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SCALING LAWS FOR PLASMA CONFINEMENT

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ABSTRACT. Scaling laws for plasma energy confinement must be invariant under any transformation which leaves the basic plasma equations themselves invariant. Hence, constraints can be placed on the scaling laws even though the calculation of energy loss may be quite intractable. These constraints are derived for several plasma models and shown to be characteristic of the model. Hence, in addition to reducing the number of parameters which have to be obtained empirically, these results could delineate which plasma models are appropriate for the calculation of losses. They show that the empirical scaling laws at present proposed for tokamaks are incompatible with many conventional plasma models.

1. INTRODUCTION

One objective of large plasma experiments, such as the tokamak series, is the determination of the relation between the overall energy confinement time τ and the parameters of the apparatus – such as radius and magnetic field. This is the so-called 'scaling law' for confinement.

Several attempts have been made to deduce empirical scaling laws from the observed data [1], most recently by Hugill and Sheffield [2]. In these attempts one assumes that the confinement time τ can be expressed as a function of certain independent physical variables, such as plasma density n, plasma temperature T, field strength B and minor radius a, together with geometrical factors such as inverse aspect ratio a/R and safety factor $q = aB_T/RB_\theta$. One then seeks a best fit of the experimentally observed confinement times to a function f(n,T,B,a,q,a/R), usually in the form of a power law, $\tau \propto (n^p T^q B^r a^s \ldots)$.

One may also attempt to obtain scaling laws purely theoretically, by calculating the loss due to, say, a non-linear collisionless-trapped-particle-drift-wave instability – though usually it is necessary to introduce some hypothesis about the non-linear saturation mechanism or about the nature of plasma turbulence.

In the present paper we approach the problem from an altogether different point of view – more general than the theoretical calculation of non-linear drift waves or the like, but still based on plasma physics rather than empirical curve fitting.

The basis of our approach is the observation that if the basic equations of plasma behaviour (which might, e.g. be the Vlasov equation together with charge neutrality) are invariant under a certain group of transformations, then any scaling law derived from these equations must be invariant under the same group of transformations. It turns out that for all the conventional plasma models this invariance property greatly circumscribes the permissible scaling laws. (Indeed in one extreme example the invariance properties alone completely determine the scaling law!) In other words, even though the necessary non-linear theory may be quite intractable, the mere fact that a scaling law is, in principle, derivable from certain basic equations already provides information about that scaling law. For example, we shall be able to show that some recently suggested empirical scaling laws [1,2] for tokamaks are incompatible with theories based on the Vlasov equation in the electrostatic limit, or indeed with many conventional plasma models!

In the following sections, we consider a number of basic plasma models and for each model we determine the constraints which are placed on the scaling laws by the invariance properties of the model. An alternative interpretation is briefly discussed in Section 3. In Section 4, we summarize these results in a form convenient for comparison with the empirically deduced scaling laws, in particular those for Ohmically heated plasmas. The comparison with empirical laws suggests that radiation plays a significant role and the influence of this on scaling laws is discussed in Section 5.

2A. COLLISIONLESS VLASOV EQUATION IN THE ELECTROSTATIC LIMIT

As a simple and archetypal example of our approach we consider first the collisionless Vlasov model. In this the plasma distribution function $f_i(x,v)$ for each species is described by

$$\frac{\partial f_i}{\partial t} + (\vec{v}\cdot\nabla)f_i + \frac{e_i}{m_i}(\vec{E} + \vec{v}\times\vec{B})\cdot\frac{\partial f_i}{\partial\vec{v}} = 0 \qquad (1)$$

The electric field is determined by charge neutrality

$$\sum_i e_i \int f_i(x,v)\,d^3v = 0 \qquad (2)$$

and, in the electrostatic limit, the magnetic field is fixed. [This model may be said to describe collisionless, low-β plasmas.]

