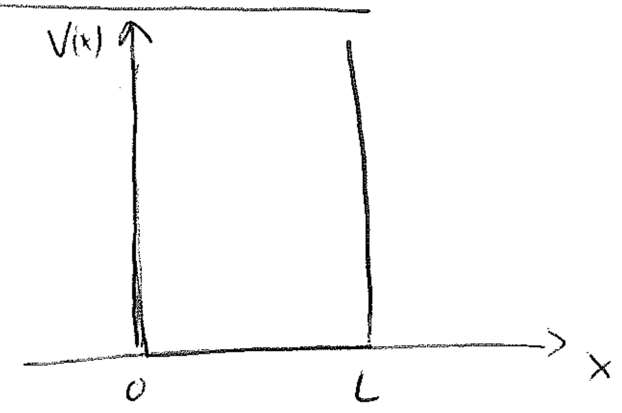
barrière de potentiel



Pourquoi étudier ces problèmes ?

- systèmes simples : solutions analytiques, ou presque.
- illustrant des effets quantiques importants :
  - états liés
  - effet tunnel
- certaines situations physique sont très proches :
  - barrière Josephson (supraconducteurs)
  - points quantiques.

Puit de Potentiel infini à 1 dimension



$$V(x) \begin{cases} = +\infty \text{ si } x < 0 \\ = 0 \text{ si } 0 < x < L \\ = +\infty \text{ si } x > L \end{cases}$$

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2m}\partial_x^2 \psi + V(x)\psi$$

1°) séparation de variables $\psi(x,t) = \chi(t)\,\varphi(x)$

$$\longrightarrow i\hbar \frac{1}{\chi}\partial_t \chi = \frac{1}{\varphi}\left(-\frac{\hbar^2}{2m}\partial_x^2 \varphi\right) + V(x) = E$$

$$\rightarrow \chi = \chi_0 e^{-i\frac{E}{\hbar}t}$$

$$\rightarrow -\frac{\hbar^2}{2m}\partial_x^2\varphi + V(x)\varphi = E\varphi$$

cdts. au bord $\varphi(0) = \varphi(L) = 0$

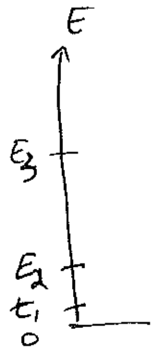
pour $0 < x < L$ $\quad -\frac{\hbar^2}{2m}\partial_x^2\varphi = E\varphi$ $\qquad k = \frac{\sqrt{2mE}}{\hbar} \rightarrow -\partial_x^2\varphi = k^2\varphi$

$$\longrightarrow \varphi = \alpha \sin(kx) + \beta\cos(kx).$$

cdt. au bord $\beta = 0$

$kL = n\pi$

$n = 1, 2, \dots$ $\qquad \boxed{k_n = \frac{n\pi}{L}} \qquad \boxed{E_n = \frac{\hbar^2}{2m}\frac{\pi^2 n^2}{L^2}}$



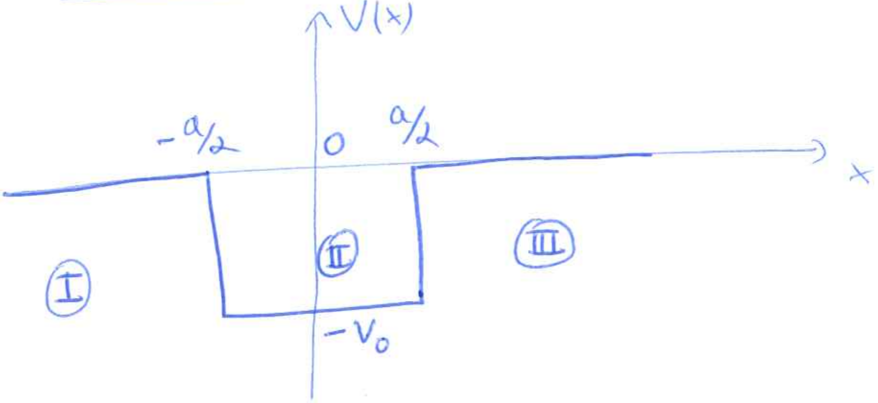
$$\psi_n(x,t) = A\,e^{-i\frac{E_n t}{\hbar}} \sin\left(\frac{n\pi}{L}x\right)$$

$\longrightarrow$ quantification de l'énergie

Solution générale $\psi(x,t) = \sum_{n=1}^{\infty} A_n e^{-i\frac{E_n t}{\hbar}} \sin\left(\frac{n\pi x}{L}\right)$

