Quick and Dirty Introduction to Intersection Theory and Chern classes

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1. Introduction.

Although intersection theory has gained a wide variety of applications, solving enumerative problems remains an important motivation for intersection theory (and certainly for the schubert package). As examples, consider the following problems:

(i) The number of points common to two plane curves of degrees m and n. (Answer: mn, provided the two curves are sufficiently general.)
(ii) The number of lines in $\mathbf{P}^3$ intersecting 4 given lines in general position (2).
(iii) The number of lines on a general cubic surface (27).
(iv) The number of conics tangent to five given conics in general position (3264).
(v) The number of lines (resp. conics) on a general quintic hypersurface in $\mathbf{P}^4$ (2875 resp. 609250).

Here is a rough idea of how to approach (ii), for example. Let $G(1,3)$ be the Grassmannian of lines in $\mathbf{P}^3$, and for a given line L, let $\Sigma(L) \subseteq G(1,3)$ be the set of lines meeting L. Then our problem is to determine the cardinality of $\bigcap_{i=1}^4 \Sigma(L_i)$ for four given lines L_i.

Intersection theory can be said to consist of turning intersections into multiplication in a certain ring, thus transforming a geometric problem like the above into the study of the algebraic properties of this ring. We want to describe the ring in terms of generators and relations, and a means of translating the original intersection problem into an algebraic computation in the intersection ring. Here is where Chern classes play their part.

Throughout this note, all varieties are nonsingular and projective over the ground field $\mathbf{C}$ of complex numbers.

The primary reference for intersection theory is W. Fulton's book [1]. Chapters 19, 14, and 3 are closest to the material presented here.

2. The numerical intersection ring.

Let X be a variety of dimension d. There are several rings one can associate to X and reasonably call an intersection ring [1, ch. 19]. The one described here is

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the coarsest possible that can be used for enumerative problems, and possibly the easiest to define: the numerical equivalence ring with rational coefficients.

Let $Z(X)$ be the free $\mathbf{Q}$-vector space generated by all irreducible subvarieties $V \subseteq X$. This has the structure of a graded group if we let a generator V have degree $\operatorname{codim}(V, X)$. Thus $Z(X) = \oplus_{k=0}^{d} Z^k(X)$ (we put $Z^n(X) = 0$ for $n < 0$ and for $n > \dim(X)$). An element of Z^{d-k} is called a k-cycle. There is a partially defined bilinear map

$$m\colon Z(X) \times Z(X) \to Z(X) \tag{1}$$

as follows: First of all, by bilinearity it suffices to define $m(V, W)$ where V and W are irreducible subvarieties (*prime cycles*). Suppose that $V \cap W = \bigcup_{i=1}^{n} R_i$ is a transversal intersection (i.e., transversal in the generic point of each irreducible component R_i of the intersection). Then we say the $m(V, W)$ is defined and equals $\sum_i R_i$. If $V \in Z^p(X)$ and $W \in Z^q(X)$, the intersection $V \cap W$ has pure codimension $p+q$, hence m is a *graded* bilinear (partially defined) map. Notation: we usually write $\alpha \cdot \beta$ instead of $m(\alpha, \beta)$.

There is a map

$$\int_X : Z(X) \to \mathbf{Q} \tag{2}$$

given by $\int_X P = 1$ for any point P, and 0 on all prime cycles of positive dimension. Define an equivalence relation $\equiv$ on $Z(X)$ by saying that

$$\alpha \equiv \beta \text{ if and only if } \int_X \alpha \cdot \gamma = \int_X \beta \cdot \gamma$$

for all $\gamma \in Z(X)$ for which both products are defined. This generates an equivalence relation known as *numerical equivalence*. An important fact which lies at the heart of enumerative geometry, is that any two cycles which belong to an algebraic family over a connected base are numerically equivalent.

Let $\operatorname{Num}(X) = \oplus \operatorname{Num}^k(X)$ consist of all cycles numerically equivalent to zero, and put $N(X) = Z(X)/\operatorname{Num}(X)$. If V is a prime cycle, we denote its class in $N(X)$ by $[V]$.