Instead of dealing directly with the confinement time it is more convenient to work with the energy loss per unit area and unit time, given by

$$\vec{Q} = \int \vec{v}\,v^2 f(x,v)\,d^3v \qquad (3)$$

We assume that this flux can be expressed, in principle, as a function of n,T,B,a and the geometrical ratios a/R, q. In fact, nothing can be deduced by the present methods about the dependence of confinement on the geometrical ratios and we therefore concentrate on

$$Q = Q(n,T,B,a) \qquad (4)$$

We now seek all the linear transformations of the independent and dependent variables

$$f \to \alpha f,\ v \to \beta v,\ x \to \gamma x,\ B \to \delta B,\ t \to \epsilon t,\ E \to \eta E \qquad (5)$$

which leave the basic equations (1) and (2) of the problem invariant. There are three, and only three, such transformations

$A_1: f \to \alpha f$

$A_2: v \to \beta v,\ B \to \beta B,\ t \to \beta^{-1} t,\ E \to \beta^2 E$

$A_3: x \to \gamma x,\ B \to \gamma^{-1} B,\ t \to \gamma t,\ E \to \gamma^{-1} E$

Under these combined transformations the heat flux transforms as $Q \to \alpha\beta^6 Q$, the temperature as $T \to \beta^2 T$ and the plasma density as $n \to \alpha\beta^3 n$. Consequently, if the heat flux is expressed as

$$Q = \Sigma c_{pqrs} n^p T^q B^r a^s \qquad (6)$$

the requirement that it remain invariant under the transformations $A_1 - A_3$ imposes the following restrictions on the exponents:

$$p = 1,\ 3p + 2q + r = 6,\ r - s = 0$$

so that the general expression for Q is restricted to

$$Q = \Sigma c_q na^3B^3(T/a^2B^2)^q = na^3B^3F(T/a^2B^2) \qquad (7)$$

where F is some unknown function. The corresponding energy confinement time τ is proportional to naT/Q and so is restricted to the form

$$B\tau = F(T/a^2B^2) \qquad (8a)$$

An alternative expression for this is

$$B\tau = F(\beta/N) \qquad (8b)$$

where $\beta = nT/B^2$ and $N = na^2$ is the "line density". [The exact relation between τ and Q involves the temperature and density profiles. However, these are themselves governed by the basic equations so that the scaling laws 8a, 8b are exact consequences of the model as long as the boundary conditions do not introduce any dominant additional physical effects. Indeed, the scaling laws for confinement time could be obtained directly from the invariance principle.]

Thus we see that if a scaling law can be derived, by however tortuous a route, from the collisionless Vlasov equation in the electrostatic limit, then it must take the form (8). Consequently, if the scaling is assumed to follow a power law, then it must be

$$B\tau \sim (T/a^2B^2)^q \qquad (9)$$

and *can contain only one free exponent.*

Before we examine more complex plasma models, we note a remarkable extension of the present result. Suppose that instead of assuming a scaling law for confinement time, we made the stronger assumption that a local transport coefficient exists, i.e. $\kappa = \kappa(n,T,B)$, such that $Q = -\kappa\partial T/\partial x$. In such an event the confinement time would be proportional to a^2 and this must be reflected in the scaling law. However, in conjunction with Eq.(8), this is sufficient to completely determine the scaling, which becomes $\tau \sim a^2 B/T$. This represents the ubiquitous "Bohm diffusion coefficient" which is now seen to be the *only* local transport coefficient compatible with the collisionless Vlasov equation in the electrostatic limit! [A similar result was derived in two-dimensional guiding-centre plasmas[3], but it is now seen to be a more general conclusion.]

2B. COLLISIONAL VLASOV EQUATION IN THE ELECTROSTATIC LIMIT

If the plasma is described by the Vlasov equation including collisions, that is by the Boltzmann equation,

$$\frac{\partial f_i}{\partial t} + \vec{v} \cdot \nabla f_i + \frac{e_i}{m_i} (\vec{E} + \vec{v} \times \vec{B}) \cdot \frac{\partial f_i}{\partial \vec{v}} = C(f,f) \qquad (10)$$

(where C represents the Coulomb collision operator) together with charge neutrality and the electrostatic approximation (a model used for all collisional, low-β plasma studies), the discussion proceeds in a very similar way.

