

Data Reduction

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- ▶ Intead, we often make parametric assumptions about the DGP that allow us to focus on specific statistics calculated from the sample.
- ▶ Here we focus on two conceptual quantities that we can calculate from our sample:
 - ▶ Sufficient statistics
 - ▶ The likelihood

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- ▶ By making assumptions about the DGP, we can calculate statistics like $T(\mathbf{x})$ or $T(\mathbf{y})$ that include all the information we need to make statistical inference.
- ▶ The drawback is that we can have very different vectors \mathbf{x} and \mathbf{y} that give us the same values for $T(\mathbf{x})$ and $T(\mathbf{y})$.
- ▶ The value of this approach is computational efficiency. The drawback is that our inferences are only as good as our assumptions.

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Sufficient statistics

- ▶ The core concept is that we want to determine a reduced form of the data that will tell us about the DGP.
- ▶ First we make a parametric assumption about the DGP, which allows us to characterize it in terms of a set of parameters θ

If $T(\mathbf{X})$ is a sufficient statistic for θ , then any inference about θ should depend on the sample \mathbf{X} only through the value of $T(\mathbf{X})$.

Formal definition

A statistic $T(\mathbf{X})$ is a sufficient statistic if the conditional distribution of the sample \mathbf{X} given the value of $T(\mathbf{X})$ does not depend on θ .

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A statistic $T(\mathbf{X})$ is a sufficient statistic if the conditional distribution of the sample \mathbf{X} given the value of $T(\mathbf{X})$ does not depend on θ .

- ▶ In words, this means that the conditional distribution of our data does not change for any value of θ once we know $T(\mathbf{X})$

Establishing sufficiency

- ▶ Calculate $p(\mathbf{x}|\theta)$
- ▶ Choose some candidate for the sufficient statistic $T(\mathbf{X}|\theta)$
- ▶ Calculate $q(T(\mathbf{x})|\theta)$
- ▶ Calculate

$$\frac{p(\mathbf{x}|\theta)}{q(T(\mathbf{x})|\theta)}$$

- ▶ If this quantity does not depend on θ , it is sufficient.

Why?

- ▶ If the distribution of \mathbf{X} does not depend on θ then
- ▶

$$P(\mathbf{X} = \mathbf{x} | T(\mathbf{X}) = T(\mathbf{x})) = \frac{P(\mathbf{X} = \mathbf{x} \text{ and } T(\mathbf{X}) = T(\mathbf{x}))}{P(T(\mathbf{X}) = T(\mathbf{x}))}$$

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- ▶ Which can be re-written as

$$\frac{p(\mathbf{x}|\theta)}{q(T(\mathbf{x})|\theta)}$$

Example 6.2.3: Binomial sufficient statistic

Let

$$X_1, \dots, X_n$$

be iid Bernoulli random variables with parameter θ . Show that $T(\mathbf{X}) = \sum X_i$ is a sufficient statistic for θ .

Example 6.2.4

Let X_1, \dots, X_n be iid $N(\mu, \sigma^2)$ where σ^2 is known. Show that the sample mean is a sufficient statistic for μ .

Logistic distribution

Let X_1, \dots, X_n be iid logistic where $f(x|\theta) = \frac{e^{-(x-\theta)}}{1+e^{-(x-\theta)^2}}$. Show that the order statistics are a sufficient statistic for θ .

The exponential family

- ▶ A number of very common distributions can be “factored” in such a way that they can be re-represented as having a common family form.
- ▶ This is useful because we can then prove results for this broader family without having to prove it for each individual distribution.

Defining the exponential family

Suppose X_1, \dots, X_n is a random sample from a pdf or pmf $f(x|\theta)$.

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Suppose X_1, \dots, X_n is a random sample from a pdf or pmf $f(x|\theta)$. We say this is an exponential family if we can factor the distribution such that:

$$f(x|\theta) = h(x)c(\theta) \exp\left(\sum_{i=1}^k w_i(\theta)t(x)\right)$$

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An equivalent way to write this is:

$$f(x|\theta) = h(x) \exp(\eta' T(x) - A(\eta))$$

Exercises

- ▶ Show that the normal distribution with known variance σ can be written as a member of the exponential family.
- ▶ Show that the poisson distribution is a member of the exponential family.

