Variable Selection

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Variable/feature selection

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- Various approaches include:
    1. Use theory
    2. Use model fit
    3. Regularization (e.g., ridge)
    4. Don't
- Do *not* use stepwize regression (or any proxy)

Variable selection is even harder than you think

- Variables that are highly correlated are hard to think about
- Computationally "hard" in that the problem is generally NP-complete
- What if you have more variables than observations?
- What if missingness varies by variable?

Model fit: Things to avoid

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- ▶ Stepwise
    - ▶ Some combination of these two
    - ▶ Often done in case a variable was "lost" at early stages but would be significant later.

Why this is bad

- Can easily miss the optimal model since you are not considering anything like the entire model space.
- **p-values don't mean what you think they mean**
- Tends to lead to models that over-estimate the significance of the variables that remain
- Just. Don't. Do. It.

A little better: Criteria-based

- Choose some model fit criteria (e.g., AIC, BIC, adjusted R-squared, etc.) and choose the model that fits best.
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- Risks over-fitting, but you can hope the complexity penalty will help.
- Probably better to use out-of-sample criteria in conjunction or instead.
- Different "search" algorithms have been proposed for only considering "likely" models when all 2^p are infeasible.

Deviance criteria

- One way we have already covered is to look at deviance.
- At a minimum the model should do better than the null model.
- In some other cases we can set up two models as perfectly "nested" within each other. Then we can formally do a LR-test.
- But in most cases neither of these are really sufficient.

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- AIC $= -2\ell(\hat{\boldsymbol{\beta}}|\mathbf{y}) + 2p$ where $\ell(\hat{\boldsymbol{\beta}}|\mathbf{y})$ is the maximized model log likelihood value and p is the number of explanatory variables in the model (including the constant).

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- AIC has a bias towards models that overfit with extra parameters since the penalty component is obviously linear with increases in the number of explanatory variables, and the log likelihood often increases more rapidly

Bayesian Information Criterion (BIC)

- BIC $= -2\ell(\hat{\beta}|\mathbf{y}) + p\log(n)$ where n is the sample size.
- There is also a Deviance Information Criterion (DIC) used in Bayesian MCMC estimation.

Problems and recommendations

Problems:

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- ▶ You can get widely different answers based on choices of criteria

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Recommendations:

- ▶ Theory (and literature) first.
- ▶ Can use automated criteria methods, but only as a first step.
- ▶ If your subsantive results change, you are in a danger zone.
- ▶ Radically different implications from seemingly inocuous changes in control variables mean you have serious post-treatment/colinearity problems.

Regularization 1: Lasso and Ridge

Regularization 2: Elastic Net

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- It tended to choose only one of any set of related variables.
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- Solution is to combine ridge and Lasso

$$\text{argmin} \left\{ \sum_{i=1}^{n}(y_i - \beta_0 - \sum_{j=1}^{p} x_{ij}\beta_j)^2 + \sum_{j=1}^{p} \lambda_2|\beta_j|^2 + \sum_{j=1}^{p} \lambda_1|\beta_j| \right\}$$

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- Equivalent to: $\text{argmin}|y - X\beta|^2$ where $(1 - \alpha)|\beta| + \alpha|\beta|^2 \leq t$ and $\alpha = \lambda_2/(\lambda_1 + \lambda_2)$
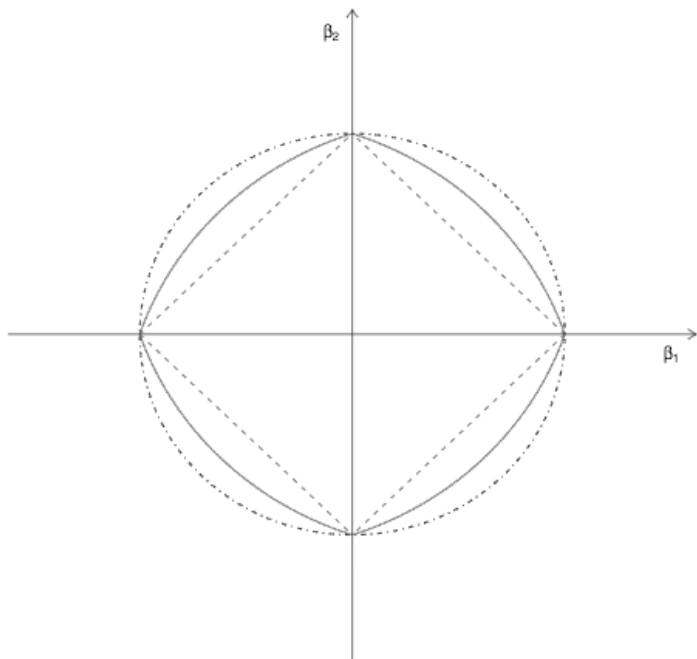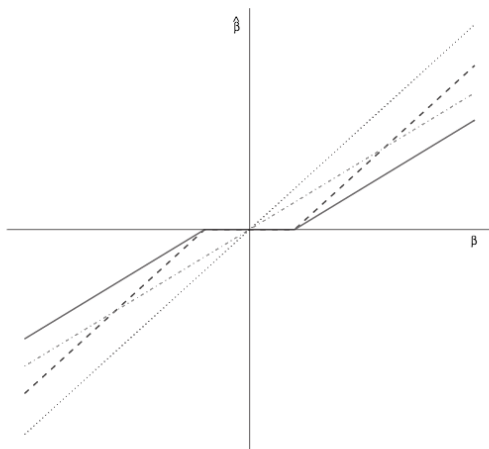
Fig. 2. Exact solutions for the lasso (-------), ridge regression (· · · · ·) and the naïve elastic net (———) in an orthogonal design (· · · · · ·, OLS): the shrinkage parameters are $\lambda_1 = 2$ and $\lambda_2 = 1$

