Nonparametric Estimation

Jacob M. Montgomery

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- All of these methods are based on a “model” of the DGP.
- Today, we will move away from this approach and instead try and learn from our data without assuming much of anything.
  1. Find some pointwise estimate
  2. Calculate it for the whole sample.
  3. Do something to the sample and re-calculate.
  4. Repeat and summarize.

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  - Thinking broadly, *most* possible estimands do not have known asymptotic standard errors.
  - The best we can do in other circumstances is use the delta method (or higher-order Taylor approximations).
- (You will see a lot of estimands that can be "non-parametrically" identified. Always think about where their error bars came from.)
- But what we really need are more general ways to estimate standard errors.

Example: The Jackknife

- We have $X_1, \ldots, X_n$ that are iid samples from some unknown function or function space

$$X_i \sim F$$

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- We calculate some statistic based on the data $\hat{\theta} = t(\mathbf{x})$. This is our estimate.
- Let $\mathbf{x}_{(i)}$ be the sample with the i^{th} observation removed.

$$x_{(i)} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)'$$

- Let $\hat{\theta}_{(i)} = t(\mathbf{x}_{(i)})$

- Then the jackknife estimate of the standard error of $\hat{\theta}$ is

$$\hat{se}_{jack} = \left[\frac{n-1}{n}\sum_{1}^{n}\left(\hat{\theta}_{(i)} - \hat{\theta}_{(\cdot)}\right)^2\right]^{1/2}$$

- Where $\theta_{(\cdot)} = \sum_1^n \theta_{(i)}/n$

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- $\hat{\theta}_{(i)} = (n\bar{x} - x_i)/(n-1)$ and $\hat{\theta}_{(\cdot)} = \bar{x}$
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- $\hat{se}_{jack} = \frac{s}{\sqrt{n}}$

Example: Correlation coefficient

- Imagine we have two iid random samples of the same size, $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$. The correlation between these two variables is:

$$\frac{\sum_i^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i^n (x_i - \bar{x})^2 \sum_i^n (y_i - \bar{y})^2}}$$

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- For a sample of

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x=c(2, 4, 5, 3, 8, 10)
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- Plot the two variables
- Estimate the correlation (use the cor() function)
- Find the jackkknife standard error

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- This is a truly non-parametric method.
- There is a hidden assumption here that $s(\mathbf{x})$ will not deviate wildly when sample size is $n-1$. Can be weird for things like medians (defined differently when sample size is odd or even).
- Estimates of standard errors can be upwardly biased.
- Behaves poorly where local derivatives are eratic.

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- From a frequentist perspective, we would like to re-sample from our distribution $\mathcal{F}$ and then re-calculate over and over and over again. But how?
- What if we used the empirical distribution we have to estimate $\mathcal{F}$ itself and then sampled from that distribution?

Estimating an unknowing distribution

The **empirical distribution function** $\hat{\mathcal{F}}$ is a CDF that puts a mass of $\frac{1}{n}$ at each data point X_i.

$$\hat{\mathcal{F}}(x) = \frac{\sum_{i=1}^{n} I(X_i \leq x)}{n}$$

where $I(\cdot)$ is an indicator function.

Class exercise: Estimate $\mathcal{F}$ for the following dataset

```
x=c(2, 4, 5, 3, 8, 10, -2, 1, -1, 2, 5, 5)
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Discussion of the empirical estimation of an unknown funciton

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2. $$Var(\hat{\mathcal{F}}(x)) = \frac{F(x)\,(1 - F(x))}{n}$$

3. $$MSE = \frac{F(x)\,(1 - F(x))}{n} \to 0,$$

4. $\hat{\mathcal{F}}(x)$ converges in probability to $F(X)$.

Moving from the estimate of the distribution to the bootstrapped standard error

- Now that we have an estimate of $\mathcal{F}$, we want to imagine "sampling" repeatedly from it.
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- In some sense, this is the same "plug in" estimation method we have used before.
- The idea is that we sample from from $\hat{\mathcal{F}}$
- We can then calculate our estimate over and over and over again and via this method get an approximation of the frequentist standard error without making any parametric assumptions.

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- Calculate $\hat{\theta}^* = t(\mathbf{x}^*)$
- Repeat

- Let B indicate the total number of bootstrap samples and b index each such that:

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- For some large number B (B=500 will do for calculating standard errors)

$$\hat{se}_{boot} = \left[\frac{\sum_{b=1}^{B} (\hat{\theta}^{*(b)} - \hat{\theta}^{*\cdot})^2}{B-1} \right]^{1/2},$$

- Where

$$\hat{\theta}^{*\cdot} = \frac{\sum_{1}^{B} \hat{\theta}^{*(b)}}{B}$$

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- Estimate the correlation (use the cor() function)
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Discussion

- This is a completely non-parametric approach
- More dependable (but more computationally intensive) than the jackknife
- We could just as easily have calculated the absolute bias, the MSE or anything else.

Class exercise

- For a sample of