We again seek all transformations similar to (5) which leave the equations of the collisional Vlasov model unchanged. There are two such transformations

$$B_1: f \to \beta f,\ v \to \beta v,\ B \to \beta B,\ t \to \beta^{-1} t,\ E \to \beta^2 E$$

$$B_2: f \to \gamma^{-1} f,\ x \to \gamma x,\ B \to \gamma^{-1} B,\ t \to \gamma t,\ E \to \gamma^{-1} E$$

Under these transformations $Q \to \beta^7 \gamma^{-1} Q$, $T \to \beta^2 T$, $n \to \beta^4 \gamma^{-1} n$ and so for this model the exponents in Eq.(6) are subject to the restrictions

$4p + 2q + r = 7$, and $s - r - p = -1$

so that

$$Q = na^3 B^3 F\left(\frac{n}{B^4 a^3}, \frac{T}{a^2 B^2}\right) \qquad (11)$$

In a collisional low-β plasma, therefore, any scaling law for confinement times must take the form

$$B\tau = F\left(\frac{n}{B^4 a^3}, \frac{T}{a^2 B^2}\right) \qquad (12)$$

or if a power law is adopted

$$B\tau \sim (n/B^4 a^3)^p \, (T/a^2 B^2)^q \qquad (13)$$

Note that in this case the power-law scaling contains only two free exponents.

2C. COLLISIONLESS VLASOV EQUATION AT HIGH-β

Even at high-β, a high-temperature plasma may be described by the collisionless Vlasov equation, Eq.(1), but the electric and magnetic fields must be self-consistently determined from the Maxwell equations

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}; \quad \nabla \times \vec{B} = 4\pi \vec{j} \qquad (14)$$

with $\vec{j} = \int \vec{v} f(x,v) d^3 v$. Charge neutrality may still be assumed if the Debye length is negligible. (This collisionless, high-β plasma model may be that appropriate in the thermonuclear reactor regime.)

In this case there are again two independent transformations under which the full set of basic equations are invariant. These are

$$C_1: f \to \beta^{-3} f,\ v \to \beta v,\ B \to \beta B,\ t \to \beta^{-1} t,\ E \to \beta^2 E,\ j \to \beta j$$

$$C_2: f \to \gamma^{-2} f,\ x \to \gamma x,\ B \to \gamma^{-1} B,\ t \to \gamma t,\ E \to \gamma^{-1} E,\ j \to \gamma^{-2} j$$

Proceeding in the now familiar way, we find that in a collisionless high-β plasma the heat flux is restricted to the form

$$Q = na^3 B^3 F\left(na^2, \frac{T}{a^2 B^2}\right) \qquad (15)$$

and the confinement time must be

$$B\tau = F\left(na^2, \frac{T}{a^2 B^2}\right) = F(N, \beta) \qquad (16)$$

Note that if a power-law is adopted then the scaling law can again contain at most two free exponents.

2D. COLLISIONAL VLASOV EQUATION, HIGH-β

The most complete plasma model of any normally used for plasma confinement studies is one in which the plasma distribution function is described by the Vlasov equation including collisions (Eq.10) while the electromagnetic field is determined self-consistently from the Maxwell equations (14), and charge neutrality, Eq.(2). There is only one transformation which leaves this set of equations invariant;

$$D_1: f \to \beta^5 f,\ v \to \beta v,\ x \to \beta^{-4} x,\ B \to \beta^5 B,\ t \to \beta^{-5} t,$$

$$E \to \beta^6 E,\ j \to \beta^9 j.$$

Consequently, the scaling law for a collisional high-β plasma is subject to one constraint and must be of the form

$$Q = na^3 B^3 F(na^2, Ta^{1/2}, Ba^{5/4}) \tag{17}$$

and

$$B\tau = F(na^2, Ta^{1/2}, Ba^{5/4}) \tag{18}$$

Note that in this model a power-law scaling for confinement time may contain three free exponents, more than allowed by any of the simpler models but still fewer than the general expression (6).

