Analytical Option Prices for the Black-Karasinski Short Rate Model

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Introduction

- The Black Karasinski short rate model falls under the family of one-factor lognormal models. Its specificity is that it has an exogenous speed of mean reversion compared to a mean reversion driven by the volatility term in the BDT model.
- In this project, we would like to dive into this model by studying asymptotic analytical ways to price options. It would essentially be based on the eponymous article by Horvath, Jacquier and Turfus.
- The project would consist of two main parts:
  1) Derivation of analytical results
  2) Application of those methods to price interest rate products and compare results with prices based on Monte-Carlo method.

Modeling Assumptions - I

SDE of short rate can be written as

$$d \ln r_t = \alpha_r \big(\ln\big(\bar{r}(t) + r^*(t)\big) - \ln r_t\big) dt + \sigma_r dW_t$$

where we assume α_r and σ_r are constant parameters, $\bar{r}(t)$ represents the instantaneous forward rate and can be computed using the current yield curve and $r^*(t)$ also needs to be calibrated for zero coupon bonds.

We introduce an ancillary process with the following SDE

$$dx_t = -\alpha_r x_t dt + \sigma_r dW_t \qquad (1)$$

With the ancillary process above, we get the following relation to the short rate r_t

$$r_t = \big(\bar{r}(t) + r^*(t)\big)\mathcal{E}(x_t, t) \qquad (2) \qquad \text{where } \mathcal{E}(x,t) = \exp\left(x - \frac{\sigma_r^2}{4\alpha_r}\left(1 - e^{-2\alpha_r t}\right)\right)$$

For future notations, we will call $D(t, v)$ the deterministic discount factor given by

$$D(t, v) = \exp\left(-\int_t^v \bar{r}(s) ds\right)$$

Modeling Assumptions - II

We will consider European-style security which comes to expiry at time T and gives a payout of $P(x_t)$. The price of this security at time t will be given by f_t = $f(x_t, t)$. With this formalism, we recall that $f(x,t)$ should follow the Kolmogorov backwards diffusion equation:

$$\frac{\partial f}{\partial t} - \alpha_r x \frac{\partial f}{\partial x} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 f}{\partial x^2} = r(x,t) f$$

with the final condition $f(x,t)$ = $P(x)$ and $r(x,t)$ = $r_t \mid x_t$ = x.

This ODE can be rewritten as

$$\left(\frac{\partial}{\partial t} + \mathcal{L} - h(x,t)\right) f(x,t) = 0$$

where $\mathcal{L} = \frac{1}{2}\sigma_r^2 \frac{\partial^2}{\partial x^2} - \alpha_r x \frac{\partial}{\partial x} - \bar{r}$

and $h(x,t) = h(x,t,t)$ with

$$h(x,t,u) = \big(\bar{r}(u) + r^*(u)\big)\mathcal{E}(x,t,u) - \bar{r}(u)$$

Model Calibration to Yield Curve

When we assume that the mean reversion speed and the volatility of our short rate model are constants, we still need to calibrate $\bar{r}$ and r^*. The calibration of $\bar{r}$ is well know using the current yield curve.

$$\bar{r}(t) = \frac{f_0^T}{\partial T}$$

with f_0^T the zero-coupon bond price with maturity T at time 0.

To calibrate r^*, we need to take a closer look at our model. Indeed, we have to get the zero-coupon bond pricing formula. As there is no closed formula for the Black-Karasinski model we will need to work in the lox volatility world and express the different quantities in comparison to $\epsilon^2 = {\sigma_r}^2/_{\alpha}$. This ϵ is small for rates that goes back quickly to their mean reversion value. We can then rewrite r^* as an asymptotic expansion

$$r^*(t) = \epsilon r_1(t) + \epsilon^2 r_2(t) + \epsilon^3 r_3(t) + O(\epsilon^4)$$

More generally speaking, every quantity we are dealing with can be rewritten as an asymptotic expansion according to epsilon.

