Lecture 16: Multiple Regression

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Quantitative Political Methodology

Multiple regression

Roadmap

- **Before**: Regression with one explanatory variable
- **Today** we will learn how to:
  - Draw the best (hyper)plane through the data
  - Interpret multivariate regression results

Class business

- PS is due on Wed.
- Take notes on this one

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- Introducing multivariate regression
  - An example (time for change model)
  - (Hyper)planes in (hyper)space
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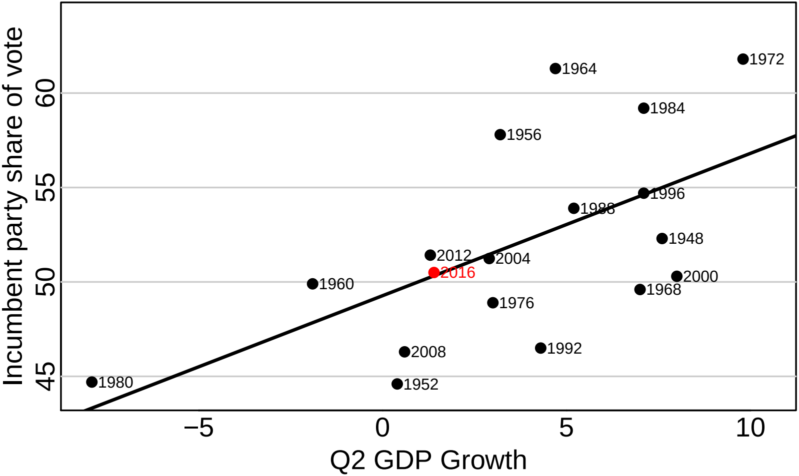
So far we have looked at data like this



But what if it is time for change?

Success of Incumbent Party Candidate in Presidential Electios by Type of Election, 1948-2016

Results	First-Term	Second- or Later
Won	8	2
Lost	1	8
Average vote	55.3	49.3

Accounting for time in office

Estimate a more complex equation:

$$\mu_y = \beta_0 + \beta_1 x_1 + \beta_2 x_2$$

`where:`

- μ_y is mean presidential vote share
- β_0 is the y-intercept ("constant"")
- β_1 is the slope ("coefficient") for Q2 GDP growth
- x_1 is Q2 GDP growth in the election year
- β_2 is the slope ("coefficient"") for TFC ("time for a change"")
- x_2 is an indicator ("dummy") variable for TFC (1=first term; 0=second term or later)

Equation for the graph:

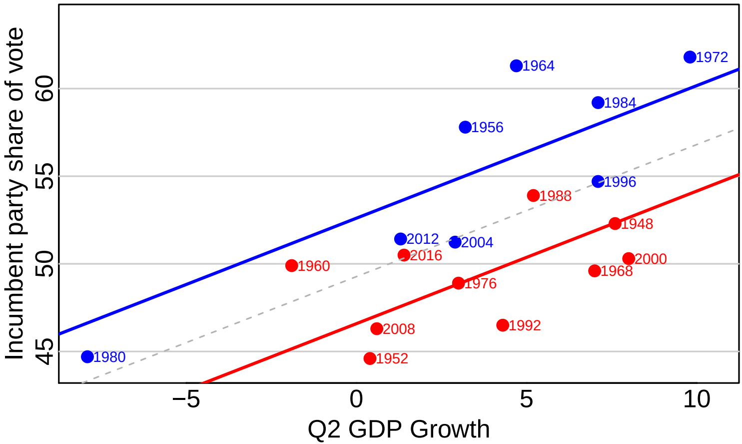
$$\text{Vote share} = 46.59 + 0.76 \times \text{Q2 GDP} + 6.02 \times \text{FirstTermInc}$$

or

$$\text{Vote share}_{\text{TFC}} = 46.59 + 0.76 \times \text{Q2 GDP}$$

$$\text{Vote share}_{\text{Not TFC}} = 52.61 + 0.76 \times \text{Q2 GDP}$$

Multivariate regression



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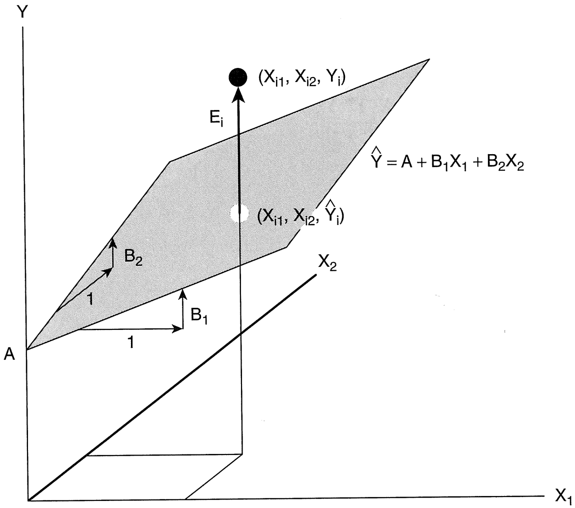