2E. FLUID MODELS

Fluid descriptions of plasma can be derived from appropriate Vlasov or Boltzmann equation models under certain additional approximations. However, because of their widespread use and since the additional approximations lead to extra constraints on the scaling law, it is convenient to analyse them as independent models in their own right.

The fluid equations comprise the continuity equation relating the density ρ and the macroscopic velocity $\vec{v}$,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{19}$$

the momentum equation

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}\right) + \nabla p - \vec{j} \times \vec{B} = 0 \tag{20}$$

where the pressure p is related to the temperature T through $p = nT$ and, e.g. satisfies an energy equation such as

$$\frac{\rho^\gamma}{\gamma - 1}\left(\frac{\partial}{\partial t} + \vec{v} \cdot \nabla\right)(p\rho^{-\gamma}) = \eta\, |\vec{j}|^2 \tag{21}$$

with the resistivity $\eta \sim T^{-3/2}$. The fluid model is completed with Ohm's law

$$\vec{E} + \vec{v} \times \vec{B} = \eta \vec{j} \tag{22}$$

and the Maxwell equations (14) for the electromagnetic field.

In the ideal fluid we ignore the resistivity η; then the equations are invariant under the three transformations

$$E_1: n \to \alpha n,\ B \to \alpha^{1/2} B,\ E \to \alpha^{1/2} E,\ p \to \alpha p,\ j \to \alpha^{1/2} j$$

$$E_2: v \to \beta v,\ t \to \beta^{-1} t,\ B \to \beta B,\ E \to \beta^2 E,\ p \to \beta^2 p,$$

$$j \to \beta j,\ T \to \beta^2 T$$

$$E_3: x \to \gamma x,\ t \to \gamma t,\ j \to \gamma^{-1} j$$

In a fluid model the energy loss transforms like nvT, so any scaling laws which are invariant under the above transformations must have the form

$$Q = T^{1/2} B^2 F(nT/B^2) \tag{23}$$

and

$$B\tau = (na^2)^{1/2} F(nT/B^2) = N^{1/2} F(\beta) \tag{24}$$

Thus in the fluid model only *one* combination of the two quantities (na^2) and $(T/a^2 B^2)$ occurs whereas they arise independently in the scaling law of the collisionless high-β Vlasov model (Eqs (15) and (16)). Correspondingly, only one free exponent is allowed by the ideal fluid model when a power law is assumed.

For the resistive fluid there are two transformations which leave the full set of equations invariant:

F_1: $n \to \alpha n$, $B \to \alpha^{1/2}B$, $E \to \alpha^{1/2}E$, $p \to \alpha p$, $j \to \alpha^{1/2}j$

F_2: $v \to \beta v$, $x \to \beta^{-4}x$, $t \to \beta^{-5}t$, $B \to \beta B$, $E \to \beta^2 E$,

$p \to \beta^2 p$, $j \to \beta^5 j$, $T \to \beta^2 T$

These transformations lead to the scaling laws

$$Q = T^{1/2}B^2 F\left(\frac{na^2}{B^2 a^{5/2}}, Ta^{1/2}\right)$$

$$\frac{B\tau}{n^{1/2}a} = F\left(\frac{na^2}{B^2 a^{5/2}}, Ta^{1/2}\right) = F(\beta, Ta^{1/2}) \qquad (25)$$

which are also special cases of the corresponding laws for the collisional Vlasov high-β model, (Eqs (17), (18)), with the quantities na^2 and $(Ba^{5/4})$ appearing only in the combination $(na^2/a^{5/2}B^2)$. A power law scaling similarly contains only two free exponents.