Green’s Function Definition

Under this formalism, we have that that the Green function is a solution for the Kolmogorov's equation, meaning

$$\left(\frac{\partial}{\partial t} + \mathcal{L} - h(x,t)\right) G(x,t;\xi,v) = \delta(x-\xi)\delta(t-v)$$

with $G(x,t;\xi,v) = \sum_{n=0}^{\infty} G_n(x,t;\xi,v)$

$$G_n(x,t;\xi,v) \; = \; \sum \int_t^v \int_{-\infty}^{+\infty} G_0(x,t;x_1,t_1) h_i(x_1,t_1) G_{n-i}(x_1,t_1\xi,v) dx_1 \, dt_1$$

and $G_0(x,t;\xi,v) = D(t,v) \frac{\partial}{\partial \xi} N\left(\frac{\xi - \phi_r(t,v)x}{\sqrt{I_r(t,v)}}\right)$

with $N(\cdot)$ is the standard Gaussian cumulative distribution.

This result can be proven quite easily by studying the partial sum.

Zero-Coupon Bond Pricing - I

We deduce by standard means that the T-maturity zero coupon bond price $f(x_t, t)$ will be governed under the money market numéraire by the following backward diffusion equation:

$$\frac{\partial \widehat{f_T}}{\partial t} - \alpha_r \hat{x} \frac{\partial \widehat{f_T}}{\partial \hat{x}} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 \widehat{f_T}}{\partial \widehat{x^2}} = r(\hat{x}, t)\widehat{f_T}$$

In the absence of exact closed form solutions to Eq. (7), we seek an approximate solution under a weak volatility assumption. To this end we rescale both $\hat{x}$ and $\sigma_x(t)$ by an asymptotic parameter ϵ defined by:

$$\epsilon^2 = \frac{1}{\alpha(T_m - t_0)} \int_{t_0}^{T_m} \frac{{\sigma_r}^2}{|\bar{r}(t)|^{2\beta}} dt$$

which we take to be small. Here we take T_m to be the longest maturity trade for which we wish our model to be calibrated. Thus, we define new scaled variables $\hat{x}$ and $\sigma_x(t)$ by

$$x_t \coloneqq \epsilon^{-1} \widehat{x_t}$$

$$\sigma_x(t) \coloneqq \epsilon^{-1} \sigma_r(t)$$

Performing the rescaling we obtain

$$\frac{\partial f_T}{\partial t} - \alpha_r x \frac{\partial f_T}{\partial x} + \frac{1}{2}\sigma_r^2 \frac{\partial^2 f_T}{\partial x^2} = r(x, t) f_T$$

Zero Coupon Bond Pricing - II

The above equation can be re-written as

$$\mathcal{L}[f_T(x,t)] = h(x,t) f_T(x,t)$$

where $\mathcal{L}[.]$ is a standard forced diffusion operator given by

$$\mathcal{L} = \frac{\partial}{\partial t} - \alpha x \frac{\partial}{\partial x} + \frac{1}{2}\sigma_r^2(t)\frac{\partial^2}{\partial^2 x} - \bar{r}(t) \tag{3}$$

and the asymptotically small forcing function is given by

$$h(x,t) = \big(\bar{r}(t) + r^*(t)\big)\mathcal{E}\left(\frac{\epsilon x_t}{|\bar{r}(t)|^\beta}\right) - \bar{r}(t) \tag{4}$$

Solving the above equation, we get an expression for a zero-coupon bond as (5)

$$f_T(x_t,t) = f_T^*(x_t,t) - \epsilon^2 D(t,T)\Big(x_t^2 - I_x(t_0,t)\Big)B_2^*(t,T) + O(\epsilon^3)$$

where
$$f^*(x_t,t) \coloneqq D(t,T)\mathcal{E}\left(-\left(\epsilon x_t + \epsilon^2 I_x^{(1)}(t)\right)B_1^*(t,T)\right)$$

$$B_j^*(t_1,t_2) \coloneqq \frac{(1-\beta)^{j-1}}{j!}\int_{t_1}^{t_2} e^{-j\alpha(u-t_1)}\frac{\bar{r}(u)}{|\bar{r}(u)|^{j\beta}}\,du \qquad I_x^{(1)}(t) \coloneqq \int_{t_0}^{t_1} e^{-\alpha(t-u)}\, I_x(t_0,u)\frac{\bar{r}(u)}{|\bar{r}(u)|^\beta}\,du$$