Then one may prove that

PROPOSITION 1. *The partially defined map m in (1) descends to an everywhere defined bilinear map*

$$m\colon N(X) \times N(X) \to N(X). \tag{3}$$

Under this multiplication, $N(X)$ is a commutative and associative graded ring with identity element $1 = [X]$. It is finitely generated as a $\mathbf{Q}$-algebra. Furthermore,

(i) *$N^0(X) = \mathbf{Q}$, generated by $[X]$ (which is the ring identity),*
(ii) *$N^d(X) = \mathbf{Q}$, generated by the class $[P]$ of a point,*
(iii) *the pairings $N^p(X) \times N^{d-p}(X) \to N^d(X) = \mathbf{Q}$ are non-degenerate.*

In fancy terminology, $N(X)$ is a Gorenstein 0-dimensional graded $\mathbf{Q}$-algebra with socle in degree $d = \dim(X)$.

Example. Consider $X = \mathbf{P}^n$. Bezout's theorem states that if V and W are prime cycles of complementary dimension and they intersect transversally, then the number of intersection points is $\deg(V) \cdot \deg(W)$. It follows immediately that any two prime cycles of the same degree are numerically equivalent, and vice versa. Therefore $N^k(\mathbf{P}^n) \simeq \mathbf{Q}$, generated by the class of an $(n-k)$-dimensional hyperplane. The ring structure is $N(X) \simeq \mathbf{Q}[h]/h^{n+1}$, where $h \in N^1(X)$ is the class of a hyperplane.

We may thus interpret Bezout's theorem as a description of $N(\mathbf{P}^n)$, similarly we may view intersection theory as a generalization of Bezout's theorem.

Exercise. Let $x_1, \ldots, x_n$ be homogenous algebra generators of $N(X)$, and use them to write $N(X)$ as a quotient of a polynomial algebra $R = \mathbf{Q}[X_1, \ldots, X_n]$. Show that the structure of $N(X)$ can be reconstructed from the functional

$$v \colon R_d \xrightarrow{\text{canonical}} N^d(X) \xrightarrow{\int_X} \mathbf{Q}$$

on the vector space of forms of (weighted) degree d in R. (This is essentially how $N(X)$ is represented in the schubert package.)

Connection with singular cohomology. There are so-called cycle maps

$$\mathrm{cl}_X \colon Z^k(X) \to H^{2k}(X, \mathbf{Q})$$

taking a subvariety V to the Poincaré dual of the homology class of V in X. Under these maps, $\mathrm{cl}_X(\alpha \cdot \beta) = \mathrm{cl}_X(\alpha) \cup \mathrm{cl}_X(\beta)$. Let $\mathrm{Hom}(X) = \{\alpha \in Z(X) \mid \mathrm{cl}_X(\alpha) = 0\}$, these cycles are called homologically equivalent to zero. By the compatibility between intersections and cup products, it follows that numerical equivalence is coarser than homological equivalence, i.e., $\mathrm{Hom}(X) \subseteq \mathrm{Num}(X)$. Whether they are equal in general is an open problem; they are if the Hodge conjecture holds for X. In any case, this shows that $N(X)$ is a finite vector space over $\mathbf{Q}$, since it is a quotient of $\mathrm{Im}(\mathrm{cl}_X)$ which in turn sits inside $H^*(X, \mathbf{Q})$, which has finite dimension since X is compact.

Functoriality. To a morphism $f \colon X \to Y$ of nonsingular projective varieties there are associated maps

$$f^* \colon N(Y) \to N(X) \quad \text{and} \quad f_* \colon N(X) \to N(Y). \tag{4}$$

The first of these, the pullback f^*, is a homomorphism of graded $\mathbf{Q}$-algebras, in particular, it makes $N(X)$ an $N(Y)$-module. With this module structure, the pushforward f_* is a morphism of graded $N(Y)$-modules which lowers degrees by

$\dim(X) - \dim(Y)$. In more concrete terms, this implies that f^* preserves codimension, f_* preserves dimension, and they are related by the projection formula

$$f_*(x \cdot f^*(y)) = y \cdot f_*(x) \quad \text{for } x \in N(X) \text{ and } y \in N(Y), \tag{5}$$

this is exactly the $N(Y)$-linearity of f_*. The pullback and pushforward maps are compatible with compositions of morphisms, making N both a covariant and a contravariant functor.