Finally, we comment on the inclusion of additional effects in the fluid equations. Adding finite-Larmor-radius (FLR) effects to the ideal fluid leads, not surprisingly, to the same scaling law (17), (18), as the collisionless Vlasov model at high β, while including FLR, electron thermal conductivity or viscosity in the resistive fluid equations leads to the same law as the collisional high-β Vlasov model.

3. INTERPRETATION

At this point it is useful to give an interpretation of the present results from the point of view of Kadomtsev's discussion of scaling laws using dimensional analysis [4]. From the variables n,T,B,a one can construct four independent dimensionless parameters. The choice of these is arbitrary, but if we select the set

β, a_i/a, ν/ω_c, λ_D/a

where a_i is the ion Larmor radius and λ_D the electron Debye length, then the confinement time can be written

$$B\tau = \frac{m}{e}\ F(\beta, a_i/a, \nu/\omega_c, \lambda_D/a)$$

A basis for the preceding results is now apparent. The most general expression for confinement time is a function of four variables (or, in power law form, contains four arbitrary exponents). For all the conventional plasma models, however, charge neutrality is assumed so that one of these functions, or exponents, (that relating to the Debye length) is removed. For the collisionless Vlasov models a second function or exponent (that relating to collision frequency) is removed leaving the scaling law as a function of two variables. Finally, the electrostatic assumption removes a third variable or exponent – that referring to the parameter β – with a corresponding further limitation of the possible scaling law.

However, this interpretation of the results depends on a judicious choice of the independent parameters. This is illustrated by the fact that if the dimensionless line density, $(e^2/mc^2)na^2 \sim N$ were adopted as one of the variables instead of a_i^2/a^2, then β appears in the scaling law (8b), although it refers to a plasma model in which β is neglected. Furthermore, as we shall see in Section 5, the present method can be used to discuss dependence on some dimensionless quantities such as atomic number.

In the case of the ideal and resistive fluid models, which do not refer to particle aspects, we have only the dimensionless parameters β and τ_A/τ_R, where $\tau_A = a\sqrt{4\pi\rho}/B$ and $\tau_R = a^2/\eta$ are the Alfvén and resistive time scales. Thus we can write

$$\frac{B\tau}{n^{1/2}a} = F\left(\beta, \frac{\tau_A}{\tau_R}\right)$$

so that confinement time is a function of two variables, or, in the case of a power law, has two free exponents. Restriction to ideal MHD removes one of the variables or exponents, that referring to the resistive time scale, and the scaling law then contains only one free variable or exponent.

4. COMPARISON WITH EXPERIMENT AND ITS IMPLICATIONS

To facilitate comparison between the restricted scaling laws of each model and any experimental data we now summarize the theoretical results. If the confinement time is written

$$B\tau = n^p T^q B^r a^s \qquad (26)$$

then each model imposes its own characteristic constraints on the exponents as in Table I.

TABLE I. SCALING LAWS FOR CONVENTIONAL PLASMA MODELS

Plasma model	Scaling law for $B\tau$	Constraints on power law scaling (26)	Free exponents
Collisionless low-β	$F(T/a^2B^2)$	$p = 0, r = s = -2q$	1
Collisional low-β	$F\left(\frac{T}{a^2B^2}, \frac{na^2}{B^4a^5}\right)$	$3p + 2q + s = 0$ $4p + 2q + r = 0$	2
Collisionless high-β	$F\left(na^2, \frac{T}{a^2B^2}\right)$	$2p - 2q - s = 0$ $r + 2q = 0$	2
Collisional high-β	$F(na^2, Ta^{1/2}, Ba^{5/4})$	$2p + \frac{q}{2} + \frac{5r}{4} - s = 0$	3
Ideal MHD	$(na^2)^{1/2}F(nT/B^2)$	$p = q + 1/2$ $r + 2q = 0$ $s = 1$	1
Resistive MHD	$(na^2)^{1/2}F(nT/B^2, Ta^{1/2})$	$p - q + 2s - \frac{5}{2} = 0$ $2p + r - 1 = 0$	2

