Solution - Adjustable Robust optimization

14-04-2020

Production planning continued

Assume the following robust program (1a) - (1g) for the production example from the exercise on Robust Optimization (31-03-2020).

Here the full problem statement as reminder:
You work for a production company and support them with optimizing their capacity and production schedule for a new factory.

The company has $p \in P$ different products that are produced on different machine types $m \in M$. Not each product can be produced on each machine, i.e., parameter $a_{p,m} = 1$, if product p can be produced on machine type m and $a_{p,m} = 0$ otherwise. As you are opening a new factory, you also have to decide how many machines of type m you want to buy. The price is c^M_m for one machine of type $m \in M$. Each machine of type $m \in M$ provides T_m hours of production.

The production costs are c^P_p for each $p \in P$. The targeted production quantities d_p for each product $p \in \mathcal{P}$ for the next year are given and should at least be covered. Because we consider the entire year, we approximate the production quantities as continuous values.

The production time of product $p \in P$ is uncertain. You know that the expected production time is $\bar{t}_p$ and the deviation (positive and negative) can be up to t^{dev}_p. From experience from other factories, we can conclude that for each machine type $m \in M$ not more than 30% of the products that can be produced on machine type m will have a deviation from the expected production time.

Use the following robust formulation using a box-uncertainty set.

y_m Number of machines of type m

$x_{p,m}$ Production amount of product p on machine type m

$$\min \sum_{m \in M} \left[c^M_m y_m + \sum_{p \in P} c^P_p x_{p,m} \right] + \sum_{p \in P} \Phi \delta_p(\zeta_p) \qquad -1 \leq \zeta_p \leq 1 \tag{1a}$$

$$s.t. \sum_{m \in M} x_{p,m} + \delta_p(\zeta_p) \geq d_p \qquad \forall p \in P, -1 \leq \zeta_p \leq 1 \tag{1b}$$

$$\sum_{p \in P} \bar{t}_p x_{p,m} + \sum_{p \in P} t^{\text{dev}}_p \zeta_p x_{p,m} \leq T_m y_m \qquad \forall m \in M, -1 \leq \zeta_p \leq 1 \tag{1c}$$

$$x_{p,m} \leq BigM_{m,p} a_{p,m} y_m \qquad \forall m \in M, p \in P \tag{1d}$$

$$y_m \geq 0 \text{ and integer} \qquad \forall m \in M \tag{1e}$$

$$x_{p,m} \geq 0 \qquad \forall m \in M, p \in P \tag{1f}$$

$$\delta_p(\zeta_p) \geq 0 \qquad \forall p \in P, -1 \leq \zeta_p \leq 1 \tag{1g}$$

where ζ_p describes the uncertainty in the production time of product p. $BigM_{m,p}$ is a large constant (here: $\frac{T_m}{\bar{t}_p - t^{\text{dev}}_p}$). The objective function (1a) minimizes the cost. Constraints (1b) ensure that the targeted production amounts are fulfilled. Constraints (1c) limits the production time and constraints (1d) model the compatibility between products and machine types.

We now allow the demand constraint in (1b) to be violated, i.e., not fulfilling the demand is allowed but penalized with Φ per unit in the objective function. δ_p is the amount of products of type p that we are missing.

Task

1. Transform constraint (1c) to a linear robust formulation by transforming the box uncertainty set.
2. The missing amount δ_p is a recourse decision and based on the uncertainty of the production time ζ_p. Therefore, you need to formulate a Linear Decision Rule dependent on the uncertain parameter $-1 \leq \zeta_p \leq 1$. Reformulate the necessary parts of the model based on your Linear Decision Rule.

3. Solve the model for different values of the penalty term $\Phi = \{100, 150, 200, 300\}$ using the model in the Julia file `production_ARO.jl` as a basis. The file has deterministic model of the problem above, i.e., a placeholder variable for δ_p is used and constraint (1c) uses the mean value of the uncertain parameter.

y_m	Number of machines of type m
$x_{p,m}$	Production amount of product p on machine type m

Solution

1)

Transformation of constraint (1c)

$$\sum_{p \in P} \bar{t}_p x_{p,m} + \sum_{p \in P} t_p^{\text{dev}} \zeta_p x_{p,m} \leq T_m y_m \qquad \forall m \in M, -1 \leq \zeta_p \leq 1$$