Caplet Pricing - I

Consider a caplet which pays the positive difference between tenor-τ Libor and a strike K on a unit notional, based on a payment period $[T-\tau, T]$, which rate we shall denote $L(\tau, T)$. For simplicity we assume no spread between forward Libor rates and the equivalent risk-free rates inferred from Eq. (2). However, it is not difficult to introduce an assumed deterministic spread by adjusting the value of the strike accordingly in the formulae derived below. We denote the (stochastic) value at time t of the caplet by $C_{T,K}(x,t)$.

$$Payoff_T = max\big((L(\tau,T)-K)\delta(T-\tau,T),0\big)$$

where $\delta(t_1,t_2)$ is the day count faction calculated according to the relevant convention (usually actual/360 or actual/365). We note in particular that, under our assumptions, the realized Libor rate is related to the stochastic zero-coupon bond price $f_T(x,t)$ calculated above by

$$1+L(\tau,T)\delta(T-\tau,T)=\frac{1}{f_T(x,T-\tau)}$$

whence $$Payoff_T(x)=max\{f_T(x,T-\tau)^{-1}-\big(1+K\delta(T-\tau,T)\big),0\}$$

If we consider an equivalent payoff payment made at time $T-\tau$, this must be discounted by precisely the T-maturity zero coupon bond price observed at time $T-\tau$, whence we can write

$$Payoff_{T-\tau}(x)=\kappa^{-1}max\{\kappa-f_T(x,T-\tau),0\} \tag{6}$$

where $$\kappa:=\frac{1}{1+K\delta(T-\tau,T)}$$

Caplet Pricing - II

In other words, we should consider a put option on the bond price. Writing the price of this option as $C_{T,K}(x,t)$, we see this will satisfy

$$\mathcal{L}\left[C_{T,K}(x,t)\right] = h(x,t)C_{T,K}(x,t)$$

with $\mathcal{L}[.]$ and $h(x,t)$ given by Eqs. (3) and (4) above. The final condition satisfied will be

$$C_{T,K}(x,T-\tau) = Payoff_{T-\tau}(x)$$

We note in passing that, if we remove the max condition from Eq. (6) and instead use

$$Payoff_T \coloneqq (L(\tau,T) - K)\delta(T-\tau,T)$$

we obtain the price of a short position in a Libor forward rate agreement, which we denote $F_{T,K}(x,t)$. Standard no-arbitrage shows this to be given by

$$F_{T,K}(x,t) = f_{T-\tau}(x,t) - \kappa^{-1} f_T(x,t)$$

which can be evaluated asymptotically making use of Eq. (5). Also, setting κ = 1 in the above yields the t-value of the forward Libor contract.

Caplet Pricing - III

To solve for $C_{T,K}(0,t_0)$, we pose formally:

$$C_{T,K}(x,t) = C_0(x,t) + \epsilon C_1(x,t) + \epsilon^2 C_2(x,t) + O(\epsilon^3)$$

Although we choose not to make it explicit in our notation, each of the $C_i(x,t)$ is expected to have a weak dependence on ϵ but to be bounded independently of ϵ as $\epsilon \to 0$. Substituting the above expansion into Eq. and proceeding as previously we obtain

$$C_{T,K}(0,t_0) = D(t_0, T-\tau)N(-d_2) - \kappa^-1 D(t_0,T)\big(N(-d_1) + I_r(t_0, T-\tau)B_2^*(T-\tau,T)d_1 N'(-d_1)\big) + O(\epsilon^3) \quad (7)$$

where N(.) is a unit normal cumulative distribution function and we have defined

$$d_1 := \frac{ln\left(\kappa^{-1}D(T-\tau,T)\right)+\frac{1}{2}B_1^*(T-\tau,T)^2 I_r(t_0,T-\tau)}{B_1^*(T-\tau,T)\sqrt{I_r(t_0,T-\tau)}}$$

$$d_2 := d_1 - B_1^*(T-\tau,T)\sqrt{I_r(t_0,T-\tau)}$$

As can be seen, Eq. (7) takes the form of the standard Black formula with an asymptotically small adjustment, effectively at $O(\epsilon^3)$.

