5. Faddeev-Leverrier Method for Eigenvalues

Faddeev-Leverrier Method

Let $\mathbf{A}$ be an $n \times n$ matrix. The determination of eigenvalues and eigenvectors requires the solution of

(1) $$\mathbf{A}\mathbf{X} = \lambda\mathbf{X}$$

where λ is the eigenvalue corresponding to the eigenvector $\mathbf{X}$. The values λ must satisfy the equation

(2) $$\det(\mathbf{A} - \lambda\mathbf{I}) = 0.$$

Hence λ is a root of an nth degree polynomial $P(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$, which we write in the form

(3) $$P(\lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_{n-2}\lambda^2 + c_{n-1}\lambda + c_n.$$

The Faddeev-Leverrier algorithm is an efficient method for finding the coefficients c_k of the polynomial $P(\lambda)$. As an additional benefit, the inverse matrix $\mathbf{A}^{-1}$ is obtained at no extra computational expense.

Recall that the trace of the matrix $\mathbf{A}$, written $\mathrm{Tr}[\mathbf{A}]$, is

(4) $$\mathrm{Tr}[\mathbf{A}] = a_{1,1} + a_{2,2} + \ldots + a_{n,n}.$$

The algorithm generates a sequence of matrices $\{\mathbf{B}_k\}_{k=1}^{n}$ and uses their traces to compute the coefficients of $P(\lambda)$,

(5)

$$\begin{array}{lll}
\mathbf{B}_1 = \mathbf{A} & \text{and} & p_1 = \mathrm{Tr}[\mathbf{B}_1] \\
\mathbf{B}_2 = \mathbf{A}(\mathbf{B}_1 - p_1\mathbf{I}) & \text{and} & p_2 = \frac{1}{2}\mathrm{Tr}[\mathbf{B}_2] \\
\quad\vdots & \ldots & \quad\vdots \\
\mathbf{B}_k = \mathbf{A}(\mathbf{B}_{k-1} - p_{k-1}\mathbf{I}) & \text{and} & p_k = \frac{1}{k}\mathrm{Tr}[\mathbf{B}_k] \\
\quad\vdots & \ldots & \quad\vdots \\
\mathbf{B}_n = \mathbf{A}(\mathbf{B}_{n-1} - p_{n-1}\mathbf{I}) & \text{and} & p_n = \frac{1}{n}\mathrm{Tr}[\mathbf{B}_n]
\end{array}$$

Then the characteristic polynomial is given by

(6) $$P(\lambda) = \lambda^n - p_1 \lambda^{n-1} - p_2 \lambda^{n-2} - \ldots - p_{n-1} \lambda - p_n .$$

In addition, the inverse matrix is given by

(7) $$\mathbf{A}^{-1} = \frac{1}{p_n} \left(\mathbf{B}_{n-1} - p_{n-1} \mathbf{I} \right) .$$

Example 1. Use Faddeev's method to find the characteristic polynomial and inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} .$$

Solution 1.

Example 2. Use Faddeev's method to find the characteristic polynomial and inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} 8 & -1 & 3 & -1 \\ -1 & 6 & 2 & 0 \\ 3 & 2 & 9 & 1 \\ -1 & 0 & 1 & 7 \end{pmatrix} .$$

Solution 2.

Example 1. Use Faddeev's method to find the characteristic polynomial and inverse of the matrix

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}.$$

Solution 1.

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$p_1 = \mathrm{Tr}[B_1] = 6$$

$$B_2 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} \left(\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} - (6) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} -6 & 1 & -3 \\ 3 & -8 & -3 \\ -1 & 1 & -8 \end{pmatrix}$$

$$p_2 = \frac{1}{2}\mathrm{Tr}[B_2] = \frac{1}{2}(-22) = -11$$

$$B_3 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} \left(\begin{pmatrix} -6 & 1 & -3 \\ 3 & -8 & -3 \\ -1 & 1 & -8 \end{pmatrix} - (-11) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$p_3 = \frac{1}{3}\mathrm{Tr}[B_3] = \frac{1}{3}(18) = 6$$

$$P[\lambda] = \lambda^3 - \sum_{i=1}^{3} p_i \lambda^{n-i}$$

$$P[\lambda] = -6 + 11\lambda - 6\lambda^2 + \lambda^3$$

We can compare this result with the method of determinants for finding the characteristic polynomial.

$$A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$M = \begin{pmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & 1 \\ 1 & -1 & 2-\lambda \end{pmatrix}$$

Q[λ] = Det[M] = $6 - 11\lambda + 6\lambda^2 - \lambda^3$

Solve Q[λ] = 0 get

$\lambda \to 1$

$\lambda \to 2$

$\lambda \to 3$

The polynomials are different only in the fact that P[λ] = − Q[λ] .