Beyond two dimensions

Incumbent party vote share

	Model 1	Model 2
Intercept	49.27	49.35
	(1.35)	(4.51)
2nd Qtr GDP	0.754	0.451
	(0.248)	(0.161)
June Polling		0.147
		(0.085)
Multiple R-Squared	0.366	0.781

Standard errors are in parentheses. N=18.

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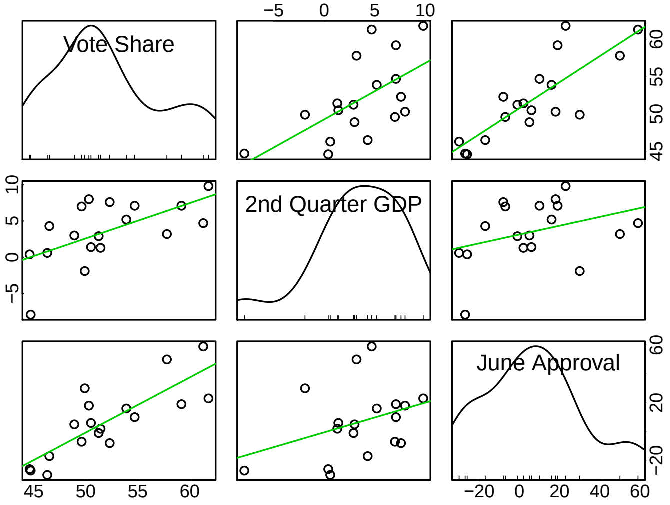
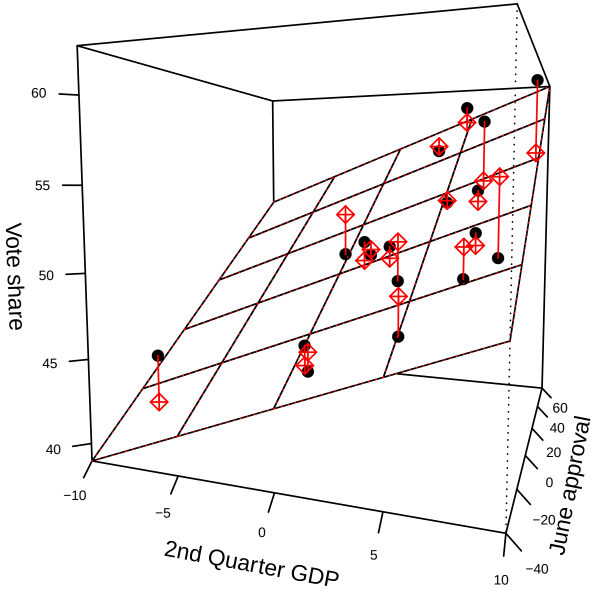
Two questions to try to understand:

- What do the coefficients (and standard errors) mean?
- Why did the "2nd Quarter GDP" coefficient change?

Now we need to think about data like this



Or even better . . . . this



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Residuals:

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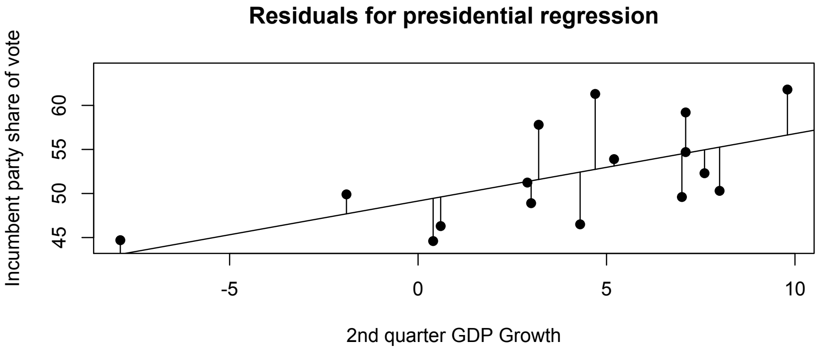

