Assignment 2 – AC-OPF with convex relaxations (SDP-OPF)

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Outline

- Semidefinite Relaxation of AC-OPF
- Notes on the Implementation
- Assignment 2: AC-OPF with convex relaxations (SDP-OPF)

Mathematical Reformulation of AC-OPF

We introduce the variable transformation of complex bus voltages V:

$$X := [\Re\{V\}\,\Im\{V\}]^T$$

Mathematical Reformulation of AC-OPF

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Then, we can write the bus power injections P_{inj_k} for bus k as:

$$\begin{aligned}
\text{Nodal power injections} \quad P_{\mathsf{inj}_k} &= \mathrm{Tr}\{X^T \mathbf{Y}_k X\} \\
\text{Multiplicity property of trace operator} \quad &= \mathrm{Tr}\{\mathbf{Y}_k \underbrace{X X^T}\} \\
\text{Introduce matrix variable } W \quad &= \mathrm{Tr}\{\mathbf{Y}_k W\}
\end{aligned}$$

The term Tr{A} denotes the trace operator which is the summation of the diagonal elements of matrix A. The matrix $\mathbf{Y}_k$ is an auxiliary variable resulting from the admittance matrix Y of the power grid.

Mathematical Reformulation of AC-OPF

The $2n_{\text{bus}}$ - dimensional vector X is transformed to a $2n_{\text{bus}} \times 2n_{\text{bus}}$ - dimensional matrix W

$$W = \begin{bmatrix} V_1^r V_1^r & V_1^r V_2^r & \cdots & V_1^r V_n^r & V_1^r V_1^i & V_1^r V_2^i & \cdots & V_1^r V_n^i \\ V_2^r V_1^r & V_2^r V_2^r & \cdots & V_2^r V_n^r & V_2^r V_1^i & V_2^r V_2^i & \cdots & V_2^r V_n^i \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ V_n^r V_1^r & \cdots & \cdots & V_n^r V_n^r & V_n^r V_1^i & \cdots & \cdots & V_n^r V_n^i \\ V_1^i V_1^r & V_1^i V_2^r & \cdots & V_1^i V_n^r & V_1^i V_1^i & V_1^i V_2^i & \cdots & V_1^i V_n^i \\ V_2^i V_1^r & V_2^i V_2^r & \cdots & V_2^i V_n^r & V_2^i V_1^i & V_2^i V_2^i & \cdots & V_2^i V_n^i \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ V_n^i V_1^r & \cdots & \cdots & V_n^i V_n^r & V_n^i V_1^i & \cdots & \cdots & V_n^i V_n^i \end{bmatrix}$$

Convex Relaxation of AC-OPF

. . . for each node k and line lm:

Convex Relaxation of AC-OPF

…for each node k and line lm:

Minimize Generation Cost $$\sum_{k\in\mathcal{G}}\{c_{k2}(\mathrm{Tr}\{\mathbf{Y}_k W\} + P_{D_k})^2 + c_{k1}(\mathrm{Tr}\{\mathbf{Y}_k W\} + P_{D_k}) + c_{k0}\}$$

Matrices $\mathbf{Y}_k$, $\bar{\mathbf{Y}}_k$ and $\mathbf{Y}_{lm}$ are auxiliary variables resulting from the admittance matrix Y of the system.

Convex Relaxation of AC-OPF

…for each node k and line lm:

Minimize Generation Cost	$\sum_{k\in\mathcal{G}}\{c_{k2}(\mathrm{Tr}\{\mathbf{Y}_k W\}+P_{D_k})^2+ c_{k1}(\mathrm{Tr}\{\mathbf{Y}_k W\}+P_{D_k})+c_{k0}\}$
s. t. Active Power Balance	$P_k^{\mathrm{min}} \leq \mathrm{Tr}\{\mathbf{Y}_k W\} \leq P_k^{\mathrm{max}}$
Reactive Power Balance	$Q_k^{\mathrm{min}} \leq \mathrm{Tr}\{\bar{\mathbf{Y}}_k W\} \leq Q_k^{\mathrm{max}}$
Bus Voltages	$(V_k^{\mathrm{min}})^2 \leq \mathrm{Tr}\{M_k W\} \leq (V_k^{\mathrm{max}})^2$
Active Branch Flow	$-P_{lm}^{\mathrm{max}} \leq \mathrm{Tr}\{\mathbf{Y}_{lm} W\} \leq P_{lm}^{\mathrm{max}}$
Apparent Branch Flow	$\mathrm{Tr}\{\mathbf{Y}_{lm} W\}^2+\mathrm{Tr}\{\bar{\mathbf{Y}}_{lm} W\}^2 \leq (S_{lm}^{\mathrm{max}})^2$

