Technical University of Denmark

02435 Decision-Making Under Uncertainty

Assignment 3

Jorge Montalvo Arvizu, s192184

Spring 2020

Task 1 L-shaped method

Mathematical model of the planning problem to optimize the production of a small district heating company:

$$\text{Min} \quad \sum_{t\in T}\left[\sum_{j\in J}\left(c_j^{CHP}q_{j,t}^{CHP} - \sum_{s\in S}\pi_s e_{t,s}\frac{1}{\phi_j}q_{j,t}^{CHP}\right) + \sum_{i\in I}\sum_{s\in S}\pi_s c_i^H q_{i,t,s}^H\right] \tag{1}$$

$$\text{s.t.} \quad q_{j,t}^{CHP} \le Q_j^{CHP} \qquad \forall j\in J, t\in T \tag{2}$$

$$q_{i,t,s}^{H} \le Q_i^{H} \qquad \forall i\in I, \forall t\in T, s\in S \tag{3}$$

$$\sum_{j\in J} q_{j,t}^{CHP} + \sum_{i\in I} q_{i,t,s}^{H} = d_{t,s} \qquad \forall t\in T, s\in S \tag{4}$$

$$q_{j,t}^{CHP} \ge 0 \qquad \forall j\in J, t\in T \tag{5}$$

$$q_{i,t,s}^{H} \ge 0 \qquad \forall i\in I, t\in T, s\in S \tag{6}$$

1.1 Artificial full recourse

Artificial full recourse should be introduced to solve the two-stage stochastic program without complete recourse, i.e. when not all master problem solutions lead to feasible subproblem solutions. In this case, constraint (4) contains both first-stage and second-stage variables and it could be the case that by solving the master problem and $q_{j,t}^{CHP}$ is obtained for some time t, constraint (4) might not be feasible in the case that the resulting total heat production of all CHP units at time t is too low and the sum of all heat production (both from CHP units and heat-only units) will not equal the demand at time t for scenario s since $q_{j,t}^{CHP}$ and $q_{i,t,s}^{H}$ are constrained by their maximum heat production capacities per hour through constraints (2) and (3). Thus, the artificial full recourse is introduced via the implementation of two artificial variables to measure the infeasibility of the subproblems and penalize it in the objective function through penalty constants; in this case, $v_{t,s}^{+}$ and $v_{t,s}^{-}$ represent the slack between the total heat production and total demand, thus finding the optimal solution by penalizing the difference between the total heat production and demand. The following equations are added (by replacing some of the previous equations):

$$\text{Min} \quad \sum_{t\in T}\left[\sum_{j\in J}\left(c_j^{CHP}q_{j,t}^{CHP} - \sum_{s\in S}\pi_s e_{t,s}\frac{1}{\phi_j}q_{j,t}^{CHP}\right) + \sum_{i\in I}\sum_{s\in S}\pi_s \left(c_i^H q_{i,t,s}^H + \phi^+ v_{t,s}^{+} + \phi^- v_{t,s}^{-}\right)\right] \tag{7}$$

$$\text{s.t.} \quad \sum_{j\in J} q_{j,t}^{CHP} + \sum_{i\in I} q_{i,t,s}^{H} + v_{t,s}^{+} - v_{t,s}^{-} = d_{t,s} \qquad \forall t\in T, s\in S \tag{8}$$

$$v_{t,s}^{+} \ge 0 \qquad \forall t\in T, s\in S \tag{9}$$

$$v_{t,s}^{-} \ge 0 \qquad \forall t\in T, s\in S \tag{10}$$

with the following equation mappings:

- (1) → (7)
- (4) → (8)
- (9) and (10) are new constraints

and the following new variables and parameters:

New parameters	
ϕ^+	Penalty cost for positive slack (constant, e.g. 10000) [EUR/MWh-heat]
ϕ^-	Penalty cost for negative slack (constant, e.g. 10000) [EUR/MWh-heat]
New positive variables	
$v^+_{t,s}$	Positive slack variable in period t and scenario s [MWh-heat]
$v^-_{t,s}$	Negative slack variable in period t and scenario s [MWh-heat]

1.2 Master and subproblem

Given that the infeasibility of the subproblems is now solved, then it is possible to proceed to divide the problem into master and subproblems with $\mathcal{L}$ general optimality cuts.

