Lecture notes for *EE 123: Digital Signal Processing* lectured by Kannan Ramchandran, Spring 2021

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Lecture 1

Introduction

Digital signal processing is a broad field that has existed for quite some time but remains hugely important to date. Prof. Lustig likes to call it the "swiss-army knife" for scientists and engineers, and also says you can do a lot of badass things with DSP like steal cars.
This course will extend off of EE 120 to cover many aspects of digital signal processing, in both theory and implementation. There will be a significant lab and project component of this course, although we won't be doing any HAM radio things as usual.

1.1 Review of Common Signals

We will first talk about common signals you should be familiar with. But before then, let's make sure we remember the difference between continuous time, discrete time, and digital signals.

Definition 1.1.1. Continuous-time signals are functions of a continuous independent variable. These will be written as $x(t)$.

Definition 1.1.2. Discrete-time signals are sequences of numbers. These will be written as $x[n]$.

Definition 1.1.3. Digital signals are sequences of numbers with quantized amplitudes.

Here are some common signals you should be familiar with

- unit impulse
- unit step
- sinusoid
  - concept of period for discrete and continuous time

1.2 Systems Review

In EE 120, we dealt mainly with LTI, linear time-invariant, systems. Let's define these terms.

Definition 1.2.1. A **linear** system T must satisfy $T\{x + y\} = T\{x\} + T\{y\}$ for any inputs x, y.

Definition 1.2.2. A **time-invariant** system T must satisfy $T\{x(n - \tau)\} = T\{x\}(n - \tau)$ for any input x and delay τ.

There are other important classifations of these systems as well.

Definition 1.2.3. Causal sytems produce an output that is only dependent on current and prior values of the input.

Definition 1.2.4. Memory-less sytems produce an output that is only dependent on the current value of the input.

Definition 1.2.5. A signal x is **bounded** if and only if we can define some finite value c such that $\forall t, |x(t)| \leq c$

Definition 1.2.6. A sytem is said to be **BIBO stable** if there does not exist any bounde input to the system that will produce an unbounded output.

Let's look at some examples.

Example 1.2.1. Consider the system $T\{x\}(n) = x(n - \tau)$. This system is LTI, causal, not memory-less, and BIBO stable.

Example 1.2.2. Consider the system $T\{x\}(n) = \sum_{i=-\infty}^{n} x(i)$. This system is LTI, causal, not memory-less, and BIBO stable.

Example 1.2.3. Consider the system $T\{x\}(n) = x[m \cdot t]$ for some scalar m. This system is linear, not time-invariant, not causal, not memory-less, and BIBO stable.

Lecture 2

120 Review Cont.

2.1 LTI Recap

2.1.1 Linearity and Time Invariance

As discussed previously, we can prove linearity of a system by proving that it satisfies properties of superposition and scaling.

Example 2.1.1. Consider a 3 point moving average filter, $y[n] = \frac{1}{3}(x[n-1] + x[n] + x[n+1])$. This represents a linear system.

Example 2.1.2. Consider a 3 point median filter, $y[n] = \text{Median}(x[n-1], x[n], x[n+1])$. This represents a non-linear system. Note that this non-linear filter still is used a lot in practice, since it's good at removing "shot noise".

Recall that we also need time-invariance to hold for a system to be LTI.

2.1.2 Impulse Response

Definition 2.1.1. The **impulse response** of a system is the output of the system when the input is the unit impulse $\delta[n]$. We often label this output as $h[n]$.

LTI systems are completely characterized by their impulse response.

Output in terms of Impulse Reponse

Proof. Start □

Stability in terms of Impulse Response

Theorem 2.1.1. *BIBO stability in LTI Systems* $\iff$ *Impulse response is absolutely summable* $\sum_{n=-\infty}^{\infty} |h[n]|$

Causality and Memoryless Properties

We can also show that these properties of LTI systems can be determiend through analysis of the impulse response.

2.1.3 Eigenfunctions of LTI systems

Recall the concept of eigenvectors Similarly, we may define eigenfunctions of LTI systems to. In contrast to linear algebra, we may identify all eigenfucntions of an LTI system quite easily since they all take on the form $e^{j\omega n}$

Definition 2.1.2. The **frequency response** of an LTI system is the function $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$. This is also sometimes refered to as the **transfer function**.

