Tree-based models for political science data

November 30, 2017

Flexible modeling of data

- Handling sparcity

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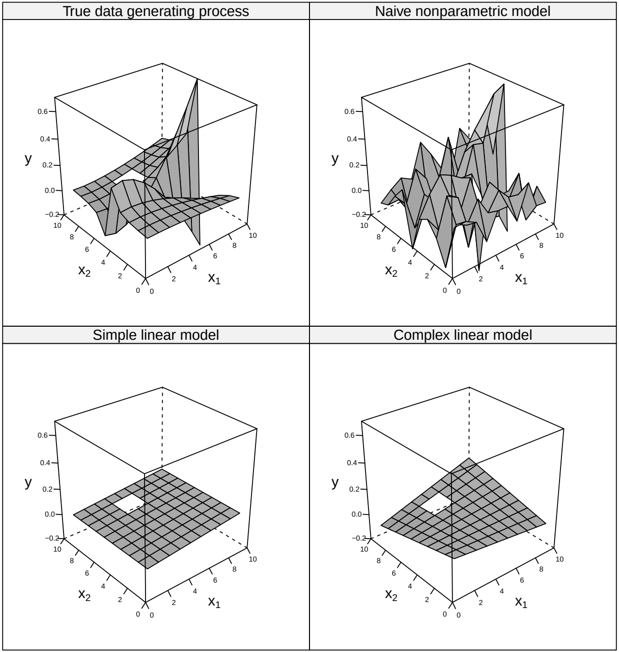
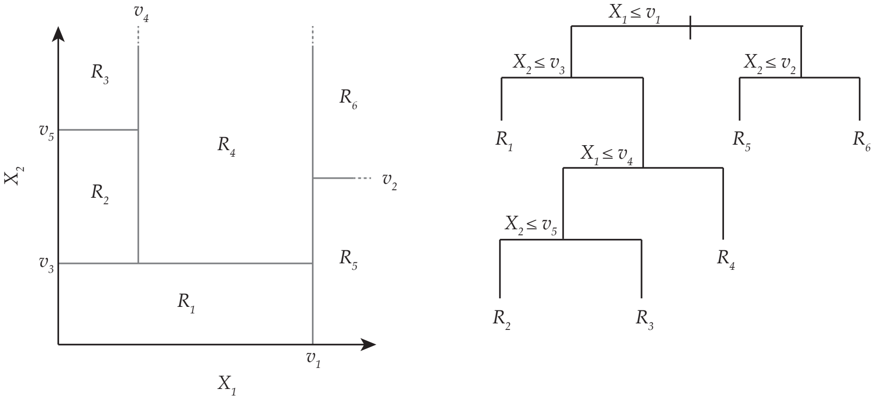
- Regularization

X_1 \ X_2	1	2	3	4	5	6	7	8	9	10
10	4	9	6	9	4	5	6	5	5	3
9	7	8	4	7	4	13	12	5	4	3
8	5	3	7	7		8	6	8	3	6
7	2	4	6	6	5	13	12	4	5	5
6	7	5	6	8	4	7	4	6	4	5
5	2	6	11	11	2	11	4	2	3	6
4	5	6	6	5	3	9	5	4	3	5
3	7	6	11	5	5	4	2	4	8	5
2	4	7	8	7	10	7	9	8	2	7
1	8	10	5	7	5	5	13	2	9	7


True data generating process
Naive nonparametric model
Simple linear model
Complex linear model
y
x_1
x_2

Partitioning the covariate space using binary trees



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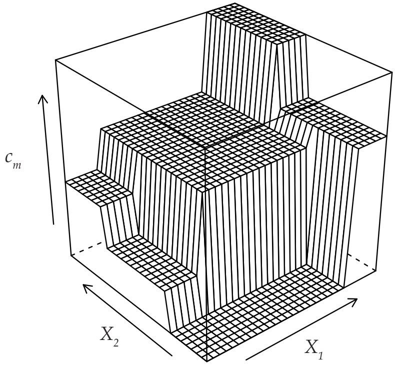

$$f(X_i) = T(X_i;\Theta) \equiv \sum_{b=1}^{B} c_b I(X_i \in R_b), \tag{1}$$

The basics of single-tree models: Optimization

$$f(X_i) = T(X_i; \Theta) \equiv \sum_{b=1}^{B} c_b I(X_i \in R_b) \tag{2}$$

$$\hat{\Theta} = \arg\min_{\Theta} \sum_{b=1}^{B} \sum_{X_i \in R_b} L(y_i, c_b) \tag{3}$$

Where $L(.)$ is some loss function, e.g., $\sum_{i:X_i \in R_b} (y_i - c_b)^2$.