```r
x=c(2, 4, 5, 3, 8, 10, -2, 1, -1, 2, 5, 5)
```

1. Find the bootstrapp/jackknife SE for the median.
2. Estimate the absolute error $E(|\hat{\theta} - \theta|)$ for the median and the mean using the bootstrap.

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- One of the assumptions we did *not* get rid of was the iid assumption.
- It is important realize that *everything* we have done in this session rests on the assumption that we can collect truly independent draws.
- In many (most?) cases, our data do not meet this assumption.
- In these cases, the bootstrap must be used carefully or not at all.

Moving blocks bootstrap

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- We can then construct a bootstrap sample by sampling from these blocks rather than from the "raw" sample.
- An even more complex scheme must be developed for time-series cross-sectional datsets, and insufficient work has been done in this area to recommend a "clean" solution.

Additional bootstrap schemes

- Bayesian bootstraps
- "Robust" estimation for quantities such as the trimmed mean to reduce the effect of out outliers
- See Efron and Hastie Chapter 10 for additional details

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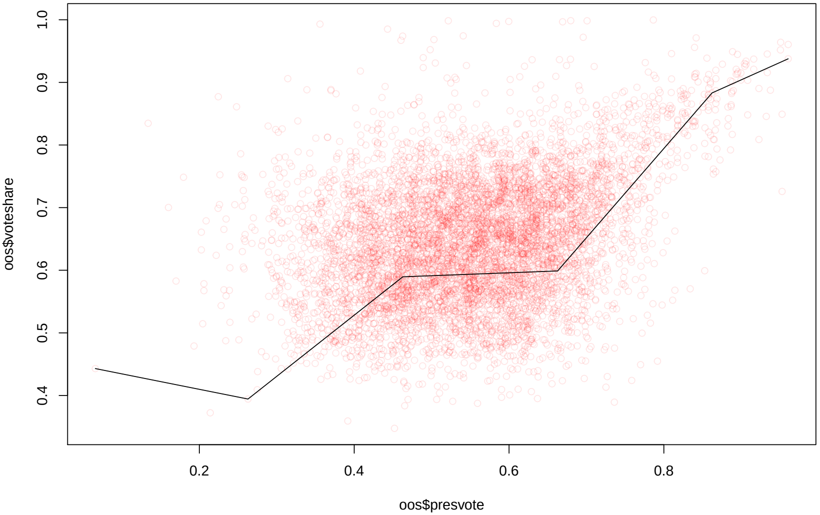
- The tricubic kernal used by default in R is:

$$K_s(x_0, x_i) = \begin{cases} (1 - u_i^3)^3 \text{ if } u_i \leq 1 \\ 0 \text{ otherwise} \end{cases}$$

```r
oos<-read.csv(file =
url("https://jmontgomery.github.io/ProblemSets/incumbents.csv"))
myFit<-lowess(oos$voteshare~oos$presvote, delta=.2 )
plot(oos$presvote, oos$voteshare, col=rgb(1,0,0, alpha=.1))
points(myFit, type="l")
```



For problem set exercise

1. Use the bootstrap method (assuming iid) to estimate the 95% CI for this curve.
2. Add the CI to the plot.
3. Re-do this for several different values of delta (small and big).
4. What is driving this result?