Practice Midterm

QPM II

October 3, 2018

1. We observe iid data $X_1, \ldots, X_n$ that represents the number of faculty in a department who leave each year due to retirement, failed retentions, and tenure denials. Your dean tells you that you should model this using a Possion distribution.

$$X_i \sim \frac{\lambda^x e^{-\lambda}}{x!}$$

a. Show that this data is in the exponential family of distributions of the form $h(x)\exp\left(\eta(\theta)T(x) - A(\eta)\right)$

b. Calculate the likelihood.

c. Calculate the log-likelihood.

d. Find the MLE for λ.

e. Show that the MLE is unbiased and consistent.

f. Prove (directly) that the MLE is a sufficient statistic for λ, given that $\text{Pois}(\alpha) + \text{Pois}(\beta) \sim \text{Pois}(\alpha + \beta)$. **Do not use the factorization theorem, or your knowledge of the exponential family form**.

g. Find the asymptotic distribution of the MLE for λ and show that it is asymptotically efficient.

h. Calculate a 95% confidence interval for the MLE.

i. Using the delta method, find the asymptotic distribution of $2\sqrt{\lambda}$.

2. Your dean points out that the asymptotic properties of the MLE are not relevant because the sample size is so small. Your data consists of the following:

$$\mathbf{x} = (2, 2, 1, 3, 2, 1, 0, 4)$$

a. Estimate the standard error of the asymptotic distribution using the non-parametric bootstrap. Set your number of bootstraps to 10^4.

b. Unrelated to the dean's request, you're interested in running a computational check for your answer to Problem 1-i. Estimate the standard error of the asymptotic distribution of $2\sqrt{\lambda}$ using the parametric bootstrap, given $\lambda = 4$. Set your number of bootstraps to 10^4. Use n = 100 for each bootstrap.

3. No luck. The dean's office remains skeptical of your ability to estimate the asymptotic distribution using such a small sample. You decide to pull out your BBG (big Bayesian guns). You decide to use a Bayesian prior with

$$\pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}$$

   a. Given $\alpha = 10$ and $\beta = 5$, find the posterior distribution of λ

   b. Using this posterior, find the $E(\lambda|\mathbf{x})$ and the 95% highest posterior density. (Partial credit for the 95% credible interval.)