Master problem:

$$\text{Min} \quad \sum_{t\in T}\sum_{j\in J}\left(c_j^{CHP}q_{j,t}^{CHP} - \sum_{s\in S}\pi_s e_{t,s}\frac{1}{\phi_j}q_{j,t}^{CHP}\right) + \theta \tag{11}$$

$$\text{s.t.} \quad q_{j,t}^{CHP} \leq Q_j^{CHP} \qquad \forall j \in J, t \in T \tag{12}$$

$$\sum_{t\in T} E_{t,l}\sum_{j\in J} q_{j,t}^{CHP} + \theta \geq \sum_{t\in T} e_{t,l} \qquad \forall l \in \mathcal{L} \tag{13}$$

$$q_{j,t}^{CHP} \geq 0 \qquad \forall j \in J, t \in T \tag{14}$$

$$\theta \in \mathbb{R} \tag{15}$$

Subproblems:

$$\text{Min} \quad \sum_{t\in T}\sum_{i\in I}\sum_{s\in S}\left[\pi_s\left(c_i^H q_{i,t,s}^H + \phi^+ v_{t,s}^+ + \phi^- v_{t,s}^-\right)\right] \tag{16}$$

$$\text{s.t.} \quad \sum_{j\in J}\{q_{j,t}^{CHP}\}_{fixed} + \sum_{i\in I} q_{i,t,s}^H + v_{t,s}^+ - v_{t,s}^- = d_{t,s} \qquad \forall t \in T, s \in S \tag{17}$$

$$q_{i,t,s}^H \leq Q_i^H \qquad \forall i \in I, \forall t \in T, s \in S \tag{18}$$

$$q_{i,t,s}^H \geq 0 \qquad \forall i \in I, t \in T, s \in S \tag{19}$$

$$v_{t,s}^+ \geq 0 \qquad \forall t \in T, s \in S \tag{20}$$

$$v_{t,s}^- \geq 0 \qquad \forall t \in T, s \in S \tag{21}$$

with the following new variables and sets:

New sets	
$\mathcal{L}$	Set of optimality cuts
New parameters	
E_l	Cut coefficient [-]
e_l	Cut coefficient (RHS) [-]
New real variables	
θ	Optimality cut value (bound) [EUR]

and variable $q_{j,t}^{CHP}$ fixed in the subproblems.

1.3 Dual variable and objective

Based on the subproblems from equations (16)-(21), the necessary dual variables are obtained from equation (17) with indices $\{t, s\}$ and equation (18) with indices $\{i, t, s\}$, as follows:

$$\sum_{j\in J}\{q^{CHP}_{j,t}\}_{fixed} + \sum_{i\in I} q^{H}_{i,t,s} + v^{+}_{t,s} - v^{-}_{t,s} = d_{t,s} \quad : \quad \lambda_{t,s} \qquad \forall t \in T, s \in S \tag{22}$$

$$q^{H}_{i,t,s} \le Q^{H}_{i} : \quad \nu_{i,t,s} \qquad \forall i \in I, \forall t \in T, s \in S \tag{23}$$

Thus defining the following dual subproblem objective function **for each scenario** s:

$$\text{Max} \quad \sum_{t\in T}\left[\left(d_{t,s} - \sum_{j\in J}\{q^{CHP}_{j,t}\}_{fixed}\right)\lambda_{t,s} + \sum_{i\in I}\left(Q^{H}_{i}\right)\nu_{i,t,s}\right] \qquad \forall s \in S \tag{24}$$

with the following variable:

Variable	
$\lambda_{t,s}$	Dual variable obtained from equation (17) of the subproblem [EUR/MWh-heat]
$\nu_{i,t,s}$	Dual variable obtained from equation (18) of the subproblem [EUR/MWh-heat]

1.4 Cut coefficients

From the previous dual variable, the cut coefficients are calculated by taking the expected value of the recourse function over all scenarios as follows:

$$E_{t,l} = \sum_{s\in S}\pi_s\lambda_{t,s} \qquad \forall t \in T, \forall l \in \mathcal{L} \tag{25}$$

$$e_{t,l} = \sum_{s\in S}\sum_{i\in I}\pi_s\left[\lambda_{t,s}d_{t,s} + \nu_{i,t,s}Q^{H}_{i}\right] \qquad \forall t \in T, \forall l \in \mathcal{L} \tag{26}$$

1.5 Cut constraint

Thus, new values of the recourse function is bounded from below (via θ), resulting in the following cut constraint:

$$\sum_{t\in T}\left[\sum_{s\in S}\pi_s\lambda_{t,s}\right]\sum_{j\in J}q^{CHP}_{j,t} + \theta \ge \sum_{t\in T}\sum_{s\in S}\sum_{i\in I}\pi_s\left[\lambda_{t,s}d_{t,s} + \nu_{i,t,s}Q^{H}_{i}\right] \qquad \forall l \in \mathcal{L} \tag{27}$$

