Confinement studies on WEST tokamak

Content

Note: with bold font type, you will find indications about the work to carry out.

I. Confinement time

The quality of confinement is usually expressed in term of the *energy confinement time*, defined by

$$\tau_E = \frac{W}{P_{loss}}$$

where W is the total energy of the plasma and P_{loss} is the energy loss rate. This is the time scale on which plasma energy would decay if heating were switched off. In a steady state P_{loss} is equal to the heating power, but if the plasma is evolving some care is required in evaluating the confinement time. For a general derivation of the confinement time expression, see section 8.7 in "Collisional Transport of Magnetized Plasmas" by Helander and Sigmar. They show that the general expression of the confinement time for a moving plasma is

$$\tau_E = \frac{W}{P_{aux} + V_{loop}I_p - \dot{W} - X}$$

with

$$V_{loop}I_p = P_{ohm}$$

$$\dot{W} = \frac{dW}{dt}$$

$$X = \frac{1}{\mu_0}\int_U \left[B_p\dot{B}_p + B_\phi^{plasma}\left(\dot{B}_\phi^{vacuum} + \dot{B}_\phi^{plasma}\right)\right] dV.$$

Here U is the region occupied by the plasma.

In our case, we will limit the present study to quasi-steady-states where we can approximate the confinement time by

$$\tau_E \approx \frac{W}{P_{aux} + V_{loop}I_p} = \frac{W}{P_{tot}}$$

with

$$P_{tot} = P_{aux} + P_{ohm}.$$

The total power P_{tot} is easy to compute knowing the injected auxiliary power, V_{loop} and I_p. The plasma energy W is more difficult to calculate and there exist different methods allowing its computation. In the present exercise, we will see one method. This method consist on solving the static equilibrium equation, the Grad-Shafranov equation, finding the total plasma pressure p and integrating over the plasma volume to compute W,

$$W = \frac{3}{2}\int_V p \, dV.$$

Other methods used to compute W are diamagnetic measurements, measuring the magnetic toroidal flux using several diamagnetic loops (windings in the poloidal direction around the vacuum vessel). Also using kinetic profiles, profiles of electron density n_e, ion and electron temperatures (respectively T_i and T_e). In this case we say that we measure the plasma *thermal energy* W_{th},

$$W_{th} = \frac{3}{2}\int_V p_{th} \, dV = \frac{3}{2}\int_V n_e(T_e + T_i) \, dV.$$

This expression assumes a single ion and quasi-neutrality, that is ion and electron density are almost identical $n_e \approx n_i$.

In the next section, we will detail the computation of W via the Grad-Shafranov equation.

II. Magnetic computation of plasma energy using Grad-Shafranov equation

It is possible to compute the energy stored in the plasma using different methods. One of them uses the MHD (magneto-hydro-dynamic) reconstruction of the plasma equilibrium. The fundamental concept behind MHD is that magnetic fields can induce currents in a moving conductive fluid, which in turn polarizes the fluid and reciprocally changes the magnetic field itself. Starting from hypothesis that a thermonuclear plasma can be naturally considered as a mixture of at least two gases (ions and electrons) the set of equations that describe MHD are a combination of the Navier–Stokes equations of fluid dynamics (continuity, momentum and energy balance equations) and Maxwell's equations of electromagnetism.

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \quad \mathit{continuity\ equation} \\ \rho \left(\dfrac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla \right) \boldsymbol{v} = \boldsymbol{J} \times \boldsymbol{B} - \nabla P \quad \mathit{momentum\ equation} \\ \dfrac{d}{dt} \left(\dfrac{p}{\rho^{\gamma}} \right) = 0 \quad \mathit{energy\ equation} \\ \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0 \quad \mathit{Ohm's\ law} \\ \dfrac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E} \quad \mathit{Faraday's\ law} \\ \mu_0 \mathbf{J} = \nabla \times \boldsymbol{B} \quad \mathit{Ampere'slaw} \\ \nabla \cdot \boldsymbol{B} = 0 \quad \mathit{magnetic\ divergence} \end{cases} \tag{1}$$

The plasma can be confined by a proper equilibrium of plasma pressure and magnetic field.

