Lecture 4: Probability

Jacob M. Montgomery

Quantitative Political Methodology

Lecture 4

Class business

- PROBLEM SET 1 IS DUE RIGHT NOW
- Problem set 2 will be distributed today via the syllabus

Facebook and survey

- Sign up for our Facebook group: https://www.facebook.com/groups/1071702902960687/
- Take the class survey! Can't assign teams until you all do. https://wustl.az1.qualtrics.com/jfe/form/SV_6rpSYD3xxmbRe5v

Roadmap

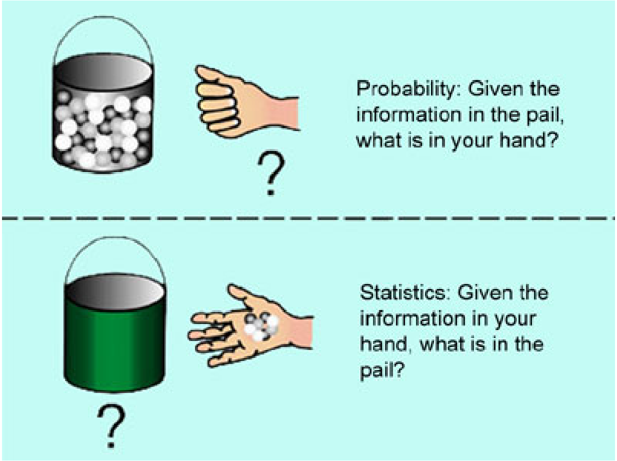
Last time:

- Visualizing data
- Measures of central tendency and spread

This time:

- Understand core concepts of probability
- Understand concept of a “parameter”
- Introduce some probability distributions

Why are we studying this?



Probability defined

Imagine tossing a coin…

- Can you predict the outcome of a single coin toss?

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AF p. 73: "For a particular possible outcome for a random phenomenon, the probability of that outcome is the proportion of times that the outcome would occur in a very long sequence of observations."

Example

Imagine you were rolling two six-sided dice.

1. Write down all possible scores.
2. Calculate the probability of each score
   - What is the probability of rolling a 2?

36 possible outcomes for the two dice:

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

How many outcomes will generate a total score of 2?

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

P(roll=2) $= \frac{1}{36} = 0.028$.

Putting this all together

y_k	$Pr(Y = y_k)$
2	$1/36$
3	$2/36$

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Probability *is the relative frequency of occurrence for some particular outcome if a process is repeated a large number of times under similar conditions*

- If I flip a coin three times, what is the probability that I will get exactly two heads?
- If I roll two dice, what is the probability of getting a two?
- If I take a random sample of 100 Wash U students, what is the probability that less than 40% of the sample will be male?

Frequency distributions

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- $$\sum_{k=1}^{K} p(y_k) = 1$$

We already made one of these

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Expected value

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Mean of probability distribution (**expected value**)

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Mean of probability distribution (**expected value**)

$$\mu = \sum_{k=1}^{K} y_k Pr(Y = y_k) = 2(1/36) + 3(2/36) + \ldots + 12(1/36)$$

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$$\sigma^2 = E(Y - \mu)^2$$

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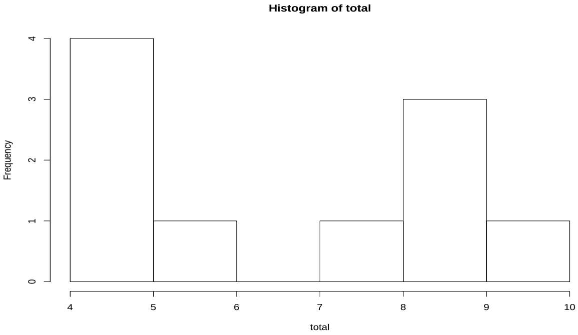
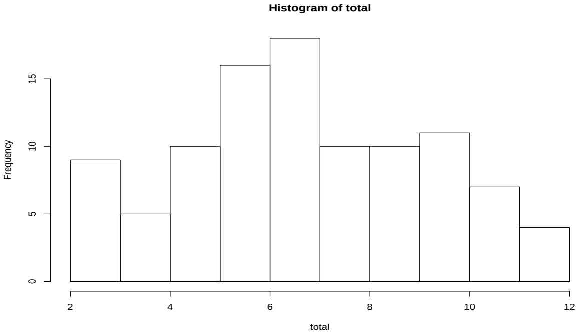
$\sigma^2 = E(Y - \mu)^2$ requires extra calculations