Additional notes

- The non-naive estimates work to get rid of the double shrinkage by gettig rid of the ridge shrinkage:

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    1. Start with all coefficients at zero
    2. Find the predictor most correlated with the outcome, and increase coefficient estimate until some other variable more correlated with the residuals.
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- Thus, in a weird way is related to the the other procedures
- Can use the same package as just discusssed.

Don't select variables: Bayesian Model Averaging

Fearon and Laitin (2003) wish to model civil conflict

Constant	-6.731	-7.019
	(0.736)	(0.751)
Prior war	-0.954	-0.916
	(0.314)	(0.312)
Per capita income	-0.344	-0.318
	(0.072)	(0.071)
log (population)	0.263	0.272
	(0.073)	(0.074)
log (% mountainous)	0.219	0.199
	(0.085)	(0.085)
Noncontiguous state	0.443	0.426
	(0.274)	(0.272)
Oil exporter	0.858	0.751
	(0.279)	(0.278)
New state	1.709	1.658
	(0.339)	(0.342)
Instability	0.618	0.513
	(0.235)	(0.242)
Democracy (Polity IV)	0.021	
	(0.017)	
Ethnic fractionalization	0.166	0.164
	(0.373)	(0.368)
Religious fractionalization	0.285	0.326
	(0.509)	(0.506)
Anocracy		0.521
		(0.237)
Democracy		0.127
		(0.304)
Wars in neighboring countries		

"When we add dummy variables for countries that have an ethnic or religious majority and a minority of at least 8% of the country's poulation, both are incorrectly signed and neither comes close to statistical significance. This finding does not depend on which other variables are included in the model."

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> ethnic or religious majority and a minority of at least 8%
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> the model."

We often want to show that our results are "robust" to modeling
choices:

- Online appendix includes 18 additional tables
- At least 74 possible explnatory variables discussed
- $2^{74} = 2 \times 10^{22} = 20$ sextillion

The basics

The model:

- Let **X** denote the $n \times p$ matrix of all the independent variables theorized to be predictors of outcome Y.
- $Y = \mathbf{X}\beta + \epsilon$, where $\epsilon \sim N(0, \sigma^2 \mathbf{I})$.
- $\mathcal{M} = [\mathcal{M}_1, \mathcal{M}_2, \ldots \mathcal{M}_q]$ where $q = 2^p$.

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Priors:

- $\pi(\beta_k | M_k)$
- $\pi(\sigma | M_k)$.
- We also put a prior probability $\pi(M_k)$ on the model space.

BMA as an Hierarchical Mixture model (1)

$$\mathcal{M}_k \sim \pi(\mathcal{M}_k). \tag{1}$$

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$$\beta_\omega | \mathcal{M}_k \sim \pi(\beta_\omega | \mathcal{M}_k, \sigma^2). \tag{3}$$

where $\Omega = \omega_1, \ldots, \omega_p$ represents a vector of zeroes and ones indicating the inclusion (or exclusion) of variables in model M_k.