Then, assuming an azimuthal symmetry about the z-axis of a cylindrical coordinate system (r, ϕ, z), it is possible to derive the Grad-Shafranov equation. This equation is a two-dimensional, nonlinear, elliptic partial differential equation. Solving the equation analytically, it is possible to compute the plasma pressure needed to calculate the energy stored inside the plasma, known as W_{MHD}.

A. How we can derive the Grad-Shafranov equation?

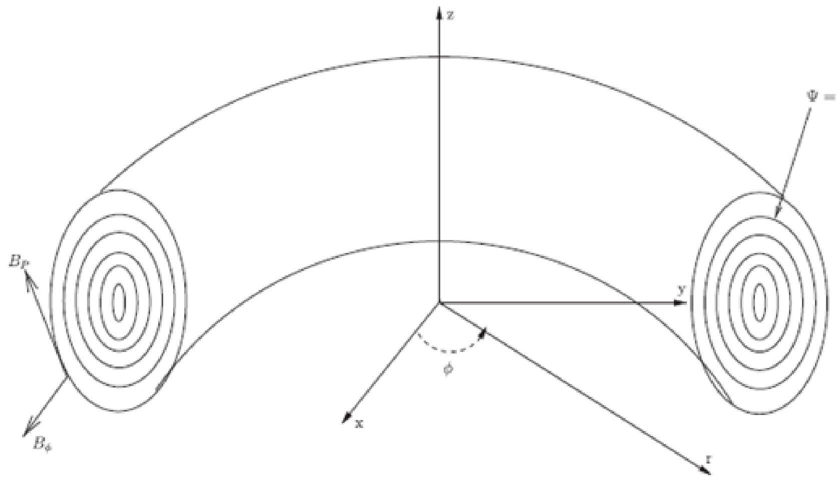

FIGURE 1

Thanks to the axisymmetry, there is no dependence with respect to the toroidal direction, therefore, the spatial derivatives of physical quantities with respect to ϕ are zero, $\frac{\partial f}{\partial \phi} = 0$. Moreover, the vectors in ϕ direction are expressed in terms of $\nabla\phi = \frac{\hat{\phi}}{r}$.

Due to the divergence-free nature of magnetic field, it can be expressed as a curl of a vector potential:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{\partial A_\phi}{\partial z}\hat{r} + \frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}\hat{z} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\phi}.$$

The term toroidal denotes vectors in the $\hat{\phi}$ direction, while the term poloidal denotes vectors in the (r,z) plane. Therefore, it is possible to define the poloidal flux function as:

$$\psi(\mathrm{r}, \mathrm{z}) = \mathrm{r}A_\phi(r, z).$$

The poloidal components of the magnetic field are:

$$\begin{cases} B_r = -\frac{1}{r}\frac{\partial\psi}{\partial z} \\ B_z = \frac{1}{r}\frac{\partial\psi}{\partial r} \end{cases}$$

Defining $g = rB_\phi$ (r,z), the toroidal component of the magnetic field is written as: $B_\phi = B_\phi\hat{\phi} = \frac{g}{r}\hat{\phi} = g\nabla\phi$

Therefore, the general form for magnetic field is:

$$B = B_r\hat{r} + B_z\hat{z} + B_\phi\hat{\phi} = -\frac{1}{r}\frac{\partial\psi}{\partial z}\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial r}\hat{z} + g\nabla\phi = \frac{1}{r}\nabla\psi \times \hat{\phi} + g\nabla\phi = \nabla\psi \times \nabla\phi + g\nabla\phi.$$

Since $B = B_{tor} + B_{pol}$ is possible to split the total magnetic field in the poloidal and toroidal component:

$$\begin{cases} B_{tor} = B_\phi \hat{\phi} = \dfrac{\mu_0 I}{2\pi} \nabla\phi = \dfrac{\mu_0 I}{2\pi r} \hat{\phi} \\ B_{pol} = \dfrac{1}{2\pi} (\nabla\psi \times \nabla\phi) \end{cases}$$

Remember that the value of B_ϕ is obtained from the Ampere's law: $\oint B\, dl = \mu_0 I_T \rightarrow 2\pi R B_\phi = \mu_0 I_T$, with R the major radius and I_T the total current in the coils around the torus.