Puit de potentiel fini

C. p 76
complement H_I



$$-\frac{\hbar^2}{2m}\partial_x^2 \varphi + V(x)\varphi = E\varphi \qquad \underline{-V_0 < E < 0}$$

en I : $\varphi(x) = B_1 e^{\rho x}$ $\qquad \boxed{\rho = \sqrt{\frac{-2mE}{\hbar^2}}}$

en III : $\varphi(x) = B_3' \, e^{-\rho x}$

en II : $\varphi(x) = A_2 e^{ikx} + A_2' e^{-ikx}$ $\qquad \boxed{k = \sqrt{\frac{2m(E+V_0)}{\hbar^2}}}$

conditions de raccord : $\varphi(x)$ continue et $\partial_x\varphi(x)$ continue

en $x = -a/2$

$$① \begin{cases} B_1 e^{-\rho a/2} = A_2 e^{-ika/2} + A_2' e^{ika/2} \\ \rho B_1 e^{-\rho a/2} = ik\left(A_2 e^{-ika/2} - A_2' e^{ika/2}\right) \end{cases}$$

en $x = a/2$

$$② \begin{cases} B_3' e^{-\rho a/2} = A_2 e^{ika/2} + A_2' e^{-ika/2} \\ -\rho B_3' e^{-\rho a/2} = ik\left(A_2 e^{ika/2} - A_2' e^{-ika/2}\right) \end{cases}$$

(4 équations linéaires à 4 inconnues $\longrightarrow$ solution non triviale si det(Matrice) = 0)

$$
① \Rightarrow \left\{ \begin{array}{l} A_2 = \dfrac{\rho + ik}{2ik} e^{(-\rho + ik)a/2} B_1 \\ A'_2 = -\dfrac{\rho - ik}{2ik} e^{-(\rho + ik)a/2} B_1 \end{array} \right\} ③
$$

$$
② \Rightarrow \quad A_2 (\rho + ik) e^{ika/2} + A'_2 (\rho - ik) e^{-ika/2} = 0
$$

$$
③ \Rightarrow \quad \frac{B_1}{2ik} \left[(\rho + ik)^2 e^{ika} - (\rho - ik)^2 e^{-ika} \right] = 0
$$

$$
B_1 \neq 0 \Rightarrow \left(\frac{\rho - ik}{\rho + ik} \right)^2 = e^{2ika}
$$

$$
\Rightarrow \frac{\rho - ik}{\rho + ik} = - e^{ika}
$$

↳ # complexe de phase $\pi + ka$

complexe de norme 1 de phase $-2 \, tg^{-1}\left(\frac{k}{\rho}\right)$

$$
\longrightarrow tg^{-1}\left(\frac{k}{\rho}\right) = -\frac{\pi}{2} + \frac{ka}{2}
$$

$$
\frac{k}{\rho} = tg\left(-\frac{\pi}{2} + \frac{ka}{2}\right) = \frac{1}{tg \, ka/2}
$$

