P-Values and NHPT

Jacob M. Montgomery

2017



What are they good for?

- Last class we:
 - Established what a test is
 - Discussed properties of a good test in terms of errors

What are they good for?

- Last class we:
 - Established what a test is
 - Discussed properties of a good test in terms of errors
- ► This class we:
 - Discuss p-values as a particular approach
 - Consider the Bayesian critique that maybe we should not be doing this at all

A p-values does not mean **any** of the things below:

▶ The probability that the alternative hypothesis is true

- ▶ The probability that the alternative hypothesis is true
- ▶ The probability that the null hypothesis is false

- ▶ The probability that the alternative hypothesis is true
- ▶ The probability that the null hypothesis is false
- ▶ The probability that your finding is a false positive

- ▶ The probability that the alternative hypothesis is true
- ▶ The probability that the null hypothesis is false
- ▶ The probability that your finding is a false positive
- The probability that the finding was produced by random error

- ▶ The probability that the alternative hypothesis is true
- ▶ The probability that the null hypothesis is false
- ▶ The probability that your finding is a false positive
- The probability that the finding was produced by random error
- The effect size was large,

- ▶ The probability that the alternative hypothesis is true
- ▶ The probability that the null hypothesis is false
- ▶ The probability that your finding is a false positive
- The probability that the finding was produced by random error
- The effect size was large, small,

- ▶ The probability that the alternative hypothesis is true
- ▶ The probability that the null hypothesis is false
- ▶ The probability that your finding is a false positive
- The probability that the finding was produced by random error
- ▶ The effect size was large, small, important, or unimportant

A p-values does not mean **any** of the things below:

- ▶ The probability that the alternative hypothesis is true
- ▶ The probability that the null hypothesis is false
- ▶ The probability that your finding is a false positive
- ▶ The probability that the finding was produced by random error
- ▶ The effect size was large, small, important, or unimportant

So what does it mean?

Defining a p-value broadly

Suppose that for every $\alpha \in (0,1)$ we have a size α test with rejection region R_{α} . Then,

$$p$$
-value = $inf \{ \alpha : t(\mathbf{x}) \in R_{\alpha} \}$

▶ In words this means that the p-value is the smallest level at which we can reject H_0 .

Getting down to business

- First we set up a null hypothesis.
- ▶ Then we assume that the null hypothesis is true.
- ▶ Then we consider a set of tests with size $\alpha \in (0,1)$
- ▶ The p-value is the size of the largest test at which we can still reject H_0 .

A critical value perspective

- ▶ Suppose that the size α test is of the form "reject H_0 iff $t(\mathbf{x}) \geq c_{\alpha}$ "
- ▶ In this case the p-value is

$$\sup_{\theta \in \Theta_0} P(t(\mathbf{X}) \geq t(\mathbf{x})))$$

where \mathbf{X} is the random variable and \mathbf{x} is the observed value.

More intuitively

► A p-value is a measure of surprise

More intuitively

- ► A p-value is a measure of surprise
- ▶ It is the probability of observing a value of some statistic ASSUMING THE NULL HYPOTHESIS IS TRUE that is the same *or more extreme* than what was actually observed.

We are going to try and support our research hypothesis using a technique called *proof by contradiction*.

First, we set up a null hypothesis (determined sceptic).

- First, we set up a null hypothesis (determined sceptic).
- Second, we use a sample statistic to show that the data we have observed would be very unlikely if the null hypothesis were true.

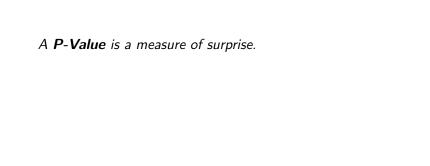
- First, we set up a null hypothesis (determined sceptic).
- Second, we use a sample statistic to show that the data we have observed would be very unlikely if the null hypothesis were true.
- ► Third, we (kind of) conclude that our research hypothesis is true. This is usually called the alternative hypothesis.

- First, we set up a null hypothesis (determined sceptic).
- Second, we use a sample statistic to show that the data we have observed would be very unlikely if the null hypothesis were true.
- Third, we (kind of) conclude that our research hypothesis is true. This is usually called the alternative hypothesis.
- ▶ To do this, we first need to specify our hypotheses.

We calculate a statistic that summarizes how much our data diffe	rs
from what we would have expected to observe if the null hypothes	

were true.

from wh	at we would	tic that summ have expected is is somethin	to observe i	f the null hype	othesis



A **P-Value** is a measure of surprise. We ask, "If the null

hypothesis is true, how likely is it that we would observe a

test-statistic this extreme **or more**?"

A **P-Value** is a measure of surprise. We ask, "If the null hypothesis is true, how likely is it that we would observe a test-statistic this extreme **or more**?"

- ▶ P-Values are very difficult for many people to understand.
- ▶ Smaller P-Values more strongly contradict the null.

How surprised would you have to be in order to conclude that the *null hypothesis* is false?