The basics of single-tree models: Regularization

	Purpose	Description
1	Calculate optimal splits	For each covariate j, calculate the optimal point (v) to create a new split.
2	Choose optimal covariate	Select the covariate and split rule that minimize $L(\cdot)$ using the average y_i in the corresponding regions as c_b.
3	Check stopping rules for new leaves	Check whether the tree has reached pre-specified level of complexity.
4	Repeat steps 1-3	For each new leaf, if the stopping rule has not been reached, add a new split.

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Prune by:

- Find subtree T that minimizes $C_\alpha(T) = \sum_{b=1}^{B} L(y_{i:X_i \in R_b}, c_b) + \alpha B$
- B is the number of terminal nodes
- $\alpha \geq 0$ is user specified

Pros and cons of single-tree models

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Solution is to use ensembles of trees

Tree ensembles: Bagging, boosting, and BART

$$f(X_i) = \sum_{m=1}^{M} T_m(X_i; \Theta_m), \tag{4}$$

- M is the number of trees, and Θ_m are the parameters that define tree T_m

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- During recursive binary splitting, use only $a < j$ randomly selected covariates.

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For each new tree, we now maximize:

$$\hat{\Theta}_m = \arg\min_{\Theta_m} \sum_{i=1}^{N} L\big(y_i, f_{m-1}(X_i) + T_m(X_i; \Theta_m)\big) \tag{6}$$

Since this cannot be calculated directly, we approximate using:

$$\hat{\Theta}_m = \arg\min_{\Theta_m} \sum_{i=1}^{N} \big(-g_{im} - T_m(X_i, \Theta_m)\big)^2, \tag{7}$$

where $\mathbf{g}_m$ is the gradient of the loss function.

Gradient boosting machines: Regularization

To avoid over-fitting the data:

- Set B very low (although this also determines the degree of assumed interactions in the model)
- Shrinkage: $f(X_i) = f_{m-1}(X_i) + \nu T(X_i; \Theta_m)$

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GBM is:

- wicked fast
- very flexible
- requires bootstrapping to get uncertainty estimates

BART

$$y_i = \sum_{m=1}^{M} T_m(X_i; \Theta_m) + \epsilon_i, \qquad \text{with } \epsilon_i \sim N(0, \sigma^2) \tag{8}$$

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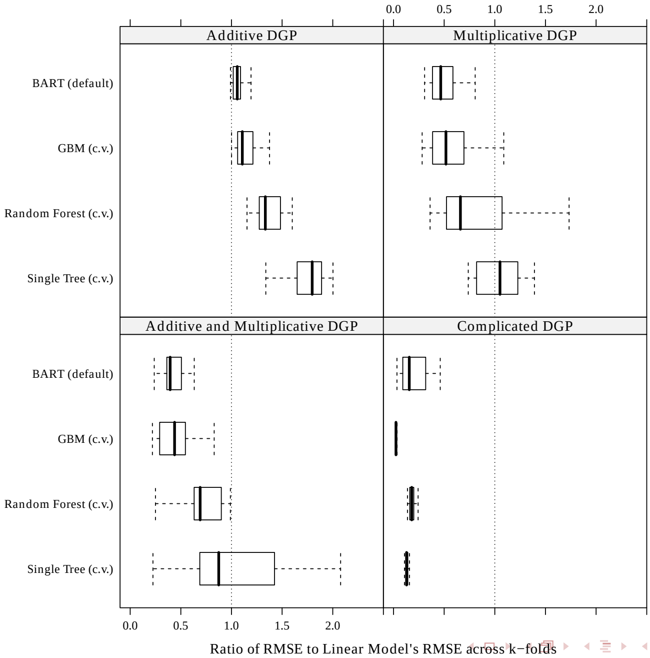
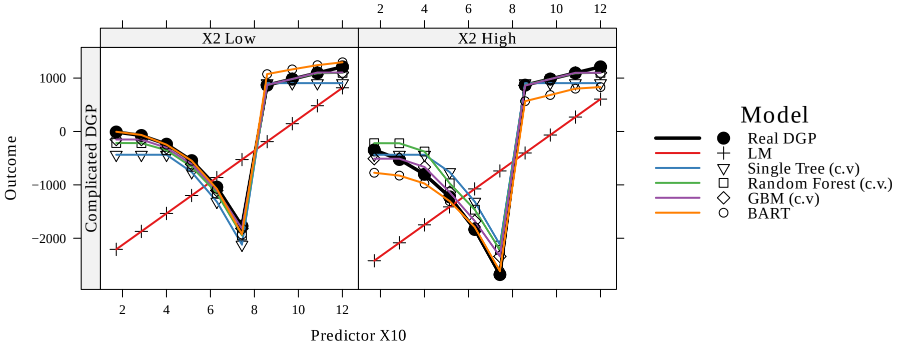
- Samples from a posterior of ensemble models
- Provides uncertainty estimates
- Default priors seem to work very well (no cross validation)
- Not as flexible and SLOW