Formulate worst case:

$$\sum_{p \in P} \bar{t}_p x_{p,m} + \sum_{p \in P} \max_{-1 \leq \zeta_p \leq 1} \{t_p^{\text{dev}} \zeta_p x_{p,m}\} \leq T_m y_m \qquad \forall m \in M$$

Reformulate to absolute value (due to box uncertainty):

$$\sum_{p \in P} \bar{t}_p x_{p,m} + \sum_{p \in P} |t_p^{\text{dev}} x_{p,m}| \leq T_m y_m \qquad \forall m \in M$$

Drop absolute value due $x_{p,m} \geq 0 \forall p \in P, m \in M$ and $t_p^{\text{dev}} \geq 0$:

$$\sum_{p \in P} \bar{t}_p x_{p,m} + \sum_{p \in P} t_p^{\text{dev}} x_{p,m} \leq T_m y_m \qquad \forall m \in M$$

2)

Definition of linear decision rule

We use the linear decision rule $\delta_p(\zeta_p) = z_p + Q_p \zeta_p$ with $p \in P$, $z_p \in \mathbb{R}$, $Q_p \in \mathbb{R}$ and $-1 \leq \zeta_p \leq 1$.

Transformation of constraint (1b)

$$\sum_{m \in M} x_{p,m} + \delta_p(\zeta_p) \geq d_p \qquad \forall p \in P, -1 \leq \zeta_p \leq 1$$

Insert linear decision rule:

$$\sum_{m \in M} x_{p,m} + z_p + Q_p \zeta_p \geq d_p \qquad \forall p \in P, -1 \leq \zeta_p \leq 1$$

Formulate worst-case:

$$\sum_{m \in M} x_{p,m} + z_p + \max_{-1 \leq \zeta_p \leq 1} Q_p \geq d_p \qquad \forall p \in P$$

Reformulate to absolute value (due to box uncertainty) and negative sign due to $\geq$-constraint:

$$\sum_{m \in M} x_{p,m} + z_p - |Q_p| \geq d_p \qquad \forall p \in P$$

Linearize absolute value using new variable $\alpha_p \geq 0, \forall p \in P$:

$$\sum_{m \in M} x_{p,m} + z_p - \alpha_p \geq d_p \qquad \forall p \in P$$

$$-\alpha_p \leq Q_p \leq \alpha_p \qquad \forall p \in P$$

Transformation of constraint (1g)

$$\delta_p(\zeta_p) \geq 0 \qquad \forall p \in P, -1 \leq \zeta_p \leq 1$$

Insert linear decision rule:

$$z_p + Q_p\zeta_p \geq 0 \qquad \forall p \in P, -1 \leq \zeta_p \leq 1$$

Formulate worst-case:

$$z_p + \min_{-1 \leq \zeta_p \leq 1} \{Q_p\zeta_p\} \geq 0 \qquad \forall p \in P$$

Reformulate to absolute value (due to box uncertainty) with negative sign due to $\geq$-constraint:

$$z_p - |Q_p| \geq 0 \qquad \forall p \in P$$

Use α as absolute value from before.

$$z_p - \alpha_p \geq 0 \qquad \forall p \in P$$

Transformation of objective function (1a)

$$\min \sum_{m \in M} \left[c_m^M y_m + \sum_{p \in P} c_p^P x_{p,m}\right] + \sum_{p \in P} \Phi\delta_p(\zeta_p) \qquad -1 \leq \zeta_p \leq 1$$

Reformulate uncertain part as constraint and addition variable $\beta \in \mathbb{R}$:

$$\min \sum_{m \in M} \left[c_m^M y_m + \sum_{p \in P} c_p^P x_{p,m}\right] + \beta$$

$$\sum_{p \in P} \Phi\delta_p(\zeta_p) \leq \beta \qquad -1 \leq \zeta_p \leq 1$$

Insert linear decision rule:

$$\sum_{p \in P} \Phi(z_p + Q_p\zeta_p) \leq \beta \qquad -1 \leq \zeta_p \leq 1$$

$$\sum_{p \in P} \Phi z_p + \sum_{p \in P} \Phi Q_p\zeta_p \leq \beta \qquad -1 \leq \zeta_p \leq 1$$