- ▶ Usually, $p \le 0.05 \Rightarrow$ statistically significant result
- ▶ We would observe a test-statistic this extreme or more 1/20 times if the null hypothesis was true.
- ▶ More generically, we want $p \le \alpha$

How surprised would you have to be in order to conclude that the *null hypothesis* is false?

- ▶ Usually, $p \le 0.05 \Rightarrow$ statistically significant result
- ▶ We would observe a test-statistic this extreme or more 1/20 times if the null hypothesis was true.
- times if the null hypothesis was true • More generically, we want $p < \alpha$
- ▶ We "reject the null" and conclude that the evidence supports the alternative hypothesis.

Example

According to a union agreement, the mean income for all senior-level assembly-line workers in a large company equals \$525 per week. A representative of a women's group decides to analyze whether the mean income μ for female employees matches this norm. For a random sample of 36 female employees, $\bar{\gamma} = \$495$ and s = \$75.

- ▶ Calculate the test using $\alpha = .01$.
- ▶ Draw a picture that represents the p value.

Example

We flip a coin 20 times and observe 18 heads. Find the p-value for the null hypothesis that we have a fair coin.

- ► The p-value you calculate is a function of the *null hypothesis* you set up.
- You can get a significant value because:
 - ▶ Your null hypothesis is false

- ► The p-value you calculate is a function of the *null hypothesis* you set up.
- You can get a significant value because:
 - ▶ Your null hypothesis is false
 - You are underpowered

- ► The p-value you calculate is a function of the *null hypothesis* you set up.
- You can get a significant value because:
 - ▶ Your null hypothesis is false
 - You are underpowered (BEWARE SMALL SAMPLES)
 - You tested a lot of hypotheses

- ► The p-value you calculate is a function of the *null hypothesis* you set up.
- You can get a significant value because:
 - ► Your null hypothesis is false
 - You are underpowered (BEWARE SMALL SAMPLES)
 - You tested a lot of hypotheses
 - You estimated the sampling distribution wrong





The new true False True False negatives False positives 1. Of hypotheses 2. The tests have a 3. Not knowing false positive rate what is false and interesting enough to test, of 5%. That means what is not, the perhaps one in they produce 45 researcher sees ten will be true. 125 hypotheses as false positives (5% true, 45 of which So imagine tests of 900). They have a power of 0.8, so are not. on 1,000 they confirm only hypotheses. The negative 100 of which 80 of the true results are much more reliable—but are true. hypotheses, producing 20 false unlikely to be negatives. published.

Source: The Economist

This is kind of a big deal

- ▶ We are increasingly aware that we are in a "replication crisis"
- P-values are at the heart of it (and there is no easy solution)
- As a practical matter:
 - ▶ Pre-registeration
 - Lower acceptable p-value standards for new findings
 - ▶ Wait for the meta analysis
 - DO NOT TRUST UNDERPOWERED STUDIES
 - ► Publish replications
 - ▶ Don't believe everything you read (but don't be a jerk about it)

This is kind of a big deal

- ▶ We are increasingly aware that we are in a "replication crisis"
- P-values are at the heart of it (and there is no easy solution)
- As a practical matter:
 - ▶ Pre-registeration
 - Lower acceptable p-value standards for new findings
 - Wait for the meta analysis
 - DO NOT TRUST UNDERPOWERED STUDIES
 - ► Publish replications
 - ▶ Don't believe everything you read (but don't be a jerk about it)
- Maybe don't do hypothesis testing? But what could we do instead?

► Consider the case where θ is a scalar and we are testing $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$.

- Consider the case where θ is a scalar and we are testing $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$.
- ▶ From a Bayesian perspective, we want to directly summarize the strength of the evidence for H_1 relative to H_0 .

- Consider the case where θ is a scalar and we are testing $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$.
- ▶ From a Bayesian perspective, we want to directly summarize the strength of the evidence for H_1 relative to H_0 .
- ▶ Using Bayes' rule, we get

$$\frac{\Pr(H_1|\mathbf{x})}{\Pr(H_0|\mathbf{x})} = \frac{f(\mathbf{x}|H_1)}{f(\mathbf{x}|H_0)} \times \frac{\pi(H_1)}{\pi(H_0)} \equiv \text{Bayes Factor} \times \text{Prior odds}$$

- Consider the case where θ is a scalar and we are testing $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$.
- ▶ From a Bayesian perspective, we want to directly summarize the strength of the evidence for H_1 relative to H_0 .
- ▶ Using Bayes' rule, we get

$$\frac{\Pr(H_1|\mathbf{x})}{\Pr(H_0|\mathbf{x})} = \frac{f(\mathbf{x}|H_1)}{f(\mathbf{x}|H_0)} \times \frac{\pi(H_1)}{\pi(H_0)} \equiv \mathsf{Bayes} \; \mathsf{Factor} \times \mathsf{Prior} \; \mathsf{odds}$$

