Ordered and Multinomial Logit

Implementation and interpretation

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Introduction

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Ordered Logit

Motivation Ordered

- Finite and discrete values with more than two outcomes

Motivation Ordered

- Finite and discrete values with more than two outcomes
- Data with meaningful sequential values
  - income levels

    $(0{,}10000],(10000{,}30000],(30000,\ \infty]$
  - Likert-type scale

Motivation Ordered

- Finite and discrete values with more than two outcomes
- Data with meaningful sequential values
  - income levels
    $(0,10000],(10000,30000],(30000,\infty]$
  - Likert-type scale
  - ~~Gender~~
  - ~~Party affiliation~~
  - ~~Education~~

Mechanics Ordered

Proportional Odds Assumption[2]

The assumption that the explanatory variables have the same effect on the odds regardless of the threshold.

poor, $\log \frac{p_1}{p_2+p_3+p_4+p_5}$, 0

poor or fair, $\log \frac{p_1+p_2}{p_3+p_4+p_5}$, 1

poor,fair, or good,, $\log \frac{p_1+p_2+p_3}{p_4+p_5}$, 2

poor, fair, good, or very good, $\log \frac{p_1+p_2+p_3+p_4}{p_5}$, 3

Math Ordered

Formula

$$y_i^* = \alpha + \mathbf{x}_i'\beta + \epsilon_i = \alpha + Z_i + \epsilon_i$$

y_i^* = latent utility
where ϵ_i Logistic(0,σ^2)

Distribution

$$F(Z_i) = exp(Z_i)/1 + exp(Z_i)$$

$$\begin{aligned} y_i &= 1, \quad if \quad Z_i \leq \eta_1 \quad \text{(Region 1)} \\ y_i &= j, \quad if \quad \eta_{j-1} < Z_i \leq \eta_j \quad \text{(Region j)} \\ y_i &= J, \quad if \quad \eta_{J-1} < Z_i \quad \text{(Region J)} \end{aligned}$$

where $\eta_1 < \eta_2 < \eta_3, .., \eta_n$ & $\eta_1 \geq 0$,
parameters known as *thresholds* or *cutpoints*

Math Ordered

Marginal Effects:

Obtained by evaluating the appropriate density functions at the relevant points and multiplying by the associated coefficient [1]

Continuous:

$$\frac{d}{dx}\left[\frac{\exp x}{1+\exp x}\right]$$

$$= \frac{[1+\exp(x)]\exp(x) - [\exp(x)]^2}{[1+\exp(x)]^2}$$

Math Intuition Ordered

$$logit[P(Y \leq j)] = \alpha + \mathbf{x}_i'\beta, j - 1, ...J - 1$$

Prob. answering specific level consv. given party

	Democrat[1]	Republican[0]
Very Liberal[1]	0.1832505	0.07806044
Slightly Liberal[2]	0.1942837	0.10819225
Moderate[3]	0.3930552	0.37275214
Slightly Conservative[4]	0.1147559	0.18550357
Very Conservative[5]	0.1146547	0.25549160

Call:
polr(formula=pol.ideology party, data = dat

Coefficients:

	Value	Std. Error	t-value
partyDem	-0.9745	0.1292	-7.545
VeryLiberal|Slightly Liberal	-2.4690	0.1318	-18.7363
Slightly Liberal|Moderate	-1.4745	0.1090	-13.5314
Moderate|Slightly Conservative	0.2371	0.0942	2.5165
Slightly Conservative|Very Conservative	1.0695	0.1039	10.2923

If we wanted to find the odds a Democrat identifies as 'Slightly Liberal' or less:

$$\text{logit}[\text{P}(\text{Y} \leq 2)] = -1.4745 - -0.9745(1) = -0.5$$

$$P(Y \leq j) = \frac{exp(\alpha+\mathbf{x}_i'\beta)}{1+exp(\alpha+\mathbf{x}_i'\beta)} \Rightarrow \frac{exp(-0.5)}{1+exp(-0.5)} = .378$$