Formulate worst case:

$$\sum_{p \in P} \Phi z_p + \sum_{p \in P} \max_{-1 \leq \zeta_p \leq 1} \{\Phi Q_p\zeta_p\} \leq \beta \qquad -1 \leq \zeta_p \leq 1$$

Reformulate using absolute value:

$$\sum_{p \in P} \Phi z_p + \sum_{p \in P} |\Phi Q_p| \leq \beta \qquad -1 \leq \zeta_p \leq 1$$

Linearize using variable $\sigma_p \geq 0, \forall p \in P$:

$$\sum_{p \in P} \Phi z_p + \sum_{p \in P} \sigma_p \leq \beta \qquad -1 \leq \zeta_p \leq 1$$

$$-\sigma_p \leq \Phi Q_p \leq \sigma_p \qquad \forall p \in P$$

3)

Full adjustable robust model:

$$\min \sum_{m \in M} \left[c^M_m y_m + \sum_{p \in P} c^P_p x_{p,m} \right] + \beta \tag{2a}$$

$$s.t. \sum_{p \in P} \Phi z_p + \sum_{p \in P} \sigma_p \leq \beta \qquad -1 \leq \zeta_p \leq 1 \tag{2b}$$

$$-\sigma_p \leq \Phi Q_p \leq \sigma_p \qquad \forall p \in P \tag{2c}$$

$$\sum_{m \in M} x_{p,m} + z_p - \alpha_p \geq d_p \qquad \forall p \in P \tag{2d}$$

$$-\alpha_p \leq Q_p \leq \alpha_p \qquad \forall p \in P \tag{2e}$$

$$\sum_{p \in P} \bar{t}_p x_{p,m} + \sum_{p \in P} t^{\text{dev}}_p x_{p,m} \leq T_m y_m \qquad \forall m \in M \tag{2f}$$

$$x_{p,m} \leq BigM_{m,p} a_{p,m} y_m \qquad \forall m \in M, p \in P \tag{2g}$$

$$z_p - \alpha_p \geq 0 \qquad \forall p \in P \tag{2h}$$

$$y_m \geq 0 \text{ and integer} \qquad \forall m \in M \tag{2i}$$

$$x_{p,m} \geq 0 \qquad \forall m \in M, p \in P \tag{2j}$$

$$\alpha_p, \sigma_p \geq 0 \qquad \forall p \in P \tag{2k}$$

$$\beta, Q_p, z_p \in \mathbb{R} \qquad \forall p \in P \tag{2l}$$

The results of the models with different values of Φ are given in Table 1. It can be observed that with increasing cost for missing products, the number of machines and with this the production capacity are increased. At a penalty cost of $\Phi = 300$, missing products are so expensive that the entire is covered. All results cover the worst case of the production times.

Φ	**100**	**150**	**200**	**300**
Objective	17771600.00	26111182.00	29486263.07	29508532.00
Machines of type 1	0	13	24	24
Machines of type 2	0	0	0	0
Machines of type 3	0	0	12	13
Machines of type 4	0	0	3	3
Production of product 1	0.00	5844.00	5844.00	5844.00
Production of product 2	0.00	0.00	9313.00	9313.00
Production of product 3	0.00	0.00	5725.00	5725.00
Production of product 4	0.00	0.00	8511.00	8511.00
Production of product 5	0.00	0.00	9465.00	9465.00
Production of product 6	0.00	27866.00	27866.00	27866.00
Production of product 7	0.00	0.00	28396.00	28396.00
Production of product 8	0.00	0.00	27394.00	27394.00
Production of product 9	0.00	25545.33	27590.00	27590.00
Production of product 10	0.00	0.00	26909.18	27612.00
Missing demand of product 1	5844.00	0.00	0.00	0.00
Missing demand of product 2	9313.00	9313.00	0.00	0.00
Missing demand of product 3	5725.00	5725.00	0.00	0.00
Missing demand of product 4	8511.00	8511.00	0.00	0.00
Missing demand of product 5	9465.00	9465.00	0.00	0.00
Missing demand of product 6	27866.00	0.00	0.00	0.00
Missing demand of product 7	28396.00	28396.00	0.00	0.00
Missing demand of product 8	27394.00	27394.00	0.00	0.00
Missing demand of product 9	27590.00	2044.67	0.00	0.00
Missing demand of product 10	27612.00	27612.00	702.82	0.00

Table 1: Solution values for different values of Φ