Structural Models/Recursive Strategic Models

Modeling Utility Functions

William O'Brochta

Summary

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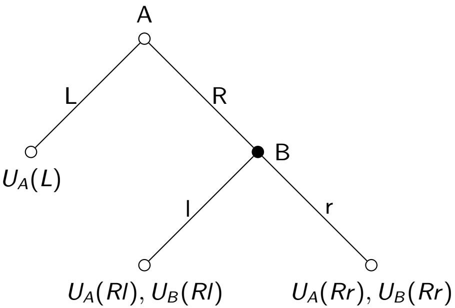
- Previous research has developed direct tests, but implementation was complicated.
- Goal: use logit models to "backward induct" estimates of utility from data.

A Simple Game

Figure: Decision Tree



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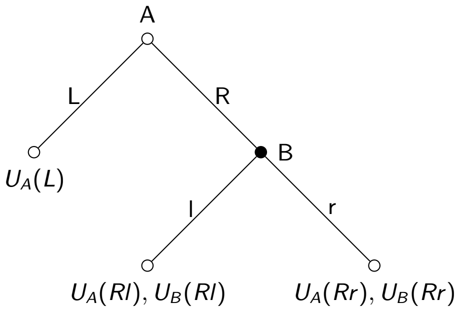

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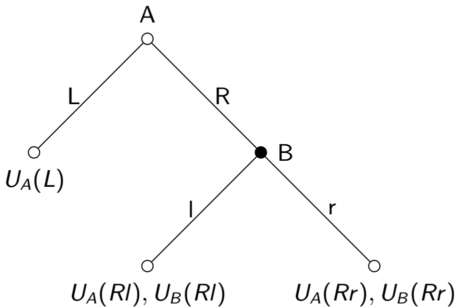

- Player A chooses R or L.
- If Player A chooses R, Player B chooses r or l.

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    - Thus, we write $p_r = \frac{e^{U_B(Rr)}}{e^{U_B(Rl)} + e^{U_B(Rr)}}$ (assuming $\lambda = 1$)
    - $p_l = \frac{e^{U_B(Rl)}}{e^{U_B(Rl)} + e^{U_B(Rr)}}$

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  - $p_L = \frac{e^{U_A(L)}}{e^{U_A(L)} + e^{E[U_A(R)]}} = \frac{e^{U_A(L)}}{e^{U_A(L)} + e^{p_l U_A(Rl) + p_r U_A(Rr)}}$
  - $p_R = \frac{e^{E[U_A(R)]}}{e^{U_A(L)} + e^{E[U_A(R)]}} = \frac{e^{p_l U_A(Rl) + p_r U_A(Rr)}}{e^{U_A(L)} + e^{p_l U_A(Rl) + p_r U_A(Rr)}}$
- These probabilities are the link between player B and player A.

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- This is *not* just a nested multinomial logit. Why? Because different players are involved.

Remember This?

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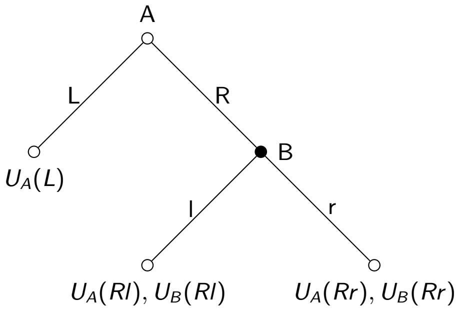

Simplifying Utilities

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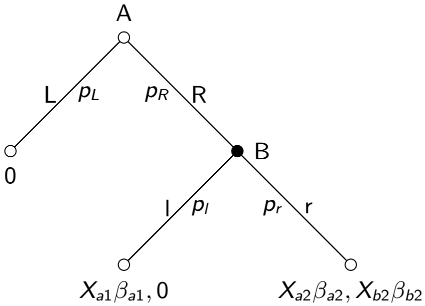