Task 2 (Meta-)heuristics - General questions

As opposed to exact methods, where the computing time increases with the problem size (combinatorial optimization problems) making them inapplicable to time-constrained applications, heuristics methods give a solution with a shorter runtime. However, these methods cannot guarantee finding an optimal solution compared to exact methods. Regarding the solution structure, heuristics methods are problem-specific, contrary to exact methods which are problem-independent. For example, in local search the neighbor solutions depend on each problem thus the neighborhood has to be independently developed for each of them, also for simulated annealing there's a need for parameter tuning, where these are hard to decide and directly affects the problem. Another problem might be the possible convergence into local minima since it only searches through a part of the whole solution space. This local minima problem could be addressed with metaheuristics, where the solution is improved via different methods that operate on intensification or diversification strategies, looking for a better solution. Finally, it is important to note that some of these methods, if poorly applied, might lead to errors or long computing time to find a good solution or, especially when handling models with uncertainty - nonetheless, they provide a wide range of modelling possibilities that could be applied to hard problems

Task 3 Robust optimization - General questions

1. First, to find only feasible solutions within all possible realizations of the uncertainty in the data. Second, to find the best solution for the worst-case realization of the uncertain data. Both of these purposes are addressed with Robust Optimization (RO) since it aims at solving problems where data contains uncertainty given measurement/estimation errors (or implementation errors) that could heavily affect the quality of the problem's solution.

2. The main difference is that RO solves for the worst-case scenario and is feasible for all uncertainty cases, thus is more conservative - risk-averse - than Stochastic Optimization (SO). RO uses a continuous set of uncertainty that is computationally tractable and used through, for example, box uncertain sets or polyhedral uncertain sets. In contrast, SO uses a discretization of the uncertainty set and the computational tractability is often very hard to obtain its underlying assumptions/simplifications, for example, the hard problem of using the true distribution of the uncertainty which is often times unknown.

3. Adjustable robust optimization is considered less conservative than normal robust optimization because it relaxes the assumption that all variables are here-and-now decisions and makes use of recourse decisions as in SO. Therefore, these 'new' wait-and-see variables can freely adjust its value after the uncertainty is resolved. Hence, since not all variables are here-and-now and aren't considered in a single set, the worst-case scenario is relaxed.

Task 4 Reformulation to robust linear models

Considering a basis model, three model cases are reformulated to a robust linear model.

4.1 $\tilde{a}$ and $\tilde{b}$ as uncertain:

$$\begin{aligned}
\text{Max } & 10x_1 + 20x_2 + 15x_3 && (28)\\
\text{s.t.} \quad & 5x_1 + 3x_2 + \tilde{a}x_3 \le 10 && (29)\\
& 7x_1 - 2x_2 - \tilde{b}x_3 \ge 5 && (30)\\
& -2 \le x_1 \le 10 && (31)\\
& 0 \le x_2 \le 15 && (32)\\
& -10 \le x_3 \le 10 && (33)
\end{aligned}$$

$\tilde{a}$ and $\tilde{b}$ are uniform distributed with $\tilde{a} \sim \mathcal{U}(1,5)$ and $\tilde{b} \sim \mathcal{U}(1,3)$.

1. Reformulating the uncertainty sets using the mean value and range (box uncertainty sets):

$$\tilde{a} = \bar{a} + P^{\tilde{a}}\zeta^{\tilde{a}}$$
$$\tilde{b} = \bar{b} + P^{\tilde{b}}\zeta^{\tilde{b}}$$

with the following values obtained from the mean and range of each distribution:

$$\bar{a} = 3 \qquad P^{\tilde{a}} = 2 \qquad |\zeta^{\tilde{a}}| = 1$$
$$\bar{b} = 2 \qquad P^{\tilde{b}} = 1 \qquad |\zeta^{\tilde{b}}| = 1$$

2. Replacing the reformulated uncertainty sets into the constraints:

$$5x_1 + 3x_2 + 3x_3 + 2\zeta^{\tilde{a}}x_3 \le 10 \qquad \forall|\zeta^{\tilde{a}}| \le 1$$
$$7x_1 - 2x_2 - 2x_3 - \zeta^{\tilde{b}}x_3 \ge 5 \qquad \forall|\zeta^{\tilde{b}}| \le 1$$

3. Assuming worst-case scenario:

$$5x_1 + 3x_2 + 3x_3 + \max_{-1\le\zeta\le1}\left\{2\zeta^{\tilde{a}}x_3\right\} \le 10$$
$$7x_1 - 2x_2 - 2x_3 - \max_{-1\le\zeta\le1}\left\{\zeta^{\tilde{b}}x_3\right\} \ge 5$$

The second equation has a negative sign therefore the worst-case scenario should happen at the maximum value even though there's a $\geq$ sign.