The integration of the poloidal magnetic field over the area of the circle of radius r and center in axial location z gives the poloidal flux at location r, z:

$$\int_0^r B_{pol} ds = \int_0^r \frac{1}{2\pi} \nabla\psi \times \nabla\phi \cdot 2\pi r' \hat{z} dr' = \psi(\mathrm{r}, \mathrm{z}).$$

Thanks to the axisymmetry, the curl of the toroidal vector is poloidal:

$$\nabla \times B_{tor} = \frac{\mu_0}{2\pi} \nabla I \times \nabla\phi.$$

Moreover, the curl of the poloidal vector is toroidal:

$$B_{pol} = \nabla \times (B_r \hat{r} + B_z \hat{z}) = \hat{\phi} \left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \right).$$

The curl of a poloidal magnetic field is a Laplacian-like operator on ψ, in fact:

$$\nabla\phi \cdot \nabla \times B_{pol} = \nabla \cdot \left(B_{pol} \times \nabla\phi \right) = \nabla \cdot \left(\frac{1}{2\pi} (\nabla\psi \times \nabla\phi) \times \nabla\phi \right) = -\frac{1}{2\pi} \nabla \cdot \left(\frac{1}{r^2} \nabla\psi \right).$$

Therefore:

$$\nabla \times B_{pol} = -\frac{r^2}{2\pi} \nabla \cdot \left(\frac{1}{r^2} \nabla\psi \right) \nabla\phi.$$

Using the Ampere's law, is possible to derive the toroidal and the poloidal currents:

$$\mu_0 \mathrm{J} = \nabla \times B = \begin{cases} \mu_0 \mathrm{J_r} = \dfrac{1}{r} \dfrac{\partial \mathrm{B_z}}{\partial \phi} - \dfrac{\partial \mathrm{B}_\phi}{\partial z} \\ \mu_0 \mathrm{J}_\phi = \dfrac{\partial \mathrm{B_r}}{\partial z} - \dfrac{\partial \mathrm{B}_z}{\partial r} \\ \mu_0 \mathrm{J_z} = \dfrac{1}{r} \dfrac{\partial (r \mathrm{B}_\phi)}{\partial r} - \dfrac{\partial \mathrm{B_r}}{\partial \phi} \end{cases} = \begin{cases} J_{tor} = -\dfrac{r^2}{2\pi\mu_0} \nabla \cdot \left(\dfrac{1}{r^2} \nabla\psi \right) \nabla\phi \\ J_{pol} = \dfrac{1}{2\pi} \nabla I \times \nabla\phi \end{cases}$$

Starting from the momentum equation for the plasma:

$$\nabla P = \boldsymbol{j} \times \boldsymbol{B}$$

and decomposing the magnet field and current into toroidal and poloidal components, the pressure gradient is:

$$\nabla P = J_{pol} \times B_{tor} + J_{tor} \times B_{pol} + J_{pol} \times B_{pol}$$

$J_{pol} \times B_{pol}$ must vanish because the physical quantities are independent of ϕ. This implies:

$$(\nabla I \times \nabla\phi) \times (\nabla\psi \times \nabla\phi) = 0.$$

In other word, ∇I bust be parallel to $\nabla\psi$. This means that an arbitrary displacement dr results in respective changes in current and poloidal flux. Thus I must be function of ψ, in fact:

$$\frac{dI}{d\psi} = \frac{dr\nabla I}{dr\nabla\psi} \rightarrow \nabla I(\psi) = I'(\psi)\nabla\psi.$$

The poloidal current, and therefore the pressure gradient, can be expressed in terms of the poloidal flux function as:

$$J_{pol} = \frac{I'}{2\pi}\nabla\psi \times \nabla\phi$$

$$\begin{aligned} \nabla P &= \frac{I'}{2\pi}(\nabla\psi \times \nabla\phi) \times \frac{\mu_0 I}{2\pi}\nabla\phi - \frac{r^2}{2\pi\mu_0}\nabla\cdot\left(\frac{1}{r^2}\nabla\psi\right)\nabla\phi \times \frac{1}{2\pi}(\nabla\psi \times \nabla\phi) \\ &= -\left[\frac{\mu_0 I' I}{(2\pi r)^2} + \frac{1}{(2\pi)^2\mu_0}\nabla\cdot\left(\frac{1}{r^2}\nabla\psi\right)\right]\nabla\psi. \end{aligned}$$

This gives the important result that ∇P is parallel to $\nabla\psi$, which implies that $P = P(\psi)$ and $\nabla P(\psi) = P'(\psi)\nabla\psi$. Now, the original vector equation reduces to the scalar equation:

$$\nabla\cdot\left(\frac{1}{r^2}\nabla\psi\right) + 4\pi^2\mu_0 P' + \frac{\mu_0^2}{r^2}I'I = 0$$

Show this last equation starting from the force balance equation: $\nabla \mathrm{P} = \mathrm{j} \times \mathrm{B}$.