Model	Tuning parameters	Usual Values
Single tree (`rpart`)	Min. nr. of observations in any terminal node (`minbucket`)	3, 5, 10
	Max. tree depth (`maxdepth`)	3, 5, 15
	Minimum factor decrease in lack-of-fit measure (`cp`)	Inf., 0.1, 0.01, 0.001
Random forest (`randomForest`)	Nr. of trees (`ntree`)	100, 500, 1000
	Nr. of candidate variables sampled at each split (`mtry`)	2, 10
	Min. nr. of observations in any terminal node (`nodesize`)	3, 5, 10
	Nr. of observations sampled when forming each tree (`sampsize`)	360, 720
Gradient boosting (`gbm`)	Nr. of trees (`n.trees`)	100, 500, 1000
	Max. tree depth (`interaction.depth`)	3, 5, 10
	Learning rate (or factor shrinkage) of each tree (`shrinkage`)	0.001, 0.005, 0.01
BART (`BayesTree`)	Degrees of freedom for error variance prior (`sigdf`)	3, 10
	Quantile for error variance prior (`sigquant`)	0.9, 0.99, 0.75
	Nr. of trees (`ntree`)	50, 200
	Factor multiplying the prior range of the outcome variable (`k`)	1, 2, 3, 4


Additive DGP
Multiplicative DGP
Additive and Multiplicative DGP
Complicated DGP
BART (default)
GBM (c.v.)
Random Forest (c.v.)
Single Tree (c.v.)
0.0
0.5
1.0
1.5
2.0
Ratio of RMSE to Linear Model's RMSE across k−folds

Recovering relationships: Synthetic data



Predicted values of outcome variable as a function of x_{10} under the complicated DGP for tree-based models and a linear model, conditional on two values of x_2. The thicker, black line with solid circles represents the true effect of x_{10}.

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- We fit the model using GBM (cv) and BART (default)

Interpreting results

1. Does the variable contribute to the model's explanatory power? (i.e., How much does this variable improve the fit of the overall model?)
2. What is the relationship between the covariate and the outcome? (i.e., Does the conditional relationship between the variable and the outcome of interest match theoretical expectations?)

Which variables are important?

$$\mathcal{I}_j^{Improve} = \left(\frac{1}{M} \sum_{m=1}^{M} \sum_{k:\in K_{mj}} i_k^2 \right)^{0.5} \tag{9}$$

