Problem Set 3

QPM II - Fall 2018

10/3/2018

Maximum Likelihood: Closed-Form Estimation

1. Find the MLE of θ for
$$X \sim \text{Poisson}(\theta) = \frac{\theta^x \exp(-\theta)}{x!}$$
Does the second derivative of the log-likelihood indicate that the MLE occurs at a maximum? Does it attain the Cramer-Rao Lower Bound?

2. Find the MLE of θ for
$$X \sim \text{Rayleigh}(\theta) = \frac{x}{\theta^2} \exp\left(\frac{-x^2}{2\theta^2}\right)$$

3. Find the MLE of $\theta = \begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix}$ for
$$X \sim N(\theta_1, \theta_2) = \frac{1}{\sqrt{2\pi\theta_2^2}} \exp\left(\frac{-(x-\theta_1)^2}{2\theta_2^2}\right)$$
and evaluate the Hessian.

4. Prove that the MLE of θ for
$$X \sim \text{Weibull}(\beta, \theta) = \frac{\beta}{\theta^\beta} x^{\beta-1} \exp\left(\left(-\frac{x}{\theta}\right)^\beta\right) \text{ , for } \beta \text{ and } \theta > 0$$
is $\hat{\theta} = \left(\frac{1}{n}\sum x_i^\beta\right)^{1/\beta}$

Maximum Likelihood 2: Computational Approximation

5. Prove that the score function w.r.t β for
$$X \sim \text{Weibull}(\beta, \theta) = \frac{\beta}{\theta^\beta} x^{\beta-1} \exp\left(\left(-\frac{x}{\theta}\right)^\beta\right) \text{ , for } \beta \text{ and } \theta > 0$$
is $\sum^n \left(\frac{1}{\beta} - \log(\theta) + \log(x) - \log\left(\frac{x}{\theta}\right)\left(\frac{x}{\beta}\right)^\beta\right)$.

6. We can see above that $\hat{\beta}$ lacks a neat, closed-form solution, therefore we need to estimate the MLE of β by computational approximation. One method is by root-finding, where we find the value of β such that the score function of the log-likelihood equals 0. Since we've already evaluated the first derivative of the log-likelihood w.r.t β, it means (with some algebra, and treating $\hat{\theta}$ as a plug-in estimator) that the score function equals 0 when
$$0 = \left[\frac{\sum x^\beta \log(x)}{\sum x^\beta} - \frac{\sum \log(x)}{n}\right]^{-1} - \beta$$
Therefore to find the MLE of β via root-finding, do the following (**showing the code for each step**)
   - Generate 10^6 random draws from Weibull($\beta = 5, \theta = 7$). This is your data.

- Code the score function directly above as a function of β, x_i (your randomly generated data), and n (# observations).
- Apply a root finding algorithm to your function, given your observed data (in `R` use `uniroot()`; for non-`R` users, find an analogous function).
- Return the result (hint: if it's not ≈ 5, you've done something wrong)

7. Find the log-likelihood, its gradient, and its second derivative for

$$X \sim \text{Frechet}(\alpha, scale = 1, location = 0) = \alpha\,(x)^{-1-\alpha} \exp\left(-x^{-\alpha}\right)$$

8. Then write out the generic Newton-Raphson update step (do not simplify the ratio of the gradient to the second derivative).
9. Finally, do the following (**showing the code for each step**)
   - Generate 10^6 random draws from $\text{Frechet}(\alpha = 3, scale = 1, location = 0)$ (in `R`, this distribution is contained in `library(evd)`; beyond `R`, Google it). This is your data.
   - Code the Newton-Raphson update step as a function of α and x_i (your randomly generated data).
   - Run the update step 10 times, using $\alpha_0 = 1$ as a start value
   - Return the result (hint: if it's not ≈ 3, you've done something wrong)
10. Re-run the Newton-Raphson alogirthm, again using $\alpha_0 = 1$ as the start value. This time, do not return the result, but plot a tangent line to the likelihood function at each step (please include all steps on a single plot). You'll need to code the likelihood function in order to plot it.
11. Write out the generic gradient descent update step for α (just keep γ as a generic value)
12. Re-do Problem 9, but this time using the gradient descent algorithm, instead of Newton-Raphson. Set $\gamma = 10^-6$, $\alpha_0 = 1$, and run 50 iterations.
13. Perform a parametric bootstrap for $X \sim \text{Poisson}(\lambda = 5)$. Calculate the asymptotic standard error of λ^2. **Show your code**.
14. Perform a parametric bootstrap for $X \sim N(\mu = 5, \sigma^2 = 1)$. What are the mean and standard error of the asymptotic distribution of $\frac{1}{\mu}$? **Show your code**.