Relating back to sufficiency: Factorization theorem

Let $f(\mathbf{x}|\theta)$ denote the joint pdf or pmf of a sample \mathbf{X} . A statistic $T(\mathbf{X})$ is a sufficient statistic for θ iff the pmf/pdf can be re-written as

$$f(\mathbf{x}|\theta) = g(T(\mathbf{x}|\theta))h(\mathbf{x})$$

Theorem 6.2.10

Let $f(\mathbf{x}|\theta)$ denote the joint pdf or pmf that belongs to an exponential family given by

$$f(x|\theta) = h(x)c(\theta) \exp \left(\sum_{i=1}^k w_i(\theta) t_i(x) \right)$$

,

where $\theta = (\theta_1, \theta_2, \dots, \theta_d)$, where $d \leq k$. Then

$$T(\mathbf{X}) = \left(\sum_{j=1}^n t_1(X_j), \dots, \sum_{j=1}^n t_k(X_j) \right)$$

is a sufficient statistic for θ .

Example 6.2.9: Normal sufficient statistic, both parameters unknown

Assume that X_1, \dots, X_n are iid $N(\mu, \sigma^2)$ where neither parameter is known, such that $\theta = (\mu, \sigma^2)$. Use the factorization theorem to show that \bar{x} and s^2 are sufficient statistics for this distribution.

The likelihood function

- ▶ As we have seen, in some cases simply handling a sufficient statistic may be inadequate since a sufficient statistic may be the entire dataset.
- ▶ Moreover, for several types of statistical inference we will not rely on sufficient statistics at all.
- ▶ For both of these reasons, we often switch to calculate a statistic called the *likelihood*.

Defining the likelihood function

Let $f(\mathbf{x}|\theta)$ denote the joint pdf of pmf of the sample $\mathbf{X} = (X_1, \dots, X_n)$. Then, given that $\mathbf{X} = \mathbf{x}$ is observed, the function of θ defined by

$$L(\theta|\mathbf{x}) = f(\mathbf{x}|\theta)$$

is called the *likelihood function*.

Thinking about the likelihood function

- ▶ We seem to be defining the likelihood the same as the pdf/pmf.
- ▶ The only difference is how we will think about θ and \mathbf{x} .
 - ▶ For $f(\mathbf{x}|\theta)$ we consider \mathbf{x} as the variable and θ to be fixed.
 - ▶ For $L(\theta|\mathbf{x})$ we consider \mathbf{x} to be the observed sample and θ to be varying over all possible parameter values.
- ▶ Bayesian thinking will consider θ as a variable. Other approaches tend to think of θ as a fixed but unknown parameter.

Poisson likelihood.

Let

$$X_1, \dots, X_n$$

be iid Poisson random variables with parameter θ . Assume that the observed values of \mathbf{X} are $\mathbf{x} = (4, 17, 4)$.

- ▶ Find $L(\theta|\mathbf{x})$.
- ▶ Write out the generic version for any (non-empty) observed data \mathbf{x}

Binomial Likelihood.

Let

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be iid Bernoulli random variables with parameter θ . Find $L(\theta|\mathbf{x})$.

Normal likelihood.

Let X_1, \dots, X_n by iid $N(\mu, \sigma^2)$ where σ^2 is known.

- ▶ Find $L(\theta|\mathbf{x})$.
- ▶ Can it be represented in terms of the sufficient statistic $T(\mathbf{x})$?

The likelihood principal

If x and y are two sample points such that $L(\theta|x)$ is proportional to $L(\theta|y)$, that is, there exists a constant $C(x, y)$ such that

$$L(\theta|x) = C(x, y)L(\theta|y)\forall\theta,$$

then the conclusion drawn from x and y should be identical.

Thinking about the likelihood principal

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- ▶ If we instead observe $L(\theta_2|y)$ and $L(\theta_1|y)$, then θ_2 should still be twice as likely such that $L(\theta_2|y) = 2L(\theta_1|y)$.

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- ▶ Imagine if we knew that $L(\theta_1|x) = 4L(\theta_1|y)$ and $L(\theta_2|x) = 4L(\theta_2|y)$ but somehow concluded $L(\theta_1|x) > L(\theta_2|x)$ and $L(\theta_1|y) > L(\theta_2|y)$

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- ▶ This seems almost tautologically true, but we shall see that frequentist approaches to inference actually break this rule.