4. Linearizing the equations:

$$5x_1 + 3x_2 + 3x_3 + |2x_3| \leq 10$$
$$7x_1 - 2x_2 - 2x_3 - |x_3| \geq 5$$

then:

$$\begin{aligned} 5x_1 + 3x_2 + 3x_3 + \alpha &\leq 10 \\ 7x_1 - 2x_2 - 2x_3 - \beta &\geq 5 \\ -\alpha \leq 2x_3 &\leq \alpha \\ -\beta \leq x_3 &\leq \beta \\ \alpha &\geq 0 \\ \beta &\geq 0 \end{aligned}$$

Variables α and β are new positive variables used to linearize the absolute term of the equations.

5. Replacing the equations into the final model:

$$\begin{aligned} &\text{Max } 10x_1 + 20x_2 + 15x_3 && (34) \\ \text{s.t.} \quad & 5x_1 + 3x_2 + 3x_3 + \alpha \leq 10 && (35) \\ & 7x_1 - 2x_2 - 2x_3 - \beta \geq 5 && (36) \\ & -\alpha \leq 2x_3 \leq \alpha && (37) \\ & -\beta \leq x_3 \leq \beta && (38) \\ & -2 \leq x_1 \leq 10 && (39) \\ & 0 \leq x_2 \leq 15 && (40) \\ & -10 \leq x_3 \leq 10 && (41) \\ & \alpha \geq 0 && (42) \\ & \beta \geq 0 && (43) \end{aligned}$$

where equations (34)-(43) replaces equations (28)-(33). The solution is:

```
Variable values:
x1: 1.129
x2: 1.452
x3: 0.000

Objective value: 40.323
```

4.2 $\tilde{c}$ as uncertain:

$$\begin{aligned} &\text{Max } 10x_1 + \tilde{c}x_2 + 15x_3 && (44) \\ \text{s.t.} \quad & 5x_1 + 3x_2 + 3x_3 \leq 10 && (45) \\ & 7x_1 - 2x_2 - 2x_3 \geq 5 && (46) \\ & -2 \leq x_1 \leq 10 && (47) \\ & 0 \leq x_2 \leq 15 && (48) \\ & -10 \leq x_3 \leq 10 && (49) \end{aligned}$$

$\tilde{c}$ is uniform distributed with $\tilde{c} \sim \mathcal{U}(5, 35)$.

1. Reformulating the uncertainty sets using the mean value and range (box uncertainty sets):

$$\tilde{c} = \bar{c} + P^{\tilde{c}}\zeta^{\tilde{c}}$$

with the following values obtained from the mean and range of each distribution:

$$\bar{c} = 20 \qquad P^{\tilde{c}} = 15 \qquad |\zeta^{\tilde{c}}| = 1$$

2. Replace the uncertainty set with a new variable in the objective function and replacing the reformulated uncertainty set into the new constraint:

$$\text{Max} \qquad 10x_1 + \gamma + 15x_3 \tag{50}$$

$$20x_2 + 15\zeta^{\tilde{c}}x_2 \geq \gamma \qquad \forall|\zeta^{\tilde{c}}| \leq 1 \tag{51}$$

where equation (50) replaces the objective function, including a new variable ($\gamma \in \mathbb{R}$), and a new constraint based on the uncertainty set in the objective function.

3. Assuming worst-case scenario:

$$20x_2 + \min_{-1\leq\zeta\leq1} \left\{15\zeta^{\tilde{c}}x_2\right\} \geq \gamma$$

4. Linearizing the equations:

$$20x_2 - |15x_2| \geq \gamma$$

then:

$$\begin{aligned} 20x_2 - \eta &\geq \gamma \\ -\eta \leq 15x_2 &\leq \eta \\ \eta &\geq 0 \end{aligned}$$

Note that it is also possible to avoid the extra variable η since the variable x_2 is always positive, so the absolute term can be dropped.

5. Replacing the equations into the final model:

$$\text{Max } 10x_1 + \gamma + 15x_3 \tag{52}$$
$$\text{s.t.} \quad 5x_1 + 3x_2 + 3x_3 \leq 10 \tag{53}$$
$$7x_1 - 2x_2 - 2x_3 \geq 5 \tag{54}$$
$$20x_2 - \eta \geq \gamma \tag{55}$$
$$-\eta \leq 15x_2 \leq \eta \tag{56}$$
$$-2 \leq x_1 \leq 10 \tag{57}$$
$$0 \leq x_2 \leq 15 \tag{58}$$
$$-10 \leq x_3 \leq 10 \tag{59}$$
$$\eta \geq 0 \tag{60}$$
$$\gamma \in \mathbb{R} \tag{61}$$

where equations (52)-(61) replaces equations (44)-(49). The solution is:

```
Variable values:
x1: 1.129
x2: 0.000
x3: 1.452

Objective value: 33.065
```