In the special case where Y is a point, we may identify $N(Y)$ with $\mathbf{Q}$ via $\int_Y$. Under this identification, $f_*(x) = \int_X x$. More generally, composing f with the structure map $Y \to \mathrm{Spec}(\mathbf{C})$ and using the functoriality property of pushforward maps, we arrive at

$$\int_X x = \int_Y f_* x \qquad \text{for all } x \in N(X). \tag{6}$$

To define pullback and pushforward maps, consider the partially defined map $f^* \colon Z(Y) \to Z(X)$ and the everywhere defined map $f_* \colon Z(X) \to Z(Y)$ given on prime cycles by

$$\begin{aligned} f^*(W) &= f^{-1}(W) && \text{if } W \text{ is transverse to } f, \text{ and} \\ f_*(V) &= \begin{cases} [k(V) : k(f(V))]V & \text{if } \dim(f(V)) = \dim(V) \\ 0 & \text{if } \dim(f(V)) < \dim(V). \end{cases} \end{aligned}$$

Here $[k(V) : k(f(V))]$ is the degree of $f|V \colon V \to f(V)$. One can show that these maps preserve numerical equivalence and descend to the maps f^* and f_* as above.

Exercise. It is instructive to try to verify the projection formula on the cycle level, or even the underlying set-theoretical level.

Exercise. Let $\{\alpha_i\}$ and $\{\beta_i\}$ be homogenous $\mathbf{Q}$-bases for $N(Y)$, dual with respect to the bilinear form $(y, z) \mapsto \int_Y y \cdot z$. Using (5) and (6), show that

$$f_*(x) = \sum_i \left(\int_X x \cdot f^* \alpha_i \right) \beta_i \qquad (x \in N(X)) \tag{7}$$

This is how the pushforward map (for a general morphism) is computed in the schubert package.

3. Chern classes and relations.

To a vector bundle E (= locally free sheaf) on X there is associated a sequence of *Chern classes* $c_i(E) \in N^i(X)$, for $i = 0, \ldots, \dim(X)$. The sum

$$c(E) = \sum_{i=0}^{\dim(X)} c_i(E)$$

is called the *total Chern class* of E. If t is an indeterminate,

$$c_t(E) = \sum_{i=0}^{\dim(X)} c_i(E)\, t^i$$

is called the *Chern polynomial* of E. There are several ways to go about defining these classes, but here we will just list some of their properties.

(i) $c_0(E) = 1$ for any E.
(ii) If $0 \to E' \to E \to E'' \to 0$ is exact, then $c(E) = c(E') \cdot c(E'')$ (Whitney formula.)
(iii) If $f\colon Y \to X$ is a morphism, then $f^*(c(E)) = c(f^*E)$.
(iv) If $E = \mathcal{O}_X(D)$ for some divisor D, then $c(E) = 1 + [D]$.
(v) $c_i(E) = 0$ for $i > \operatorname{rank}(E)$.

By (i) and (ii), the total Chern class is a homomorphism from the Grothendieck group $K(X)$ of locally free sheaves on X to the multiplicative group of the ring $N(X)$. Since X is nonsingular, any coherent sheaf on X gives an element of $K(X)$, hence we may define Chern classes for general coherent sheaves. However, properties (iii) and (v) do not necessarily hold if E is not locally free.
Remark. These Chern classes are compatible with the topological Chern classes in $H^*(X, \mathbf{Q})$.

Projective bundles and Segre classes. For a vector bundle E on X of rank r, the Chern classes of E are closely related to the structure of $N(\mathbf{P}(E))$ as an $N(X)$-algebra. We use Grothendieck's $\mathbf{P}$ here: its points are rank-1 quotient bundles of E. Let $\pi\colon P = \mathbf{P}(E) \to X$ be the projection, denote by $\mathcal{O}_P(1)$ the tautological line bundle, and let $\tau \in N^1(P)$ be its first Chern class.