ou bien $\dfrac{\rho - ik}{\rho + ik} = e^{ika}$

$$
\hookrightarrow tg^{-1}\left(\frac{k}{\rho}\right) = -\frac{ka}{2}
$$

$$
\hookrightarrow tg\left(\frac{ka}{2}\right) = -\frac{k}{\rho}
$$

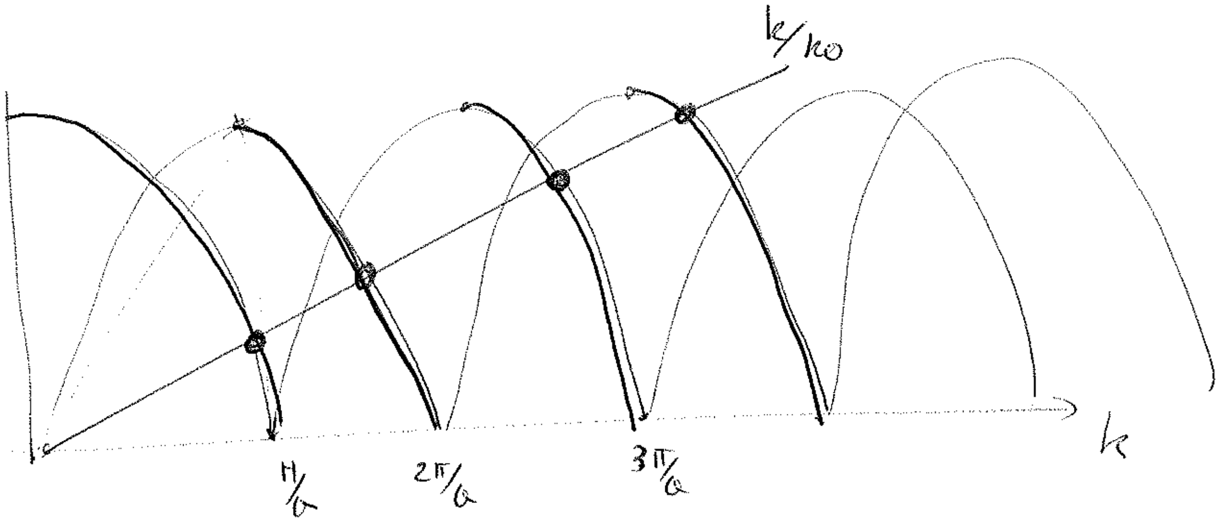
II.15 $\frac{5}{16}$

$$k_0 = \sqrt{\frac{2mV_0}{\hbar^2}} = \sqrt{k^2+\rho^2} \quad \text{indép. de } E$$

$$\frac{1}{\cos^2(ka/2)} = 1 + \mathrm{tg}^2\frac{ka}{2} = \frac{k^2+\rho^2}{k^2} = \frac{k_0^2}{k^2}$$

1°) $\Leftrightarrow \begin{cases} \left|\cos\frac{ka}{2}\right| = \frac{k}{k_0} \\ \mathrm{tg}\left(\frac{ka}{2}\right) > 0 \end{cases}$

2°) $\Leftrightarrow \begin{cases} \left|\sin\frac{ka}{2}\right| = \frac{k}{k_0} \\ \mathrm{tg}\,\frac{ka}{2} < 0 \end{cases}$



si $k_0 \leq \pi/a$

$\Leftrightarrow V_0 \leq V_1 = \frac{\pi^2\hbar^2}{2ma^2} \longrightarrow$ 1 état lié.

$\frac{\pi}{a} \leq k_0 \leq 2\pi/a \qquad \frac{\pi^2\hbar^2}{2ma^2} \leq V_0 \leq 4\frac{\pi^2\hbar^2}{2ma^2}$ 2 états liés.

etc...

si $V_0 \gg V_1$ et les états d'énergie les + bas sont

$\simeq k = n\pi/a \qquad E \simeq \frac{\pi^2\hbar^2 n^2}{2ma^2} - V_0$

Potentiel en escalier

$$\frac{d^2\varphi}{dx^2} + \frac{2m}{\hbar^2}(E-V)\varphi = 0$$

1°) $E > V \longrightarrow \varphi = A e^{ikx} + A' e^{-ikx}$ $k^2 = \frac{2m}{\hbar^2}(E-V) \geq 0$

ondes progressives

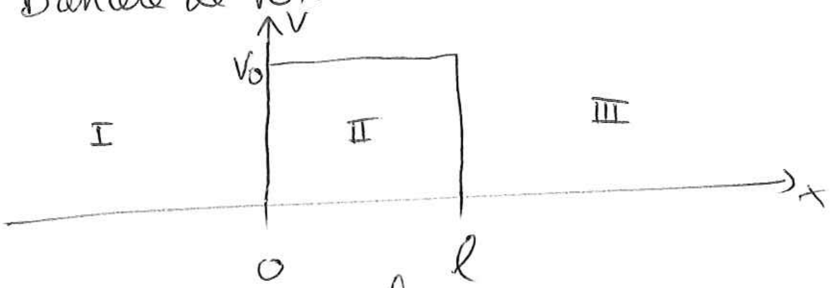
2°) $E < V \longrightarrow \varphi = B e^{\rho x} + B' e^{-\rho x}$ $\rho^2 = \frac{2m}{\hbar^2}(V-E) \geq 0$

effet tunnel.

3°) là ou V est discontinu : φ est continu

$\partial_x \varphi$ est continu

Barrière de Potentiel



Onde incidente de la gauche

1°) $E > V_0$

$$\begin{cases} \varphi_I = A_1 e^{ik_1x} + A_1' e^{-ik_1x} \\ \varphi_{II} = A_2 e^{ik_2x} + A_2' e^{-ik_2x} \\ \varphi_{III} = A_3 e^{ik_3x} \end{cases}$$

