Test of entropic uncertainty relations using continuous measurements in circuit QED

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The goal of these working notes is to dissect the entropic uncertainty relations described in Yunger Halpern *et. al.*, *Reconciling two notions of quantum operator disagreement: Entropic uncertainty relations and information scrambling, united through quasiprobabilities* (arXiv:1806.04147). To understand the importance of entropic uncertainty relations, recall the usual formulation for the uncertainty relation for two observables A and B:

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} |\langle [A, B] \rangle|^2, \tag{1}$$

Which relates the expectation values for the dispersion of A and B. The expectation values are calculated for a particular state, so this relation is applied to a particular state on both sides of the inequality, and the dispersions are different for different states that are considered. In contrast, entropic uncertainty relations take a more general form,

$$\begin{pmatrix} \text{quantity true for} \\ \text{all physical states} \end{pmatrix} \geq \begin{pmatrix} \text{quantity that does} \\ \text{not depend on the state} \end{pmatrix}, \tag{2}$$

Which is more general than the usual uncertainty relationship.

First we'll try to unpack the notation and then see how the test could be implemented with our measurement setup.

I. ENTROPIC UNCERTAINTY RELATIONS

We start by defining three operators $\mathcal{I},\ A,\ F$ which can be three qubit Pauli operators e.g. σ_z, σ_y, σ_x. The basic experimental protocol will be to measure $\mathcal{I}$ strongly to prepare an initial state, measure A weakly and then postselect on an outcome for F. This type of experiment is the usual recipe for obtaining weak values for the measurement of A.

An example of an entropic uncertainty relation is

$$\boxed{H_{\mathrm{vN}}(\mathcal{I})_\rho + H_{\mathrm{vN}}(\bar{A}F)_\rho \geq f_{\mathrm{wk}}} \tag{3}$$

There are two sides to this inequality, the LHS has to do with quantities that are true for any physical state, and the RHS had to do with quantities that do not depend on the state at all. The von-Neumann entropy $H_{\mathrm{vN}}(\mathcal{I})_\rho$ should be understood as follows, given a state ρ, imagine performing measurements $\mathcal{I}$ (e.g.), projective measurements of σ_z. The entropy is the usual Shannon entropy associated with the outcomes, $\sum_i p_i \log p_i$.

The LHS of (3) contains two terms that are meant to be evaluated for any state ρ: given ρ, calculate $H_{\mathrm{vN}}(\mathcal{I})_\rho$ and also calculate $H_{\mathrm{vN}}(\bar{A}F)_\rho$ given the same state. Experimentally, we would simply prepare a state ρ and then make many measurements $\mathcal{I}$ and and use the probabilities of the outcomes that we sample to determine $H_{\mathrm{vN}}(\mathcal{I})_\rho$, then we'd also make many measurements of A and then F to determine $H_{\mathrm{vN}}(\bar{A}F)_\rho$.

Since the POVM satisfies the completeness relation,

$$1 = \sum_{r=-\infty}^{\infty} M_r M_r^\dagger \cong \sum_{r=-50}^{50} M_r M_r^\dagger$$

$$H_{\mathrm{vN}}(\mathcal{A}\mathcal{F})_\rho = -\sum_{r=\frac{-50}{l}}^{\frac{50}{l}} \mathrm{Tr}\left(\Pi_{-1}^F M_r \rho M_r^\dagger\right) l \log(\mathrm{Tr}\left(\Pi_{-1}^F M_r \rho M_r^\dagger\right) l)$$

$$-\sum_{r=\frac{-50}{l}}^{\frac{50}{l}} \mathrm{Tr}\left(\Pi_{1}^F M_r \rho M_r^\dagger\right) l \log(\mathrm{Tr}\left(\Pi_{1}^F M_r \rho M_r^\dagger\right) l)$$

We need to multiply each probability density $\mathrm{Tr}\left(\Pi_1^F M_r \rho M_r^\dagger\right)$ by a bin width ℓ. We also need to specify a value for ℓ.

Also,

$$H_{\rm vN}(\mathcal{I})_\rho = -\sum_{z=\pm 1} p(z|\rho)\log_2 p(z|\rho) \tag{4}$$

$$\tag{5}$$

We can also use the relationship

$$p(i{=}z|\rho) = \mathrm{Tr}\left(\Pi^Z_z \rho\right) \tag{6}$$

$$\tag{7}$$

Thus,

$$H_{\rm vN}(\mathcal{I})_\rho = -\mathrm{Tr}\left(\Pi^Z_{-1}\rho\right)\log_2(\mathrm{Tr}(\Pi^Z_{-1}\rho)) - \mathrm{Tr}\left(\Pi^Z_1\rho\right)\log_2(\mathrm{Tr}(\Pi^Z_1\rho))$$

Therefore,

$$H_{\rm vN}(\mathcal{I})_\rho = \log_2 2 = 1$$

Now we turn to the quantity $f_{\rm wk}$,

$$\boxed{f_{\rm wk} := \min_{i,j,f}\left\{-\log_2\left(p^A_j \mathrm{Tr}(\Pi^F_f \Pi^{\mathcal{I}}_i)\right) - \frac{2\mathrm{Tr}(\Pi^{\mathcal{I}}_i)}{\ln 2\sqrt{p^A_j}}\mathrm{Re}\left(\bar{g}^A_j A_{\rm wk}(i,f)\right)\right\}} \tag{8}$$

which should be true in the weak measurement limit where terms that are second order in the coupling to the detector g can be ignored. depends only on the choices for measurements $\mathcal{I}$, A, F. Here, i, j, f label the outcomes of the three different measurements. The term p^A_j labels the probability of getting outcome j in the weak measurement sequence, which we will discuss later. The final term involves $\mathrm{Re}\left(\bar{g}^A_j A_{\rm wk}(i,f)\right)$ which involves g which is related to the strength of the coupling to the detector, and the weak value.

In an experiment, the outcomes $jl \equiv r$ will be discretized and binned. Let ℓ denote the bin width. The probability density p^A_j will be replaced with the probability $p^A_j\ell$ associated with the bin centered on j. Similarly, each of $\sqrt{p^A_j}$ and $\bar{g}^A_r$ will be multiplied by $\sqrt{\ell}$. These $\sqrt{\ell}$'s will cancel.