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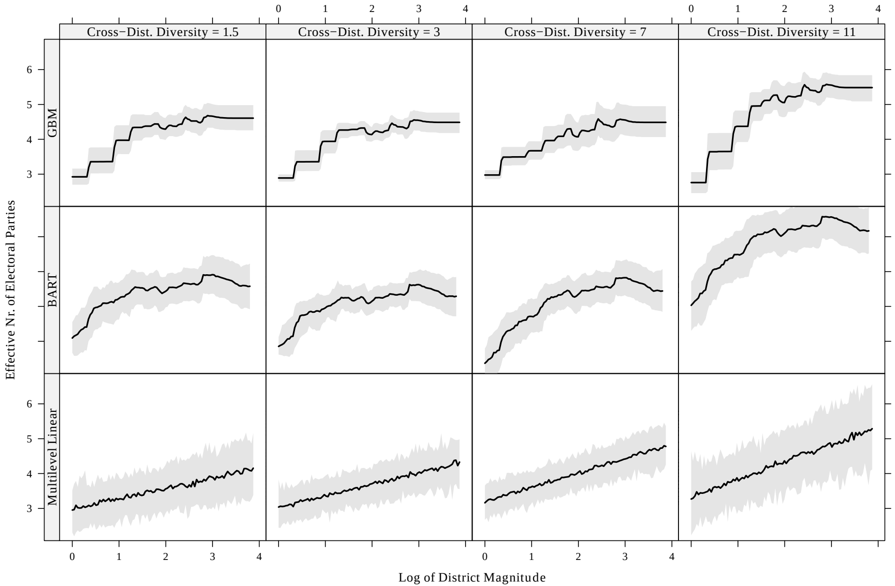
$$\mathcal{I}_j^{Use} = \frac{1}{S} \sum_{s=1}^{S} z_{js}, \tag{10}$$

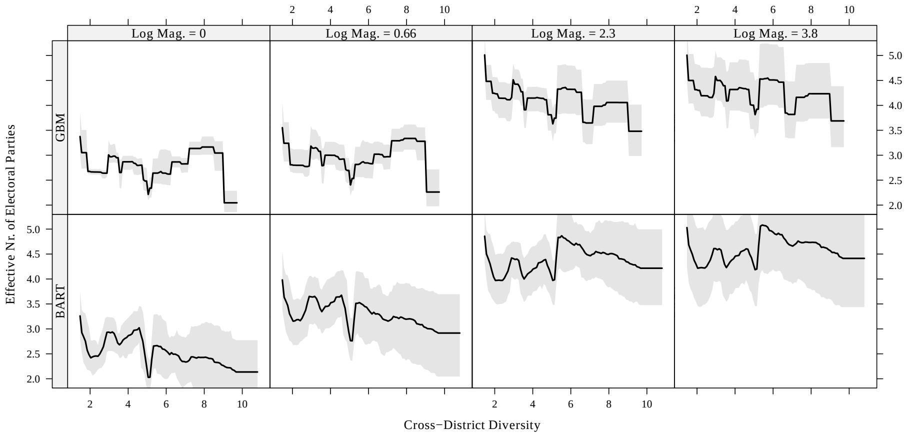
Importance indicators for covariates determining district-level effective number of parties

	$I_j^{Improve}$	I_j^{Use}
Model	GBM	BART
District magnitude	1.00	0.23
Cross-district diversity	0.453	0.21
Age of democratic system	0.291	0.23
District diversity	0.033	0.16
Mixed system	0.019	0.16
Out-of-sample RMSE	0.67	0.69
Out-of-sample R^2	0.57	0.55
n	1581	1581

Partial dependence plots



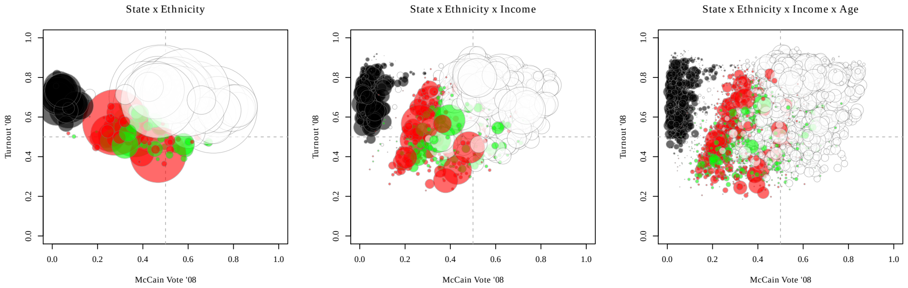
Partial dependence plots: Removing Brazil/Italy