4.3 $\tilde{a}_2$ and $\tilde{a}_3$ as uncertain:

$$
\begin{aligned}
\text{Max } & 10x_1 + 20x_2 + 15x_3 && (62)\\
\text{s.t. } & 5x_1 + \tilde{a}_2 x_2 + \tilde{a}_3 x_3 \leq 10 && (63)\\
& 7x_1 - 2x_2 - \tilde{b}x_3 \geq 5 && (64)\\
& -2 \leq x_1 \leq 10 && (65)\\
& 0 \leq x_2 \leq 15 && (66)\\
& -10 \leq x_3 \leq 10 && (67)
\end{aligned}
$$

$\tilde{a}_2$ and $\tilde{a}_3$ can take values in the uncertainty set $U = \{\tilde{a}_2 + \tilde{a}_3 \leq 8, \tilde{a}_2 + \tilde{a}_3 \geq 2, \tilde{a}_2, \tilde{a}_3 \geq 0\}$. Given the uncertainty set definition, polyhedral uncertainty has to be applied.

1. Representing the worst-case scenario of the constraint involving the uncertainty:

$$5x_1 + \max\{\tilde{a}_2 x_2\} + \max\{\tilde{a}_3 x_3\} \leq 10$$

2. Rewritting the subproblem with the maximization of the variables with uncertainty:

$$
\begin{aligned}
\text{Max} \quad & \tilde{a}_2 x_2 + \tilde{a}_3 x_3 & \\
\text{s.t.} \quad & \tilde{a}_2 + \tilde{a}_3 \leq 8 & : \lambda_1\\
& \tilde{a}_2 + \tilde{a}_3 \geq 2 & : \lambda_2\\
& \tilde{a}_2, \tilde{a}_3 \geq 0 &
\end{aligned}
$$

3. Evaluating the dual problem, considering the dual variables of the inequality constraints (λ_1 and λ_2) of the primal problem.

$$
\begin{aligned}
\text{Min} \quad & 8\lambda_1 - 2\lambda_2\\
\text{s.t.} \quad & \lambda_1 - \lambda_2 = x_2\\
& \lambda_1 - \lambda_2 = x_3\\
& \lambda_1, \lambda_2 \geq 0
\end{aligned}
$$

4. Including the dual into the primal replacing equation (63) with the following constraints:

$$
\begin{aligned}
5x_1 + \min\{8\lambda_1 - 2\lambda_2\} \leq 10\\
\lambda_1 - \lambda_2 = x_2\\
\lambda_1 - \lambda_2 = x_3\\
\lambda_1, \lambda_2 \geq 0
\end{aligned}
$$

Then:

$$
\begin{aligned}
5x_1 + 8\lambda_1 - 2\lambda_2 \leq 10\\
\lambda_1 - \lambda_2 = x_2\\
\lambda_1 - \lambda_2 = x_3\\
\lambda_1, \lambda_2 \geq 0
\end{aligned}
$$

since the minimization is included in a $\leq$-constraint: if one value fulfils the constraint the minimum will too, therefore the minimization can be dropped.

5. Including everything in the final model:

$$\begin{aligned}
\text{Min} \quad & 10x_1 + 20x_2 + 15x_3 && (68)\\
\text{s.t.} \quad & 5x_1 + 8\lambda_1 - 2\lambda_2 \leq 10 && (69)\\
& 7x_1 - 2x_2 - 2x_3 \geq 5 && (70)\\
& \lambda_1 - \lambda_2 = x_2 && (71)\\
& \lambda_1 - \lambda_2 = x_3 && (72)\\
& -2 \leq x_1 \leq 10 && (73)\\
& 0 \leq x_2 \leq 15 && (74)\\
& -10 \leq x_3 \leq 10 && (75)\\
& \lambda_1, \lambda_2 \geq 0 && (76)
\end{aligned}$$

where equations (68)-(76) replaces equations (62)-(67). Finally, the solution is:

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Variable values:
x1: 1.053
x2: 0.592
x3: 0.592

Objective value: 31.250
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4.4 $\tilde{a}_1$, $\tilde{a}_2$ and $\tilde{a}_3$ as uncertain:

$$\begin{aligned}
\text{Max } & 10x_1 + 20x_2 + 15x_3 && (77)\\
\text{s.t.} \quad & \tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \tilde{a}_3 x_3 \leq 10 && (78)\\
& 7x_1 - 2x_2 - \tilde{b} x_3 \geq 5 && (79)\\
& -2 \leq x_1 \leq 10 && (80)\\
& 0 \leq x_2 \leq 15 && (81)\\
& -10 \leq x_3 \leq 10 && (82)
\end{aligned}$$

$\tilde{a}_1$, $\tilde{a}_2$ and $\tilde{a}_3$ are uniform distributed with $\tilde{a}_1 \sim \mathcal{U}(1,8)$, $\tilde{a}_2 \sim \mathcal{U}(2,9)$, $\tilde{a}_3 \sim \mathcal{U}(1,5)$. At most two of the three parameters will deviate from their mean value. Therefore, budget of uncertainty has to be applied with $\Gamma = 2$.