This is the Grad-Shafranov equation. It has the peculiarity that ψ shows up as both an independent variable and as a dependent one. It can be substituted in the toroidal current equation and the total current can be expressed as:

$$J = J_{pol} + J_{tor} = \frac{1}{2\pi}\nabla I \times \nabla\phi + \left(2\pi r^2 P' + \frac{\mu_0}{r^2}I'I\right)\nabla\phi = 2\pi r^2 P'\nabla\phi + I'B$$

The last term is called the force-free current because it is parallel to the magnetic field and so provides no force, while the first term on the right hand side is the diamagnetic current.

B. How to solve the Grad-Shafranov equation?

Grad-Shafranov equation is non-linear equation in ψ because it involves three independent quantities ψ, $P(\psi)$ and $I(\psi)$, therefore, it cannot be solved analytically. There exist a limited number of analytic solutions, one of them is the Solov'ev solution.

Two assumptions are involved:

- The pressure is assumed to be a linear function of ψ, $P = P_0 + \lambda\psi$
- The plasma current is assumed to be constant inside the plasma: $I' = 0$

The Grad-Shafranov equation is reduced to:

$$\mathrm{r}\frac{\partial}{\partial \mathrm{r}}\left(\frac{1}{r}\frac{\partial \psi}{\partial \mathrm{r}}\right) + \frac{\partial^2 \psi}{\partial^2 z} + 4\pi^2 r^2 \mu_0 \lambda = 0.$$

Which as the exact solution:

$$\psi(\mathrm{r}, \mathrm{z}) = \psi_0 \frac{r^2}{r_0^4}(2r_0^2 - r^2 - 4\alpha^2 z^2),$$

where ψ_0, r_0 and α are constants. For the case of $\alpha = 1$, the contour plot of ψ shows open curves going to $z = \pm\infty$ (for any negative value of ψ), concentric closed curves (for any positive value of ψ) and a single curve called the separatrix (if $r^2 + 4\alpha^2 z^2 = 2r_0^2$).

Substituting the poloidal flux function in the Grad-Shafranov equation:

$$\lambda = \frac{2\psi_0}{\pi^2 r_0^4 \mu_0}(1 + \alpha^2),$$

then, the pressure is:

$$P(\psi) = P_0 + \frac{2\psi_0}{\pi^2 r_0^4 \mu_0}(1 + \alpha^2)\psi \rightarrow P(\psi) = \frac{2\psi_0}{\pi^2 r_0^4 \mu_0}(1 + \alpha^2)\left[\psi - \psi_{\mathrm{edge}}\right].$$

Considering the pressure linear with ψ and the plasma current constant inside the plasma, as done here, re-derivate the pressure expression $P(\psi)$.

III. Scaling laws for tokamak confinement time

Due to the complexity of the processes determining heat and particle transport in thermonuclear plasmas, it is not yet possible to provide a first principle derivation of the dependence of energy confinement properties on plasma parameters. The description of the global energy confinement time by empirical scaling that are based on relevant datasets within specific operating regimes such as L-mode or H-mode has, therefore, become a key tool to model plasma transport and is used in extrapolating plasma performance to future tokamak devices.

