Convex Relaxations of AC Optimal Power Flow

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Questions?

Outline

- Motivation
- Mathematical Reformulation of Non-Convex AC-OPF
- Semidefinite Relaxation of AC-OPF
- Assignment 2: AC-OPF with convex relaxations (SDP-OPF)

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[1] Javad Lavaei and Steven H Low. "Zero duality gap in optimal power flow problem". In: *IEEE Transactions on Power Systems* 27.1 (2012), pp. 92–107

Motivation

- AC-OPF problem non-linear & non-convex
  $\Rightarrow$ No guarantee obtained solution from non-convex solver is global optimum
  $\Rightarrow$ Distance to global optimum cannot be specified (generation cost)
  $\Rightarrow$ Relaxation provides absolute lower bound on control effort

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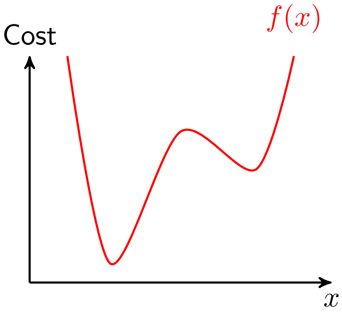

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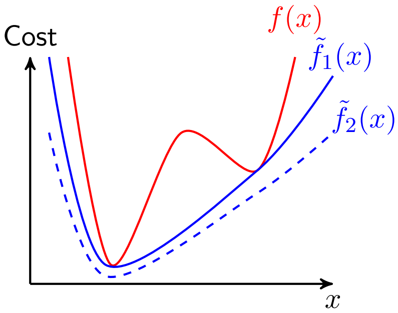

⇒ Under certain conditions, obtained solution is the global optimum to the original AC-OPF (Zero relaxation gap (exact) in work by Lavaei and Low[1])

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Motivation

- "Zero duality gap in optimal power flow problem" from Lavaei and Low has been published in 2011 $\Rightarrow > 950$ citations
- For many test systems, the semi-definite relaxation works, i.e. it is exact.
- Relaxations can be used to verify if the non-convex solver has achieved the global optimum.
- Different convex relaxations have emerged (linear, second-order cone, higher order moment).
- Ongoing research: The semi-definite relaxation is extended in several directions (e.g. security-constrained AC-OPF, AC-OPF under uncertainty[2]).

[2] A. Venzke et al. "Convex Relaxations of Chance Constrained AC Optimal Power Flow". In: *IEEE Transactions on Power Systems* 33.3 (2018), pp. 2829–2841

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Mathematical Reformulation of Non-Convex AC-OPF

AC-OPF[1]

obj.function	$\min c^T P_G$
AC flow	$S_G - S_L = diag(\overline{V})\overline{Y}^*_{\text{bus}}\overline{V}^*$
Line Current	$\lvert\overline{Y}_{\text{line},i\rightarrow j}\overline{V}\rvert \leq I_{line,max}$
	$\lvert\overline{Y}_{\text{line},j\rightarrow i}\overline{V}\rvert \leq I_{line,max}$
or Apparent Flow	$\lvert\overline{V}_i\overline{Y}^*_{\text{line},i\rightarrow j,\text{i-row}}\overline{V}^*\rvert \leq S_{i\rightarrow j,max}$
	$\lvert\overline{V}_j\overline{Y}^*_{\text{line},j\rightarrow i,\text{j-row}}\overline{V}^*\rvert \leq S_{j\rightarrow i,max}$
Gen. Active Power	$0 \leq P_G \leq P_{G,max}$
Gen. Reactive Power	$-Q_{G,max} \leq Q_G \leq Q_{G,max}$
Voltage Magnitude	$V_{min} \leq V \leq V_{max}$
Voltage Angle	$\delta_{min} \leq \delta \leq \delta_{max}$

[1] All shown variables are vectors or matrices. The bar above a variable denotes complex numbers. $(\cdot)^*$ denotes the complex conjugate. To simplify notation, the bar denoting a complex number is dropped in the following slides. Attention! The *current* flow constraints are defined as *vectors*, i.e. for all lines. The apparent power *line* constraints are defined *per line*.

Complex Power Injections

- Complex power balance for all buses writes:

$$S_G - S_L = diag(\bar{V})\bar{Y}_{bus}^{*}\bar{V}^{*}$$

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- Formulating the complex power balance for bus k yields:

$$[S_G - S_L]_k = \bar{V}^T e_k e_k^T \bar{Y}_{bus}^* \bar{V}^*$$

$\Rightarrow$ The vectors e_k are unit vectors that have a $\{1\}$ at the k-th entry. Otherwise, their entries are $\{0\}$.

