Problem Set 2

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The Pareto distribution

Let

$$X_1, X_2, \ldots, X_n$$

be a simle random sample of a Pareto random variables with density

$$f(x|\theta) = \frac{\theta}{x^{\theta+1}}$$

where $x > 1$.

The first and second central moments are defined as

$$\mu = \frac{\theta}{\theta - 1},$$

and

$$\sigma^2 = \frac{\theta}{(\theta - 1)^2(\theta - 2)}$$

1. Use the method of moments to estimate θ.
2. Find the MLE for θ.
3. Use the delta method to find the approximate asymptotic distribution of the Methods of Moments estimator.
4. Use the parametric bootstrap to find the approximate asymptotic distribution of the Methods of Moments estimator.
5. A conjugate prior for θ is a gamma distribution

   $$\propto \theta^{\alpha-1} e^{-\beta\theta}.$$

   Find the posterior distribution of θ.
6. Calculate a 95% posterior credible interval and 95% HPD.
7. Re-do steps questions 5-6 using the uninformative prior

   $$\pi(\theta) = \frac{1}{\theta}$$

   .

MLE for normal data

8. Recall that in the last problem set you found the first and second derivatives for data generated from a normal distribution with unknown mean and variance. Find the MLE for μ and σ^2.
9. Find the asymptotic distribution for the vector $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)$.
10. What is the covariance of σ^2 and μ?
11. Assume that σ is known. Using a normal prior for μ, find the posterior distribution. Write out our point estimate and calculate a 95% CI.

Negative binomial

In the so-called negative binomial experiment we continue a Bernoulli trial with parameter θ until we obtain s successes, where x is fied in advance. We assume that s is known. Let x be the number o trials needed. The pdf is then

$$f(x|\theta, s) = \binom{x-1}{s-1} \theta^s (1-\theta)^{x-s}$$

. Assume that we observe the following observations where $s = 1$:

$$23, 14, 24, 17, 4, 40, 17, 13, 31, 24$$

11. If $X_1, \ldots, X_n$ are iid negative binomial, find the MLE for θ using the Newton-Rhapson method.
12. In Bayesian methods, we can calculate the posterior distributions using only numerical integration methods.
$$E(\theta) = \frac{\int \theta \pi(\theta) L(\theta) d\theta}{\int \pi(\theta) L(\theta) d\theta}$$
Use the **integrate** function in R to find the EAP estimate of θ. Use a beta distribution for the prior with parameters $\alpha = \beta = 1$.
13. Now execute a similar method to find the posterior variance of θ.
14. Show that these results are sensitive to choices of α and β. Plot different priors, calculate the posterior mean and sd using the numerical method above, and explain the changes.
15. Analytically find the MLE and its asymptotic distribution.
16. Analytically find the posterior distribution of θ using the original prior specified in 12.
17. Construct a 95% CI using each method. Compare them and tell me how each should be interpreted.