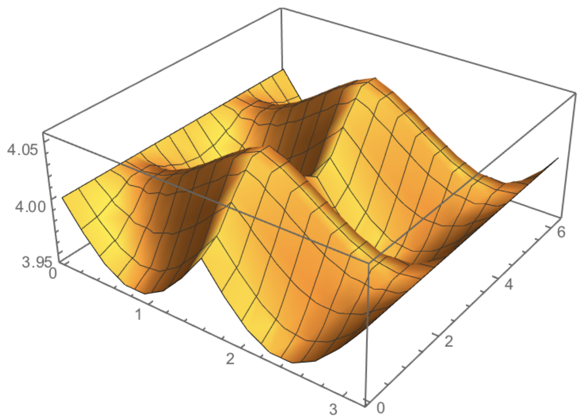
The constant-in-g term's $-\log(\ell)$ appears to have no counterpart to cancel with. This negative log of a tiny number is large. The largeness might concern us. But the uncertainty relation's LHS counterbalances this term:

$$H_{\rm vN}(\bar{A}F) = -\sum_{r,f} p(r,f|\rho)\ell \log(p(r,f|\rho)\ell) \quad = -\log(\ell)\sum_{r,f} p(r,f|\rho) - \ell \sum_{r,f} p(r,f|\rho)\log(p(r,f|\rho)). \tag{9}$$

The first sum equals one, by the probability's normalization. Hence this entropy contains a $-\log(\ell)$ that cancels with the $-\log(\ell)$ on the uncertainty relation's RHS.

We know that RHS has the minimum value when $r = \frac{\mathrm{Re}(A_{\rm wk}(i,f))}{\ln 2}$

Let us first consider the case when $\frac{\delta t}{t} = 0.1$, $\tilde{\theta} = \frac{\pi}{2}, \tilde{\phi} = 0$. We need to find out the minimum value of $f_{\rm wk}$ depending on the four cases $(i = \pm 1, f = \pm 1)$. We used a Mathematica to figure out the minimum of $f_{\rm wk}$, and once we get the minimum of $f_{\rm wk}$, we plotted 3D graph.



(x-axis : $0 < \theta < \pi$, y-axis : $0 < \phi < 2\pi$, z-axis :$f_{\rm wk}$)

II. PROPORTIONALITY BETWEEN $\langle r \rangle$ AND $A_{\rm wk}$

Let all symbols be defined as in "Reconciling" and in our working notes. Consider preparing the system of interest in a state ρ, coupling the detector to the system observable $A = \sum_a a|a\rangle\langle a|$, measuring the detector strongly, and measuring the system-of-interest observable $F = \sum_f f|f\rangle\langle f|$ strongly. Let f denote the F measurement's outcome. The measurement of the detector yields a random variable r. Averaging r yields a value $\langle r \rangle_{\rho,f}$ proportional to the weak value of A. I derive the exact proportionality in this note.

The probability density associated with obtaining the outcomes r and f equals

$$p(r, f|\rho) = \mathrm{Tr}\left(\Pi_f^F M_r \rho M_r^\dagger\right). \tag{10}$$

Since $M_r = \sqrt{p_r^A}\,1 + \bar{g}_j^A A$,

$$p(r, f|\rho) = p_r^A + 2\sqrt{p_r^A}\,\Re\left[\bar{g}_r^A \mathrm{Tr}\left(\Pi_f^F A\rho\right)\right] + O(g^2). \tag{11}$$

The weak value is defined as

$$A_{\mathrm{wk}}(\rho, f) := \frac{\mathrm{Tr}\left(\Pi_f^F A\rho\right)}{\mathrm{Tr}\left(\Pi_f^F \rho\right)}. \tag{12}$$

We solve for the numerator, then substitute into Eq. (11):

$$p(r, f|\rho) = p_r^A + 2\mathrm{Tr}\left(\Pi_f^F \rho\right)\sqrt{p_r^A}\,\Re\left[\bar{g}_r^A A_{\mathrm{wk}}(\rho, f)\right] + O(g^2). \tag{13}$$

Let us condition this probability. Consider preparing ρ and measuring f. The measurement has a probability $p(f|\rho) := \mathrm{Tr}\left(\Pi_f^F \rho\right)$ of yielding f. Using $p(f|\rho)$ and Eq. (13), we calculate the conditional probability that, if one prepares ρ, weakly measures A, measures F, and obtains f from the final measurement, the weak measurement yielded r:

$$p(r|\rho, f) = \frac{p(r, f|\rho)}{p(f|\rho)} = p_r^A + 2\Re\left[\bar{g}_r^A A_{\mathrm{wk}}(\rho, f)\right] + O(g^2). \tag{14}$$

Assume that the detector is calibrated as follows: Imagine preparing the detector and measuring it strongly without having coupled the detector to the system. The average outcome r vanishes: $\int dr\; r\, p_r^A = 0$. This calibration assumption simplifies the proportionality of interest.

Consider integrating r against the conditional probability (14):

$$\int dr\; r\, p(r, f|\rho) = 2\int dr\; r\sqrt{p_r^A}\,\Re\left[\bar{g}_r^A A_{\mathrm{wk}}(\rho, f)\right] + O(g^2). \tag{15}$$

We define the left-hand side as $\langle r\rangle_{\rho,f}$. Suppose that $\bar{g}_r^A$ is real. The real function in Eq. (15) is $\Re\left[\bar{g}_r^A A_{\rm wk}(\rho,f)\right] = \bar{g}_r^A\, \Re[A_{\rm wk}(\rho,f)]$, and

$$\langle r\rangle_{\rho,f} = 2\,\Re[A_{\rm wk}(\rho,f)] \int dr\; r\sqrt{p_r^A}\;\bar{g}_r^A + O(g^2)\,. \tag{16}$$

In our experimental setup, $\sqrt{p_r^A} = \left(\frac{\delta t}{2\pi\tau}\right)^{1/4} \exp\left[-\frac{\delta t}{4\tau}\left(r^2+1\right)\right]$, and $\bar{g}_r^A = \frac{r}{\pi^{1/4}}\left(\frac{\delta t}{2\tau}\right)^{5/4} \exp\left[-\frac{\delta t}{4\tau}\left(r^2+1\right)\right]$. The integral in Eq. (16) evaluates to $\frac{1}{2}\exp\left(-\frac{\delta t}{2\tau}\right)$. The conditioned expectation value becomes

$$\boxed{\langle r\rangle_{\rho,f} = \Re\left[A_{\rm wk}(\rho,f)\right]\, e^{-\delta t/(2\tau)}}\,. \tag{17}$$

The exponential comes from the $+1$ in the exponentials in $\sqrt{p_r^A}$ and $\bar{g}_r^A$. Since the coupling is weak, $\frac{\delta t}{\tau} \ll 1$, the exponential ≈ 1, and $\langle r\rangle_{\rho,f} \approx \Re\left[A_{\rm wk}(\rho,f)\right]$.