A little simulation

```
posVal<-c(1,2,3,4,5,6)
numRoll<-10
die1<-sample(x = posVal, size=numRoll, replace=TRUE)
die2<-sample(x = posVal, size=numRoll, replace=TRUE)
total<-die1+die2
hist(total)
```



```r
posVal<-c(1,2,3,4,5,6)
numRoll<-100
die1<-sample(x = posVal, size=numRoll, replace=TRUE)
die2<-sample(x = posVal, size=numRoll, replace=TRUE)
total<-die1+die2
hist(total)
```



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posVal<-c(1,2,3,4,5,6)
numRoll<-1000
die1<-sample(x = posVal, size=numRoll, replace=TRUE)
die2<-sample(x = posVal, size=numRoll, replace=TRUE)
total<-die1+die2
hist(total)
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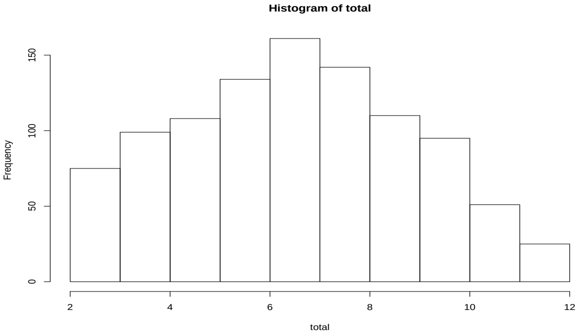
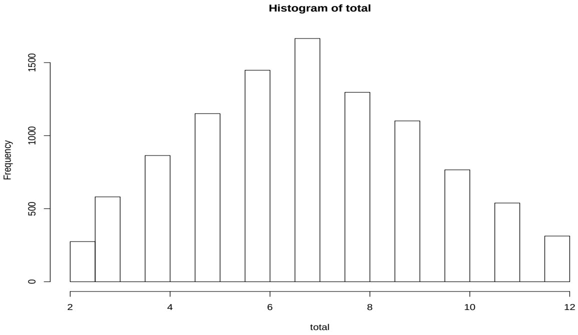


```r
posVal<-c(1,2,3,4,5,6)
numRoll<-10000
die1<-sample(x = posVal, size=numRoll, replace=TRUE)
die2<-sample(x = posVal, size=numRoll, replace=TRUE)
total<-die1+die2
hist(total)
```



End of part 1

- What is a probability?
- What is a frequency distribution?
- What are the two most important parameters for characterizing a distribution?

Example: The Binomial Distribution

Imagine tossing a fair coin in the air three times, where we are interested in the number of heads.

Coin 1	Coin 2	Coin 3	# Heads
H	H	H	3
T	H	H	2
H	T	H	2
T	T	H	1
H	H	T	2
T	H	T	1
H	T	T	1
T	T	T	0

This can be re-written as

y_k	$Pr(Y = y_k)$
0	1/8
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- This table represents a *binomial distribution* where we have $n = 3$ trials and the probability of success is $p = 0.5$.
- We can make similar tables for any value of n or p.

Parameters of the binomial distribution

- p = Probability of "success"
- n = # of "trials"

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Mean and variance of the binomial