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Convex Relaxation of AC-OPF

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Active Branch Flow	$-P_{lm}^{\text{max}} \leq \mathrm{Tr}\{\mathbf{Y}_{lm} W\} \leq P_{lm}^{\text{max}}$
Apparent Branch Flow	$\mathrm{Tr}\{\mathbf{Y}_{lm} W\}^2 + \mathrm{Tr}\{\bar{\mathbf{Y}}_{lm} W\}^2 \leq (S_{lm}^{\text{max}})^2$
Decomposition	$W = \underbrace{[\Re\{V\}\,\Im\{V\}]^T}_{X} \underbrace{[\Re\{V\}\,\Im\{V\}]}_{X^T}$

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Convex Relaxation of AC-OPF

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s. t. Active Power Balance	$P_k^{\mathsf{min}} \leq \mathrm{Tr}\{\mathbf{Y}_k W\} \leq P_k^{\mathsf{max}}$
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Active Branch Flow	$-P_{lm}^{\mathsf{max}} \leq \mathrm{Tr}\{\mathbf{Y}_{lm} W\} \leq P_{lm}^{\mathsf{max}}$
Apparent Branch Flow	$\mathrm{Tr}\{\mathbf{Y}_{lm} W\}^2 + \mathrm{Tr}\{\bar{\mathbf{Y}}_{lm} W\}^2 \leq (S_{lm}^{\mathsf{max}})^2$
Semi-Definiteness of W	$W \succeq 0$
Rank Constraint	$\mathrm{rank}(W) = 1$

Matrices $\mathbf{Y}_k$, $\bar{\mathbf{Y}}_k$ and $\mathbf{Y}_{lm}$ are auxiliary variables resulting from the admittance matrix Y of the system.

Convex Relaxation of AC-OPF

...for each node k and line lm:

$$
\begin{aligned}
&\text{Minimize Generation Cost} && \sum_{k\in\mathcal{G}} \{c_{k2}(\mathrm{Tr}\{\mathbf{Y}_k W\} + P_{D_k})^2 + \\
& && \quad c_{k1}(\mathrm{Tr}\{\mathbf{Y}_k W\} + P_{D_k}) + c_{k0}\} \\
&\text{s. t.} \quad \text{Active Power Balance} && P_k^{\text{min}} \leq \mathrm{Tr}\{\mathbf{Y}_k W\} \leq P_k^{\text{max}} \\
&\text{Reactive Power Balance} && Q_k^{\text{min}} \leq \mathrm{Tr}\{\bar{\mathbf{Y}}_k W\} \leq Q_k^{\text{max}} \\
&\text{Bus Voltages} && (V_k^{\text{min}})^2 \leq \mathrm{Tr}\{M_k W\} \leq (V_k^{\text{max}})^2 \\
&\text{Active Branch Flow} && -P_{lm}^{\text{max}} \leq \mathrm{Tr}\{\mathbf{Y}_{lm} W\} \leq P_{lm}^{\text{max}} \\
&\text{Apparent Branch Flow} && \mathrm{Tr}\{\mathbf{Y}_{lm} W\}^2 + \mathrm{Tr}\{\bar{\mathbf{Y}}_{lm} W\}^2 \leq (S_{lm}^{\text{max}})^2 \\
&\text{Semi-Definiteness of } W && W \succeq 0 \\
&\text{Rank Constraint} && \xcancel{\mathrm{rank}(W) = 1} \Rightarrow \text{Convex Relaxation}
\end{aligned}
$$

Matrices $\mathbf{Y}_k$, $\bar{\mathbf{Y}}_k$ and $\mathbf{Y}_{lm}$ are auxiliary variables resulting from the admittance matrix Y of the system.

Auxiliary Variables

A power grid consists of $\mathcal{N}$ buses and $\mathcal{L}$ lines. The set of generator buses is denoted with $\mathcal{G}$. The following auxiliary variables are introduced for each bus $k \in \mathcal{N}$ and line $(l, m) \in \mathcal{L}$:

$$
\begin{aligned}
Y_k &:= e_k e_k^T Y \\
Y_{lm} &:= (\bar{y}_{lm} + y_{lm}) e_l e_l^T - (y_{lm}) e_l e_m^T \\
\mathbf{Y}_k &:= \frac{1}{2} \begin{bmatrix} \Re\{Y_k + Y_k^T\} & \Im\{Y_k^T - Y_k\} \\ \Im\{Y_k - Y_k^T\} & \Re\{Y_k + Y_k^T\} \end{bmatrix} \\
\mathbf{Y}_{lm} &:= \frac{1}{2} \begin{bmatrix} \Re\{Y_{lm} + Y_{lm}^T\} & \Im\{Y_{lm}^T - Y_{lm}\} \\ \Im\{Y_{lm} - Y_{lm}^T\} & \Re\{Y_{lm} + Y_{lm}^T\} \end{bmatrix} \\
\bar{\mathbf{Y}}_k &:= \frac{-1}{2} \begin{bmatrix} \Im\{Y_k + Y_k^T\} & \Re\{Y_k - Y_k^T\} \\ \Re\{Y_k^T - Y_k\} & \Im\{Y_k + Y_k^T\} \end{bmatrix} \\
M_k &:= \begin{bmatrix} e_k e_k^T & 0 \\ 0 & e_k e_k^T \end{bmatrix}
\end{aligned}
$$

The terms $\Re\{\cdot\}$ and $\Im\{\cdot\}$ denote the real and imaginary part. Matrix Y denotes the bus admittance matrix of the power grid, e_k the k-th basis vector, $\bar{y}_{lm}$ the shunt admittance of line $(l, m) \in \mathcal{L}$ and y_{lm} the series admittance.

Notes on the Convex Relaxation

- In order to obtain zero relaxation gap, i.e. an exact relaxation, include small resistance $(10^{-4}$ p.u.$)$ to each transformer
  $\Rightarrow$ connected resistive graph

[1] Javad Lavaei and Steven H Low. "Zero duality gap in optimal power flow problem". In: *IEEE Transactions on Power Systems* 27.1 (2012), pp. 92–107

Notes on the Convex Relaxation

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Lavaei and Low[1]show

- rank$(W) = 1$ or 2 solution to original OPF problem can be recovered
- rank$(W) \geq 3$ solution to original OPF problem cannot be recovered

[1] Javad Lavaei and Steven H Low. "Zero duality gap in optimal power flow problem". In: *IEEE Transactions on Power Systems* 27.1 (2012), pp. 92–107

Notes on the Convex Relaxation

If $\text{rank}(W) = 2$, then apply eigendecomposition according to Molzahn et al.[2]:

$$W_{\text{opt}} = \rho_1 E_1 E_1^T + \rho_2 E_2 E_2^T$$
$$X_{\text{opt}} = \sqrt{\rho_1^{\text{opt}}} E_1^{\text{opt}} + \sqrt{\rho_2^{\text{opt}}} E_2^{\text{opt}}$$

The terms ρ_1, ρ_2 denote the first and second largest absolute eigenvalue of W and E_1 and E_2 the corresponding eigenvectors.

[2] Daniel K Molzahn et al. "Implementation of a large-scale optimal power flow solver based on semidefinite programming". In: *IEEE Transactions on Power Systems* 28.4 (2013), pp. 3987–3998

Schur's Complement

The objective on generation cost and the constraint on apparent line flow cannot be directly implemented in the SDP.

We can use the so-called Schur's complement to reformulate polynomial equations as semidefinite constraints.

Schur's Complement

The Schur complement is defined as follows[3]. Given a matrix $X \in S^n$ which can be partitioned in the sub-matrices A, B and C:

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \tag{1}$$

If det $A \neq 0$, the matrix

$$S = C - B^T A^{-1} B \tag{2}$$

is called the Schur complement of A in X. The following statements can be made regarding the positive semi-definiteness of the matrix X:

- $X \succ 0$ if and only if $A \succ 0$ and $S \succ 0$
- If $A \succ 0$, then $X \succeq 0$ if and only if $S \succeq 0$

[3]Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004

Schur's Complement

To obtain an optimization problem linear in W, the objective function is reformulated using Schur's complement:

$$\min_{W,\alpha} \sum_{k\in\mathcal{G}} \alpha_k$$

$$\begin{bmatrix} c_{k1}\mathsf{Tr}\{\mathbf{Y}_k W\} + a_k & \sqrt{c_{k2}}\mathsf{Tr}\{\mathbf{Y}_k W\} + b_k \\ \sqrt{c_{k2}}\mathsf{Tr}\{\mathbf{Y}_k W\} + b_k & -1 \end{bmatrix} \preceq 0$$

where $a_k := -\alpha_k + c_{k0} + c_{k1}P_{D_k}$ and $b_k := \sqrt{c_{k2}}P_{D_k}$. The variable α is introduced as an additional optimization variable. In addition, the apparent branch flow constraint is rewritten:

$$\begin{bmatrix} -(\overline{S}_{lm})^2 & \mathsf{Tr}\{\mathbf{Y}_{lm} W\} & \mathsf{Tr}\{\bar{\mathbf{Y}}_{lm} W\} \\ \mathsf{Tr}\{\mathbf{Y}_{lm} W\} & -1 & 0 \\ \mathsf{Tr}\{\bar{\mathbf{Y}}_{lm} W\} & 0 & -1 \end{bmatrix} \preceq 0$$