Now, suppose that $\bar{g}_r^A$ is imaginary. The real function in Eq. (15) is $\Re[\bar{g}_r^A A_{\rm wk}(\rho,f)] = i\bar{g}_r^A \Im[A_{\rm wk}(\rho,f)]$, and

$$\langle r\rangle_{\rho,f} = 2i\,\Im[A_{\rm wk}(\rho,f)] \int dr\; r\sqrt{p_r^A}\;\bar{g}_r^A + O(g^2)\,. \tag{18}$$

For our experimental setup, $\sqrt{p_r^A} = \left(\frac{\gamma_\phi\,\delta t}{\pi}\right)^{1/4} \exp\left(-\frac{\gamma_\phi\,\delta t}{2}\,r^2\right)$, and $\bar{g}_r^A = -\frac{i}{\pi^{1/4}}\left(\gamma_\phi\,\delta t\right)^{5/4} r\, \exp\left(-\frac{\gamma_\phi\,\delta t}{2}\,r^2\right)$. The integral in Eq. (18) evaluates to $-\frac{i}{2}$. Equation (18) becomes

$$\boxed{\langle r\rangle_{\rho,f} = \Im[A_{\rm wk}(\rho,f)] + O(g^2)}\,. \tag{19}$$

III. DISPERSIVE MEASUREMENTS OF QUBIT PAULI OPERATORS

There are several different types of weak measurements that can be performed with our system, but as a test case we will study dispersive measurements of the σ_z operator. When combined with high fidelity rotations on the qubit, we can realize dispersive measurements of the qubit about any axis $\sigma\cdot\hat{\mathbf{n}}$. The POVM operator for a typical dispersive measurement is given by

$$M_r = \frac{1}{\sqrt{r_0}}\left(\frac{\delta t}{2\pi\tau}\right)^{1/4} \exp\left[-\frac{\delta t}{4\tau}\left(\frac{r}{r_0}\,\hat{1} - \sigma_z\right)^2\right] \tag{20}$$

The measurement result is labeled r. Here, δt is the measurement integration time and τ is the characteristic measurement time such that $\delta t/\tau$ characterizes the strength of the measurement and is the variance of the probability distribution for r given an eigenstate of σ_z. By decreasing τ and increasing δt sufficiently, this POVM can approximate a projective measurement of σ_z.

The POVM element is parameterized by a continuous parameter r. During data collection, r is discretized with some bin width ℓ. Moreover, r has a nontrivial dimensionality. We suppose, for concreteness, that r denotes a voltage. Let r_0 denote the unit of voltage. M_r has dimensions of $1/\sqrt{\text{voltage}}$, by the completeness condition $\int dr M_r^\dagger M_r = 1$. Hence M_r contains a $1/\sqrt{r_0}$.

The POVM 20 is the type of POVM that is implemented when we measure the quadrature of the field that conveys information about the qubit state in the z-basis, we can alternately measure the field in the other quadrature, leading to what Korotkov refers to as "realistic backaction", described by the POVM

$$M_r^\phi = \frac{1}{\sqrt{r_0}} \left(\frac{2\gamma_\phi \delta t}{2\pi} \right)^{1/4} \exp\left(-\frac{\gamma_\phi \delta t}{2} \left\{ \left[\frac{r}{r_0} \right]^2 + 2i \frac{r}{r_0} \sigma_z \right\} \right), \tag{21}$$

Where γ_ϕ is the ensemble dephasing rate due to the measurment, so $\gamma_\phi \delta t$ plays the same role as $\delta t/\tau$ in terms of the small parameter for weak measurement.

A. Expansion of the measurement operators: Z measurement

The goal here is to expand the POVM so that we have something that looks like $\sqrt{p_j^A}\hat{1} + g_j^A \sigma_z + O(g^2)$.

Assuming that $\delta t/\tau \ll 1$,

we Taylor-approximate M_r to first order:

$$\begin{aligned} M_r =& \frac{1}{\sqrt{r_0}} \left(\frac{\delta t}{2\pi\tau} \right)^{1/4} \exp\left[\left(-\frac{\delta t}{4\tau} \left[\left(\frac{r}{r_0} \right)^2 + 1 \right] \right) \left(1 + \frac{r}{2r_0} \left(\frac{\delta t}{\tau} \sigma_z \right) \right) \right] && (22) \\ =& \sqrt{p_r}\, 1 + \bar{g}_r^A \, \sigma_z + O\left(\left[\bar{g}_r^A \right]^2 \right) && (23) \end{aligned}$$

wherein

$$\boxed{\sqrt{p_r^A} := \frac{1}{\sqrt{r_0}} \left(\frac{\delta t}{2\pi\tau}\right)^{1/4} \exp\left[-\frac{\delta t}{4\tau}\left(\left(\frac{r}{r_0}\right)^2 + 1\right)\right]} \tag{24}$$

and

$$\boxed{\bar{g}_r^A := \frac{r}{\pi^{1/4}\,(r_0)^{3/2}} \left(\frac{\delta t}{2\tau}\right)^{5/4} \exp\left[-\frac{\delta t}{4\tau}\left(\left(\frac{r}{r_0}\right)^2 + 1\right)\right]} \tag{25}$$

Under discretization of r in experiments, each of $\sqrt{p_r^A}$ and $\bar{g}_r^A$ acquires a square-root of the binwidth, $\sqrt{\ell}$.