A. Dimensional power law scaling expressions

The power law scaling expressions for energy confinement time can be expressed either in engineering variables as

$$\tau = C I^{\alpha_I} B^{\alpha_B} n^{\alpha_n} P^{\alpha_P} R^{\alpha_R} \kappa^{\alpha_\kappa} \varepsilon^{\alpha_\varepsilon} M^{\alpha_M}$$

The engineering variables are:

R = major radius (geometric center) $[m]$
I = plasma current [MA]
B = toroidal magnetic field (at major radius R) $[T]$
P = loss power [MW]
n = line average density $[m^{-3}]$
κ = elongation
ε = inverse aspect ratio (a/R)
M = average ion mass [AMU]

From Chapter 2 ITER Physics Basis (Nuclear Fusion, Vol. 39, No. 12, 1999) we can find the most used exponents taking into account multi-tokamak H-mode and L-mode databases. The exponents found using these multi-machine databases are summarized in the following table

Scaling	C	α_I	α_B	α_n	α_P	α_R	α_κ	α_ε	α_M	$RMSE$
H-mode **IPB98(y,2)**	0.0562	0.93	0.15	0.41	−0.69	1.97	0.78	0.58	0.19	14.5
L-mode **ITERL96**	0.0230	0.96	0.03	0.40	−0.73	1.83	0.64	−0.06	0.20	15.8

With the *Root Mean Square Error* (RMSE):

$$RMSE = \sqrt{\sum_{i=1}^{N} \frac{\left(y_i^{predicted} - y_i^{observed}\right)^2}{N}}$$

And N is the number of obervations.

To have a relative error is useful to compute the *Mean Absolute Percentage Error* (MAPE), also known as *Mean Absolute Percentage Deviation* (MAPD):

$$MAPE = \frac{1}{N} \sum_{i=1}^{N} \frac{\left|y_i^{predicted} - y_i^{observed}\right|}{max(\epsilon, \left|y_i^{observed}\right|)}$$

where ϵ is an arbitrary small yet strictly positive number to avoid undefined results when $y_i^{observed}$ is zero. Note that the mean can be replaced by the median to make the regression metric robust to outliers.

Also we can evaluate the *coefficient of determination* R^2, it represents the proportion of variance (of observed y) that has been explained by the independent variables in the model. It provides an indication of goodness of fit and therefore a measure of how well unseen samples are likely to be predicted by the model, through the proportion of explained variance. Best possible score is 1.0 and it can be negative (because the model can be arbitrarily worse). A constant model that always predicts the mean value of observed y, disregarding the input features, would get a R^2 score of 0.0,

$$R^2 = 1 - \frac{\sum_{i=1}^{N}\left(y_i^{predicted} - y_i^{observed}\right)^2}{\sum_{i=1}^{N}\left(\bar{y} - y_i^{observed}\right)^2}$$

where $\bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i^{observed}$.

Verify that the plasma-stored energy $\mathrm{W_{MHD}}$ ***(from WEST equilibrium reconstruction using Grad-Shafranov equation) is taken at quasi-steady-states. Therefore we can use the formula below to compute WEST confinement time***

$$\tau_{WEST} = \frac{\mathrm{W_{MHD}}}{P_{tot}}.$$

Taking into account WEST data compute RMSE ***and H factor for H-mode and L-mode scalings defined:***

$$\mathrm{H_{scaling}} = \frac{\tau_{\mathrm{WEST}}}{\tau_{\mathrm{scaling}}}.$$

Plot $H_{scaling}$ ***as a function of*** $\mathrm{P_{tot}}$***. You can color the plot using other quantities you consider are the more relevant to explain the H factor distribution.***

Taking into account WEST data compute your own regression and evaluate its quality.

B. Dimensionless analysis

Dimensionless parameter scaling techniques are a robust method for obtaining qualitative and quantitative information about complex physical systems when complete mathematical descriptions of the systems cannot be found. Two distinct situations where such complete descriptions cannot be obtained are commonly encountered. The first case is when a mathematical model describing the quantity of interest does not yet exist. If a set of parameters that are believed to determine the behavior of this quantity can be written down, based on intuition or experimentation, then application of established techniques for finding appropriate dimensionless parameters can provide significant information on the relationships among the original set of empirical parameters. In the second case, the governing equations are established, but they are impossible to solve analytically or numerically in the situation of interest. Again, the use of dimensionless parameter scaling techniques can lead to the identification of a few key parameters that characterize the behavior of the system. Both of these situations are encountered routinely in plasma physics.