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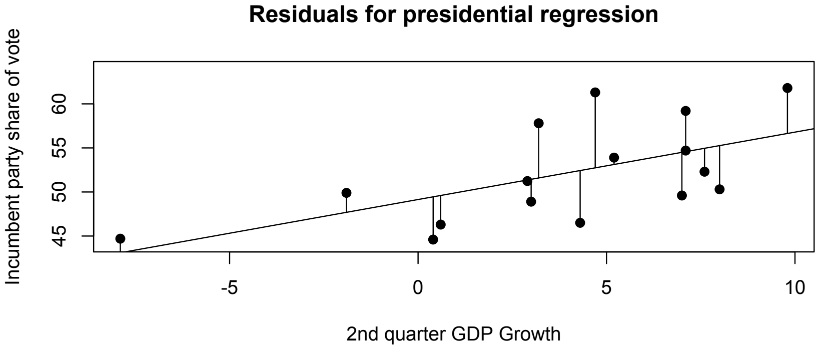

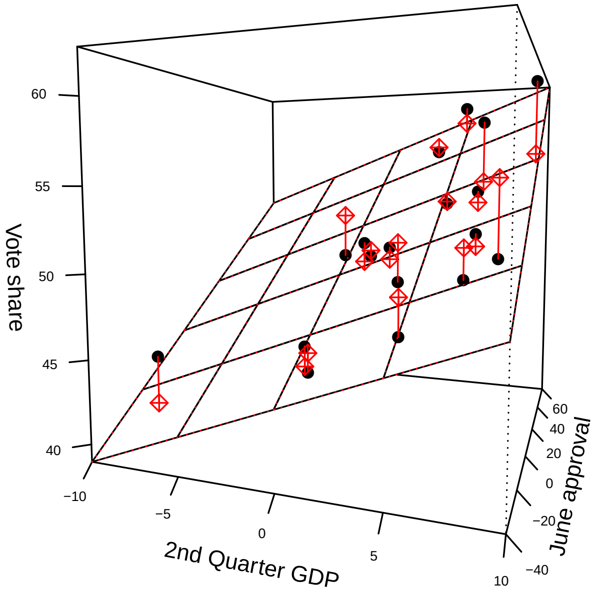

Vote share
60
55
50
45
40
2nd Quarter GDP
−10
−5
0
5
10
June approval
60
40
20
0
−20
−40

Multidimensional “linear” models

On **average**, we are hypothesizing that the **world** looks like this:

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Overall, we think that the **data** looks like this

$$Y_i = \alpha + \beta_1 X_{1,i} + \ldots + \beta_k X_{k,i} + \epsilon_i$$
$$\epsilon_i \sim N(0, \sigma^2)$$

Just like before, we need to decide on a rule to choose the best estimates:

$$\hat{\alpha}, \hat{\sigma}^2, \hat{\beta}_1, \hat{\beta}_2, \ldots$$

Residuals, SSE, and $\hat{\sigma}^2$

- Residuals

$$e_i = (Y_i - \hat{Y}_i) = (Y_i - \hat{\alpha} - \hat{\beta}_1 X_{1i} - \ldots - \hat{\beta}_k X_{ki})$$

- Sum of squared error

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- Conditional Variance: Estimate of variance around hyperplane in population

$$\hat{\sigma}^2 = \frac{SSE}{n-(k+1)} = \frac{\sum (Y_i - \hat{Y}_i)^2}{n-(k+1)} \Rightarrow \hat{\sigma} = \sqrt{\frac{SSE}{n-(k+1)}} = \sqrt{\frac{\sum (Y_i - \hat{Y}_i)^2}{n-(k+1)}}$$

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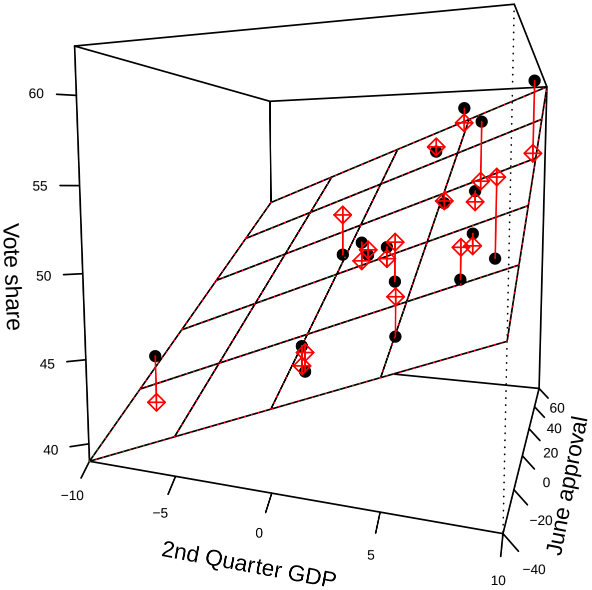
Thinking about regression 1: Planes