Theorem 2.1.2. *The output of an LTI system to input* $e^{j\omega n}$ *us* $H(e^{j\omega})e^{j\omega}$

Proof. □

2.2 Discrete-Time Fourier Transform (DTFT)

Discussion of the frequency response now brings us to the DTFT.

Definition 2.2.1. The **discrete-time fourier transform** of a signal is $\text{DTFT}\{x[n]\} = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$. We can also define the **inverse discrete-time fourier transform** as $x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega n})\, d\omega$

We can see how the DTFT definition is motivated by the definition of the frequency response from earlier.
The DTFT has a nice dual to the classical fourier series.

Definition 2.2.2. The **fourier series** of a continuous signal is $f_T(t) = \sum_{-\infty=\infty}^{k}$

Example 2.2.1. Define a signal

$$w[n] = \begin{cases} 1 & 0 \leq n \leq N-1 \\ 0 & \text{else} \end{cases}$$

This is a finite impulse train. We will compute the DTFT of this signal

$$\begin{aligned} W(e^{j\omega}) &= \sum_{k=0}^{N-1} e^{-j\omega k} \\ &= \sum_{k=0}^{N-1} \alpha^k \\ &= \frac{1-\alpha^N}{1-\alpha} \\ &= \frac{1-e^{-j\omega N}}{1-e^{-j\omega}} \\ &= \frac{e^{-j\omega \frac{N}{2}}[e^{-j\omega \frac{N}{2}} - e^{j\omega \frac{N}{2}}]}{} \end{aligned}$$

$$\boxed{W(e^{j\omega}) = | \, |}$$

Properties

- Convolution in time is multiplication in frequency

$$y[n] = (x * h)[n] \leftrightarrow Y(e^{j\omega n}) = H(e^{j\omega})X(e^{j\omega})$$

- Conjugate symmetry: If $x[n]$ is real...

$$X^*(e^j\omega) = X(e^{-j\omega n})$$

  We can actually exploit this redundancy in the requency specturm to save time and money

- Parseval's Theorem First, we state generalized Parseval's theorem

$$\sum_{n=-\infty}^{\infty} x[n] * y[n] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(e^{j\omega})Y^*(e^{-j\omega})\, \mathrm{d}\omega$$

  and then here is a form that shows how energy is preserved

- Shift property

Lecture 3

Transforms of Discrete Signals

3.1 DTFT Continued

Refer above to more in depth of these DTFT properties

- Analysis and synthesis equations
- Relationship between DTFT and FFT
- Window function and periodic sinc
- Convolution in time is multiplication in frequency
- Conjugate symmetry for real signals
- Parseval's theorem tells us that energy is conserved (with a factor of (2π)
- Shift property

Example 3.1.1. Consider an ideal low pass filter box function from $-\omega_c$ to ω_c. We get

$$h[n] = \frac{\omega_c}{\pi} \frac{sin(\omega_c n)}{\omega_c n}$$

Cool! Recall that we see the sinc function here.

$$\text{sinc}(x) = \begin{cases} \frac{sin(x)}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

Actually not cool. Looking at the impulse response We see there are two problems that make this ideal not realizable

- $h[n]$ is not causal
- $h[n]$ is infinite

We can fix this by adding a delay to our impulse response and applying a windowing function. What would the effects on the frequency response? We can use the shift property to see what the delay does, but for the multiplication by the windowing function we derive a relationship between multiplication in time and convolution in frequency

3.2 Convergence

Definition 3.2.1.

This form of convergence might be too strict and will tell us that a lot of things don't converge. Thus, we must introuce MSE convergence.

Definition 3.2.2.

Theorem 3.2.1. $\sum_{n=-\infty}^{\infty} |x[n]| < \infty \implies X(e^{j\omega})$ *converges uniformly*

Theorem 3.2.2. $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \implies X(e^{j\omega})$ *converges in MSE*

Example 3.2.1.

3.3 Z-Transform

Motvation:

- DTFT doesn't always converge, so we need another tool in our toolbox
- Solve difference equations
- Can use it for the same analysis we were doing with the DTFT

Similarly to how $e^{j\omega n}$ behaves as an eigenfunction to LTI systems, we can see that the general exponential z^n behaves the same way. The Z-transform of the impulse response gives us the transfer function of the system.

3.3.1 Region of Convergence (ROC)

Example 3.3.1. Consider a right sided sequence $x[n] = a^n u[n]$. Consider its Z-transform

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

Remembering what we know about geometric series, we see this converges to $X(z) = \frac{1}{1-az^{-1}}$ if $|\frac{a}{z}| < 1$, or $|a| < |z|$.