Math Intuition Ordered

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- Non-linear
- Overcomes OLS i.i.d. assumption
- Applicable to discrete or continuous independent variables

Shortcomings Ordered

- Vague
    - Move toward larger values of dv

Shortcomings Ordered

- Vague
  - Move toward larger values of dv
- Trivial differences
  - Quasi-normal data w/ 3-4 scale dv

Multinomial Logit

Motivation Multinomial

- Discrete, mutually exclusive, unordered dependent variables
  - Party ID
    - 0=Republican, 1=Independent, 2=Democrat

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Motivation Multinomial

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  - Party ID
    - 0=Republican, 1=Independent, 2=Democrat
  - Region
- Choice Model

Mechanics Multinomial

- M outcomes
  - M-1 binary logistic regression models
- Extension of binomial logistic regression
- Probability of success in a given category M

Mechanics Multinomial

Independence of Irrelevant Alternatives (IIA):

The **assumption** that the introduction or improvement of any alternative will have the same proportional impact on the original alternatives

Example

1= Train ; 2= Bus ; 3 = Car
IIA assumes that adding a 4^{th} option, a bike, will not have any impact on the probability of choosing your original 1-3 choices

Math Multinomial

Formula

$Z_{ij} = \sum_{r=1}^{R} \beta_{jr} X_{ir}$

Normalization

$Pr(Y = 1) = \frac{1}{1+\sum_{j=2}^{M} \exp{(Z_{ij})}}$

$Pr(Y_i = K) = \frac{\exp{(Z_{ik})}}{1+\sum_{j=2}^{M} \exp{(Z_{ij})}}$

Math Multinomial

Risk Ratio[3]:

The logarithm of the ratio of the probability of outcome m to that of outcome k

k = excluded observation

$$\left(\frac{Pr(Y_i=m)}{Pr(Y_i=1)}\right) = \exp\left(Z_{im}\right)$$

Benefits Multinomial

- Does not assume
  - normality
  - linearity
  - homoscedasticity

Benefits Multinomial

- Does not assume
  - normality
  - linearity
  - homoscedasticity
- Indepenedent variables
  - can be unbounded
  - needn't be interval

Shortcomings Multinomial

- IIA assumption

Examples

TAPS Data Multinomial

- pid
  - Generally speaking, do you usually think of yourself as [Republican/a Democrat] and independent?
    - 1-Democrat; 2-Independent; 3-Republican
- abort
  - Do you generally support or oppose a woman's right to abortion
    - 1-support; 2-oppose;
- taxes
  - Please tell me if you would favor or oppose a federal tax policy that increases income taxes for people with the highest incomes .
    - 1-support; 2-oppose;
- n = 1164

Example Multinomial

```
#library(nnet)
> m2 <- multinom(pid ~ abort + taxes, data = dat2)
> summary(m2)
```

```
Coefficients:
  (Intercept)     abort    taxes
2   -3.636725 0.8829103 2.034146
3   -6.683510 1.7884000 3.316103
```

Example Multinomial

```
>Anova(m2)
```

```
Analysis of Deviance Table (Type II tests)

Response: pid
      LR Chisq Df Pr(>Chisq)
abort    75.68  2  < 2.2e-16 ***
taxes   231.17  2  < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
>
Residual Deviance: 2086.295
AIC: 2098.295
```

Example Multinomial

Risk Ratio:

```
> exp(coef(m1))
```

```
  (Intercept)    abort      taxes
2 0.026338453 2.417926  7.645719
3 0.001251378 5.979877 27.552755
```

TAPS Data Ordered

- y
  - Do you agree or disagree that an American-born child of illegal immigrants should not be considered a U.S. citizen.
    - 1-strongly oppose; 2-oppose; 3-neutral; 4-support; 5-strongly support
- dt_strong
  - In your opinion, how well does the phrase 'is a strong leader' describe Donald Trump?
    - 1-not well at all; 2-slightly well; 3-moderately well; 4-very well; 5-extremely well
- party_id
  - Generally speaking, do you usually think of yourself as [Republican/a Democrat] and independent?
    - 1-Democrat; 2-Independent; 3-Republican
- n = 1332