We can also check if the value is normalized:

$$\int_{-\infty}^{\infty} dr\ p_r^A = \int_{-\infty}^{\infty} \frac{dr}{r_0} \left(\frac{\delta t}{2\pi\tau}\right)^{1/2} \exp\left[-\frac{\delta t}{2\tau}((\frac{r}{r_0})^2 + 1)\right] = 1. \tag{26}$$

The final equality reflects how r_0 denotes the unit of voltage: $r_0 = 1$. We have included r_0 elsewhere explicitly to facilitate dimensional analysis. To check that the POVM satisfies the completeness relation

$$M_r^\dagger M_r = \frac{1}{r_0}\left(\frac{\delta t}{2\pi\tau}\right)^{1/2} \begin{bmatrix} \exp[-\frac{\delta t}{2\tau}(\frac{r}{r_0} - 1)^2] & 0 \\ 0 & \exp[-\frac{\delta t}{2\tau}(\frac{r}{r_0} + 1)^2] \end{bmatrix} \tag{27}$$

since $M_r^\dagger M_r$ is diagonal, we can calculate as below

$$\int_{-\infty}^{\infty} \frac{dr}{r_0} M_r^\dagger M_r = \left(\frac{\delta t}{2\pi\tau}\right)^{1/2} \begin{bmatrix} \int_{-\infty}^{\infty} \frac{dr}{r_0} \exp[-\frac{\delta t}{2\tau}(\frac{r}{r_0} - 1)^2] & 0 \\ 0 & \int_{-\infty}^{\infty} \frac{dr}{r_0} \exp[-\frac{\delta t}{2\tau}(\frac{r}{r_0} + 1)^2] \end{bmatrix} \tag{28}$$

$$= \left(\frac{\delta t}{2\pi\tau}\right)^{1/2} \begin{bmatrix} (\frac{2\pi\tau}{\delta t})^{1/2} & 0 \\ 0 & (\frac{2\pi\tau}{\delta t})^{1/2} \end{bmatrix} \tag{29}$$

$$= \left(\frac{\delta t}{2\pi\tau}\right)^{1/2} \left(\frac{2\pi\tau}{\delta t}\right)^{1/2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \tag{30}$$

B. Expansion of the measurement operators: ϕ measurement

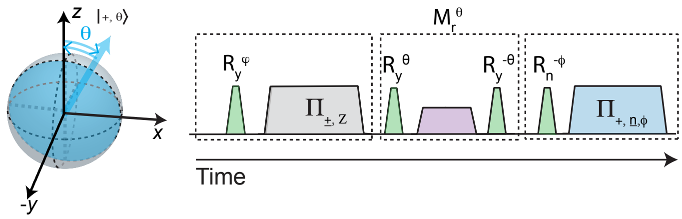
Turning to (21), we can also expand the POVM as above, finding

$$\sqrt{p_r^A} = \frac{1}{\sqrt{r_0}} \left(\frac{\gamma_\phi \delta t}{\pi} \right)^{1/4} \exp \left[-\frac{\gamma_\phi \delta t}{2} \left(\frac{r}{r_0} \right)^2 \right] \tag{31}$$

$$g_j^A = -i \, \frac{r}{(r_0)^{3/2}} \left(\frac{(\gamma_\phi \, \delta t)^5}{\pi} \right)^{1/4} \exp \left[-\frac{\gamma_\phi \delta t}{2} \left(\frac{r}{r_0} \right)^2 \right] \tag{32}$$

Where we see that the term g_j^A is imaginary in this case as is expected.

We therefore imagine an experimental sequence as depicted below, where the measurement $\mathcal{I}$ is simply a projective measurement in the σ_z basis, a weak measurement A is performed about some arbitrary axis that is at angle θ in the x—z plane, and a final projective measurement F is used to post-select the system in a final state.



The fidelities of the preselection is expected to be somewhere in the range of 99% if we preselect on the ground state, and for the final post-selection we expect it to be in the range of 95–99%. One issue with weak value experiments is that typically, the interesting cases occur very rarely, and in this case the infidelity can severely limit the type of post selections that can be performed. However, we do have some tricks that can be used to overcome these limitations.

IV. SIMPLE TEST CASE

Let's start with a simple example, where the $\mathcal{I}$, A, F measurements are simply projective measurements of σ_z, weak measurements of σ_x, and a projective measurements of σ_y respectively.

Working out the projectors for $\mathcal{I}$ and F, I find that $\mathrm{Tr}(\Pi_f^F \Pi_i^{\mathcal{I}}) = 1/2$ for any combination of $i, j = \pm$. Hence the first term in (8),

$$-\log\left(p_j^A \mathrm{Tr}(\Pi_f^F \Pi_i^{\mathcal{I}})\right) = -\log\left(\frac{p_j^A}{2}\right) \tag{33}$$

[**Taeho: work this out further given** (25)]

$$p_r^A = \frac{1}{r_0}\left(\frac{\delta t}{2\pi\tau}\right)^{1/2} \exp\left[-\frac{\delta t}{2\tau}\left(\left(\frac{r}{r_0}\right)^2 + 1\right)\right]$$

Therefore,

$$-\log\left(\frac{p_j^A}{2}\right) = -\log\left(\left(\frac{1}{2}\right)\left(\frac{\delta t}{2\pi\tau}\right)^{1/2} \exp\left[-\frac{\delta t}{2\tau}\left(\left(\frac{r}{r_0}\right)^2 + 1\right)\right]\right).$$

We have, again, invoked $r_0 = 1$.

$$= \log 2 - \log\left(\frac{\delta t}{2\pi\tau}\right)^{1/2} + \frac{\delta t}{2\tau}(j^2 + 1)$$