TABLE II. SCALING LAWS FOR OHMICALLY HEATED PLASMAS

Plasma model	Scaling law for $B\tau$	Constraints on power law scaling (28)	Free exponents
Collisionless low-β	$B\tau = F(na^7B^4)$	$z = 7x,\ y = 4x$	1
Collisional low-β	$B\tau = F(na^2, a^5B^4)$	$8x + 5y - 4z = 0$	2
Collisionless high-β	$B\tau = F(na^2, a^5B^4)$	$8x + 5y - 4z = 0$	2
Collisional high-β	$B\tau = F(na^2, a^5B^4)$	$8x + 5y - 4z = 0$	2
Ideal MHD	$B\tau = n^{1/2}a\, F\left(\frac{n^2a^4}{a^5B^4}\right)$	$4z + 2x = 5,\ y + 2x = 1$	1
Resistive MHD	$B\tau = na^{1/2}a\, F\left(\frac{n^2a^4}{a^5B^4}\right)$	$4z + 2x = 5,\ y + 2x = 1$	1

Ideally, one would now review the empirical data in the light of Table I, and so ascertain which model describes the current experiments best. Unfortunately, the majority of the experimental results available at present have been obtained with Ohmically heated plasmas, and there are probably insufficient other data to form sound conclusions. In Ohmically heated experiments, plasma temperature is no longer an independent variable but is determined by the other parameters and the confinement time. To make a valid comparison, therefore, we must first eliminate the temperature from the scaling law by using the appropriate Ohmic heating relationship:

$$T^{5/2} \sim \frac{B^2\tau}{na^2} \tag{27}$$

Combining this scaling with the results in Table I, we find that the exponents in the general law

$$B\tau = n^x B^y a^z \tag{28}$$

are subject to the constraints given in Table II.

Note that for Ohmically heated plasmas, unlike the more general situation, the same restriction on the exponents occurs in three of the four Vlasov models, despite their different basic assumptions. [This coincidence does not depend on the particular (classical) form of Ohm's law which has been adopted; it arises whenever the temperature is, in principle, determined by the collisional high-β Vlasov equations and so follows the appropriate scaling.] Similarly, both ideal and resistive MHD models produce the same constraints on the scaling law for Ohmically heated systems. [By introducing Ohmic heating into collisionless-Vlasov or ideal-fluid models one is implying that although resistivity is the source of the heating it is unimportant in the energy loss mechanism.]

An obvious application of these results is to compare Table II with the empirical laws derived from data on Ohmically heated tokamaks. According to Daughney [1], the data from ATC and other machines are well represented by Eq.(28) with $x = 1$, $y = 1/4$, $z = 3/4$, and we find that these values do not even approximately satisfy any of the criteria of Table II. From a more recent and more comprehensive survey of the available data on tokamaks, Hugill and Sheffield [2] conclude that the observations are well fitted by Eq.(28) with $x = 0.61 \pm .08$, $y = 1.89 \pm 0.13$, $z = 1.57 \pm .17$. One again finds that these values are incompatible with any of the plasma models in Table II and that the discrepancy exceeds the quoted uncertainties in x, y, z.

One possible source of the discrepancy is, of course, that the energy transport in present tokamaks is not dominated by plasma transport but by radiation due to impurities or neutrals. In the next section, therefore, we consider the effect of radiation on the permitted scaling laws.

5. RADIATION AND IMPURITIES

The effects of radiation, by bremsstrahlung or from impurities, on the scaling laws can easily be incorporated into the resistive fluid model. To achieve this Ohm's law is modified by introducing an effective charge Z into the expression for resistivity, $\eta \sim Z/T^{3/2}$, and a general radiation loss

$$P_{rad} \sim n^k T^\ell Z^m \tag{29}$$

is introduced into the energy balance equation (21).

The influence of atomic mass can also be investigated in the model by writing $\rho = An$, so that scaling laws incorporating both the dimensionless variables Z and A are obtained.