This can be re-written as:

$$\frac{L(H_1)}{L(H_0)} \frac{\pi(H_1)}{\pi(H_0)}$$

Note that if H₁ = MLE, the first part is the inverse of the likelihood ratio

- Note that if $H_1 = MLE$, the first part is the inverse of the likelihood ratio and if we give the same prior odds to both theories it is exactly the same.
- Let's unpack that a bit:

$$\frac{\Pr(H_1|\mathbf{x})}{\Pr(H_0|\mathbf{x})}$$

Jefferry's proposed the following rules for comparing two models $BF(M_1:M_2) = \frac{\Pr(M_1|\mathbf{x})}{\Pr(H_1|\mathbf{x})}$

▶
$$B(x) \ge 1 \rightarrow \mathsf{Model}\ 1$$
 supported

- ▶ $0.316 \le B(x) < 1 \rightarrow \text{Minimal evidence against Model 1 (Note } 0.316 = 10^{-1/2})$
- ▶ $0.1 \le B(x) < 0.316 \rightarrow \text{Substantial evidence against Model } 1$
- ▶ $0.01 \le B(x) < 0.510$ → Substantial evidence against Model 1

▶ $B(x) < 0.01 \rightarrow \text{Decisive evidence against Model } 1$

Problems with Bayes factors

- ▶ The flaws with this paradigm open up when we consider the case where θ is a scalar and we are testing $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$ where θ is a scalar that can take on many values.
- ► In these cases, improper (flat) priors can lead to non-defined Bayes factors

Problems with Bayes factors

- ▶ The flaws with this paradigm open up when we consider the case where θ is a scalar and we are testing $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$ where θ is a scalar that can take on many values.
- ► In these cases, improper (flat) priors can lead to non-defined Bayes factors
- ▶ Bayes factors can become seriously sensitive to priors especially when they are "vague."
- We have to integrate over the entire parameter space

Problems with Bayes factors

- ▶ The flaws with this paradigm open up when we consider the case where θ is a scalar and we are testing $H_0: \theta = \theta_0$ against $H_1: \theta \neq \theta_0$ where θ is a scalar that can take on many values.
- ► In these cases, improper (flat) priors can lead to non-defined Bayes factors
- ▶ Bayes factors can become seriously sensitive to priors especially when they are "vague."
- ► We have to integrate over the entire parameter space which may not be tractable.
- ► For more, see: http: //www.stat.cmu.edu/~kass/papers/bayesfactors.pdf

P-values and Bayes factors

One of the nice things about this setup is that

$$P(M_0|\mathbf{x}) = \frac{BF(M_0:M_1)}{1 + BF(M_0:M_1)}$$

Where the hell did that come from?

P-values and Bayes factors

One of the nice things about this setup is that

$$P(M_0|\mathbf{x}) = \frac{BF(M_0:M_1)}{1 + BF(M_0:M_1)}$$

- ▶ Where the hell did that come from?
- And what does that mean anyways?

P-values and Bayes factors

One of the nice things about this setup is that

$$P(M_0|\mathbf{x}) = \frac{BF(M_0:M_1)}{1 + BF(M_0:M_1)}$$

- Where the hell did that come from?
- And what does that mean anyways?
- ▶ This number will usually *not* correspond well with a p-value

Simulation 1

- ► Generate 100000 experiments with two variables x and y
- ▶ Both should be random normal samples of size n = 100, mu = 100, $\sigma = 20$.
- ▶ Perform a t-test and grab the p-value (difference of means)
- ▶ Plot a histogram of the p-values that result
- What does this mean? Are you surprised?

Simulation 2

- ► Generate 100000 experiments with two variables x and y
- ▶ X should be random normal samples of size n = 100, mu = 100, $\sigma = 20$.
- Y should be random normal samples of size n = 100, mu = 103, $\sigma = 20$.
- ▶ Perform a t-test and grab the p-value
- ▶ Plot a histogram of the p-values that result
- ▶ What does this mean? Are you surprised?

Simulation 3

- \blacktriangleright Let ϕ represent the proportion of cases where the null hypothesis is true
- Let α be the probability of rejecting the null hypothesis if it is false.
- Let (1β) be the probability of rejecting given that the null hypothesis is false.
- ▶ If we test many hypotheses, the false positive rate is then:

$$\frac{\alpha\phi}{\alpha\phi(1-\beta)(1-\phi)}$$

- ▶ If $\frac{\phi}{1-\phi}$ is 1/5, what is the false positive rate when $\alpha=.05$ and $\beta=.75$?
- ▶ If $\frac{\phi}{1-\phi}$ is 1/20, what is the false positive rate when $\alpha=.05$ and $\beta=.75$?
- ▶ If $\frac{\phi}{1-\phi}$ is 1/40, what is the false positive rate when $\alpha = .05$ and $\beta = .6$?

Exercise

If I, Prof. Montgomery, flipped a coin 20 times and it came up heads 18 times, what is the probability that it is a fair coin? - Set the problem up as a complete Bayesian problem. - Set the problem up as a choice of two discrete problems and solve explicitly.