In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
np.random.seed(2018)
```

In [2]:
```
### part a: generate feature vectors
num_dim = 100
num_cells = 500

v1,v2,v3 = np.random.randn(3,num_dim)
norm = np.dot(v1,v1)
v1 /= norm
norm = np.dot(v2,v2)
v2 /= norm
norm = np.dot(v3,v3)
v3 /= norm
```

In [3]:
```
### part b: check out their scalar products:
print("\nPart B: Scalar products:")
print("v_1 . v_2  = {0}".format(np.dot(v1,v2)))
print("v_1 . v_3  = {0}".format(np.dot(v1,v3)))
print("v_2 . v_3  = {0}".format(np.dot(v3,v2)))
```

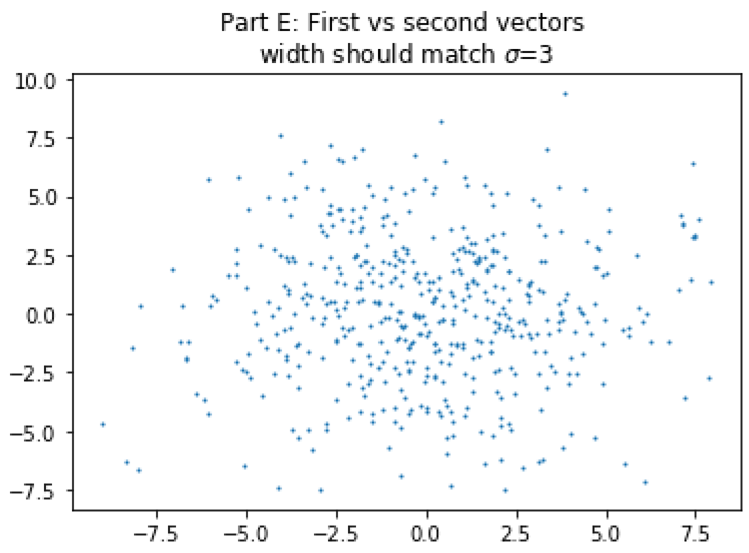
```
Part B: Scalar products:
v_1 . v_2  = -0.0007220208019868729
v_1 . v_3  = -0.00017868546616665724
v_2 . v_3  = 0.0003944704821550559
```

In [4]:
```
### part c: create a dataset using 3 secret variables (zero-mean, given variance)
a_vect = np.sqrt(20)*np.random.randn(num_cells) + 0
b_vect = np.sqrt(5)*np.random.randn(num_cells) + 0
c_vect = np.sqrt(0.5)*np.random.randn(num_cells) + 0

data = np.zeros((num_cells,num_dim))
for i in range(num_cells):
    x_i = 0 + a_vect[i]*v1 + b_vect[i]*v2 +c_vect[i]*v3
    data[i] = x_i
```

In [39]:
```
### part d: add zero-mean noise
sig = 3
noise = sig*np.random.randn(num_cells,num_dim) + 0
data += noise
```

In [40]:
```
### part e: do we see a correlation b/t the first and second features?
feat1 = data[:, 0]
feat2 = data[:, 1]
plt.figure()
plt.title(f"Part E: First vs second vectors \nwidth should match $\sigma$={sig}")
plt.scatter(feat1,feat2,s=1)
```

Out[40]:
```
<matplotlib.collections.PathCollection at 0x91b96d8>
```

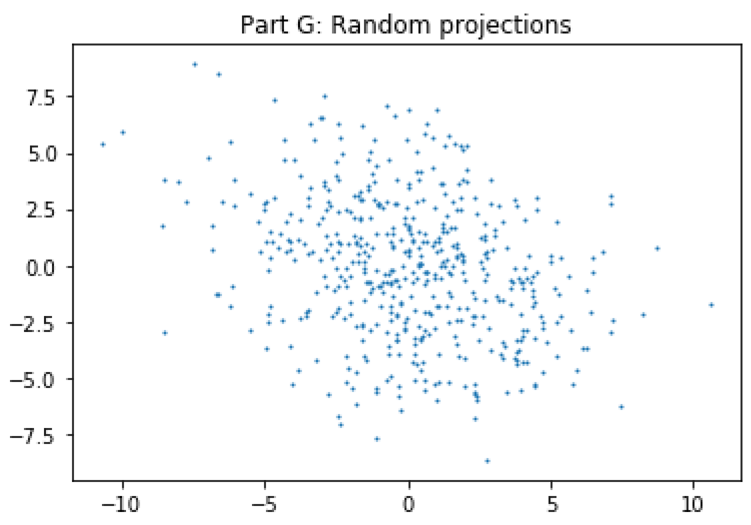


Part f: math

For a vector *x*, projected onto a unit vector *u*, the length of the projection of x onto u is $\vec{x}' = \hat{u}|x|\cos(\theta) = \hat{u}\ \vec{x}\cdot\vec{u}$ The length of this vector is simply $|\vec{x}\cdot\hat{u}|$ which lies somewhere between 0 and $|x|$

In [41]:
```
### part g: project onto new vectors
# try to choose a random direction (ie combo of features)
#   and see if this vector is representative of the data
#   ie does each of num_cells vectors 'match' with this?
unit1, unit2 = np.random.randn(2,num_dim)
unit1 /= np.sqrt( np.dot(unit1,unit1) )
unit2 /= np.sqrt( np.dot(unit2,unit2) )