Example Ordered

```
> m1 <- polr(y ~ dt_strong + party_id, data = dat)
> summary(m1)
```

```
Coefficients:
           Value  Std. Error  t value
dt_strong  0.2924    0.03800    7.693
party_id   0.6589    0.06776    9.723
```

Example Ordered

```
Intercepts:
                    Value   Std. Error  t value
s. oppose|oppose     0.4459  0.1329       3.3550
oppose|nuetral       1.7214  0.1388      12.4035
nuetral|support      2.3707  0.1466      16.1738
support|s. support   3.5660  0.1640      21.7435

Residual Deviance: 3983.457
AIC: 3995.457
```

Residual Deviance: 717.0249 AIC: 727.0249

Example Ordered

```
> pval <- Anova(m1)
> pval
```

```
Response: y
          LR Chisq  Df  Pr(>Chisq)
dt_strong   59.977   1  9.595e-15 ***
party_id    96.387   1  < 2.2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
>
```

Example Ordered

Log Odds:

```
> exp(coef(m1))
```

```
dt_strong   party_id
 1.397655   1.566525
```

Example Ordered

Predicted Probabilities:

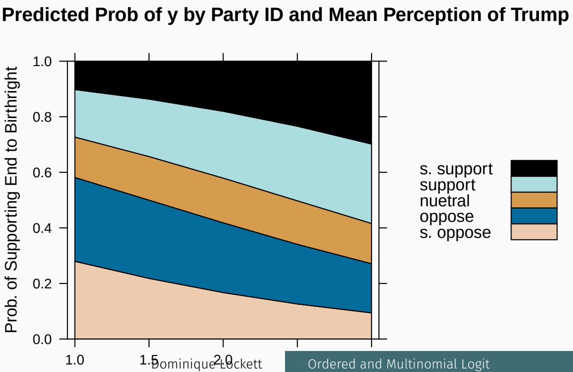
```
> > (Effect(focal.predictors = ('party_id'), m1, given.values = c(dt_strong = mean(dt_strong))))
```

```
party_id effect (probability) for s. oppose
         1        1.5          2        2.5          3
0.27964294 0.21828910 0.16727061 0.12625036 0.09415224

party_id effect (probability) for oppose
        1       1.5         2       2.5         3
0.3019306 0.2816622 0.2510610 0.2147011 0.1770585

party_id effect (probability) for nuetral
        1       1.5         2       2.5         3
0.1452534 0.1568680 0.1609246 0.1566210 0.1448073

party_id effect (probability) for support
        1       1.5         2       2.5         3
0.1710500 0.2066518 0.2405438 0.2683739 0.2858336

party_id effect (probability) for s. support
        1       1.5         2       2.5         3
0.1021230 0.1365289 0.1802000 0.2340536 0.2981484
```

Example Ordered

Predicted Probability:

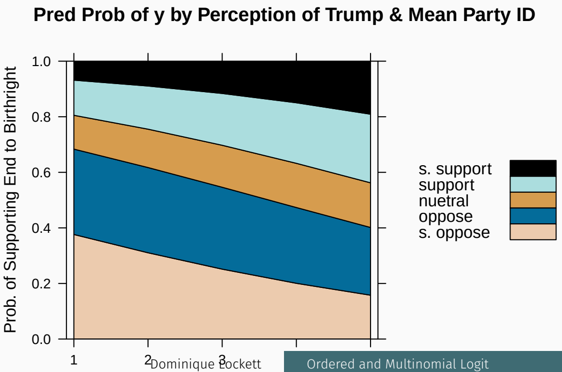
```
#require(wesanderson)
> e.out <— (Effect(focal.predictors = ('party_id'),m1, given.values = c(dt_strong = mean(dt_strong))))
> mean(dt_strong) = 2.508258
>plot(e.out, rug = F, style = 'stacked', main= 'PTitle',key.args = list(space ='right'),ylab = 'Title',
xlab = 'Title',colors = palette(wes_palette("Darjeeling2")))
```