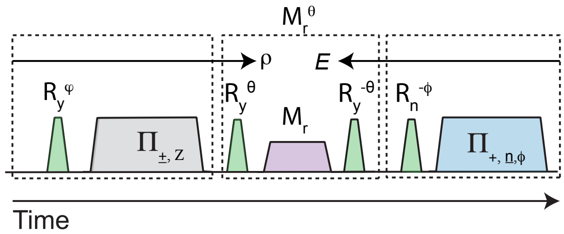
The final element to explore is the weak value in (8), which will be purely real in this case. Noting that the measurement record $r \simeq \langle\sigma_z\rangle + \text{noise}$ the average of r is a weak measurement of σ_z (or some other Pauli, given the appropriate rotations, thus the weak value is given by,

$$\langle r\rangle_{i,f} = \int r P_p(r)\, dr, \tag{34}$$

Where we have introduced notation from our previous work on past quantum states (Tan et. al. PRL 114, 090403 (2015)) and the subscript $\cdot_p$ denotes the fact that we're interested in the probability of detecting record r conditioned on the state ρ that is prepared by the measurement $\mathcal{I}$ but also on the post-selection from the measurement F.

$$P_p(r) = \frac{\mathrm{Tr}(M_r \rho M_r^\dagger E_f)}{\Sigma_r \mathrm{Tr}(M_r \rho M_r^\dagger E_f)} \tag{35}$$

The term E_f is the "effect" matrix, and it accounts for the post-selection imparted in the measurement F. For a projective measurement as a post-selection, $E = |\sigma_y, \pm\rangle\langle\sigma_y, \pm|$. For ease of calculating the weak value, we return to the sketch of the experiment,



The probability for detection of r, conditioned on the initial measurements, the various rotations, and post-selected on the final measurement can be contained in the two components of the "past quantum state", ρ, E, where ρ takes into account rotations before the weak measurement, and E takes into account the rotations that precede the post-selection measurement. Working out the probability explicitly in terms of the elements of ρ and E, (Factors of r_0 are omitted from the following calculation.)

$\mathrm{P}_p(r) \propto \mathrm{Tr}(M_r \rho M_r^\dagger E_f)$

$$\mathrm{M}_r \rho M_r^\dagger E_f = \begin{bmatrix} \exp[-\frac{\delta t}{2\tau}(r-1)^2]\rho_{00} & \exp[-\frac{\delta t}{4\tau}[(r-1)^2+(r+1)^2]]\rho_{01} \\ \exp[-\frac{\delta t}{4\tau}[(r+1)^2+(r-1)^2]]\rho_{10} & \exp[-\frac{\delta t}{2\tau}(r+1)^2]\rho_{11} \end{bmatrix} \begin{bmatrix} E_{00} & E_{01} \\ E_{10} & E_{11} \end{bmatrix} =$$

$$\begin{bmatrix} \exp[-\frac{\delta t}{2\tau}(r-1)^2]\rho_{00}E_{00} + \exp[-\frac{\delta t}{2\tau}[(r^2+1)]\rho_{01}E_{10} & \exp[-\frac{\delta t}{2\tau}(r-1)^2]\rho_{00}E_{01} + \exp[-\frac{\delta t}{2\tau}[(r^2+1)]\rho_{01}E_{11} \\ \exp[-\frac{\delta t}{2\tau}(r+1)^2]\rho_{11}E_{10} + \exp[-\frac{\delta t}{2\tau}[(r^2+1)]\rho_{10}E_{00} & \exp[-\frac{\delta t}{2\tau}(r+1)^2]\rho_{11}E_{11} + \exp[-\frac{\delta t}{2\tau}[(r^2+1)]\rho_{10}E_{01} \end{bmatrix}$$

$\mathrm{Tr}(\mathrm{M}_r \rho M_r^\dagger E_f) = \rho_{00}E_{00}e^{[-(r-1)^2\delta t/2\tau]} + \rho_{11}E_{11}e^{[-(r+1)^2\delta t/2\tau]} + (\rho_{01}E_{10} + \rho_{10}E_{01})e^{[-(r^2+1)\delta t/2\tau]}$

Therefore,

$$P_p(r) \propto \rho_{00}E_{00}e^{[-(r-1)^2\delta t/2\tau]} + \rho_{11}E_{11}e^{[-(r+1)^2\delta t/2\tau]} + (\rho_{01}E_{10} + \rho_{10}E_{01})e^{[-(r^2+1)\delta t/2\tau]} \tag{36}$$

Integrating, we have

$$\langle r\rangle_{i,f} = \frac{\rho_{00}E_{00} - \rho_{11}E_{11}}{\rho_{00}E_{00} + \rho_{11}E_{11} + \exp(\frac{-\delta t}{2\tau})(\rho_{01}E_{10} + \rho_{10}E_{01})} \tag{37}$$

which is a nice expression because it highlights how the anomalous nature of the weak value, where $\langle r \rangle_{i,f}$ exceeds the spectrum of σ_z, originates fundamentally from coherences. The usual definition of the weak value can be recovered from this expression; i.e. since the measurement signal is proportional to σ_z, which we'll denote as A, and we let $\rho = |\psi_i\rangle\langle\psi_i|$ and $E = |\psi_f\rangle\langle\psi_f|$, then the mean signal is given by,

$$\langle A_w \rangle = \mathrm{Re}\left[\frac{\langle\psi_f|A|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}\right]. \tag{38}$$

However, the formulation given in (37) is more useful for our experimental system since impurities and infidelities can be easily taken into account in the density matrix representation of the state.

V. GENERALIZATION TO ARBITRARY ANGLES

Set-up: Without loss of generality, we can choose for $\mathcal{I}$ to equal σ^z. Let $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$ denote arbitrary angles. We set A equal to the generalized Pauli operator defined by the vector $\vec{v}(\theta, \phi) := (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$:

$$A = \vec{\sigma} \cdot \vec{v}(\theta, \phi) = \sin\theta\cos\phi\,\sigma^x + \sin\theta\sin\phi\,\sigma^y + \cos\theta\,\sigma^z. \tag{39}$$

A has eigenvalues $a = \pm 1$. Onto the corresponding eigenspaces project the projectors Π_a^A. Since $A = \sum_a a\Pi_a^A$, the projectors have simple forms:

$$\Pi_a^A = \frac{1}{2}\left[1 + aA\right] \tag{40}$$

$$= \frac{1}{2}\left[1 + a(\sin\theta\cos\phi\,\sigma^x + \sin\theta\sin\phi\,\sigma^y + \cos\theta\,\sigma^z)\right]. \tag{41}$$

The Mathematica notebook in this folder confirms properties of the projectors: $\Pi_a^A \Pi_{a'}^A = \Pi_a^A\,\delta_{aa'}$, and $A\Pi_a^A A = \Pi_a^A$.