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$$Y|\beta, \sigma, \mathcal{M}_k \sim N(X_\omega \beta_\omega, \sigma^2 I) \tag{4}$$

BMA as an Hierarchical Mixture model (2)

$$E(\beta_k|Y) = \sum_{k=0}^{q} p(\mathcal{M}_k|Y)E(\beta_k|\mathcal{M}_k, Y) \tag{5}$$

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$$p(Y|\mathcal{M}_k) = \int \int p(Y|\beta_\omega, \sigma^2, \mathcal{M}_k)\pi(\beta_\omega|\sigma^2, \mathcal{M}_k)\pi(\sigma^2|\mathcal{M}_k)\, d\beta_\omega\, d\sigma^2 \tag{8}$$

Bayesian model averaging

- Y is a $n \times 1$ vector of outcomes
- X is an $n \times p$ matrix
- $Y = X\beta + \epsilon$
- $\epsilon \sim N(0, \sigma^2 I)$
- $q = 2^p$
- $\mathcal{M} = [\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_q]$

Priors and likelihood:

- $\mathcal{M}_k \sim \pi(\mathcal{M}_k)$
- $\sigma^2 | \mathcal{M}_k \sim \pi(\sigma^2 | \mathcal{M}_k)$
- $\beta_\omega | \sigma^2, \mathcal{M}_k \sim \pi(\beta_\omega | \sigma^2, \mathcal{M}_k)$
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With this, we can easily create other quantities of interest as weighted sums. For example,

$$E(\beta_k|Y) = \sum_{k=0}^{q} P(\mathcal{M}_k|Y)E(\beta|M_k, Y)$$

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Posterior computation

The great advantage of these priors is that we can quickly compute Bayes factors *exactly*.

- ▶ This makes it possible to "sample" all possible models.
- ▶ Two options are Bayesian adaptive sampling (BAS) and leaps and bounds Algorithm.
- ▶ The paper includes a detailed discussion of available software.

An aside on Bayes factors

Let's say there are just two models, and want to know the posterior probability of some model k:

$$P(M_k|\mathbf{D}) = \frac{P(\mathbf{D}|M_k)\pi(M_k)}{P(\mathbf{D}|M_1)\pi(M_1) + P(\mathbf{D}|M_2)\pi(M_2)}$$

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This implies that:

$$\frac{P(M_1|\mathbf{D})}{P(M_2|\mathbf{D})} = \frac{P(\mathbf{D}|M_1)\pi(M_1)}{P(\mathbf{D}|M_2)\pi(M_2)}$$

The limiting factor in BMA approaches has always been a combination of:

- an *extremely* high dimensional space
- intractable integrals.

Early work used "approximations" of the Bayes factors as a solution:

- $BF(M_k|M_0) = BIC_k = -2\log(L_k - L_0) + p_k\log n$
- $BF(M_K|M_0) = AIC_k = -2\log(L_k - L_0) + 2p$

Clyde (2003) and Clyde and George(2004) summarize a more comprehensive approach.

$$\pi(\mathcal{M}_k) = \gamma^{p_\omega}(1 - \gamma)^{p - p_\omega} \tag{9}$$

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Priors for posterior calculations

Prior	Formulation
g-prior	$\pi(\beta_\omega|\mathcal{M}_k, \sigma^2) \sim N_{p_\omega}(0, g\sigma^2(X_\omega'X_\omega)^{-1})$
	$\pi(\beta_0, \sigma^2|\mathcal{M}_k) \propto 1/\sigma^2$

Hyper-priors for g

Prior	Formulation
Hyper-g	$\pi(g) = \frac{a-2}{2}(1+g)^{\frac{a}{2}}$ if $g > 0$
ZS	$\pi(g) = \frac{(n/2)^{1/2}}{\Gamma(1/2)} g^{-3/2} e^{-\frac{n}{2g}}$

Example: g-prior

$$p(\mathcal{M}_k|Y) = \frac{B[\mathcal{M}_k : \mathcal{M}_0]}{\sum\limits_{k=0}^{q} B[\mathcal{M}_k : \mathcal{M}_0]} \tag{10}$$

Where:

$$B[\mathcal{M}_k : \mathcal{M}_0] = (1+g)^{(n-p_k-1)/2} \times (1 + g(1 - R_k^2))^{-(n-1)/2}$$

For coefficient j:

$$E(\beta_j|Y) = \sum_{k=0}^{q} p(\mathcal{M}_k|Y)E(\beta_j|\mathcal{M}_k, Y) \qquad (11)$$
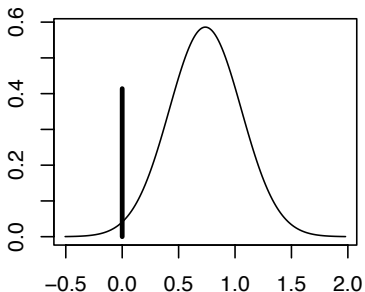
Where:

$$E(\beta_j|\mathcal{M}_k) = \frac{g}{g+1}\hat{\beta}_{j,OLS}$$
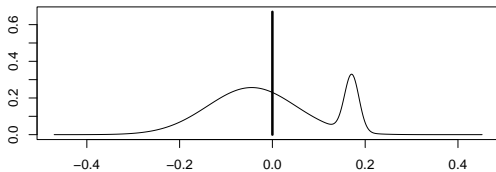
Posterior coefficient plots