From the exact sequence $0 \to R \to \pi^*E \to \mathcal{O}_P(1) \to 0$ and the Whitney formula we get $c(R) = c(\pi^*E)/(1+\tau)$. Expanding this and using that $c_r(R) = 0$, we get this relation:

$$0 = \sum_{i=0}^{r} (-1)^i \tau^i c_{r-i}(\pi^*E), \text{ or } \tau^r = \tau^{r-1}c_1 - \tau^{r-2}c_2 + \cdots + (-1)^{r-1}c_r, \qquad (8)$$

where $c_i = \pi^*(c_i(E))$. It can be shown that τ generates $N(P)$ as an $N(X)$-algebra, and that the ideal of relations is generated by (8). In particular, $N(P)$ is a free $N(X)$-module of rank r, and π^* is injective.

If X is a point, E is just a vector space, and we recover the intersection ring of projective space.

In general, (8) can be used to determine the pushforward $\pi_*\tau^j$. For dimension reasons, these are zero for $j < r-1$. Suppose for simplicity that $\mathcal{O}_P(1)$ has a global section; then on the general fiber of π, this section vanishes on a hyperplane in $\mathbf{P}^{r-1}$. Therefore τ^{r-1} induces the class of a point on each fiber. It follows that $\pi_*\tau^{r-1} = 1$.

The *Segre classes* of E are by definition the classes

$$s_j(E) = \pi_*\tau^{r-1+j} \in N^j(X).$$

Multiplying (8) by τ^{j-1}, taking π_* and using the projection formula, we get recursion relations between the Chern and Segre classes which are equivalent to the relation

$$s(E) = c(E^\vee)^{-1}.$$

Splitting principle. If a vector bundle E happens to be a direct sum of line bundles L_i, the total Chern class has the form

$$c(E) = \prod_{i=1}^{r}(1+\gamma_i), \tag{9}$$

where $\gamma_i = c_1(L_i)$. In fact, the same factorization holds if instead of being a direct sum, E admits a filtration of subbundles with linebundle quotients L_i. Note that the pull back of E to $\mathbf{P}(E)$ admits a linebundle quotient. Repeating the construction with the kernel R and so on (i.e., taking $\mathbf{P}(R)$ etc.), we may construct a morphism $f\colon F \to X$ with the property that $f^*\colon N(X) \to N(F)$ is injective, and f^*E is a succesive extension of linebundles, hence there is a factorization (9). The Chern classes of E are the elementary symmetric functions in the γ_i. The γ_i are called the *Chern roots* of E. The *splitting principle* is: When you want to prove an identity concerning Chern classes of bundles, you may without loss of generality assume that they are sums of line bundles.

Example. $c_p(E^\vee) = (-1)^p c_p(E)$, where $E^\vee$ is the dual of a bundle E. This is trivial if E decomposes, hence valid in general by the splitting principle.

Grassmannians. Again, let E be a vector bundle on X of rank r. Denote by $\pi\colon G = \mathrm{Grass}^k(E) \to X$ the Grassmannian of rank-k quotient bundles of E. It comes equipped with a universal exact sequence

$$0 \to R \to \pi^*E \to Q \to 0 \tag{10}$$

where $\mathrm{rank}(R) = r-k$ and $\mathrm{rank}(Q) = k$. It can be proved that $N(G)$ is generated as an $N(X)$-algebra by the Chern classes of Q. Considering the fact that $c(R) = c(\pi^*E)/c(Q)$ and the fact that $c_i(R) = 0$ for $i > r-k$, we get the relations

$$[\pi^*c(E)/c(Q)]_i = 0, \text{ for } i = r-k+1, \ldots, r. \tag{11}$$

Here $[c]_i$ means graded piece of degree i of a class c. These are actually sufficient to generate all relations:

