

Statistical Testing

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- ▶ The other major focus of statistics is in conducting tests.
- ▶ What is a test?
 - ▶ Definition
 - ▶ LR Test
 - ▶ Wald Test
- ▶ What criteria can we use to decide if our test is a good one?
 - ▶ Errors
 - ▶ Power/Power functions
 - ▶ Size/level

What are we testing anyways?

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A ***hypothesis*** is a statement about a population parameter.

The two complementary hypotheses in a hypothesis testing problem are called the ***null hypothesis*** and the ***alternative hypotheses***. They are denoted by H_0 and H_1 , respectively.

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- ▶ Second, we set up the alternative as the complement of this set.

$$H_1 : \theta \in \Theta_0^c$$

So what is a test?

A **hypothesis testing procedure** or **hypothesis test** is a rule that specifies:

1. For which sample values the decision is made to accept H_0 as true.
2. For which sample values H_0 is rejected and H_1 is accepted as true.

- ▶ The subset of the sample space for which H_0 will be rejected is called the rejection region.
- ▶ This is often written as

$$R = \{t(\mathbf{x}) > c\}$$

where c is some “critical value.”

- ▶ The complement of the rejection region is called the acceptance region.

$$R^c = \{t(\mathbf{x}) < c\}$$

Example 1: Likelihood ratio test

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The **likelihood ratio statistic** for testing $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_0^c$ is

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The **likelihood ratio test** is any test that has a rejection region of the form $\{\lambda(\mathbf{x}) \leq c\}$, where c is any number satisfying $0 \leq c \leq 1$.

Understanding the LRT

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- ▶ The maximum and minimum values are therefore 1 and 0. Why?
- ▶ After the midterm we will discuss why this statistic is useful, and later in this session we'll discuss how to choose c . For the moment let's practice calculating $\lambda(\mathbf{x})$.

LRT for normal data

Let X_1, \dots, X_n be a random sample from a $N(\theta, 1)$ population. We are testing the null hypothesis that $H_0 : \theta = \theta_0$. We have already established that the MLE is \bar{x} . Thus λ will be

$$\lambda(\mathbf{x}) = \frac{(2\pi)^{-n/2} \exp[-\sum_{i=1}^n (x_i - \theta_0)^2/2]}{(2\pi)^{-n/2} \exp[-\sum_{i=1}^n (x_i - \bar{x})^2/2]}$$

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- ▶ We have now found an expression of $\lambda(\mathbf{x})$, but it is pretty complicated. The test would be something like

$$\left\{ \exp[-n(\bar{x} - \theta_0)^2/2] < c \right\}$$

- ▶ So let's try and recognize and simplify so we can use a simpler function of x .

$$|\bar{x} - \theta_0| \geq \sqrt{-2(\ln(c))/n}$$

- ▶ Or even

$$|\bar{x} - \theta_0| > c^*$$

which should look pretty familiar.

Example 2: Wald test

- ▶ Let's do something simpler (but related).
- ▶ Assume we are testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$
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- ▶ $W = \frac{(\hat{\theta} - \theta_0)}{\hat{se}}$ is our statistic. What is the test?
- ▶ The size α **Wald test** is: reject H_0 when $|W| > z_{\alpha/2}$.

Two class exercises

1. Go back to our normal data example and put back in σ but still assume it is known. Show how we could use the Wald test.
2. Go back to our normal example, but now assume that σ is unknown. Find the LR test. It is useful to know that:

$$\frac{\bar{X} - \theta}{S/\sqrt{n}} \sim t(n-1)$$

Evaluating tests

- ▶ There are two criteria we use to evaluate a test.
- ▶ The **power** of a test is “power to reject.” The idea is that:
 - ▶ When the null hypothesis is false, we want a test that will reject it with a high probability (ideally one).

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 - ▶ When the null hypothesis is false, we want a test that will reject it with a high probability (ideally one).
 - ▶ When the null hypothesis is true, we want a test that will reject it with a low probability (ideally zero).
- ▶ The **size** of the test is the the maximum probability that we will reject the null hypothesis assuming that the null hypothesis is true.
- ▶ The important thing to remember here is that we use these terms to evaluate a test. The test is not usually derived based on these criteria.

