Problem Set 4

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1) LR test for the exponential function

Assume we have

$$x \sim \theta \exp(-\theta x)$$

We want to to test:

$$H_0 : \theta = \theta_0$$

against

$$H_1 : \theta > \theta_0$$

a. Find the likelihood function.
b. Find the numerator for the likelihood ratio
c. Find the first derivative of the log-likelihood $\frac{\partial \mathcal{L}}{\partial \theta}$
d. Show that $\frac{\partial \mathcal{L}}{\partial \theta}$ is an increasing function for $\theta < \frac{1}{\bar{x}}$ and a decrasing function for $\theta > \frac{1}{\bar{x}}$.
e. Use your answer to (d) to prove that:

$$\sup\{L(\theta|\mathbf{x}) : \theta \in \Theta\} = \begin{cases} \bar{x}^{-n} \exp(-n) & \text{if } \frac{1}{\bar{x}} \geq \theta_0 \\ \theta_0^n \exp(-n\theta_0 \bar{x}) & \text{if } \frac{1}{\bar{x}} < \theta_0 \end{cases}$$

Use a picture to explain this result.

f. Find the likelihood ratio statistics λ when $\frac{1}{\bar{x}} \geq \theta_0$ and $\frac{1}{\bar{x}} < \theta_0$
g. Assume that $\theta_0 > 0$. Show hat the first derivative of λ in terms of $\bar{x}$ is positive for $\bar{x} \in (0, \frac{1}{\theta_0})$. This shows that λ is a non-decrasing function of $\bar{x}$.
h. Use (g) to show that the rejection region for the LRT will then be of the form:

$$R = \{\mathbf{x} : \bar{x} \leq c\}$$

You do *not* have to figure out what c is.

2) One sample t-test

In class we worked through the LRT for normal data where we assumed that σ^2 was known. Now you are going to work through it where σ is not known. Note, however, that the null hypothesis will still only involve μ. We assume nothing about σ^2 under the null. (Not also, that σ^2 is the parameter of interest rather than σ.)

a. We have that:

$$\Theta = \left\{(\theta, \sigma^2) : \theta \in \mathcal{R}, \sigma^2 \in \mathcal{R}^+\right\}$$
$$\Theta_0 = \left\{(\theta, \sigma^2) : \theta = \theta_0, \sigma^2 \in \mathcal{R}^+\right\}$$

Find the log-likelihood under the null hypothesis.

b. Take the derivative of the log-likelihood in terms of σ^2, set it equal to zero, and solve for σ^2. We will call this σ^{2*}.

c. Substitute this into your answer for (a). Discuss how this shows that:

$$\sup\left\{L(\theta_0,\sigma^2)\right\} = \left(\frac{2\pi}{n}\sum_{i=1}^{n}(x_i-\theta_0)^2\right)^{-n/2} e^{-n/2}$$

d. Use our previous results about the MLE for normal data to find:

$$\sup\left\{L(\theta,\sigma^2)\right\}$$

e. Find the likelihood ratio statistics λ.

f. Note that

$$\begin{aligned}\sum_{i=1}^{n}(x_i-\theta_0)^2 &= \sum_{i=1}^{n}(x_i-\bar{x})+(\bar{x}-\theta_0))^2\\ &= \sum_{i=1}^{n}(x_i-\bar{x})^2+n(\bar{x}-\theta_0)^2\end{aligned}$$

Show that this allows us to re-write λ as:

$$\lambda = \left(1+\frac{n(\bar{x}-\theta_0)^2}{\sum(x_i-\bar{x})^2}\right)^{-n/2}$$

g. Show that if the critical region for the LRT is:

$$R^* = \{\mathbf{x} : \lambda(\mathbf{x}) \le k\}$$

this implies that H_0 can be rejected when the value of

$$\frac{|\bar{x}-\theta_0|}{\sqrt{\sum(x_i-\bar{x})^2}}$$

is greater than or equal to some constant.

h. Using the knowledge that

$$\frac{\bar{X}-\theta}{S/\sqrt{n}} \sim t(n-1)$$

, show how we can re-write the critical region such that

$$R = \left\{\mathbf{x} : \frac{|\bar{x}-\theta_0|}{\sqrt{\sum(x_i-\bar{x})^2}} > c\right\}$$

g. Assume that we have data $(1,3,2,4,-1,7,19,3,-4,-5,-8)$ and $\theta_0 = 0$. Conduct the likelihood ratio test for the null hypotheses using the results above. Your test should have size of 0.10.

h. Find the power function for this test.

i. Plot the power function for this test. Indicate the region where the null hypothesis is accepted and rejected.

j. Look at the power function right at the threshold of the rejection region. Would you be happy drawing results from this test?

3) Simulation

Let ϕ represent the proportion of cases where the null hypothesis is true. Let α be the probability of rejecting the null hypothesis if it is true. Let $(1-\beta)$ be the probability of rejecting given that the null hypothesis is false.

a. If we test many hypotheses, show that the false positive rate is then:

$$\frac{\alpha\phi}{\alpha\phi + (1-\beta)(1-\phi)}$$

b. If $\frac{\phi}{1-\phi}$ is 1/5, what is the false positive rate when $\alpha = .05$ and $\beta = .75$?
c. If $\frac{\phi}{1-\phi}$ is 1/20, what is the false positive rate when $\alpha = .05$ and $\beta = .75$?
d. If $\frac{\phi}{1-\phi}$ is 1/40, what is the false positive rate when $\alpha = .05$ and $\beta = .6$?

4) Bayes factors

If I, Prof. Montgomery, flipped a coin 20 times and it came up heads 18 times, what is the probability that it is a fair coin?

a. Set the problem up as a complete Bayesian problem.
b. Set the problem up as a choice of two discrete choices and solve explicitly.
c. Now imagine that Ryden Butler flipped a coin 20 times and it came up heads 18 times. Change your priors accordingly and re-do the problem (Here, use updated belief from (b) as a prior).