Allows us to focus on two questions separately:

- ▶ Does the variable contribute to the model's explanatory power?
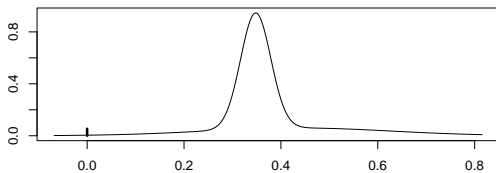- ▶ Is the variable correlated with unexplained variance when it is included?

Beware the t-test

D proximity adv.
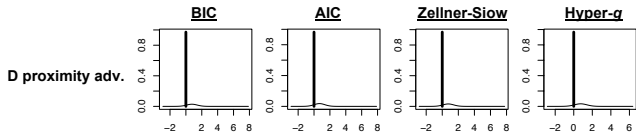
D directional adv.

Adams, Bishin, and Dow (ABD)

Adams, Bishin, and Dow (2004) compare directional to proximity voting in the context of U.S. Senate elections from 1988-1992. The construct an OLS analysis at the campaign level.

- ▶ DV: % of 2-party vote won by Democrat
- ▶ Main explanatory variables: Aggregate measures of the advantage of the Democratic candidate under the proximity and directional modeling assumptions.
- ▶ Controls: Partisan registration advantage, spending, incumbency, and candidate quality.
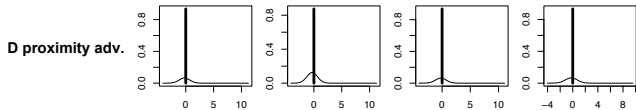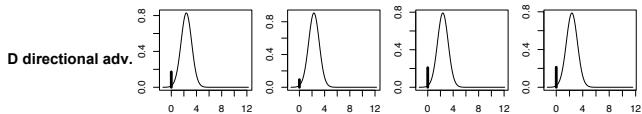- ▶ They present total of four models presented.

Adams, Bishin, and Dow (ABD)

	ABD (2004) replication			
	Reduced model		**Full model**	
	Mean cand.	Voter-specific	Mean cand.	Voter-specific
	Mean	*Mean*	*Mean*	*Mean*
	(SD)	*(SD)*	*(SD)*	*(SD)*
D proximity adv.	-3.137	-2.881	-3.053	-2.007
	(1.593)	(1.270)	(1.315)	(1.056)
D directional adv.	11.146	5.729	7.953	4.177
	(3.376)	(1.620)	(2.854)	(1.356)
D incumbency adv.	6.376	6.609	1.060	1.139
	(1.092)	(1.054)	(1.201)	(1.189)
D quality adv.	5.972	5.035	3.117	2.378
	(1.400)	(1.384)	(1.240)	(1.216)
D spending adv.			0.270	0.265
			(0.041)	(0.040)
D partisan adv.			0.055	0.060
			(0.054)	(0.054)
Constant	54.759	52.786	53.309	52.028
	(1.325)	(0.892)	(1.155)	(0.758)
N	95	95	94	94

	BMA (hyper-g)			
Mean cand.			Voter-specific	
	Cond. mean (SD)	$P(\beta \neq 0)$	*Cond. mean (SD)*	$P(\beta \neq 0)$
D proximity adv.	-0.222 (0.954)	0.067	0.745 (0.677)	0.033
D directional adv.	3.576 (2.015)	0.299	2.363 (0.840)	0.787
D incumbency adv.	1.607 (1.242)	0.194	1.295 (1.237)	0.159
D quality adv.	2.964 (1.246)	0.599	2.740 (1.223)	0.541
D spending adv.	0.3238 (0.041)	1.00	0.314 (0.037)	1.00
D partisan adv.	0.0749 (0.057)	0.201	0.0661 (0.055)	0.181
Constant	51.381 (1.032)	1.00	51.560 (0.764)	1.00
N	94		94	

	BIC	AIC	Zellner-Siow	Hyper-*g*

Voter-specific

D proximity adv.

D directional adv.

Mean placement

D proximity adv.

D directional adv.