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- ▶ Let R denote the “rejection region” for our statistic such that we reject whenever $\mathbf{x} \in R$.

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- ▶ A **Type II** error occurs if $\theta \in \Theta_0^c$ and $\mathbf{x} \in R^c$. This can be written as

$$P(\mathbf{X} \in R^c) = 1 - p(\mathbf{X} \in R)$$

The power function

- ▶ From this we can see that $P(\mathbf{X} \in R)$ contains all of the information we need to evaluate the errors.
 - ▶ If $\theta \in \Theta_0$ then $P(\mathbf{X} \in R)$ is the probability of a Type I error.
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- ▶ A good test is one with a power function near 1 when H_0 is false and near 0 when H_0 is true.
- ▶ You can think of this as the “power to reject.”

Example: Binomial power function

Let $X \sim \text{binomial}(5, \theta)$. Consider testing $H_0 : \theta \leq 1/2$ versus $H_1 : \theta > 1/2$. I propose a test that we should reject when we observe all successes.

$$\beta(\theta) = P(X \in R) = P(X = 5) = \theta^5$$

- ▶ Let's plot the power function.
- ▶ Evaluate it in terms of Type I and Type II errors.

Let $X \sim \text{binomial}(5, \theta)$. Consider testing $H_0 : \theta \leq 1/2$ versus $H_1 : \theta > 1/2$. I now propose a test that we should reject when we observe 3 or more successes.

- ▶ Find the power function
- ▶ Plot it.
- ▶ Evaluate it in terms of Type I and Type II errors.
- ▶ Which is better?

Example: Normal power function

Let X_1, \dots, X_n be a random sample from a $N(\theta, \sigma^2)$ population, where σ^2 is known. An LRT of $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$ is a test that rejects if

$$\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} > c.$$

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The power function for this test (where c is some constant) is:

$$\beta(\theta) = P\left(\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} > c\right)$$

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$$\begin{aligned}\beta(\theta) &= P\left(\frac{\bar{X} - \theta_0}{\sigma/\sqrt{n}} > c\right) \\ &= P\left(\frac{\bar{X} - \theta}{\sigma/\sqrt{n}} - \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} > c\right)\end{aligned}$$

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$$\beta(\theta) = P\left(Z > c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right)$$

- ▶ Let $\theta_0 = 0$, $n = 50$, $\sigma = 1$, and $c = 1.25$.
- ▶ Plot the power function and discuss.
- ▶ What happens when you increase c ?
- ▶ What happens when you increase n ?

Thinking about size and level

- ▶ For any fixed sample size, it is usually impossible to make both types of error arbitrarily small.
- ▶ Moreover, as we have seen, there is usually a tradeoff.
- ▶ A common approach is to choose a maximum value of Type I error we are willing to tolerate and then search for a test that has the smallest probability of Type II error that match that criteria.

Defining size and level

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- Note that many authors are very loose in using size and level and these definitions are not canonical.

Example: Size of a test for normal data

Let $X_1, \dots, X_n \sim N(\mu, \sigma)$ where σ is known. We want to test $H_0 : \mu \leq 0$ versus $H_1 : \mu > 0$. Let \bar{X} be our statistic of interest. Therefore we want to set up a test such that we reject when $\bar{X} > c$ or

$$\beta(\mu) = P(\bar{X} > c)$$

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4. This function is increasing in μ . Really?
5. Thus what is the largest value β could take on and still remain in $\theta \in \Theta_0$? That is, what value of μ will we have?
6. Substitute in that value for μ and replace $\beta(\cdot)$ with α .

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8. Now go back. This is saying that we reject if

$$\bar{X} > \frac{\sigma \Phi^{-1}(1 - \alpha)}{\sqrt{n}}$$

$$\bar{X} > \frac{\sigma z_{\alpha}}{\sqrt{n}}$$

9. This can be re-written as we reject when:

$$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > z_{\alpha}$$

More advanced topics (some to be covered later)

- ▶ For some class of tests of level α we can sometimes identify the uniformly most powerful test.
- ▶ The perils of using p-values as tests.
- ▶ Multiple tests
- ▶ Goodness of fit tests