Schur's Complement

This theorem is used to prove that the semi-definite constraint is equal to the quadratic constraint. The matrix X corresponds to

$$X = \begin{bmatrix} \overline{S}_{lm}^2 & \mathsf{Tr}\{\mathbf{Y}_{lm}W\} & \mathsf{Tr}\{\bar{\mathbf{Y}}_{lm}W\} \\ \mathsf{Tr}\{\mathbf{Y}_{lm}W\} & 1 & 0 \\ \mathsf{Tr}\{\bar{\mathbf{Y}}_{lm}W\} & 0 & 1 \end{bmatrix} \succeq 0$$

Applying Schur complement a first time, defining

$$A = \overline{S}_{lm}^2 \qquad B = \begin{bmatrix} \mathsf{Tr}\{\mathbf{Y}_{lm}W\} & \mathsf{Tr}\{\bar{\mathbf{Y}}_{lm}W\} \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

yields the following result:

Schur's Complement

$$S_1 = C - B^T A^{-1} B$$
$$= \begin{bmatrix} 1 - \dfrac{\mathrm{Tr}\{\mathbf{Y}_{lm}W\}^2}{\overline{S}_{lm}^2} & \dfrac{\mathrm{Tr}\{\mathbf{Y}_{lm}W\}\mathrm{Tr}\{\bar{\mathbf{Y}}_{lm}W\}}{\overline{S}_{lm}^2} \\ \dfrac{\mathrm{Tr}\{\mathbf{Y}_{lm}W\}\mathrm{Tr}\{\bar{\mathbf{Y}}_{lm}W\}}{\overline{S}_{lm}^2} & 1 - \dfrac{\mathrm{Tr}\{\bar{\mathbf{Y}}_{lm}W\}^2}{\overline{S}_{lm}^2} \end{bmatrix} \succeq 0$$

If Schur complement is applied a second time, the result is the initial quadratic constraint:

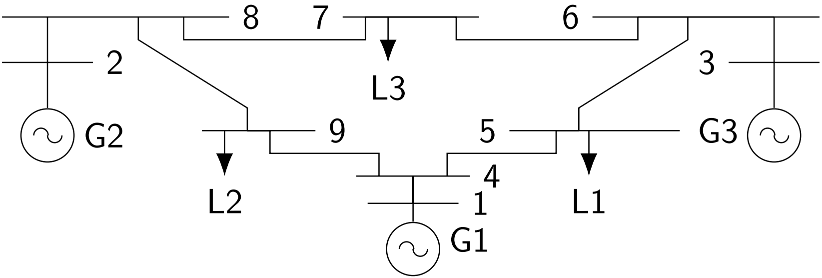
$$S_2 = \overline{S}_{lm}^2 - \mathrm{Tr}\{\mathbf{Y}_{lm}W\}^2 - \mathrm{Tr}\{\bar{\mathbf{Y}}_{lm}W\}^2 \geq 0$$

Hence, the proof is completed. In the context of semi-definite programming, Schur complement is a powerful tool, which can be used to transform polynomial constraints into semi-definite constraints.

Assignment 2: AC-OPF with convex relaxations (SDP-OPF)

1. Implement the semidefinite relaxation
2. Compare objective value and resulting P_{inj}, Q_{inj} to MATPOWER AC-OPF
3. Evaluate exactness of relaxation
4. Decompose solution matrix $\mathbf{W}$
5. Investigate exactness under varying network parameters for IEEE 9 bus system
6. Investigate exactness for 3 bus system
7. Bonus: Penalty term on reactive power injections

Assignment 2: AC-OPF with convex relaxations (SDP-OPF)



IEEE 9-bus system

Planning of Assignment 2

- Nov. 4: Deadline for Assignment 2
  - 5-10 page report (introduction including literature review, mathematical formulation, answer to the tasks of the assignment, conclusions)
  - Documented code in MATLAB/YALMIP
- Andreas and I will be out of office until end of October
  - We will always be reachable via email
  - We recommend to ask specific questions and not to send your codes.

Questions?

Feel free to write us an e-mail:

`{antosat,andven}@elektro.dtu.dk`