Now we compute the inverse matrix A^{-1}.

The inverse matrix is

$$A^{-1} = \frac{1}{p_3}(B_{3-1} - p_{3-1}\ I)$$

$$A^{-1} = \frac{1}{6}\left(\begin{pmatrix} -6 & 1 & -3 \\ 3 & -8 & -3 \\ -1 & 1 & -8 \end{pmatrix} - (-11)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)$$

$$A^{-1} = \frac{1}{6}\left(\begin{pmatrix} -6 & 1 & -3 \\ 3 & -8 & -3 \\ -1 & 1 & -8 \end{pmatrix} - \left(\begin{pmatrix} -11 & 0 & 0 \\ 0 & -11 & 0 \\ 0 & 0 & -11 \end{pmatrix}\right)\right)$$

$$A^{-1} = \frac{1}{6}\begin{pmatrix} 5 & 1 & -3 \\ 3 & 3 & -3 \\ -1 & 1 & 3 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{5}{6} & \frac{1}{6} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{6} & \frac{1}{2} \end{pmatrix}$$

Example 2. Use Faddeev's method to find the characteristic polynomial and inverse of the matrix $\mathbf{A} = \begin{pmatrix} 8 & -1 & 3 & -1 \\ -1 & 6 & 2 & 0 \\ 3 & 2 & 9 & 1 \\ -1 & 0 & 1 & 7 \end{pmatrix}$.

Solution 2.

$$A = \begin{pmatrix} 8 & -1 & 3 & -1 \\ -1 & 6 & 2 & 0 \\ 3 & 2 & 9 & 1 \\ -1 & 0 & 1 & 7 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 8 & -1 & 3 & -1 \\ -1 & 6 & 2 & 0 \\ 3 & 2 & 9 & 1 \\ -1 & 0 & 1 & 7 \end{pmatrix}$$

$$p_1 = \mathrm{Tr}[B_1] = 30$$

$$B_2 = \begin{pmatrix} 8 & -1 & 3 & -1 \\ -1 & 6 & 2 & 0 \\ 3 & 2 & 9 & 1 \\ -1 & 0 & 1 & 7 \end{pmatrix} \left(\begin{pmatrix} 8 & -1 & 3 & -1 \\ -1 & 6 & 2 & 0 \\ 3 & 2 & 9 & 1 \\ -1 & 0 & 1 & 7 \end{pmatrix} - (30) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} -165 & 22 & -42 & 18 \\ 22 & -139 & -33 & 3 \\ -42 & -33 & -175 & -17 \\ 18 & 3 & -17 & -159 \end{pmatrix}$$

$$p_2 = \frac{1}{2}\mathrm{Tr}[B_2] = \frac{1}{2}(-638) = -319$$

$$B_3 = \begin{pmatrix} 8 & -1 & 3 & -1 \\ -1 & 6 & 2 & 0 \\ 3 & 2 & 9 & 1 \\ -1 & 0 & 1 & 7 \end{pmatrix} \left(\begin{pmatrix} -165 & 22 & -42 & 18 \\ 22 & -139 & -33 & 3 \\ -42 & -33 & -175 & -17 \\ 18 & 3 & -17 & -159 \end{pmatrix} - (-319) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1066 & -106 & 146 & -70 \\ -106 & 992 & 132 & -34 \\ 146 & 132 & 1087 & 67 \\ -70 & -34 & 67 & 1085 \end{pmatrix}$$