1. Reformulating the uncertainty sets using the mean value and range:

$$\tilde{a}_1 = \bar{a}_1 + P^{\tilde{a}_1} \zeta^{\tilde{a}_1}$$
$$\tilde{a}_2 = \bar{a}_2 + P^{\tilde{a}_2} \zeta^{\tilde{a}_2}$$
$$\tilde{a}_3 = \bar{a}_3 + P^{\tilde{a}_3} \zeta^{\tilde{a}_3}$$

with the following values obtained from the mean and range of each distribution:

$$\begin{aligned}
&\bar{a}_1 = 4.5 \qquad P^{\tilde{a}_1} = 3.5 \qquad |\zeta^{\tilde{a}_1}| = 1\\
&\bar{a}_2 = 5.5 \qquad P^{\tilde{a}_2} = 3.5 \qquad |\zeta^{\tilde{a}_1}| = 1\\
&\bar{a}_3 = 3 \qquad P^{\tilde{a}_3} = 2 \qquad |\zeta^{\tilde{a}_1}| = 1
\end{aligned}$$

2. Considering the worst-case scenario:

$$\sum_j \bar{a}_j x_j + \max_{\{S \cup \{t\} | S \subseteq J, |S| \leq \lfloor \Gamma \rfloor, t \in J \setminus S\}} \left\{ \sum_{j \in S} \hat{a}_j \, |x_j| + (\Gamma - \lfloor \Gamma \rfloor) \, \hat{a}_t \, |x_j| \right\} \leq 10$$

where:

$\bar{a}_j$	Mean value of uncertain parameter a_j
$\hat{a}_j$	Variance value of uncertain parameter a_j
x_j	Set of variables
J	Set of all parameters $\{1, 2, 3\}$
$S \in J$	Set of parameters that change in the constraint
Γ	Budget of uncertainty $= 2$
$\lfloor \Gamma \rfloor$	Worst-case value
t	Parameters changing in the constraint.

3. Applying the worst-case:

$$\begin{aligned} \text{Max} \quad & \sum_{j \in J} \hat{a}_j \left| x_j \right| z_j \\ \text{s.t.} \quad & \sum_{j \in J} z_j \leq \Gamma \qquad : \lambda \\ & 0 \leq z_j \leq 1 \qquad : \mu_j \qquad \forall j \in J \end{aligned}$$

which occurs with the maximization of the uncertain parameters since it is a $\leq$ constraint.

4. Evaluating the dual problem with the dual variables λ and μ_j:

$$\begin{aligned} \text{Min} \quad & \Gamma\lambda + \sum_{j \in J} \mu_j \\ \text{s.t.} \quad & \lambda + \mu_j \geq \hat{a}_j \left| x_j \right| && \forall j \in J \\ & \lambda \geq 0 \\ & \mu_j \geq 0 \forall && \forall j \in J \end{aligned}$$

5. Then, the dual problem substitutes the relative primal in the constraint. The it has to be linearized in order to solve the absolute value on the first constraint, as shown below.

$$\begin{aligned} & \sum_{j \in J} \bar{a}_j x_j + \Gamma\lambda + \sum_{j \in J} \mu_j \leq 10 \\ \text{s.t.} \quad & \lambda + \mu_j \geq \hat{a}_j y_j && \forall j \in J \\ & -y_j \leq x_j \leq y_j && \forall j \in J \\ & \lambda \geq 0 \\ & \mu_j \geq 0 && \forall j \in J \\ & y_j \geq 0 && \forall j \in J \end{aligned}$$

since it is a $\leq$ constraint, the minimum was taken out since if one value fulfills the constraint then the minimum will too.