1. Buckingham Π theorem

Buckingham Π theorem (also known as Pi theorem) is used to determine the number of dimensional groups required to describe a phenomena. According to this theorem “the number of dimensionless groups to define a problem equals the total number of variables, n, (like density, viscosity, etc.) minus the fundamental dimensions, m, (like length, time, etc.).” On other words if there are n variables in a problem and these variables contain m primary dimensions the equation relating all the variables will have $(n-m)$ dimensionless groups. We call these dimensionless groups π_1, π_2, π_3, etc., then the equation expressing the relationship among the variables has a solution of the form

$$F(\pi_1, \pi_2, \pi_3, \dots) = 0$$

If in a problem $n = 5$ and $m = 3$ then $(n-m)$ is equal to two and the solution would be either

$$F(\pi_1, \pi_2) = 0$$

Or

$$\pi_1 = f(\pi_2)$$

As an example, we can consider the forced heat transfer through a pipe. Using Buckingham theorem, the number of variables n in the problem are 7:

1. Tube diameter D $[m]$
2. Thermal conductivity k $[W/(m\ K)]$
3. Velocity of the fluid U $[m/s]$
4. Density of the fluid ρ $[kg/m^3]$
5. Viscosity of the fluid μ $[kg/(m\ s)]$
6. Specific heat C_p $[J/(kg\ K)]$
7. Heat transfer coefficient h $[W/(m^2K)]$

The basic fundamental dimensions m are four (length, mass, time, temperature). Therefore, three $(7 - 4 = 3)$ dimensionless groups will be required to represent the problem. We can then define the nondimensional groups

$$\pi_1 = \frac{hD}{k}$$
$$\pi_2 = \frac{\rho U D}{\mu}$$
$$\pi_3 = \frac{C_p \mu}{k}$$

Verify that the three parameters given above are dimensionless.

The first dimensionless group is recognized as Nusselt number Nu. The second group is known as Reynolds number Re, while the third group is Prandtl number Pr. The functional relationship between these groups can be written as

$$Nu = f(Re, \mathrm{Pr})$$

Now with the help of experiment, one can correlate all experimental data in terms of three variables instead of the original seven. The importance of this reduction is apparent from the following experimental example.

In a series of experiments with air flowing over 2.5 cm outer diameter pipe, the heat transfer coefficient (h) has been measured at velocities (U) ranging from 0.15 to 30.5 m/s with corresponding Reynolds number from 250 to 50000. Figure 2 shows a plot between h against U, since velocity was the only variable in these tests. Resulting curve permits determination of the value of h at any velocity for the system used in the experiments but does not permit the determination of h for pipes that have different diameters to the one used in the tests. The same thing can be said about the fluid density, thermal conductivity, etc. Unless experimental data could be correlated more effectively, it would be necessary to perform separate experiments for all variables which are enormously laborious and time consuming. With the aid of dimensional analysis, however, the results of one series of experiments can be applied to other problems as illustrated in Figure 3, after replotting the data of Figure 2 in terms of dimensionless groups (Reynolds and Nusselt numbers). This correlation permits the determination of the value of h for air flowing over any size of pipe as long as Reynolds number falls under the experimental conditions. Similarly, one can correlate the effect of Prandtl number on h by plotting the data in terms of three dimensionless groups (Nu, Re, Pr) on a single plot.