Thinking about regression 2: Lines within groups

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(This provides the same inference as a t-test)

Let's look at this with nominal data

$$X_1 = \{\text{Blue}, \text{Not blue}\}, X_2 = \{\text{Brown}, \text{Not brown}\}$$

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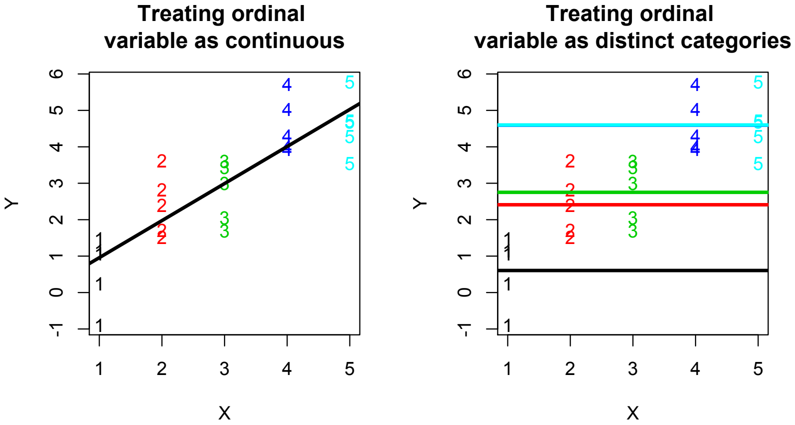
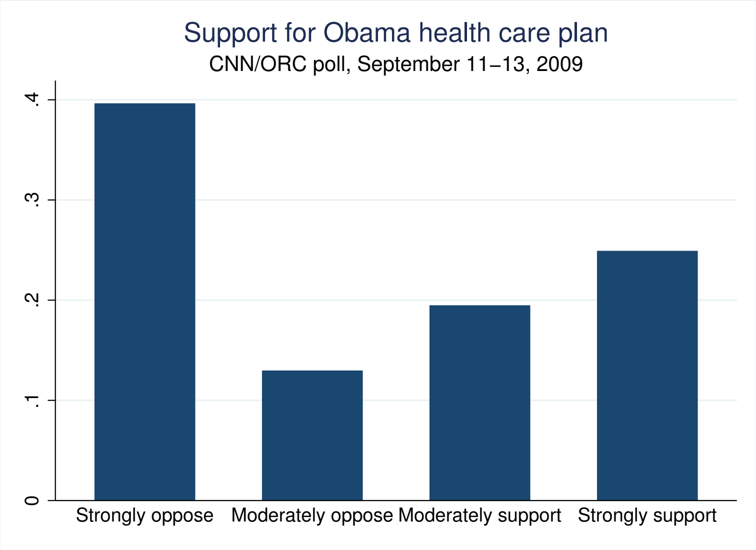
$$E(Y_i|Green) = \alpha$$

Ordinal data

$$X = \{1, 2, 3, 4, 5\}$$



Example: 2009 health care poll



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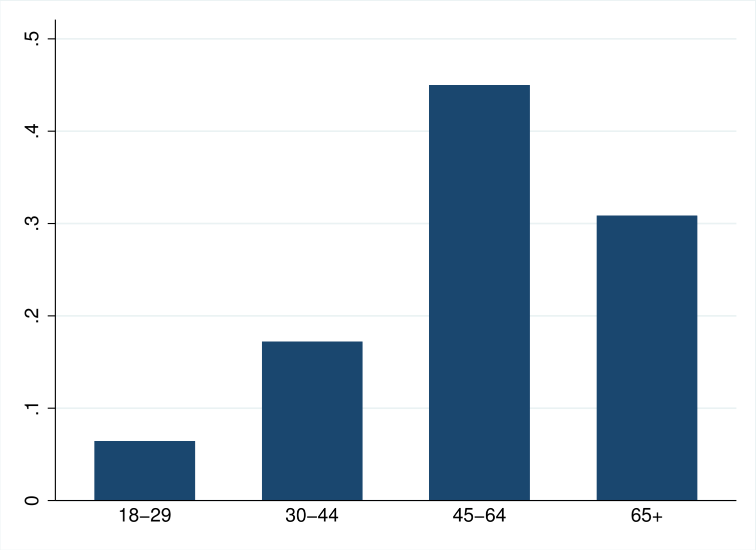