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- Introducing the trace operator (sum of the diagonal elements of a matrix) and use its multiplicity property:

$$[S_G - S_L]_k = \text{Tr}\{\bar{V}^T e_k e_k^T \bar{Y}_{bus}^* \bar{V}^*\} = \text{Tr}\{e_k e_k^T \bar{Y}_{bus}^* \bar{V}^* \bar{V}^T\}$$

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$\Rightarrow$ AC-OPF formulation in complex variables where we could substitute $\bar{V}^{*}\bar{V}^T$ with a complex W. We split the complex formulation into real and imaginary part.

Splitting into Real and Imaginary Part

- If we have two generic complex numbers $a + jb$ and $c + jd$, then we can write their product as:

$$(a + jb)(c + jd) = ac - bd + j(ad + bc)$$

- In matrix form this can be formulated as:

Splitting into Real and Imaginary Part

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$$(a + jb)(c + jd) = ac - bd + j(ad + bc)$$

- In matrix form this can be formulated as:

$$\begin{bmatrix} \mathsf{Re} \\ \mathsf{Im} \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}$$

Splitting into Real and Imaginary Part

- Following this logic, we can write the real part of the active power injections using $Y_k := e_k e_k^T Y$ as:

$$\Re\{[S_G - S_L]_k\} = \Re\{\bar{V}^T e_k e_k^T \bar{Y}_{bus}^* \bar{V}^*\}$$

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$$\begin{aligned}\Re\{[S_G - S_L]_k\} &= \Re\{\bar{V}^T e_k e_k^T \bar{Y}_{bus}^* \bar{V}^*\} \\ &= X^T \begin{bmatrix} \Re\{Y_k\} & -\Im\{Y_k\} \\ \Im\{Y_k\} & \Re\{Y_k\} \end{bmatrix} X\end{aligned}$$

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- Following this logic, we can write the real part of the active power injections using $Y_k := e_k e_k^T Y$ as:

$$
\begin{aligned}
\Re\{[S_G - S_L]_k\} &= \Re\{\bar{V}^T e_k e_k^T \bar{Y}_{bus}^* \bar{V}^*\} \\
&= X^T \begin{bmatrix} \Re\{Y_k\} & -\Im\{Y_k\} \\ \Im\{Y_k\} & \Re\{Y_k\} \end{bmatrix} X \\
&= X^T \frac{1}{2} \begin{bmatrix} \Re\{Y_k + Y_k^T\} & \Im\{Y_k^T - Y_k\} \\ \Im\{Y_k - Y_k^T\} & \Re\{Y_k + Y_k^T\} \end{bmatrix} X \\
&= X^T \mathbf{Y}_k X
\end{aligned}
$$

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$\Rightarrow$ This procedure can be similarly applied to yield the mathematical formulation of the reactive power injections and the active and apparent branch flows.

Mathematical Reformulation of AC-OPF

Based on the previous derivation, we can write the bus power injections $P_{\mathsf{inj}_k} = X^T \mathbf{Y}_k X$ for bus k as:

$$
\begin{aligned}
\text{Nodal power injections} \quad P_{\mathsf{inj}_k} &= \mathrm{Tr}\{X^T \mathbf{Y}_k X\} \\
\text{Multiplicity property of trace operator} \quad &= \mathrm{Tr}\{\mathbf{Y}_k \underbrace{X X^T}\} \\
\text{Introduce matrix variable } W \quad &= \mathrm{Tr}\{\mathbf{Y}_k W\}
\end{aligned}
$$

Mathematical Reformulation of AC-OPF

The $2n_{\text{bus}}$ - dimensional vector X is transformed to a $2n_{\text{bus}} \times 2n_{\text{bus}}$ - dimensional matrix W

$$W = \begin{bmatrix} V_1^rV_1^r & V_1^rV_2^r & \cdots & V_1^rV_n^r & V_1^rV_1^i & V_1^rV_2^i & \cdots & V_1^rV_n^i \\ V_2^rV_1^r & V_2^rV_2^r & \cdots & V_2^rV_n^r & V_2^rV_1^i & V_2^rV_2^i & \cdots & V_2^rV_n^i \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ V_n^rV_1^r & \cdots & \cdots & V_n^rV_n^r & V_n^rV_1^i & \cdots & \cdots & V_n^rV_n^i \\ V_1^iV_1^r & V_1^iV_2^r & \cdots & V_1^iV_n^r & V_1^iV_1^i & V_1^iV_2^i & \cdots & V_1^iV_n^i \\ V_2^iV_1^r & V_2^iV_2^r & \cdots & V_2^iV_n^r & V_2^iV_1^i & V_2^iV_2^i & \cdots & V_2^iV_n^i \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ V_n^iV_1^r & \cdots & \cdots & V_n^iV_n^r & V_n^iV_1^i & \cdots & \cdots & V_n^iV_n^i \end{bmatrix}$$