$$p_3 = \frac{1}{3}\mathrm{Tr}[B_3] = \frac{1}{3}(4230) = 1410$$

$$B_4 = \begin{pmatrix} 8 & -1 & 3 & -1 \\ -1 & 6 & 2 & 0 \\ 3 & 2 & 9 & 1 \\ -1 & 0 & 1 & 7 \end{pmatrix} \left(\begin{pmatrix} 1066 & -106 & 146 & -70 \\ -106 & 992 & 132 & -34 \\ 146 & 132 & 1087 & 67 \\ -70 & -34 & 67 & 1085 \end{pmatrix} - (1410) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} -2138 & 0 & 0 & 0 \\ 0 & -2138 & 0 & 0 \\ 0 & 0 & -2138 & 0 \\ 0 & 0 & 0 & -2138 \end{pmatrix}$$

$$p_4 = \frac{1}{4}\mathrm{Tr}[B_4] = \frac{1}{4}(-8552) = -2138$$

$$P[\lambda] = \lambda^4 - \sum_{i=1}^{4} p_i \lambda^{n-i}$$

$$P[\lambda] = 2138 - 1410\,\lambda + 319\,\lambda^2 - 30\,\lambda^3 + \lambda^4$$

We can compare this result with the method of determinants for finding the characteristic polynomial.

$$A = \begin{pmatrix} 8 & -1 & 3 & -1 \\ -1 & 6 & 2 & 0 \\ 3 & 2 & 9 & 1 \\ -1 & 0 & 1 & 7 \end{pmatrix}$$

$$M = \begin{pmatrix} 8-\lambda & -1 & 3 & -1 \\ -1 & 6-\lambda & 2 & 0 \\ 3 & 2 & 9-\lambda & 1 \\ -1 & 0 & 1 & 7-\lambda \end{pmatrix}$$

$Q[\lambda] = \text{Det}[M] = 2138 - 1410\lambda + 319\lambda^2 - 30\lambda^3 + \lambda^4$

Solve $Q[\lambda] = 0$ get

$$\lambda \to \frac{1}{2}\left(15 - \sqrt{37 - 4\sqrt{71}}\right)$$

$$\lambda \to \frac{1}{2}\left(15 + \sqrt{37 - 4\sqrt{71}}\right)$$

$$\lambda \to \frac{1}{2}\left(15 - \sqrt{37 + 4\sqrt{71}}\right)$$

$$\lambda \to \frac{1}{2}\left(15 + \sqrt{37 + 4\sqrt{71}}\right)$$

The polynomials are the same because the degree is even.

Now we compute the inverse matrix A^{-1}.

The inverse matrix is

$$A^{-1} = \frac{1}{p_4}(B_{4-1} - p_{4-1}\, I)$$

$$A^{-1} = -\frac{1}{2138}\left(\begin{pmatrix} 1066 & -106 & 146 & -70 \\ -106 & 992 & 132 & -34 \\ 146 & 132 & 1087 & 67 \\ -70 & -34 & 67 & 1085 \end{pmatrix} - (1410)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)$$

$$A^{-1} = -\frac{1}{2138}\left(\begin{pmatrix} 1066 & -106 & 146 & -70 \\ -106 & 992 & 132 & -34 \\ 146 & 132 & 1087 & 67 \\ -70 & -34 & 67 & 1085 \end{pmatrix} - \left(\begin{pmatrix} 1410 & 0 & 0 & 0 \\ 0 & 1410 & 0 & 0 \\ 0 & 0 & 1410 & 0 \\ 0 & 0 & 0 & 1410 \end{pmatrix}\right)\right)$$

$$A^{-1} = -\frac{1}{2138}\begin{pmatrix} -344 & -106 & 146 & -70 \\ -106 & -418 & 132 & -34 \\ 146 & 132 & -323 & 67 \\ -70 & -34 & 67 & -325 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{172}{1069} & \frac{53}{1069} & -\frac{73}{1069} & \frac{35}{1069} \\ \frac{53}{1069} & \frac{209}{1069} & -\frac{66}{1069} & \frac{17}{1069} \\ -\frac{73}{1069} & -\frac{66}{1069} & \frac{323}{2138} & -\frac{67}{2138} \\ \frac{35}{1069} & \frac{17}{1069} & -\frac{67}{2138} & \frac{325}{2138} \end{pmatrix}$$