Floorlet Pricing

As can be seen from Eq. (7) we have calculated the price of a caplet to be

$$C_{T,K}(0,t_0) = D(t_0,T-\tau)N(-d_2) - \kappa^-1D(t_0,T)\big(N(-d_1) + I_r(t_0,T-\tau)B_2^*(T-\tau,T)d_1N'(-d_1)\big) + O(\epsilon^3)$$

Following from put-call parity we have the price of a floorlet to be

$$C_{T,K}(0,t_0) = \kappa^-1D(t_0,T)\big(N(d_1) - I_r(t_0,T-\tau)B_2^*(T-\tau,T)d_1N'(d_1)\big) - \ D(t_0,T-\tau)N(d_2) + O(\epsilon^3)$$

where N(.) is a unit normal cumulative distribution function and we have defined

$$d_1 \coloneqq \frac{ln\left(\kappa^{-1}D(T-\tau,T)\right)+\frac{1}{2}B_1^*(T-\tau,T)^2I_r(t_0,T-\tau)}{B_1^*(T-\tau,T)\sqrt{I_r(t_0,T-\tau)}}$$

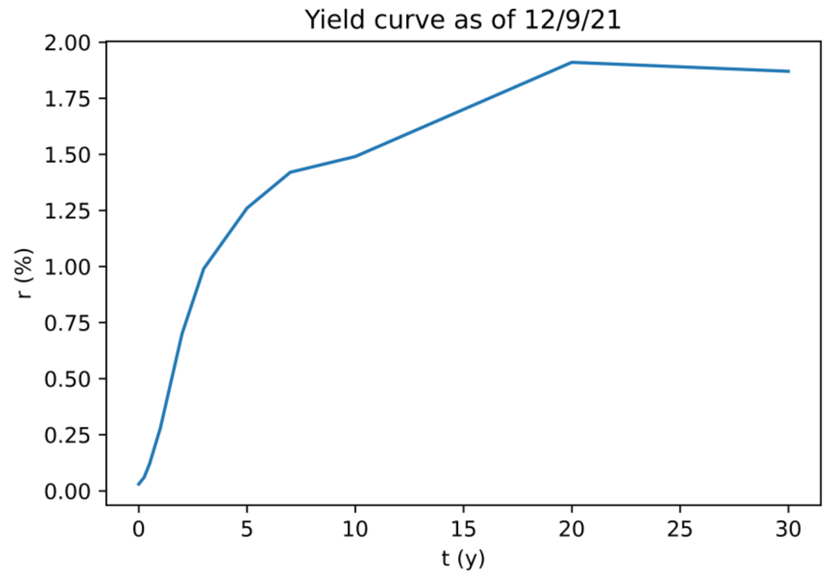
$$d_2 \coloneqq d_1 - B_1^*(T-\tau,T)\sqrt{I_r(t_0,T-\tau)}$$

Pricing Tool

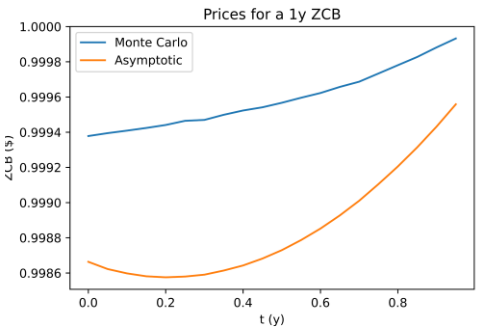
We developed a pricing tool which prices zero-coupon bonds, caplets and floorlets using the monte-carlo method as well as the asymptotic analytical formulae outlined under the Black-Karasinski short rate model in this project and consequently compares results of the two methods.

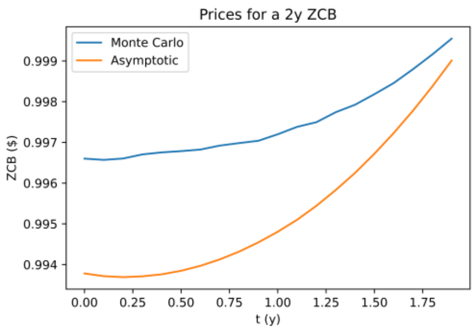
Python Notebook for pricing tool - https://colab.research.google.com/drive/149KB9tAhd-jsOaidu1PzpOXfgVTTKzmj?usp=sharing