PROPOSITION 2. *The relations (11) generate the ideal of relations between the generators $c_i(Q)$ for $N(G)$ as an $N(X)$-algebra.*

Example. Let X be a point and E a four-dimensional vector space. We denote $\mathrm{Grass}^2(E)$ also by $G(1,3)$, as it parametrizes lines in $\mathbf{P}^3 = \mathbf{P}(E)$. Then

$$N(G(1,3)) \simeq \mathbf{Q}[c_1, c_2]/(c_1^3 - 2c_1c_2, c_1^4 - 3c_1^2c_2 + c_2^2)$$

where c_i maps to $c_i(Q)$. In particular, we may deduce by pure algebra that $c_1(Q)^4 = 2c_2(Q)^2$.

For a general Grassmann bundle, the relations (11) can be used to compute the pushforward in much the same way as we got the Segre classes for a projective bundle, except that it gets a little more complicated, involving Schur functions. See [1, Ex. 14.2.2]. This is of course implemented in the schubert package.

Exercise. Let E be a rank-2 bundle and L a line bundle on X. Using the splitting principle, find the Chern classes of $E \otimes L$. Do the same if $\mathrm{rank}(E) = 3$. Ditto for general rank. (The top Chern class is easiest.)

Exercise. Speculate a little on the problem of computing Chern classes of a tensor product of bundles of arbitrary rank. What about symmetric and exterior products? Can you do better than using the splitting principle?

Geometric interpretation of Chern classes. So far we have indicated by a few examples how Chern classes may serve as generators and relations for the numerical intersection ring $N(X)$. Except for property (iv) of Chern classes, however, there remains to give interpretations of Chern classes on the level of cycles, and hence relate them to geometry. In fact, one may well argue that this presentation has things a bit backwards!

Anyway, here is the most important and direct correspondence: Assume that a vector bundle E on X of rank r is generated by its global sections. Let p be an integer between 1 and r, and choose $r - p + 1$ general sections of E. Let $D_p \subseteq X$ be the dependency locus of these sections. Then $[D_p] = c_p(E)$.

$$c_p(E) = [\text{dependency locus of } \mathrm{rank}(E) - p + 1 \text{ general global sections}]. \tag{12}$$

Example. Consider $G(1,3) = \mathrm{Grass}^2(H^0(\mathbf{P}^3, \mathcal{O}(1)))$ from the last section. We want to interpret $c_i(Q)$ for $i = 1, 2$. Now $H^0(G(1,3), Q) = E = H^0(\mathbf{P}^3, \mathcal{O}(1))$, so two general sections are the equations of a line $L \subseteq \mathbf{P}^3$, and their dependency locus as sections of Q is exactly $\Sigma(L)$. Therefore $c_1(Q) = [\Sigma(L)]$.

To understand $c_2(Q)$, choose one general section of Q. This corresponds to the equation of a hyperplane $H \subseteq \mathbf{P}^3$, and the vanishing locus of the corresponding section of Q is the locus of lines contained in H. Thus $c_2(Q) =$[lines contained in a given hyperplane].

To complete the solution of problem (ii), then, note that $c_2(Q)^2$ is the locus of lines contained in two general hyperplanes. Obviously, there is only one such line, hence $\int_{G(1,3)} c_2(Q)^2 = 1$. Above we computed that $c_1(Q)^4 = 2c_2(Q)^2$, hence the number of lines meeting four given general lines is 2.

Example. Let $\widetilde{L} \subseteq \mathbf{P}^3 \times G(1,3)$ be the universal line. Then $\widetilde{L} = \mathbf{P}(Q) \to G(1,3)$; in particular, ${pr_{G(1,3)}}_*(\mathcal{O}_{\widetilde{L}}(n)) = S_n(Q)$ for all positive integers n. Consider problem (iii) of the introduction. A cubic form $F \in H^0(\mathbf{P}^3, \mathcal{O}(3))$ gives rise to a global section $s(F)$ of the rank-4 vector bundle $S_3(Q)$ on $G(1,3)$. The zero locus of the section $s(F)$ is the locus of points in $G(1,3)$ corresponding to lines contained in the cubic surface $F = 0$. Thus the number of such lines is $\int_{G(1,3)} c_4(S_3(Q))$. Using the schubert package (what else?) to compute this we get 27.