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Semidefinite Relaxation of AC-OPF

…for each node k and line lm based on Lavaei and Low:

Semidefinite Relaxation of AC-OPF

...for each node k and line lm based on Lavaei and Low:

Minimize Generation Cost $$\sum_{k\in\mathcal{G}} \{c_{k2}(\mathrm{Tr}\{\mathbf{Y}_k W\} + P_{D_k})^2 + c_{k1}(\mathrm{Tr}\{\mathbf{Y}_k W\} + P_{D_k}) + c_{k0}\}$$

Matrices $\mathbf{Y}_k$, $\bar{\mathbf{Y}}_k$, $\mathbf{Y}_{lm}$ and $\bar{\mathbf{Y}}_{lm}$ are auxiliary variables resulting from the admittance matrix Y of the system.

Semidefinite Relaxation of AC-OPF

...for each node k and line lm based on Lavaei and Low:

$$
\begin{aligned}
&\text{Minimize Generation Cost} && \sum_{k \in \mathcal{G}} \{c_{k2}(\mathrm{Tr}\{\mathbf{Y}_k W\} + P_{D_k})^2 + \\
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&\text{s.t.}\quad \text{Active Power Balance} && P_k^{\mathsf{min}} \leq \mathrm{Tr}\{\mathbf{Y}_k W\} \leq P_k^{\mathsf{max}} \\
&\text{Reactive Power Balance} && Q_k^{\mathsf{min}} \leq \mathrm{Tr}\{\bar{\mathbf{Y}}_k W\} \leq Q_k^{\mathsf{max}} \\
&\text{Bus Voltages} && (V_k^{\mathsf{min}})^2 \leq \mathrm{Tr}\{M_k W\} \leq (V_k^{\mathsf{max}})^2 \\
&\text{Active Branch Flow} && -P_{lm}^{\mathsf{max}} \leq \mathrm{Tr}\{\mathbf{Y}_{lm} W\} \leq P_{lm}^{\mathsf{max}} \\
&\text{Apparent Branch Flow} && \mathrm{Tr}\{\mathbf{Y}_{lm} W\}^2 + \mathrm{Tr}\{\bar{\mathbf{Y}}_{lm} W\}^2 \leq (S_{lm}^{\mathsf{max}})^2
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&\text{Decomposition} && W = \underbrace{[\Re\{V\}\,\Im\{V\}]^T}_{X} \underbrace{[\Re\{V\}\,\Im\{V\}]}_{X^T}
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Semi-Definiteness of W	$W \succeq 0$
Rank Constraint	$\mathrm{rank}(W) = 1$

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Semi-Definiteness of W	$W \succeq 0$
Rank Constraint	$\cancel{\mathrm{rank}(W) = 1} \Rightarrow$ Convex Relaxation

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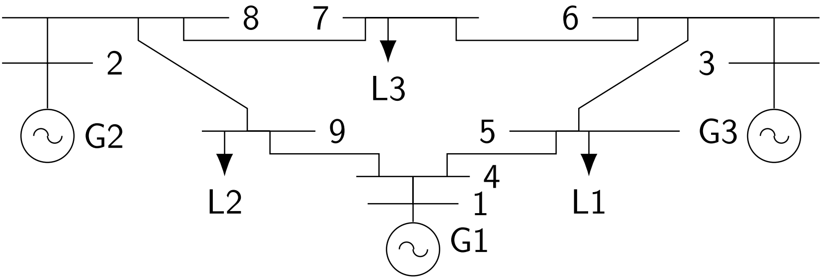
- The objectives to be achieved at the end of the assignment are the following:
  - Understand the concept of convex relaxations and the notion of exactness of a relaxation
  - Implement the semi-definite relaxation of the AC-OPF and evaluate its relaxation gap
  - Compare the solution of the relaxation to the solution of the original non-convex AC-OPF problem and evaluate its feasibility
  - Decompose the solution matrix $\mathbf{W}$ and recover the optimal voltage vector
  - Understand when a relaxation might fail and investigate possible methods to obtain a rank-1 solution

Assignment 2: AC-OPF with convex relaxations (SDP-OPF)



IEEE 9-bus system

Planning of Assignment 2

- Oct 1: Decision $\Rightarrow$ AC-OPF+project or SDP-OPF?
- Oct 9: (in two lectures from now):
  - 10am-12pm: For the groups that picked the AC-OPF+project, discussion/refinement of the topic and the project plan with each group
  - 10am-12pm: For the groups that picked the SDP-OPF: additional details from Andrea Tosatto that will help you with the implementation
- Nov. 4 @ 20.00hrs: Deadline for Assignment 2

Questions?

Feel free to write me an e-mail:

andven@elektro.dtu.dk