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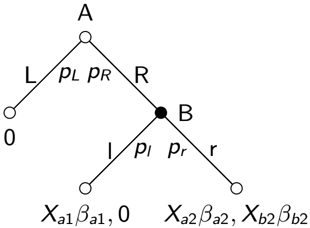

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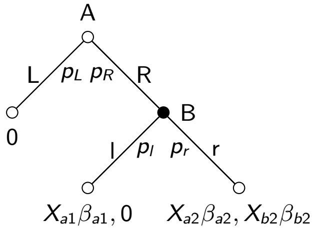

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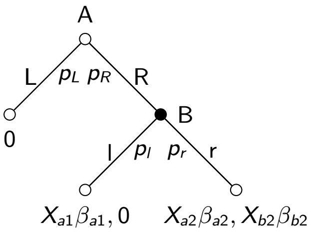

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- Substitute in data to our utility functions:
  - $y_A^* = p_l X_{a1}\beta_{a1} + p_r X_{a2}\beta_{a2} + \epsilon_A$, the utility for choosing R.
  - $y_B^* = X_{b2}\beta_{b2} + \epsilon_B$, the utility for choosing r.
  - where the error terms are distributed logistically by assumption.

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- We can bootstrap the standard errors along the way.

An Example

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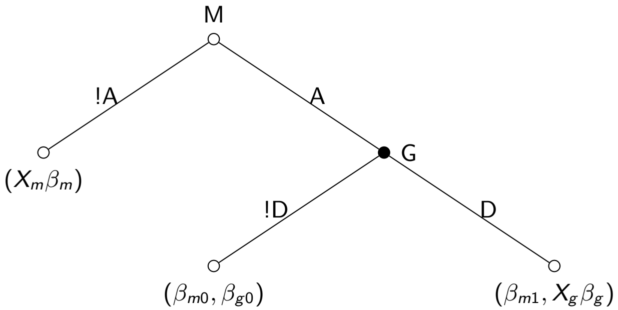

- Currency manipulation: M is market (can attack), G is government (can defend if attacked).
- If government defends, utility has two covariates.
- DV: whether an attack occurred and whether government defended.

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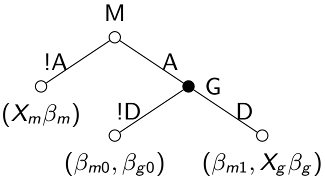

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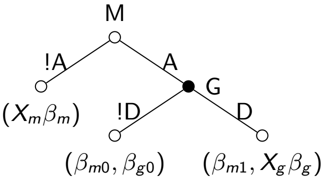

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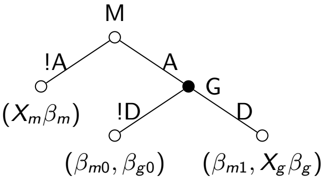

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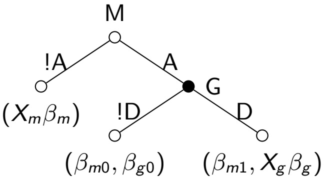

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- Calculate $p_{!D} = \frac{e^{0.20}}{e^{0.20}+e^{0.52}} = 0.57$ and $p_D = 0.43$; substitute to y_A^*.
- Knowing this, M chooses between a payoff of $-3.41(0.43) - 4.05(0.57) = -3.78$ or $X_m\beta_m = -0.46 + 0.29 = -0.17$. Thus, the equilibrium is $!A$ with payoffs $(-0.17, 0)$.

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- The sequential method described here generally means that the user has to calculate the probabilities by hand.
- There are ways to estimate all this at once, but this occurs less often since a separate piece of software (STRAT) is needed.

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- Can extend to more complex game trees and models with more possible variables contributing to the utility. Gets complicated quickly.
- Only works if you are confident that your covariates really measure players utility.
- Has frequently been used in IR, especially by David Carter (if you're looking for research ideas).