We set F equal to another Pauli operator,

$$F = \vec{\sigma} \cdot \vec{v}(\tilde{\theta}, \tilde{\phi}) \tag{42}$$

$$= \sin\tilde{\theta}\cos\tilde{\phi}\,\sigma^x + \sin\tilde{\theta}\sin\tilde{\phi}\,\sigma^y + \cos\tilde{\theta}\,\sigma^z, \tag{43}$$

defined by arbitrary angles $\tilde{\theta} \in [0, \pi]$ and $\tilde{\phi} \in [0, 2\pi)$. The eigenvalues $f = \pm 1$ correspond to the eigenprojectors

$$\Pi_f^F = \frac{1}{2} [1 + fF] . \tag{44}$$

The weak-measurement Kraus operators have the form

$$M_r = \frac{1}{\sqrt{r_0}} \left(\frac{\delta t}{2\pi\tau} \right)^{1/4} \exp\left(-\frac{\delta t}{4\tau} \left[\frac{r}{r_0} 1 - A \right]^2 \right) \tag{45}$$

$$= \sqrt{p_r}\ 1 + \bar{g}_r A + O\left([\bar{g}_r]^2 \right) . \tag{46}$$

The probability p_r and the coupling $\bar{g}_r$ equal the p_r^A and $\bar{g}_r^A$ above. Since our protocol involves only one weak measurement, the superscript is vestigial.

Let us calculate the entropic uncertainty relation's key components, the weak value $A_{\rm wk}$ and the entropies $H_{\rm vN}$.

Weak value: The eigenvalues i of $\mathcal{I}$ can easily be confused with $i = \sqrt{-1}$. I will therefore relabel the eigenvalues as $z \equiv i$. The weak value has the form

$$\boxed{A_{\rm wk}(i{=}z, f) = \frac{\mathrm{Tr}\left(\Pi_f^F A \Pi_{i=z}^{\mathcal{I}} \right)}{\mathrm{Tr}\left(\Pi_f^F \Pi_{i=z}^{\mathcal{I}} \right) \mathrm{Tr}\left(\Pi_{i=z}^{\mathcal{I}} \right)}} . \tag{47}$$

Let us evaluate each factor individually. The final trace is $\mathrm{Tr}\left(\Pi_{i=z}^{\mathcal{I}} \right) = 1$ for all $\mathcal{I}$ eigenvalues $z = \pm 1$.

The denominator's other factor is

$$\boxed{\mathrm{Tr}\left(\Pi_f^F \Pi_{i=z}^{\mathcal{I}} \right) = \frac{1}{2} \left[1 + zf \cos\tilde{\theta} \right]} \tag{48}$$

The final equality follows from five properties: (i) Every Pauli squares to the identity: $(\sigma^\alpha)^2 = 1\ \forall \alpha = x, y, z$. (ii) The Paulis compose as $\sigma^\alpha \sigma^\beta = i\epsilon_{\alpha\beta\gamma}\sigma^\gamma$, wherein $\epsilon_{\alpha\beta\gamma}$ denotes the totally antisymmetric tensor. (iii) The Paulis are traceless: $\mathrm{Tr}\left(\sigma^\alpha \right) = 0$. (iv) The identity has trace two: $\mathrm{Tr}(1) = 2$. Expression (**??**) obeys the normalization property for a conditional probability: $\sum_f p(f|i{=}z) = \sum_f \mathrm{Tr}\left(\Pi_f^F \Pi_{i=z}^{\mathcal{I}} \right) = 1$.

Let us calculate the numerator in Eq. (50). We substitute in for the operators and invoke the four properties:

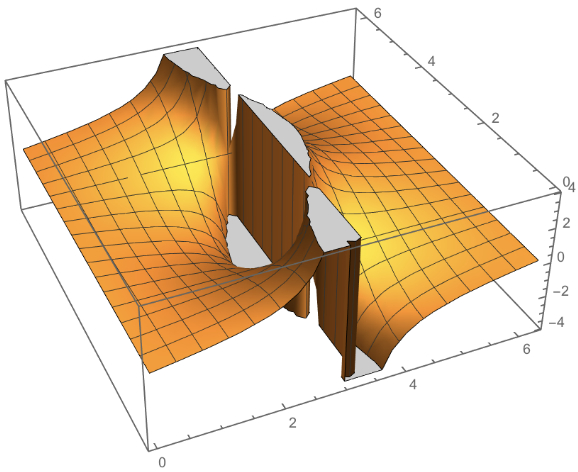
$$\boxed{\mathrm{Tr}\left(\Pi_f^F A \Pi_z^{\mathcal{I}}\right) = \frac{1}{2}\left[\cos\theta(z + f\cos\tilde{\theta}) + f\sin\theta\sin\tilde{\theta}(\cos[\phi-\tilde{\phi}] + iz\sin[\phi-\tilde{\phi}])\right]} \tag{49}$$

Finally, we can get the value of $A_{\mathrm{wk}}(z{=}i, f)$ from plugging in values to eq(31)

$$\boxed{A_{\mathrm{wk}}(i{=}z, f) = \frac{\cos\theta(z + f\cos\tilde{\theta}) + f\sin\theta\sin\tilde{\theta}(\cos[\phi-\tilde{\phi}] + iz\sin[\phi-\tilde{\phi}])}{1 + zf\cos\tilde{\theta}}} \tag{50}$$

Here, we can consider the real part of $A_{\mathrm{wk}}(i{=}z, f)$ and let the parameters to be $\phi = 0, \theta = \pi/2, z = 1, f = 1$.