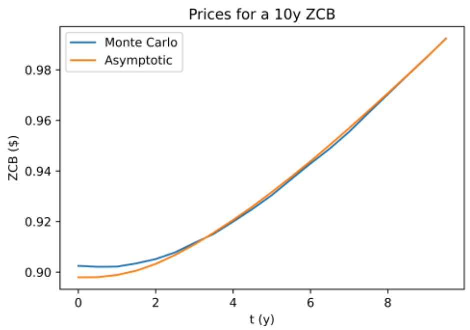
Yield Curve: Yield curve data was taken from US treasury of FRED API

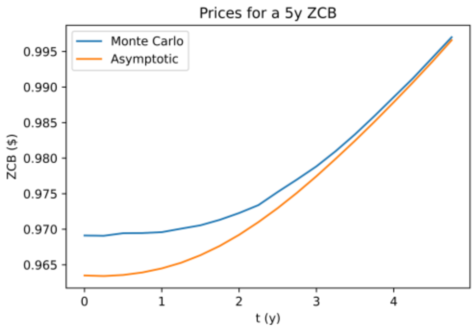


Comparison with Monte-Carlo method - I









Zero-Coupon Bond Prices

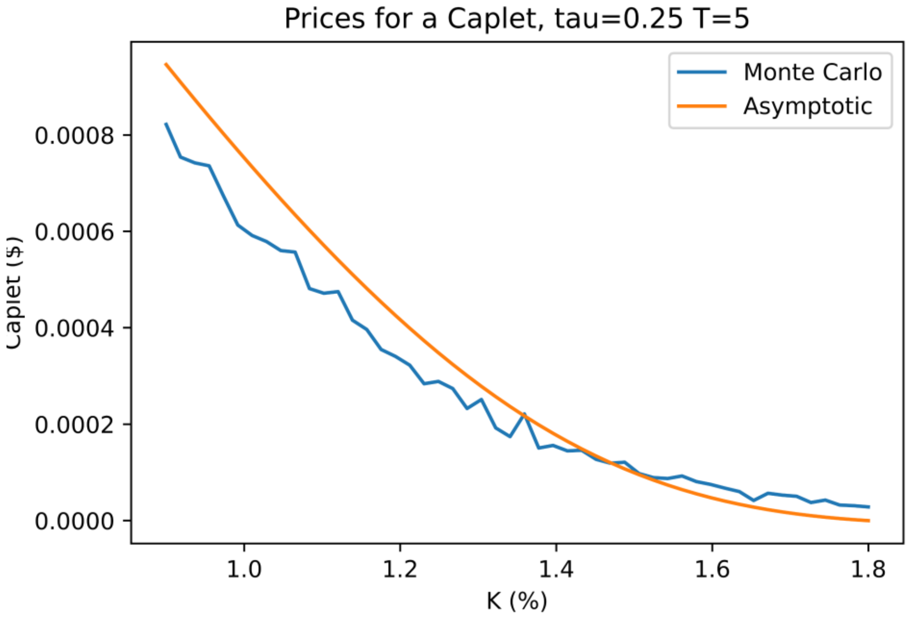
As we can see from the graphs alongside, zero-coupon bond prices from the asymptotic Black-Karasinski model deviate more from Monte-Carlo prices for bonds with smaller maturities.

For a single bond with given maturity, bond price from asymptotic Black-Karasinski model deviates more from Monte-Carlo price in the short term and gets closer to the actual price as it gets closer to maturity.

In both cases, however, the deviation is very small in percentage terms, thus falling in an acceptable range.

The asymptotic approximation under the Black-Karasinski model is much faster than Monte-Carlo method. For a 5-year zero-coupon bond, asymptotic approximation is 500 times faster than Monte-Carlo.

Comparison with Monte-Carlo method - II



Caplet Prices

We calculated the price of a 5-year maturity caplet with 4.75 years to maturity and different strike prices.

Caplet prices from asymptotic approximation in Black Karasinski model follow Monte-Carlo prices closely with an acceptable deviation.

The asymptotic approximation under the Black-Karasinski model is much faster than Monte-Carlo method. For the caplet under consideration, asymptotic approximation is 50 times faster than Monte-Carlo.

References

Lecture 5: Short-Rate Models II (Luca Capriotti)

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Analytic Option Prices for the Black-Karasinkii Short Rate Model, (B. Horvath, A. Jacquier, C.Turfus)
https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3253833

Pricing Tool - https://colab.research.google.com/drive/149KB9tAhd-jsOaidu1PzpOXfgVTTKzmj?usp=sharing