While the introduction of Z and A leads to two new scaling factors $Z \to \sigma Z$, $A \to \mu A$, there is also an additional restriction from the necessity for the radiation term to remain invariant. Consequently the resistive fluid model, with radiation, effective charge and atomic mass included, is invariant under just three independent scale transformations. These are

$$G_1: \; n \to \beta^{8b-2} n,\; v \to \beta v,\; x \to \beta^{-4} x,\; t \to \beta^{-5} t,\; B \to \beta^{4b} B,$$

$$E \to \beta^{4b+1} E,\; p \to \beta^{8b} p,\; j \to \beta^{4b+4} j,\; T \to \beta^2 T$$

$$G_2: \; n \to \sigma^{-2c} n,\; x \to \sigma x,\; t \to \sigma t,\; B \to \sigma^{-c} B,$$

$$E \to \sigma^{-c} E,\; p \to \sigma^{-2c} p,\; j \to \sigma^{-c-1} j,\; Z \to \sigma Z$$

$$G_3: \; n \to \mu^{2d-1} n,\; x \to \mu^{-3/2} x,\; t \to \mu^{-3/2} t,$$

$$B \to \mu^d B,\; E \to \mu^d E,\; p \to \mu^{2d} p,\; j \to \mu^{d+3/2} j,$$

$$T \to \mu T,\; A \to \mu A \quad .$$

where b, c, d are related to the indices of the radiation loss by

$$b = \frac{5 + 2k - 2\ell}{8(k-1)}, \quad c = \frac{m+1}{2(k-1)}, \quad d = \frac{3 + 2k - 2\ell}{4(k-1)}$$

Consequently, the scaling law for confinement must take the form

$$\frac{B\tau}{n^{1/2} A^{1/2} a} = F\left(\frac{nT}{B^2}, \frac{Ta^{1/2}}{Z^{1/2} A^{1/4}},\; Ba^b Z^{c-b} Z^{\frac{3b}{2}-d}\right) \tag{30}$$

The general power law scaling for confinement time, when Z and A are incorporated, contains six indices:

$$B\tau = n^p T^q B^r a^s Z^t A^u \tag{31}$$

but we see from relation (30) that only three of these indices can be independent if the radiation law (29) is specified. However, it may be more useful to note the converse of this, namely that if the six indices in (31) are determined empirically then these provide just sufficient information to determine the radiation loss formula (29).

6. SUMMARY AND CONCLUSIONS

We have shown that each of the conventional plasma models imposes constraints on any scaling law which can be derived from it. For general (i.e. non-Ohmically heated) plasmas these constraints are characteristic of the model (Table I) and for some models the constraints are severe. In the extreme example of a collisionless low-β model $B\tau$ can depend only on the ratio β/N and the *only* local transport coefficient possible in this regime is Bohm diffusion. In the collisionless high-β regime, which may be appropriate in reactor conditions, the constraints are less severe but $B\tau$ still depends only on β and the line density N. For Ohmically heated plasmas (Table II) in which the temperature is not an independent variable, all models show $B\tau$ to be a function of N and ($Ba^{5/4}$), and the temperature is given by $T \sim a^{-1/2} F(N, Ba^{5/4})$.

Clearly, all theoretical calculations of anomalous heat transport, such as those involved in the widely used 'six-regime model' [5], which incorporates various drift and trapped particle instabilities, must be consistent with these general results and one may use Table I to check that this is so. For example, the dissipative-trapped-ion mode has $p = 1$, $q = -7/2$, $r = 3$, $s = 4$, which satisfy the constraints appropriate to the collisional low-β Vlasov model (Table I).

By using the present results it would be possible to reduce the number of adjustable parameters which have to be determined empirically in the scaling law – provided the appropriate model were known. This possibility might become particularly attractive as the reactor regime is approached. Alternatively, one could compare the data with the scaling law of each model and so determine which models are appropriate for attempting to calculate the losses in present tokamaks and what form of transport coefficients should be sought. In view of the vast number of possible instability-driven loss processes which have been proposed, some such preliminary screening would be valuable.