proj1 = np.dot(data, unit1.T)
proj2 = np.dot(data, unit2.T)
plt.figure()
plt.title("Part G: Random projections")
plt.scatter(proj1, proj2,s=1)
print("G: std dev, {0}, {1}".format(np.var(proj1), np.var(proj2)))
```

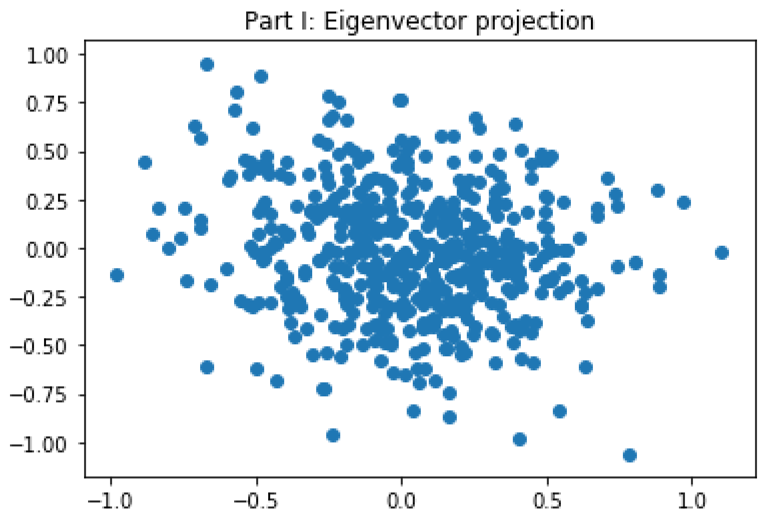
```
G: std dev, 10.085436545471369, 9.71546138822262
```



In [8]:
```
### part h: calculate eigenvectors of covariance matrix
cov = np.dot(data.T,data) /num_cells
eigenvalues, eigenvectors = np.linalg.eig(cov)

# we need to find the highest 3 vectors.
sort_indices = eigenvalues.argsort()[::-1] ## max should be first
ev = eigenvalues
e_vect_1 = eigenvectors[:, sort_indices[0]]
e_vect_2 = eigenvectors[:, sort_indices[1]]
e_vect_3 = eigenvectors[:, sort_indices[2]]

## checks:
'''
* Eigenvalues are sorted correctly.
* Eigenvectors are indexed correctly.
* Eigenvectors are orthogonal in lower dimensional space.
'''
```

In [43]:
```
### part i: projection onto primary vectors
# note, eigenvectors are already normalized
proj1 = np.dot(data, e_vect_1)
proj2 = np.dot(data, e_vect_2)
proj3 = np.dot(data, e_vect_3)
var1 = np.var(proj1)
var2 = np.var(proj2)
var3 = np.var(proj3)
print(f"\nPart I: var of (1,2,3) projection is {var1,var2,var3}")
plt.figure()
plt.title("Part I: Eigenvector projection")
plt.scatter(proj1,proj2);

## Checks
'''
* In 2 dimensions, the variance is captured by eigenvectors. (I removed the c feature and v3.)
    These principle components visually align with the major and minor axes of the ellipse.
* [failed] setting e_vect_i to vi does not clearly project into (likewise as seen in j)
    Am I therefore projecting incorrectly? Part g correctly displays sig^2 sized variance.
''';
```