$$\mu = np$$

$$\sigma^2 = np(1 - p)$$

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0	1/8
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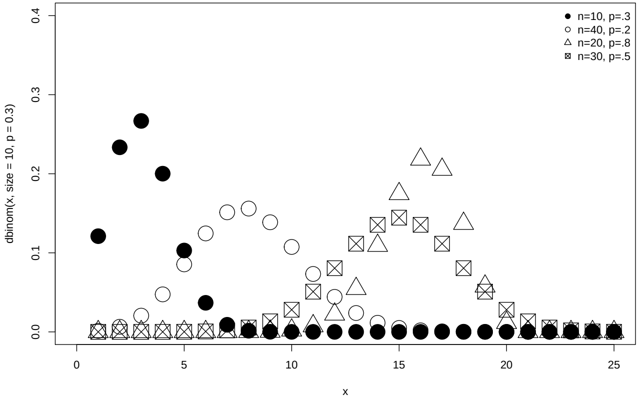
$$= 12/8 = 1.5 = np$$

```
x<-1:25
plot(x, dbinom(x, size=10, p=.3), cex=3, pch=19,
     ylim=c(0, .4), xlim=c(0, 25))
points(x, dbinom(x, size=40, p=.2), cex=3, pch=1)
points(x, dbinom(x, size=20, p=.8), cex=3, pch=2)
points(x, dbinom(x, size=30, p=.5), cex=3, pch=7)
legend("topright",
       c("n=10, p=.3", "n=40, p=.2", "n=20, p=.8", "n=30, p=.5"),
       pch=c(19,1,2,7), bty="n")
```



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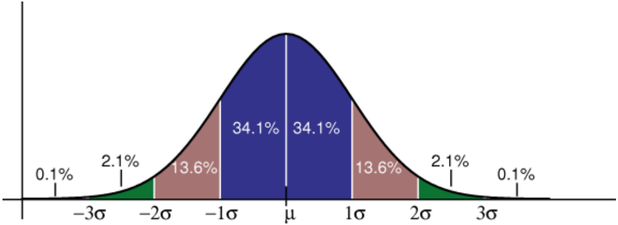
- $\sigma^2 = \ldots$

The normal distribution



- Symmetric
- Bell shaped

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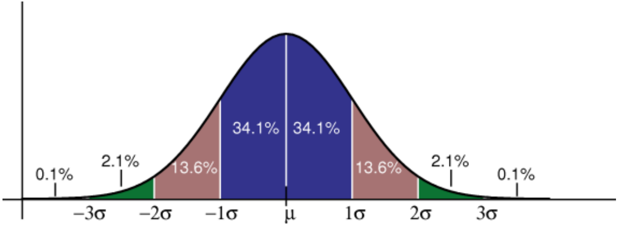

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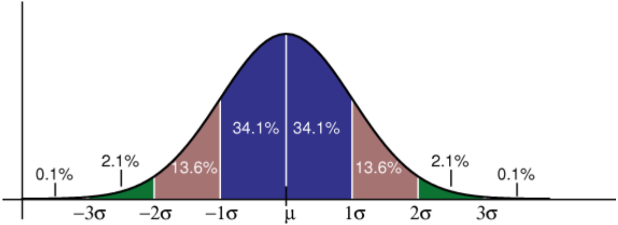

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Normal distribution

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(y-\mu)^2}$$

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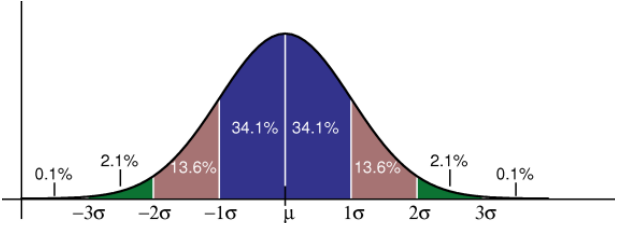

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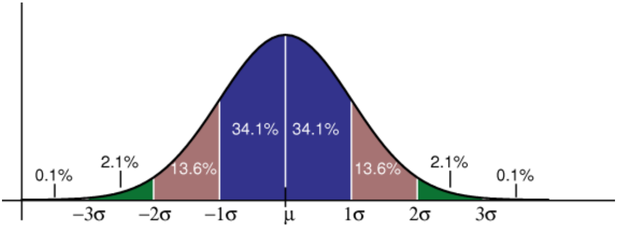
Normal distribution

$$f(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2\sigma^2}(y-\mu)^2} = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(\frac{-1}{2\sigma^2}(y-\mu)^2\right)$$



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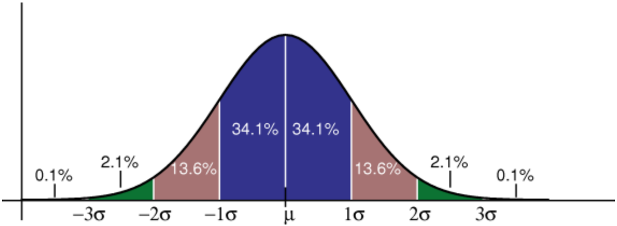