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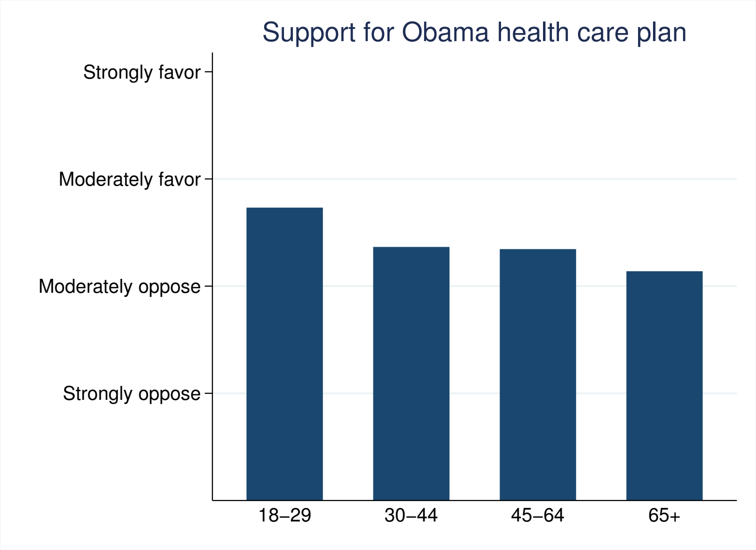

Dummy variable regression

- What is the association between age and support for HCR controlling for party?
- Goal: Recode age variable (18-29=1, 30-44=2, 45-64=3, 65+=4) into dummy variables

Equation:

$$\text{HCRS} = \beta_0 + \beta_1 \text{ Party} + \beta_2 \text{ Age 30-44} + \beta_3 \text{ 45-64} + \beta_4 \text{ 65+}$$

Dummy variable results

Variable	
Constant	1.421
	(0.116)
Party	0.914
	(0.031)
Age 30-44	0.77
	(0.13)
Age 45-64	-0.65
	(0.117)
Age 65 +	-0.16
	(0.121)
N = 981	
$R^2 = 0.4799$	

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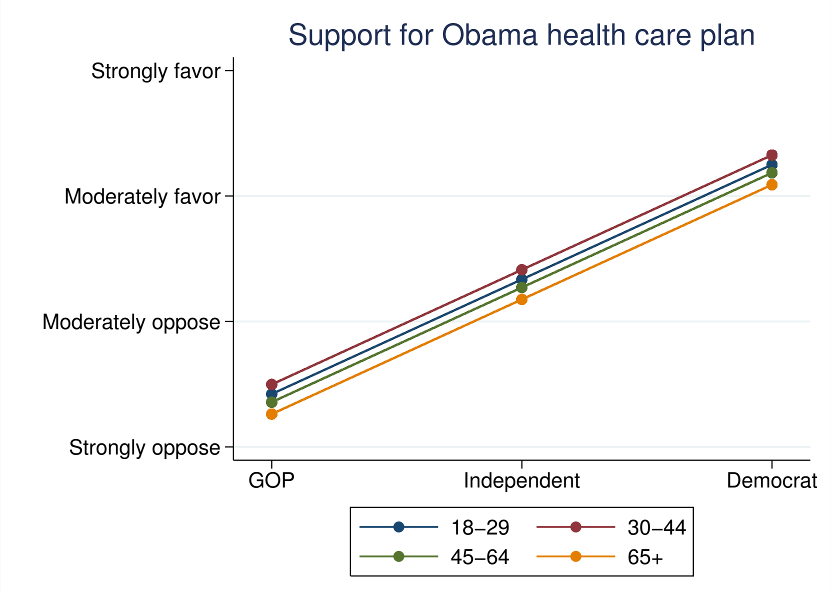
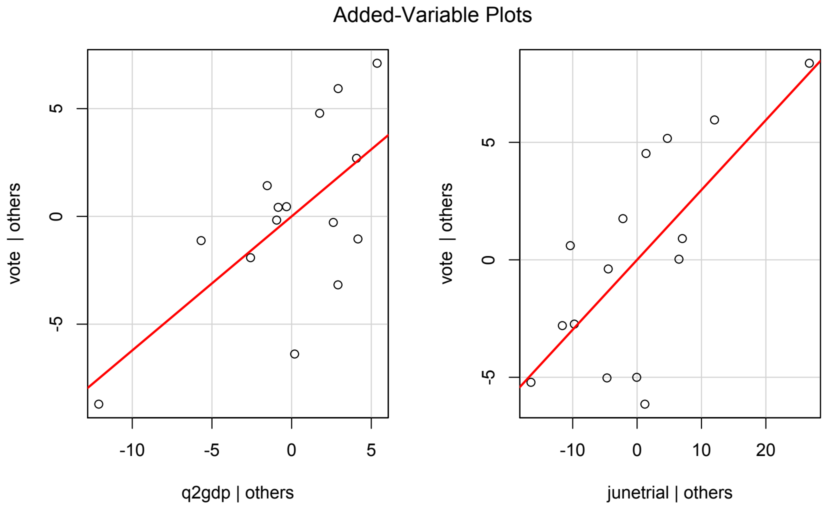