```
Part I: var of (1,2,3) projection is (0.11387565333753184, 0.10836196778962173, 10.995264465334943)
```



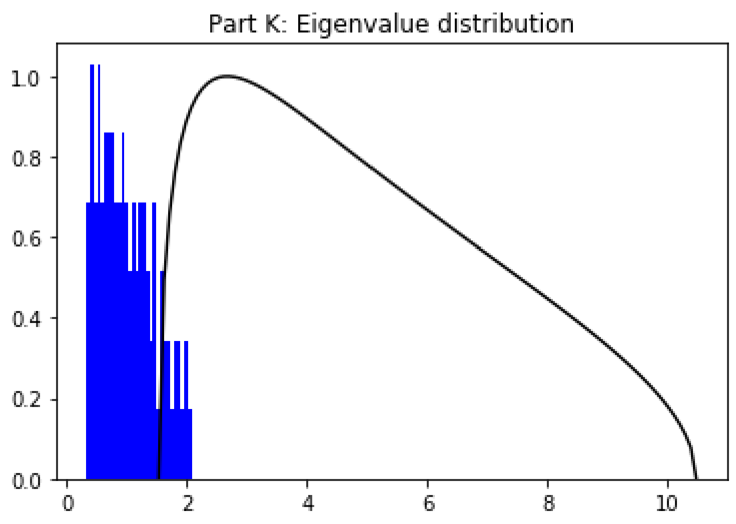
In [10]:
```
### part j: check that the output matches the input
print("\nPart J: Variable correspondence:")
print("v1.eigen1={0}".format(np.dot(e_vect_1,v1)))
print("v1.eigen2={0}".format(np.dot(e_vect_2,v1)))
print("v1.eigen3={0}".format(np.dot(e_vect_3,v1)))
```

```
Part J: Variable correspondence:
v1.eigen1=-0.02423283944634951
v1.eigen2=0.0033684664514440285
v1.eigen3=0.00702184287695051
```

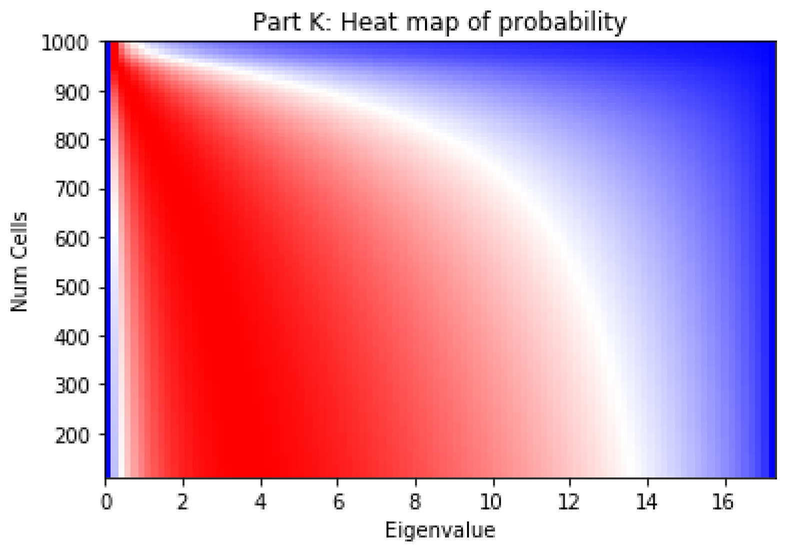
If these scalar products are +1 then our data vectors are exactly parallel to the "template vectors" implying accurate prediction by the templates. -1 indicates anti-alignment which is a trivial consequence of

While the first two components have been recovered, the third seems to have been lost. This is due to the fact that the variance, λ_c is less than the noise variance, $\sigma = 1$. Ie it is below our "noise floor" as we show next.

In [11]:
```
### part k: compare to Marchenko-Pastur distribution from random matrix theory
def p(r,size=100):
    lam_plus = sig**2 *(1+np.sqrt(r))**2
    lam_minus = sig**2 *(1-np.sqrt(r))**2
    x = np.linspace(lam_minus,lam_plus,size)
    prob = 1.0/(2*np.pi*sig) *np.sqrt((lam_plus-x)*(x-lam_minus))/r/x
    return x, prob

xs_first, ps_first = p(num_cells/num_dim);
ps_first /= max(ps_first)
plt.figure()
plt.title("\nPart K: Eigenvalue distribution")
plt.plot(xs_first,ps_first, 'k-')

# get eigenspectrum
plt.hist(eigenvalues,bins=30,color='b',density=True);
```



In [38]:
```
### part i: predict requirements for
max_num = 1000
num_datapoints_list = np.linspace(1.1*num_dim,max_num,100)
num_xs = 100
image = np.zeros((len(num_datapoints_list),num_xs))

x_min_bound = np.inf
x_max_bound = -np.inf
for i,num in enumerate(num_datapoints_list):
    xs, ps = p(num/num_dim,num_xs)
    ps /= max(ps)
    image[i, :] = ps
    x_min_bound = min(x_min_bound,min(xs))
    x_max_bound = max(x_max_bound,max(xs))
plt.imshow(image, extent=[x_min_bound,x_max_bound,1.1*num_dim,max_num],aspect='auto',cmap='bwr')
plt.title("Part K: Heat map of probability")
plt.ylabel("Num Cells")
plt.xlabel("Eigenvalue");
```



The above plot shows what I would plot if my eigenspectrum matched theory. To get down to a sensitivity associated with finding λ_c = 0.5 admist background noise of $\sigma = 1$ requires sampling enough cells in 100 dimensions to have low probability in the *higher* eigenvalue range.

```
In [ ]:
```