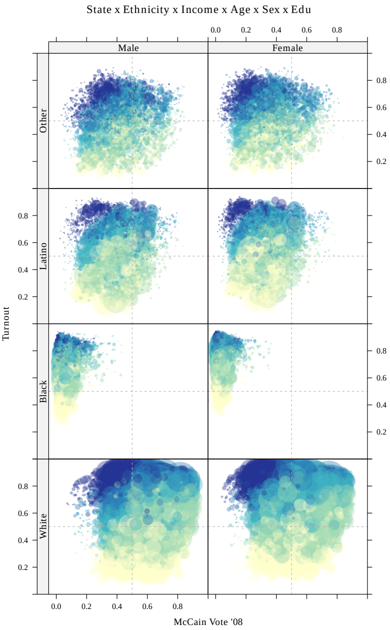
Example: Estimating opinion of demographic sub-groups

- Ghitza and Gelman (2013) use MRP with high-level interactions to model demographic sub-groups
- "By modeling deeper levels of interactions and allowing for the relationship between covariates to be non-linear and even non-monotonic" (p. 773).
- The method does not allow for all levels of interactions.
- We are going to use BART-P to discover even deeper patterns in the data.
- Data are from the 2004 (n=43,970) and 2008 (n=19,170) NAES

Replicating Ghitza and Gelman using BART



- R=0.974 for 2008 McCain Vote
- R=0.98 for 2008 Turnout


State x Ethnicity x Income x Age x Sex x Edu
Male
Female
Other
Latino
Black
White
Turnout
McCain Vote '08
0.0
0.2
0.4
0.6
0.8

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- Valuable tool for non-parametric modeling of large datasets
- Particularly valuable for detecting non-linearities, interactions, and many covariates
- Given the direction of our data resources, these methods should become a more standard tool in the discipline.

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The bad:

- Does not provide either theory or identification
- Don't be evil.

Outline

- The missing data problem
- Types of missingness
- Multiple imputation

Missing Data

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  - This is related to *case selection*.
- What is the most common method of addressing missing data?
  - Casewise deletion or list-wise deletion or complete case analysis

Mechanisms by which Data can be Missing

R is a matrix that with a dichotomous indicator valued 1 if a datum in **X** is missing and 0 if it is not. The missing data generating mechanism is described by ϕ (Little and Rubin, 2002, p.12)

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- Non-ignorable (NI)
  - Missingness depends on unobserved data
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- Non-Ignorable
  - Randomized experiment about voter turnout with a pre-election survey and a post-election survey. Non-response rate to the post-election survey is lower for the treatment group than for the control group. The missing data mechanism depends on the actual turnout of the participants of the experiment, which we don't have!

Why Case-wise Deletion is Evil

- Consider the computation of a mean μ from data $\mathbf{y}$ where some data are non-randomly missing.
- When μ_R is the mean of respondents and μ_M is the mean of missing data, we write the overall mean as:

$$\mu = \pi_R \mu_R + (1 - \pi_R)\mu_M$$

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  where π_R is the *proportion* of observed responses.
- The bias produced by casewise deletion is the expected fraction of missing data times the difference in means for observed and missing data (Little and Rubin, 2002, p.43):

$$\mu_R - \mu = (1 - \pi_R)(\mu_R - \mu_M)$$

- In the special case MCAR, $\mu_R = \mu_M$ and the statistic is unbiased, but this is commonly violated in the social sciences.

Multiple Imputation (Rubin 1979). Steps.

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- Imputation step assumes missing data is conditioned on observed values.
- Oddly, enough m = 5 to 10 is sufficient.
- Combining process uses means for coefficients and an intuitive approach for standard errors.

Rubin's rule

$$Var_{within} = \frac{\sum_{i=1}^{M} SE_i^2}{M}$$

$$Var_{between} = \frac{\sum_{i=1}^{M} (\beta_i - \bar{\beta})^2}{M-1}$$

$$Var_{total} = Var_{within} + Var_{between} + \frac{Var_{between}}{M}$$

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$$\text{Total Variance} = \mathbf{\Omega} + \mathbf{\Psi} + \mathbf{\Omega}/m$$

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- Dataset for imputation usually has more variables than the model we want to specify.
- "An imputation model does not represent causal relationships among the data." (Young and Johnson 2010)

Missing in `R`

- `mice` (van Buuren et al. 2006, van Buuren 2007)
- `Amelia`, easy to use (King, Honaker, Blackwell)
- `mi`, for multilevel data (Kropko, Gelman, others)
- `hot.deck`, for categorical variables (Cranmer and Gill 2013)