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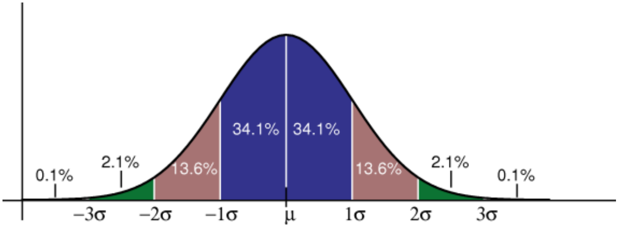

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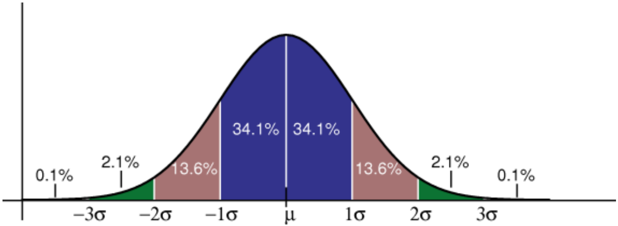

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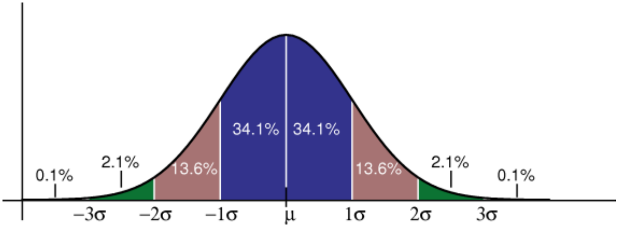

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  - tables will help you on exams
  - and `R` will help you on the homework.

T-Distribution

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- 1908 invented by William Gosset
- wanted to quickly test the quality of raw materials, testing small batches

Notes on the t-distribution

- It has “thicker”" tails than the normal distribution.
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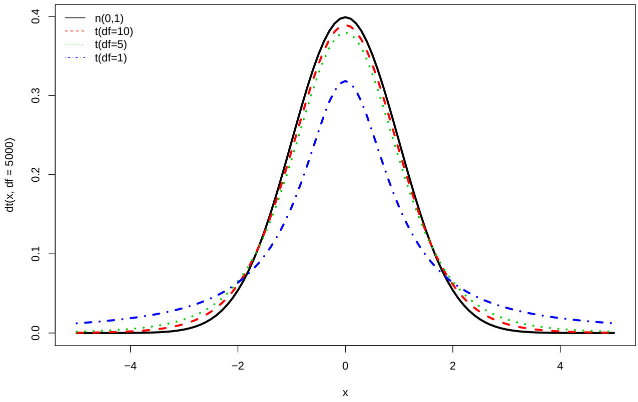
- NOTE: The use of the t-distribution is **not** related to the CLT. We are assuming the data is normally distributed.

```r
x<-seq(from=-5, to=5, by=.1)
plot(x, dt(x, df=5000), lwd=3, type="l", col=1, lty=1)
lines(x, dt(x, df=10), lwd=3, ylim=c(0, .4), col=2, lty=2)
lines(x, dt(x, df=5), lwd=3, ylim=c(0, .4), col=3, lty=3)
lines(x, dt(x, df=1), lwd=3, ylim=c(0, .4), col=4, lty=4)
legend("topleft",
       c("n(0,1)", "t(df=10)", "t(df=5)", "t(df=1)"),
       lty=c(1,2,3,4), col=c(1,2,3,4), bty="n")
```



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  - Empirical rule
- What is the t-distribution, and how is it different than a normal distribution?
  - Thicker tails
  - Degrees of freedom instead of σ^2

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- Sometimes we will talk about how certain **statistics** are distributed
  - Sampling distribution
- Sometimes we will talk about *assumptions* we make about how the **population** is distributed
  - This can sometimes influence which sampling distribution we use
- Try to keep it straight, although we will work on it

Class business

- Problem set 2 is now posted
- Review the online materials for lab (VERY IMPORTANT)
- The reading and online content for Wednesday is *absolutely essential*
- Questions or concerns?