$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$

$k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

Courant en I : $J = \frac{\hbar}{m} \mathrm{Im}\left[\bar{\varphi}\, \partial_x \varphi\right]$

$$= \frac{\hbar}{m} \mathrm{Im}\left[(\bar{A}_1 e^{-ik_1x} + \bar{A}_1' e^{ik_1x})\, ik_1 (A_1 e^{ik_1x} - A_1' e^{-ik_1x})\right]$$

$$= \frac{\hbar}{m} k_1 \left(|A_1|^2 - |A_1'|^2 + \text{terme oscillant}\right)$$

$$\simeq v_1 \left(|A_1|^2 - |A_1'|^2\right)$$

Conditions de raccord

en $x=0$: $\begin{cases} A_1 + A'_1 = A_2 + A'_2 \\ ik_1(A_1 - A'_1) = ik_2(A_2 - A'_2) \end{cases}$

en $x=\ell$: $\begin{cases} A_2 e^{ik_2\ell} + A'_2 e^{-ik_2\ell} = A_3 e^{ik_1\ell} \\ ik_2(A_2 e^{ik_2\ell} - A'_2 e^{-ik_2\ell}) = ik_3 A_3 e^{ik_1\ell} \end{cases}$

$$\longrightarrow \begin{cases} A_2 = \frac{1}{2}\left(1+\frac{k_3}{k_2}\right) e^{i(k_1-k_2)\ell} \\ A'_2 = \frac{1}{2}\left(1-\frac{k_3}{k_2}\right) e^{i(k_1+k_2)\ell} \end{cases}$$

$$\begin{cases} A_1 = \frac{1}{2}\left(1+\frac{k_2}{k_1}\right) A_2 + \frac{1}{2}\left(1-\frac{k_2}{k_1}\right) A'_2 \\ A'_1 = \frac{1}{2}\left(1-\frac{k_2}{k_1}\right) A_2 + \frac{1}{2}\left(1+\frac{k_2}{k_1}\right) A'_2 \end{cases}$$

$$\longrightarrow \begin{cases} A_1 = \left[\cos(k_2\ell) - i\,\frac{k_1^2+k_2^2}{2k_1k_2}\sin(k_2\ell)\right] e^{ik_1\ell} A_3 \\ A'_1 = i\,\frac{k_2^2-k_1^2}{2k_1k_2}\sin(k_2\ell)\, e^{ik_1\ell} A_3 \end{cases}$$

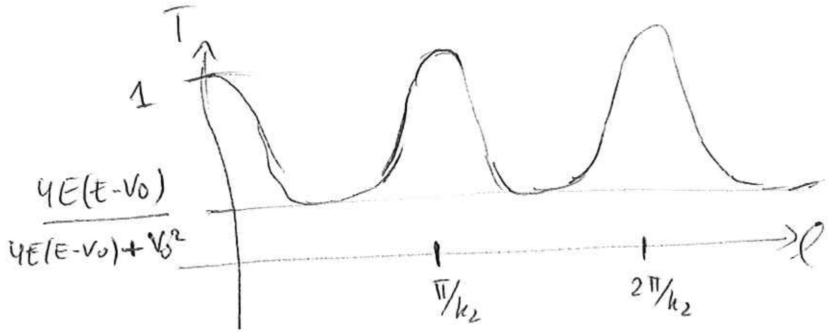
coefficient de réflexion

$$R = \frac{|A'_1|^2}{|A_1|^2} = \frac{(k_1^2-k_2^2)\sin^2 k_2\ell}{4k_1^2k_2^2 + (k_1^2-k_2^2)^2\sin^2 k_2\ell}$$

$$T = \left|\frac{A_3}{A_1}\right|^2 = \frac{4k_1^2k_2^2}{4k_1^2k_2^2 + (k_1^2-k_2^2)^2\sin^2 k_2\ell}$$

$$R+T=1$$

$$T = \frac{4E(E-V_0)}{4E(E-V_0) + V_0^2\sin^2\left(\sqrt{2m(E-V_0)}\,\ell/\hbar\right)}$$



résonances

2°) $E < V_0$ Effet Tunnel

$$k_2 \rightarrow -i\rho_2$$

$$\rho_2 = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$$

$$\sin k_2 x = \frac{e^{ik_2x} - e^{-ik_2x}}{2i}$$

$$\downarrow$$

$$\frac{1}{i}\, sh(\rho_2 x) = \frac{e^{\rho_2 x} - e^{-\rho_2 x}}{2i}$$

$$T = \frac{-4k_1^2\rho_2^2}{-4k_1^2\rho_2^2 + (k_1^2+\rho_2^2)(-)\, sh^2(\rho_2 \ell)}$$

$$T = \frac{4k_1^2\rho_2^2}{4k_1^2\rho_2^2 + (k_1^2+\rho_2^2)\, sh^2(\rho_2 \ell)}$$

$$T = \frac{4E(V_0-E)}{4E(V_0-E) + V_0^2\, sh^2(\rho_2 \ell)}$$

$$T \simeq \frac{16E(V_0-E)}{V_0^2}\, e^{-2\rho_2 \ell}$$