6. Replacing everything into the final model:

$$\begin{aligned} \text{Min} \quad & 10x_1 + 20x_2 + 15x_3 && && (83) \\ \text{s.t.} \quad & \sum_{j \in J} \bar{a}_j x_j + \Gamma\lambda + \sum_{j \in J} \mu_j \leq 10 && && (84) \\ & 7x_1 - 2x_2 - 2x_3 \geq 5 && && (85) \\ & \lambda + \mu_j \geq \hat{a}_j y_j && \forall j \in J && (86) \\ & -y_j \leq x_j \leq y_j && \forall j \in J && (87) \\ & -2 \leq x_1 \leq 10 && && (88) \\ & 0 \leq x_2 \leq 15 && && (89) \\ & -10 \leq x_3 \leq 10 && && (90) \\ & \lambda \geq 0 && && (91) \\ & \mu_j \geq 0 && \forall j \in J && (92) \\ & y_j \geq 0 && \forall j \in J && (93) \end{aligned}$$

where equations (83)-(93) replaces equations (77)-(82). Finally, the solution is:

```
Variable values:
x1: 0.878
x2: 0.209
x3: 0.365

Objective value: 18.435
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Task 5 Adjustable Robust Optimization

5.1 RO Model

Sets	
T	Set of timesteps [Month] $\{1, ..., 12\}$
I	Set of machines [Machine index] $\{1, 2\}$
Parameters	
D_t	Total demand per month t [units]
c_i	Cost per produced unit per machine i [EUR/unit] $\{4, 2\}$
K_i	Production capacity per machine i [units] $\{400k, 650k\}$
η_i	Production efficiency per machine i [between 0 and 1] $\{1, [0.6, 0.8]\}$
$\bar{\eta_2}$	Production efficiency mean for machine 2 [0.7]
η_2^{dev}	Production efficiency deviation for machine 2 [0.1]
Positive variables	
$p_{i,t}$	Production of machine i at month t [units]

Mathematical model:

$$\text{Min} \quad \sum_{t \in T} \sum_{i \in I} c_i p_{i,t} \tag{94}$$

$$\text{s.t.} \quad p_{i,t} \leq K_i \qquad \forall i \in I, \forall t \in T \tag{95}$$

$$p_{1,t} + p_{2,t}\bar{\eta_2} - p_{2,t}\eta_2^{dev} \geq D_t \qquad \forall t \in T \tag{96}$$

$$p_{i,t} \geq 0 \qquad \forall i \in I, \forall t \in T \tag{97}$$

where equation (94) represents the objective function by minimizing the cost of production of each machine i for each month t, equation (95) represents the production capacity of each machine i, equation (96) represents the demand constraint where the production of machine 1 is 1, while the production of machine 2 is based on a box uncertainty set; the demand should be at least fulfilled for each month t. Finally, equation (97) enforces the non-negativity of production for each machine i.

Note that equation (96) was obtained from the following steps:

1. Reformulating the uncertainty set of machine 2 using the box uncertainty set:

$$\eta_2 = \bar{\eta_2} + \eta_2^{dev}\zeta$$

with the following values obtained from the mean and range:

$$\bar{\eta_2} = 0.7 \qquad \eta_2^{dev} = 0.1 \qquad |\zeta| = 1$$

2. Using the reformulated uncertainty set in a constraint:

$$p_{1,t} + p_{2,t}\bar{\eta_2} + p_{2,t}\eta_2^{dev}\zeta \geq D_t \qquad \forall|\zeta^{\tilde{a}}| \leq 1, \forall t \in T$$

$$p_{1,t} + 0.7p_{2,t} + 0.1p_{2,t}\zeta \geq D_t \qquad \forall|\zeta^{\tilde{a}}| \leq 1, \forall t \in T$$

3. Assuming worst-case scenario:

$$p_{1,t} + 0.7p_{2,t} + \min_{-1\leq\zeta\leq1} \{0.1p_{2,t}\zeta\} \geq D_t \qquad \forall|\zeta^{\tilde{a}}| \leq 1, \forall t \in T$$

4. Finally, linearizing the constraint:

$$p_{1,t} + 0.7p_{2,t} - |0.1p_{2,t}| \geq D_t \qquad \forall t \in T$$

then:

$$p_{1,t} + 0.7p_{2,t} - 0.1p_{2,t} \geq D_t \qquad \forall t \in T$$

since the absolute value includes the production $p_{2,t}$ which is always positive, therefore the absolute operator can be dropped.

5.2 RO Model - Results