Problem (v) is solved essentially in the same way.

Exercise. In $N(G(1,3))$, show that [lines containing a given point]$= c_1(Q)^2 - c_2(Q)$.

Exercise. Find a pair of dual bases for $N(G(1,3))$.

4. Chern characters.

The Chern character of a vector bundle or coherent sheaf is the representation that is used in the schubert package. Before we actually define it, we list its main properties:

(i) $ch(E \oplus E') = ch(E) + ch(E')$
(ii) $ch(E \otimes E') = ch(E) \cdot ch(E')$
(iii) $ch(\mathcal{O}_X) = 1$, $ch(0) = 0$.

Hence the map $E \mapsto ch(E)$ is a ring homomorphism from the Grothendieck ring $K^0(X) \otimes \mathbf{Q}$ to $N(X)$ (which is in fact surjective). It is therefore a more convenient representation of sheaves and sheaf operations than the Chern class, which does not reflect the rank of a vector bundle. Obviously, Chern characters of tensor products are easy to compute.

The chern character can be *defined* using the splitting principle: For a line bundle L put $ch(L) = \exp(c_1(L)) = \sum_{i=0}^{\dim(X)} c_1(L)^i/i!$, and then extend this by property (i) to direct sums E of line bundles. The graded pieces $ch_i(E)$ are invariant under permutation of the Chern roots, hence by the fundamental theorem on symmetric polynomials, they can be expressed in terms of the Chern classes of E. The formula so obtained can serve as the definition for a general vector bundle.

There is however a more direct definition which may be less elegant, but much better suited to computations:

$$ch(E) = \operatorname{rank}(E) + \sum_{i=1}^{\dim(X)} p_i/i!,$$

where the p_i are related to the chern classes $c_i = c_i(E)$ by the relations (Newton's relations):

$$p_n - c_1 p_{n-1} + c_2 p_{n-2} - \cdots + (-1)^{n-1} c_{n-1} p_1 + (-1)^n n c_n = 0, \text{ for } n \geq 1. \quad (13)$$

The p_i are the power sums $\sum_{k=1}^{\mathrm{rank}(E)} \gamma_k^i$ of the Chern roots γ_k of E. The first few terms of the Chern character are:

$$\begin{aligned} ch(E) = \mathrm{rank}(E) + c_1 + 1/2(c_1^2 - 2c_2) + 1/6(c_1^3 - 3c_1c_2 + 3c_3) + \\ 1/24(c_1^4 - 4c_1^2c_2 + 2c_2^2 + 4c_1c_3 - 4c_4) + \cdots \end{aligned}$$

In the schubert package, the basic representation of a sheaf E is the Chern character polynomial

$$ch_t(E) = \mathrm{rank}(E) + \sum_{i=1}^{\dim(X)} (p_i/i!)\, t^i.$$

There are routines to convert to and from Chern classes and Segre classes by solving the relations (13) inductively.

5. The Riemann-Roch theorem.

It is impossible to introduce the Chern character without at least mentioning this celebrated theorem. The *Todd class* of X is defined as

$$\mathrm{Todd}(X) = \prod_{i=1}^{\dim(X)} \frac{\gamma_i}{1 - \exp(-\gamma_i)} \in N(X)$$

where the γ_i are the Chern roots of the tangent bundle of X. Hirzebruch's Riemann-Roch theorem states that for a coherent sheaf E on X, its Euler-Poincare characteristic is given by $\chi(X, E) = \int_X ch(E) \cdot \mathrm{Todd}(X)$. Both this theorem and its generalization by Grothendieck are implemented in the schubert package.

Exercise. Deduce the Riemann-Roch theorem for a divisor on a curve from Hirzebruch's Riemann-Roch: $\chi(C, \mathcal{O}_C(D)) = \deg(D) + 1 - g(C)$.

Bibliography

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