Example 3.3.2. Consider the sum of two right sided sequences $x[n] = (\frac{1}{2})^n + (-\frac{1}{3})^n$. The DTFT of this signal should be

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 + \frac{1}{3}z^{-1}}$$

Example 3.3.3. Consider left sided sequence $x[n] = -a^n u[-n-1]$

Example 3.3.4. Consider a two-sided sequence $x[n] = -(\frac{1}{2})^n u[n] - (-\frac{1}{3})^n u[-n-1]$

Example 3.3.5. Consider a two-sided sequence $x[n] = -(\frac{1}{2})^n u[n] - (-\frac{1}{3})^n u[-n-1]$

Example 3.3.6. Consider the finite-length M sequence $x[n] = a^n u[n] u[-n+M-1]$

General properties to remember:

- Generally, ROC will be a ring or disk in Z-plane centered at the origin
- DTFT converges $\iff$ ROC includes unit circle
- For right-sided sequences, ROC extends from outermost pole to infinity. See this in examples 1, 2.

Properties of DTFT seeem to also hold for Z transform

- $x[n-n_d] \leftrightarrow z^{-n_d} X(z)$
- $z_0^n x[n] \leftrightarrow X(\frac{z}{z_0})$
- $nx[n] \leftrightarrow -z\dfrac{\mathrm{d}X(z)}{\mathrm{d}dz}$

Rigorous inversion of z-transform relies contour integration within the ROC. We can avoid it by

- Inspection, utilizing known transforms
- Properties of the z-transform
- Power series expansion
- Parital fraction decomposition
- Residue theorem (partial fraction decomposition)

Lecture 4

Z-Transform and DFT

Readings: Chapter 3 and 8

4.1 Z-Transform Recap

4.1.1 Properties

Recall the bi-lateral z-transform

$$X(z) = \sum_{n=\infty}^{\infty} x[n]z^{-n}, z \in \text{R.O.C}(x)$$

The ROC will take the shape of a ring

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{-1} x[n]z^[-n] && \text{region of convergence} |z| < |a| \\ &= + \sum_{n=0}^{\infty} x[n]z^[-n] && \text{region of convergence } |z| > |a| \end{aligned}$$

Other notes

- ROC and inverse Z-transform
- Linearity
- Pole-zero cancelation
- Convolution

Read the book for more detail

Example 4.1.1 (Transfor function $H(z)$ of a DT system)**.**

4.1.2 Inversion of Z-Transform

There is an inverse formula, but you need complex analysis to understand it. Instead, we proceed by

- Inspection
- Properties of Z-transform
  - Delays
- Partial Fraction Decomposition (PFE)

Theorem 4.1.1. *If $H(z)$ is a rational transfer function,*

$$\textit{BIBO stability} \iff \textit{ROC includes unit circle}$$

Proof. **If**

Only If BIBO stable means that $h[n]$ is absolutely summable, so the $DTFT$ exists. Thus, Z-transform will include unit circle

□

4.2 Discrete Fourier Transform (DFT)

4.2.1 Motivaton

- DTFT is nice, we have good continuous figure of the frequency spectrum
- But not good to do on a computer
  - Infinite sum!
  - Continuous signal!
- Efficient to do on a computer!
  - Direct evaluation is $O(N^2)$
  - FFT is $O(N \log N)$
  - FFTW library
  - Use it to do convolution!
    * Direct convolution is $O(N^2)$
    * FFT-based convolution is $O(N \log N)$

4.2.2 Explanation

Assume we have the DFT definition

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}$$

This looks similar to the DTFT! In fact

$$X[k] = X(e^{j\omega})\Big|_{\omega = \frac{2\pi}{N}k}$$

We can see prove that the DFT is inherrently periodic

$$\begin{aligned} X[k + rN] &= \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k+rN)} \\ &= X[k] \cdot e^{-j\frac{2\pi}{N}rN} \\ &= X[k] \end{aligned}$$

4.2.3 Inverse

We will start with a formula for the DFT inverse

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk}$$

and prove that it works

Proof.

$$\begin{aligned} \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}nk} &= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk} e^{j\frac{2\pi}{N}nk} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} x[n] \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}nk} e^{j\frac{2\pi}{N}nk} \end{aligned}$$

□

To simplify things, we will define

$$W_n = e^{-j\frac{2\pi}{N}}$$