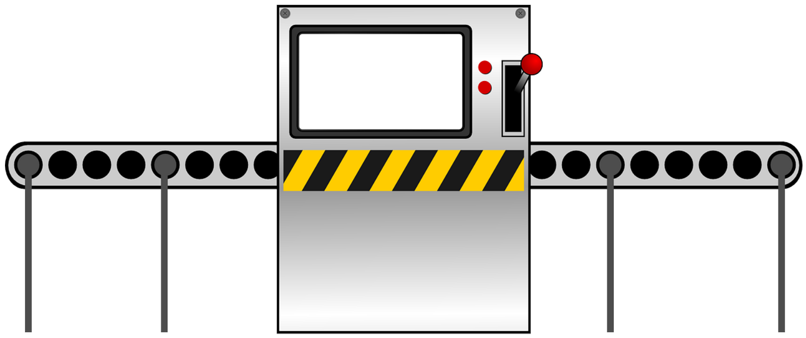
Solution - Robust optimization

31-03-2020

Production planning



You work for a production company and support them with optimizing their capacity and production schedule for a new factory.

The company has $p \in P$ different products that are produced on different machine types $m \in M$. Not each product can be produced on each machine, i.e., parameter $a_{p,m} = 1$, if product p can be produced on machine type m and $a_{p,m} = 0$ otherwise. As you are opening a new factory, you also have to decide how many machines of type m you want to buy. The price is c^M_m for one machine of type $m \in M$. Each machine of type $m \in M$ provides T_m hours of production.

The production costs are c^P_p for each $p \in P$. The targeted production quantities d_p for each product $p \in \mathcal{P}$ for the next year are given and should at least be covered. Because we consider the entire year, we approximate the production quantities as continuous values.

The production time of product $p \in P$ is uncertain. You know that the expected production time is $\bar{t}_p$ and the deviation (positive and negative) can be up to t^{dev}_p. From experience from other factories, we can conclude that for each machine type $m \in M$ not more than 30% of the products that can be produced on machine type m will have a deviation from the expected production time.

Have a look at the following model formulation

y_m	Number of machines of type m
$x_{p,m}$	Production amount of product p on machine type m

$$\min \sum_{m\in M} \left[c^M_m y_m + \sum_{p\in P} c^P_p x_{p,m} \right] \tag{1a}$$

$$s.t. \sum_{m\in M} x_{p,m} \geq d_p \qquad \forall p \in P \tag{1b}$$

$$\sum_{p\in P} \tilde{t}_p x_{p,m} \leq T_m y_m \qquad \forall m \in M, \tilde{t}_p \in [\bar{t}_p - t^{\text{dev}}_p, \bar{t}_p + t^{\text{dev}}_p] \tag{1c}$$

$$x_{p,m} \leq BigM_{m,p} a_{p,m} y_m \qquad \forall m \in M, p \in P \tag{1d}$$

$$y_m \geq 0 \text{ and integer} \qquad \forall m \in M \tag{1e}$$

$$x_{p,m} \geq 0 \qquad \forall m \in M, p \in P \tag{1f}$$

where $\tilde{t}_p$ represents the uncertain production time and $BigM_{m,p}$ a large enough constant (here: $\frac{T_m}{\bar{t}_p - t^{\text{dev}}_p}$).

The objective function (1a) minimizes the cost. Constraints (1b) ensure that the targeted production amounts are fulfilled. Constraints (1c) limits the production time and constraints (1d) model the compatibility between products and machine types.

Task

The model above is the robust counterpart that does not consider the uncertainty set in a linear formulation. Furthermore, it does not include a budget of uncertainty.
Your tasks is to transform the model to a robust linear formulation using the budget of uncertainty Γ_m per machine type $m \in M$. (Hint: We are mainly talking about constraint (1c).)

Solve your robust model with Julia JuMP and Gurobi using the file `production.jl` which already has some data input and part of the model formulation. The part about the production time is missing. You have to add the missing constraints and variable needed for the budget of uncertainty formulation.

- no deviation from the expected production time is included in the model ($\Gamma_m = 0$)
- full deviation from the expected production time is included in the model (all products on that machine type can deviate at the same time)

Solution

The model (1a)-(1f) does not model that maximum 30% of the products per machine type take their worst case values. In the remainder, we use a budget of uncertainty $\Gamma_m = 0.3 \sum_{p\in P} a_{p,m}$ per machine type.