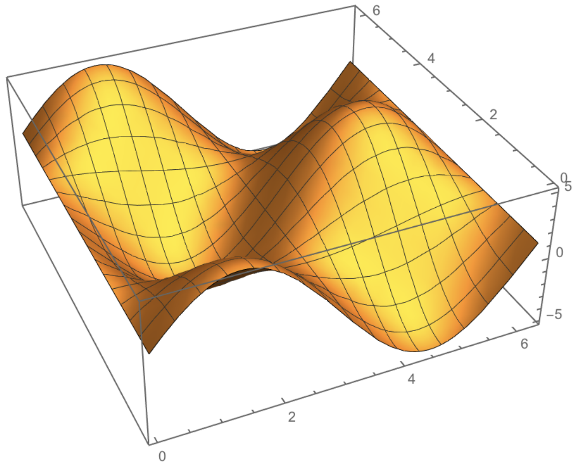
Then, we can draw the 3D plot of $A_{\mathrm{wk}}(i{=}z, f)$ in the region of $0 < \tilde{\theta} < 2\pi, 0 < \tilde{\phi} < 2\pi$



Also, in the case of

$$\tilde{\phi} = 0, \tilde{\theta} = 7\pi/8, z = 1, f = 1$$

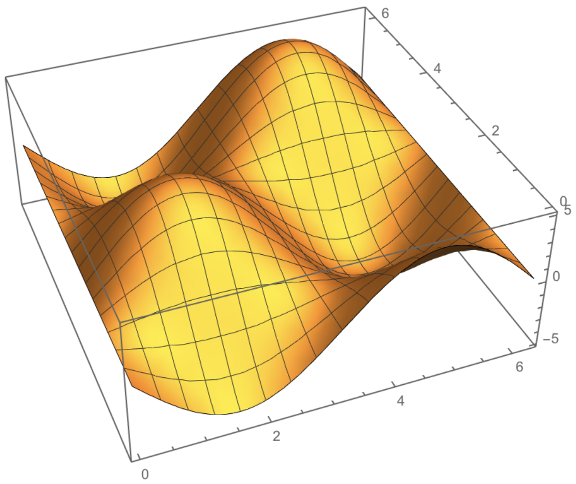
in the region of $0 < \theta < 2\pi, 0 < \phi < 2\pi$, the 3D plot of $A_{\mathrm{wk}}(i{=}z, f)$ can be drawn as below.



Also, we can consider the imaginary part of $A_{\rm wk}(i{=}z, f)$ and let the parameters to be

$$\tilde{\phi} = 0, \tilde{\theta} = 7\pi/8, z = 1, f = 1$$

Let us draw the 3D plot of $A_{\rm wk}(i{=}z, f)$ in the region of $0 < \theta < 2\pi, 0 < \phi < 2\pi$.



First von Neumann entropy, $H_{\rm vN}(\mathcal{I})_\rho$: Qn arbitrary qubit state has the form

$$\rho = \frac{1}{2}\left(1 + \vec{w}\cdot\vec{\sigma}\right). \tag{51}$$

The vector $\vec{w} = (w_x, w_y, w_z)$ is of probability weights $w_\alpha \in [0, 1]$: $\sum_{\alpha=x,y,z} w_\alpha = 1$.

Consider measuring the $\mathcal{I} = \sigma^z$ of ρ. The probability of obtaining the eigenvalue $i \equiv z$ equals

$$p(i{=}z|\rho) = \mathrm{Tr}\left(\Pi_z^Z \rho\right) \tag{52}$$

$$= \frac{1}{4}\,\mathrm{Tr}\left(\left[1 + (-1)^{z+1}\sigma^z\right]\left[1 + \sum_\alpha w_\alpha \sigma^\alpha\right]\right) \tag{53}$$

$$= \frac{1}{2}\left[1 + (-1)^{z+1} w_z\right]. \tag{54}$$

These probabilities are normalized: $\sum_{z=\pm 1} p(i{=}z|\rho) = 1$. This distribution has the von Neumann entropy

$$\boxed{H_{\mathrm{vN}}(\mathcal{I})_\rho} = -\sum_{z=\pm 1} p(z|\rho) \log p(z|\rho) \tag{55}$$

$$\boxed{= -\frac{1}{2}\left\{(1 + w_z)\log\left(\frac{1}{2}\left[1 + w_z\right]\right) + (1 - w_z)\log\left(\frac{1}{2}\left[1 - w_z\right]\right)\right\}}. \tag{56}$$

Second von Neumann entropy, $H_{\mathrm{vN}}(AF)_\rho$: Consider measuring A weakly, then measuring F strongly. This protocol has a probability density

$$p(r, f|\rho) = \mathrm{Tr}\left(\Pi_f^F M_r \rho M_r^\dagger\right) \tag{57}$$

associated with yielding the outcomes r and f. To calculate a probability associated with an outcome within ℓ of r, we would multiply $p(r, f|\rho)$ by ℓ. After the weak measurement, the system occupies the (unnormalized) state

$$M_r \rho M_r^\dagger \approx \left(\sqrt{p_r}\,1 + \bar{g}_r A\right)\rho\left(\sqrt{p_r}\,1 + \bar{g}_r A\right) \tag{58}$$

$$= p_r \rho + \sqrt{p_r}\,\bar{g}_r(\rho A + A\rho) + O\left([\bar{g}_r]^2\right). \tag{59}$$

The Hermiticity of M_r informs the first equality. Substituting into Eq. (57) yields

$$p(r, f|\rho) = \frac{1}{2}\,\mathrm{Tr}\Big(\left\{1 + [-1]^{f+1}\sigma^z\right\}\left[p_r\rho + \sqrt{p_r}\,\bar{g}_r(\rho A + A\rho)\right]\Big) + O\left(g^2\right) \tag{60}$$

$$= \frac{1}{2}\Big[p_r + 2\sqrt{p_r}\,\bar{g}_r \mathrm{Tr}(\rho A) + (-1)^{f+1} p_r \mathrm{Tr}(\sigma^z\rho) + 2(-1)^{f+1}\sqrt{p_r}\,\bar{g}_r \Re\left(\mathrm{Tr}(\sigma^z\rho A)\right)\Big] + O(g^2). \tag{61}$$

Let us evaluate the expression piece by piece. First,

$$\rho A = \frac{1}{2}\left(1 + \sum_\alpha w_\alpha \sigma^\alpha\right)(\sin\theta\cos\phi\;\sigma^x + \sin\theta\sin\phi\;\sigma^y + \cos\theta\;\sigma^z). \tag{62}$$