Example Ordered

Predicted Probability:

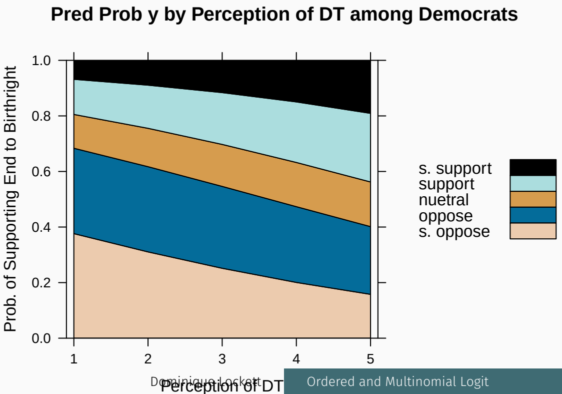
```
#require(wesanderson)
> e.out2 <— (Effect(focal.predictors = ('dt_strong'), m1, given.values = c(party_id = mean(party_id))))
> mean(party_id) = 2.916667
>>plot(e.out2, rug = F, style = 'stacked', main = 'Title', key.args = list(space = 'right'),
xlab = 'Title', ylab = 'Title', colors = palette(wes_palette("Darjeeling2")))
```



Example Ordered

Predicted Probability:

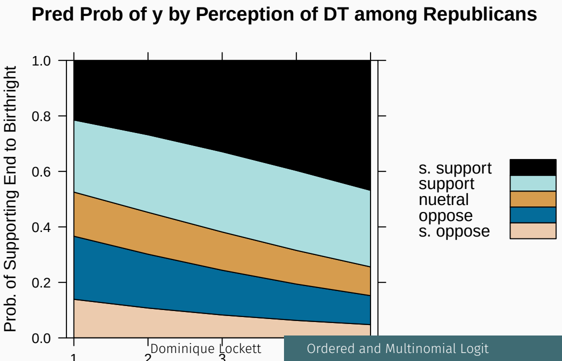
```
#require(wesanderson)
>e.out3 <— (Effect(focal.predictors = ('dt_strong'), m1, given.values = c(party_id = 1)))
>plot(e.out3, rug = F, style = 'stacked',main = 'title', key.args = list(space = 'right'),
xlab = 'title', colors = palette(wes_palette("Darjeeling2")))
```



Example Ordered

Predicted Probability:

```
#require(wesanderson)
> e.out4 <— (Effect(focal.predictors = ('dt_strong'), m1, given.values = c(party_id = 3)))
>plot(e.out4, rug = F, style = 'stacked', main = 'Title', key.args = list(space = 'right'),
xlab = 'Title', ylab = 'Title',
colors = palette(wes_palette("Darjeeling2")))
xlab = 'title', colors = palette(wes_palette("Darjeeling2")))
```



Example Ordered

```
# library(brant)
> brant(model)
```

```
Test for^^IX2^^Idf^^Iprobability
Omnibus^^I^^I12.11^^I6^^I0.06
dt_strong^^I7.36^^I3^^I0.06
party_id^^I6.03^^I3^^I0.11
```

A significant test statistic provides evidence that the parallel regression assumption has been violated.

Conclusion

Summary

- Feasible alternatives to linear regression
- Interpreted in log odds
- Useful and basic

Questions?

References i

V. K. Borooah.
Logit and probit: ordered and multinominal models.
Sage, 2001.

W. N. Venables and B. D. Ripley.
Modern Applied Statistics with S.
Springer, New York, fourth edition, 2002.
ISBN 0-387-95457-0.

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