Things to note

- For k levels of your categorical variable, you need to create $k-1$ dummy variables.
- The choice of baseline is arbitrary, but you need to know which is the baseline category in order to interpret the results correctly
- All effects are relative to the baseline category
- If you don't include them as separate dummies, you are assuming that the intercepts are equidistant and ordered.

Thinking about regression #3: Added variable plots



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- It is difficult to decide on the "right" variables, but DO NOT use stepwise methods.
- When in doubt, use theory.

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- Just read these values off of the tables
- But watch your degrees of freedom.

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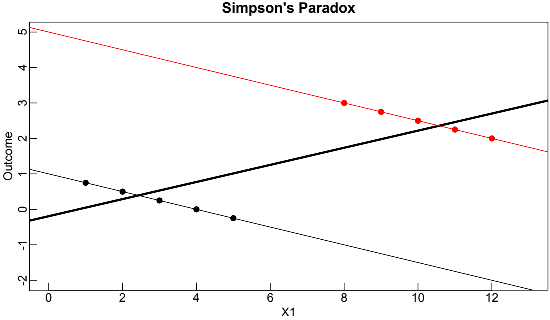
Controlling for a variable can change the sign



$$E(Y) = 1 - 0.25X_1 + 2X_2$$

- Relationship between X_1 and Y is the same across groups.
- We can solve: $X_2 = 0$ for black observations, $X_2 = 2$ for red.

Applied example: Income in presidential voting

income	clinton	trump	other/no answer
under $30,000 **17%**	**53%**	41%	6%
$30k-$49,999 **19%**	**51%**	42%	7%
$50k-$99,999 **31%**	46%	**50%**	4%
$100k-$199,999 **24%**	47%	**48%**	5%
$200k-$249,999 **4%**	48%	**49%**	3%
$250,000 or more **6%**	46%	**48%**	6%
24537 respondents			

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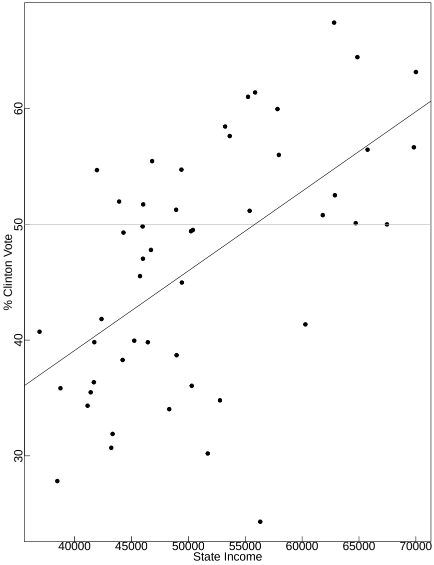

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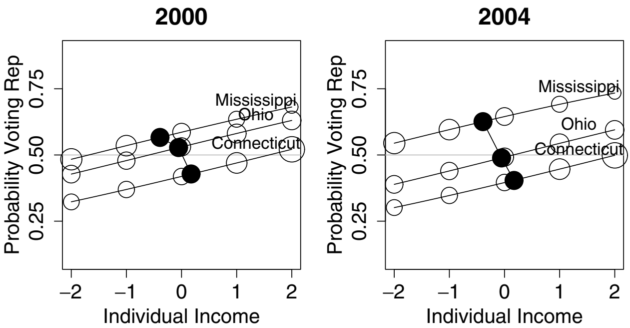

Gelman et al. (2007):

- Rich states more likely to vote D (solid circles)
- Rich *within* states more likely to vote GOP (open circles)