Unfortunately, when we examine the empirical laws which have so far been proposed we find them to be incompatible with the conventional plasma models. The most probable reason is that radiation from impurities is important and we considered the effect of this on the scaling laws of the resistive fluid model in Section 5. In this connection, we note that the empirical law would be distorted if, in the experiments, Z (or indeed some other variable such as q) varied systematically with n, B or a. We also note that although our constraints on the scaling laws depend only on the basic equations, so that profile effects are automatically incorporated, the boundary conditions may introduce additional physical processes, such as influx of neutral particles, which could affect the scaling.

On the theoretical side it is possible that the assumption of charge neutrality might be invalid, as it would be if plasma oscillations, excited by runaway electrons, were an important source of energy loss. However, Hugill and Sheffield explicitly rejected data taken in the "slide-away" regime so that this is unlikely to be significant. Nevertheless, for completeness, the scaling laws for models which do not assume charge neutrality are described in the Appendix.

Finally, we note that although we have emphasized energy confinement in this paper, similar scaling laws apply to particle confinement times, and since these will not be influenced by radiation, comparison with experiment may be more rewarding.

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Appendix

THE INFLUENCE OF CHARGE FLUCTUATIONS ON THE SCALING LAWS

In this paper, we have concentrated on models which assume charge neutrality but there are situations, such as low-density tokamak discharges and, more commonly, mirror machines, where this assumption may be invalid. The purpose of this Appendix is to re-consider the various plasma models when we include Poisson's equation:

$$\nabla \cdot \vec{E} = -4\pi \sum_i e_i \int f_i(x,v)\, d^3v \qquad \text{(A1)}$$

TABLE III. MODIFIED SCALING INCLUDING CHARGE FLUCTUATIONS

Plasma model	Scaling law for $B\tau$	Constraints on power law scaling (26)	Free exponents
Collisionless low-β	$F\left(\frac{n}{B^2}, \frac{T}{a^2B^2}\right)$	$2p + 2q + r = 0$ $2q + s = 0$	2
Collisional low-β	$F\left(\frac{n}{B^2}, \frac{T}{a^2B^2}, Ba^{3/2}\right)$	$6p + 2q + 3r - 2s = 0$	3
Collisionless high-β	$F\left(\frac{n}{B^2}, \frac{T}{a^2B^2}, aB\right)$	$2p + r - s = 0$	3
Collisional high-β	$F(n,T,B,a)$	–	4

in the description of the plasma, instead of assuming charge neutrality.

The requirement that the transformations of any one of the plasma models also leave Poisson's equation (A1) invariant reduces the number of these transformations by one in each case. We illustrate this for the collisionless Vlasov, low-β model, and tabulate the results for the others.

In the collisionless Vlasov, low-β model we require transformations G which leave the Vlasov equation (1) and Poisson's equation (A1) invariant. There are now only two such transformations

$$H_1: \; f \to \beta^{-1} f, \; v \to \beta v, \; B \to \beta B, \; t \to \beta^{-1} t, \; E \to \beta^2 E$$

$$H_2: \; f \to \gamma^{-2} f, \; x \to \gamma x, \; B \to \gamma^{-1} B, \; t \to \gamma t, \; E \to \gamma^{-1} E$$

and these lead to the scaling laws

$$Q = na^3B^3F\left(\frac{n}{B^2}, \frac{T}{a^2B^2}\right), \; B\tau = F\left(\frac{n}{B^2}, \frac{T}{a^2B^2}\right) \quad (A2)$$

so that if the scaling is assumed to follow a power law, there are two free exponents. This additional freedom can be interpreted physically as allowing the ratio of Debye length and plasma size to influence the plasma loss mechanism. In Table III, we list the modified scaling laws for all the plasma models when the charge neutrality restriction is removed.

It is interesting to note that Kadomtsev's theory of turbulent losses in mirror machines [6], despite invoking convection, a boundary scrape-off layer and turbulent mixing, still accords with this collisionless low-β model modified for the lack of charge neutrality.

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