```
Objective value: € 29772000.0 [EUR]
Variable values:
p[1, 1]: 317000.0 [units]
p[1, 2]: 363000.0 [units]
p[1, 3]: 334000.0 [units]
p[1, 4]: 394000.0 [units]
p[1, 5]: 309000.0 [units]
p[1, 6]: 153000.0 [units]
p[1, 7]: 174000.0 [units]
p[1, 8]: 132000.0 [units]
p[1, 9]: 303000.0 [units]
p[1, 10]: 353000.0 [units]
p[1, 11]: 370000.0 [units]
p[1, 12]: 341000.0 [units]
p[2, 1]: 650000.0 [units]
p[2, 2]: 650000.0 [units]
p[2, 3]: 650000.0 [units]
p[2, 4]: 650000.0 [units]
p[2, 5]: 650000.0 [units]
p[2, 6]: 650000.0 [units]
p[2, 7]: 650000.0 [units]
p[2, 8]: 650000.0 [units]
p[2, 9]: 650000.0 [units]
p[2, 10]: 650000.0 [units]
p[2, 11]: 650000.0 [units]
p[2, 12]: 650000.0 [units]
```

5.3 ARO Model

Sets	
T	Set of timesteps [Month] $\{1, ..., 12\}$
I	Set of machines [Machine index] $\{1, 2, 3\}$
Parameters	
D_t	Total demand per month t [units]
c_i	Cost per produced unit per machine i [EUR/unit] $\{4, 2, 3\}$
K_i	Production capacity per machine i [units] $\{400k, 650k, 200k\}$
η_i	Production efficiency per machine i [between 0 and 1] $\{1, [0.6, 0.8], 0.8\}$
$\bar{\eta}_2$	Production efficiency mean for machine 2 [0.7]
η_2^{dev}	Production efficiency deviation for machine 2 [0.1]
Positive variables	
$p_{i,t}$	Production of machine i at month t [units]
p_t^0	Base production for machine 3 after uncertainty at month t [units]
Q_t	Variable production for machine 3 after uncertainty at month t [units]
β_t	Production cost for machine 3 at month t [EUR]

Mathematical model:

$$\text{Min} \quad \sum_{t\in T} \sum_{i\in I\setminus 3} c_i p_{i,t} + \beta_t \tag{98}$$

$$\text{s.t.} \quad p_{i,t} \le K_i \qquad \forall i \in I, \forall t \in T \tag{99}$$

$$p_{1,t} + p_{2,t}\bar{\eta}_2 + p_t^0\eta_3 - p_{2,t}\eta_2^{dev} - \eta_3 Q_t \ge D_t \qquad \forall t \in T \tag{100}$$

$$p_{3,t} = p_t^0 + Q_t \qquad \forall t \in T \tag{101}$$

$$c_3 p_{3,t} \le \beta_t \qquad \forall t \in T \tag{102}$$

$$p_t^0 - Q_t \ge 0 \qquad \forall t \in T \tag{103}$$

$$p_{i,t} \ge 0 \qquad \forall i \in I, \forall t \in T \tag{104}$$

$$p_t^0 \ge 0 \qquad \forall t \in T \tag{105}$$

$$Q_t \ge 0 \qquad \forall t \in T \tag{106}$$

$$\beta_t \ge 0 \qquad \forall t \in T \tag{107}$$

where equation (98) represents the objective function by minimizing the production cost and including the variable β as the production cost of the virtual machine 3. Equation (99) enforces the production to always be less than the maximum capacity per month t, equation (100) represents the demand constraint with the uncertainty taken into account, equation (101) represents the equality constraint of the total production of virtual machine 3 as the sum of the base and variable production, equation (102) represents the production cost of virtual machine 3, equation (103) represents the minimum production of virtual machine 3, equations (104)-(107) enforces the non-negativity of the variables (since negative production/cost can't happen in this problem).

5.4 ARO Model - Results

```
Production cost: € 29300750.0 [EUR]

Production quantity:
p[1, 1]: 157000.0 [units]
p[1, 2]: 203000.0 [units]
p[1, 3]: 174000.0 [units]
p[1, 4]: 234000.0 [units]
```

```
p[1, 5]: 149000.0 [units]
p[1, 6]: 0.0 [units]
p[1, 7]: 14000.0 [units]
p[1, 8]: 0.0 [units]
p[1, 9]: 143000.0 [units]
p[1, 10]: 193000.0 [units]
p[1, 11]: 210000.0 [units]
p[1, 12]: 181000.0 [units]
p[2, 1]: 650000.0 [units]
p[2, 2]: 650000.0 [units]
p[2, 3]: 650000.0 [units]
p[2, 4]: 650000.0 [units]
p[2, 5]: 650000.0 [units]
p[2, 6]: 650000.0 [units]
p[2, 7]: 650000.0 [units]
p[2, 8]: 650000.0 [units]
p[2, 9]: 650000.0 [units]
p[2, 10]: 650000.0 [units]
p[2, 11]: 650000.0 [units]
p[2, 12]: 650000.0 [units]
p[3, 1]: 200000.0 [units]
p[3, 2]: 200000.0 [units]
p[3, 3]: 200000.0 [units]
p[3, 4]: 200000.0 [units]
p[3, 5]: 200000.0 [units]
p[3, 6]: 191250.0 [units]
p[3, 7]: 200000.0 [units]
p[3, 8]: 165000.0 [units]
p[3, 9]: 200000.0 [units]
p[3, 10]: 200000.0 [units]
p[3, 11]: 200000.0 [units]
p[3, 12]: 200000.0 [units]
```