Therefore, by the four properties mentioned above,

$$\mathrm{Tr}(\rho A) = \vec{w} \cdot \vec{v}(\theta, \phi), \tag{63}$$

and $\Re(\mathrm{Tr}(\sigma^z \rho A)) = \cos\theta$. Substituting into Eq. (61) yields

$$\boxed{p(r, f|\rho) = p_r \left[\frac{1}{2} + (-1)^{f+1} w_z\right] + \sqrt{p_r}\, \bar{g}_r \left[\vec{w} \cdot \vec{v}(\theta, \phi) + (-1)^{f+1}\, \cos\theta\right]}. \tag{64}$$

Whether this result obeys normalization or contains an error is questionable:

$$\int_{-\infty}^{\infty} dr \sum_{f=\pm 1} p(r, f|\rho) = 1 + \vec{w} \cdot \vec{v}(\theta, \phi) \int dr \sqrt{p_r}\, \bar{g}_r. \tag{65}$$

VI. EXPERIMENTAL CONSIDERATIONS

Testing the entropic uncertainty relation will involve some kind of measurements, and these measurements have certain infidelities, typically around the 95% to 99% level. This affects the purity of what kind of states can be prepared and post-selected, and even very high fidelities can lead to serious problems. For example, anomalous weak values tend to occur when the pre-selection and the post-selection are nearly orthogonal; experimentally one pre-selects an initial state, makes a weak measurement, and then post-selects a final state - the weak value is given by the average of the weak measurement result (and in some cases this has to be scaled by the coupling factor). If there is a 99% fidelity for the postselected state, but the expected post-selection success is only 0.1% (i.e. $|\langle\psi_i|\psi_f\rangle|^2 = 0.001$), then the number of wrong postselections will outnumber the correct ones 10 to 1—which will wash out any anomaly in the weak value.

The goal of the project is to test the uncertianty relation, for which we'd like to measure both sides of the inequality and explore where the bound is tight. The left hand side of Equation (3) contains the entropies. These are quantites that can be sampled experimentally; i.e. we'd prepare a certain state and then make appropriate measurements and based on the relative probabilities of the measurement outcomes, we determine the entropies. Here's where experimental imperfections will come in; if our state ρ is such that one of the entropy terms is vanishing, the finite fidelity will cause us to measure a larger than actual entropy.

A. How an overly wide Gaussian POVM alters the entropies in the weak-measurement uncertainty relation

The lab's detectors have an efficiency $\eta \approx 0.5$. The experimentally inferred $p(r|f)$ will therefore have about double the width of the $p(r|f)$ detected with perfect efficiency. This doubling will increase the entropy $H_{\text{vN}}(\bar{A}F)_\rho$ in the entropic uncertainty relation's left-hand side. How much does this entropy change if noise "broadens" the weak measurement? By only about a bit, or a term $+\log 2$.

VII. ZEROTH-ORDER APPROXIMATION

Let all symbols be defined as in "Reconciling" and our working notes. Consider preparing ρ, weakly measuring A, strongly measuring F, and conditioning on the strong-measurement outcome f. With what probability has the weak measurement outputted r? The probability density has a Gaussian form, according to Kater:

$$p(r|f) \approx \frac{1}{\sqrt{2\pi\sigma^2}} \, e^{-(r-r_0)^2/(2\sigma^2)}. \tag{66}$$

r_0 denotes the mean, and σ denotes the standard deviation. Noise tends to broaden the distribution. Suppose that noise doubles the standard deviation: $\sigma \mapsto \sigma' = 2\sigma$. How much does $H_{\text{vN}}(\bar{A}F)_\rho$ change?

In a simple analysis, we neglect the F in H: We approximate the change $\Delta H_{\text{vN}}(\bar{A}F)_\rho$ with the change $\Delta H_{\text{Sh}}\big(p(r|f)\big)$. The Shannon entropy is denoted by H_{Sh}. This approximation is expected to be reasonable if all the distributions $\{p(r|f)\}_r$ broaden similarly, despite having different f values.

To zeroth order, a Gaussian is a rectangle of width 2σ. Normalization requires that the height be $1/(2\sigma)$. This flat distribution's Shannon entropy equals the log of the size of the distribution's support: $H_{\rm Sh}(p(r|f)) \approx \log(2\sigma)$. Doubling the standard deviation doubles the support's size: $\Delta H_{\rm Sh}(p(r|f)) \approx \log(4\sigma) - \log(2\sigma) = \log 2$. Hence

$$\boxed{\Delta H_{\rm vN}(\bar{A}F)_\rho \approx \log 2}\,. \tag{67}$$

This change is small, $\Delta H_{\rm vN}(\bar{A}F)_\rho \ll H_{\rm vN}(\bar{A}F)_\rho \approx \log(2\sigma)$, if $p(r|f)$ has substantial weight on $\gg 2$ values of r.

VIII. SHANNON ENTROPY OF CONTINUOUS PROBABILITY DENSITY

The probability distribution inferred from experiments will be discrete. But less us calculate the Shannon entropy of the approximation (66):

$$H_{\rm Sh}(p(r|f)) = -\int_{-\infty}^{\infty} dr\; p(r|f)\;\log p(r|f) \tag{68}$$

$$= -\frac{1}{\sqrt{2\pi\sigma^2}}\left(\int_{-\infty}^{\infty} dr\;\frac{(r-r_0)^2}{2\sigma^2}\;e^{-(r-r_0)^2/2\sigma^2} + \frac{1}{2}\;\log(2\pi\sigma^2)\int_{-\infty}^{\infty} dr\; e^{-(r-r_0)^2/2\sigma^2}\right). \tag{69}$$

The final integral, a simple Gaussian integral, equals $\sqrt{2\pi\sigma^2}$. The first integral, the second moment of a Gaussian, equals $\sqrt{\frac{\pi}{2}}\,\sigma$. Substituting in yields

$$H_{\rm Sh}(p(r|f)) = \log\sigma + \frac{1}{2}\,[1 + \log(2\pi)]. \tag{70}$$

The entropy scales with the standard deviation logarithmically, as expected from the foregoing section. Discretizing $p(r|f)$, I expect, will (i) scale σ by 1/(bin width) and (ii) possibly change the constants. Cutting the integral off at finite r-values should introduce extra terms.