Reformulate constraint (1c) with budget of uncertainty:

$$\sum_{p\in P} \bar{t}_p x_{p,m} + \max_{\{S_m\cup\{s_m\}|S_m\subseteq P, |S_m|\leq \Gamma_m, s_m\in P\setminus S_m\}} \left\{ \sum_{p\in S_m} (t_p^{\text{dev}}|x_{p,m}|) + (\Gamma_m - \lfloor \Gamma_m \rfloor) t_{s_m}^{\text{dev}} |x_{s_m,m}|) \right\} \leq T_m y_m \qquad \forall m \in M \tag{2}$$

Formulate subproblem in (2) as LP for each $m \in M$, which S_m and s_m being the subsets of products that lead to the worst case:

$$\max_{\{S_m\cup\{s_m\}|S_m\subseteq P, |S_m|\leq \Gamma_m, s_m\in P\setminus S_m\}} \left\{ \sum_{p\in S_m} (t_p^{\text{dev}}|x_{p,m}|) + (\Gamma_m - \lfloor \Gamma_m \rfloor) t_{s_m}^{\text{dev}} |x_{s_m,m}|) \right\}$$

$$\Leftrightarrow$$

$$\max \quad \sum_{p\in P_m} t_p^{\text{dev}} |x_{p,m}| z_{p,m} \tag{3a}$$

$$s.t. \sum_{p\in P} z_{p,m} \leq \Gamma_m \qquad : \lambda_m \tag{3b}$$

$$0 \leq z_{p,m} \leq 1 \qquad \forall p \in P \qquad : \mu_{p,m} \tag{3c}$$

Dual of (3a)-(3c):

$$\min \quad \Gamma_m \lambda_m + \sum_{p\in P} \mu_{p,m} \tag{4a}$$

$$s.t. \lambda_m + \mu_{p,m} \geq t_p^{\text{dev}} |x_{p,m}| \qquad \forall p \in P \tag{4b}$$

$$\lambda_m \geq 0 \tag{4c}$$

$$\mu_{p,m} \geq 0 \qquad \forall p \in P \tag{4d}$$

Replace subproblem in (2) with (4a)-(4d) to get the overall robust model.

$$\min \sum_{m\in M} \left[c_m^M y_m + \sum_{p\in P} c_p^P x_{p,m} \right] \tag{5a}$$

$$s.t. \sum_{m\in M} x_{p,m} \geq d_p \qquad \forall p \in P \tag{5b}$$

$$\sum_{p\in P} \bar{t}_p x_{p,m} + \Gamma_m \lambda_m + \sum_{p\in P} \mu_{p,m} \leq T_m y_m \qquad \forall m \in M \tag{5c}$$

$$\lambda_m + \mu_{p,m} \geq t_p^{\text{dev}} x_{p,m} \qquad \forall m \in M, p \in P \tag{5d}$$

$$x_{p,m} \leq BigM_{m,p} a_{p,m} y_m \qquad \forall m \in M, p \in P \tag{5e}$$

$$y_m \geq 0 \text{ and integer} \qquad \forall m \in M \tag{5f}$$

$$x_{p,m} \geq 0 \qquad \forall m \in M, p \in P \tag{5g}$$

$$\lambda_m \geq 0 \qquad \forall m \in M \tag{5h}$$

$$\mu_{p,m} \geq 0 \qquad \forall m \in M, p \in P \tag{5i}$$

Note that we can remove the absolute value in constraint (5d), because $x_{p,m}$ is always positive.

The solution values for the three cases are given below. The production amounts are the same for all three solutions, because we have to cover the demand in each case, which does not change. However, the number of machines per type are different, because the production times are uncertain and can deviate more or less in the different cases. In case with no deviations (factor 0.0), the production time is always the expected production time and we do not have increased production times in the worst case. Therefore, we need less machines. In case with no restrictions on the deviations (factor 1.0) more machines are needed to cover the demand also in the worst case (with long production times). The solution for the restricted budget of uncertainty (factor 0.3) lies in between and leads to less cost and machines compared the very conservative case.

		$\Gamma_m = 0.3 \sum_{p \in P} a_{p,m}$	$\Gamma_m = 0.0 \sum_{p \in P} a_{p,m}$	$\Gamma_m = 1.0 \sum_{p \in P} a_{p,m}$
Obj		29458532	29378532	29508532
Machines	m1	22	20	24
	m2	0	0	0
	m3	12	10	13
	m4	3	3	3
Production	p1	5844	5844	5844
	p2	9313	9313	9313
	p3	5725	5725	5725
	p4	8511	8511	8511
	p5	9465	9465	9465
	p6	27866	27866	27866
	p7	28396	28396	28396
	p8	27394	27394	27394
	p9	27590	27590	27590
	p10	27612	27612	27612