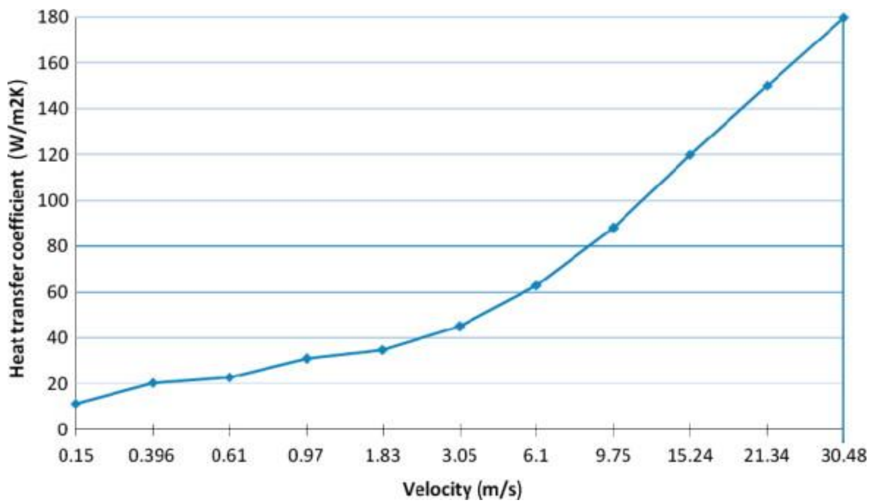

FIGURE 2

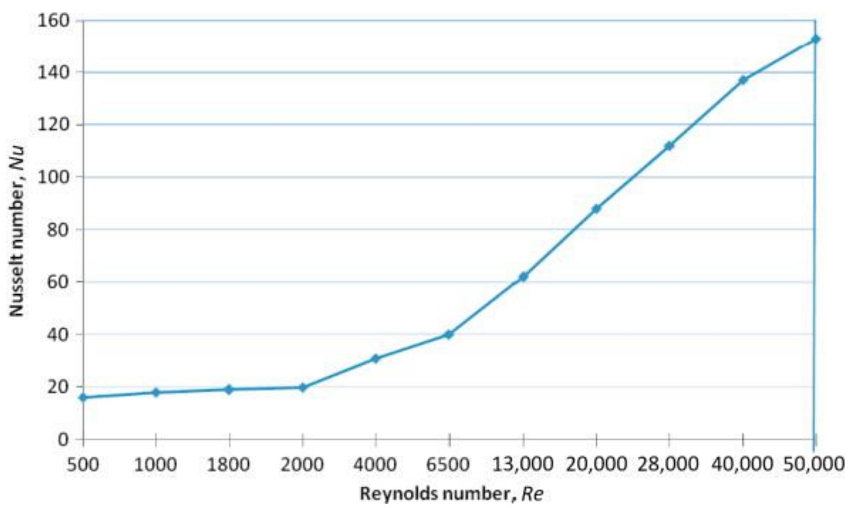

FIGURE 3

2. Applying Buckingham Π theorem to plasmas [Luce et al. PPCF 2008]

Dimensional analysis of the type demonstrated in the previous example was first applied to tokamak plasmas by Kadomtsev (papers from 1975 and 1992). He chose the following set of dimensional parameters to describe tokamak plasma behavior $\{a, R, B, B_p, c, e, m_e, m_i, T_e, T_i, n\}$. Where

1. Minor radius of the tokamak a $[m]$
2. Major radius R $[m]$
3. Toroidal magnetic field B $[T]$
4. Poloidal magnetic field B_p $[T]$
5. Speed of light in vacuum c $[m/s]$

6. Charge of the electron e $[A\ s]$
7. Mass of the electron m_e $[kg]$
8. Mass of the ion (assuming a single component plasma) m_i $[kg]$
9. Electron temperature T_e $[eV]$
10. Ion temperature T_i $[eV]$
11. Plasma density n $[m^{-3}]$

First, we need to extract the quantities with the same dimensional units. In this case, we have can define four dimensionless ratios

$$\{r\} = \{R/a,\ aB_p/(RB)\,,\ m_e/\ m_i\,, T_e/\ T_i\}$$

Notice that the second quantity $aB_p/(RB)$ is the safety factor in the cylindrical approximation. We have then decrease our number of independent variables by four. We have seven left $\{a, B, T_e, n, c, e, m_e\}$. The number of dimensions is three in CGS units[1] (mass, length and time). Applying Buckingham theorem we can reduce the seven quantities into four dimensionless variables. We will retain the ones chosen by Kadomtsev:

1. Poloidal ion Larmor radius normalized to machine size $\rho_* = \rho_i/a$
2. Normalized collision frequency: $\nu_{mfp} = qR/\lambda_c$ with λ_c the electron mean free path considering Coulomb collisions
3. Ratio of kinetic to poloidal magnetic pressure $\beta_p = 2\mu_0 nT/B_p^2$
4. Number of particles in Debye sphere $N = (4\pi/3)\lambda_D^3 n$ (in general for tokamak plasmas N is varying weakly and $1/N$ is small, therefore N can be neglected and can be dropped from the dimensionless considered quantities)

Remark: these dimensionless parameters along with the ratio of like quantities fully describe plasma physics but not atomic physics. Atomic physics can be neglected when radiation is low enough and neutrals particles do not penetrate deep into plasma.

3. Scale invariance [Luce et al. PPCF 2008]

If the equations governing the behavior of a plasma are assumed to be known, then the technique of scale invariance can be used to derive the necessary dimensionless relationships between the variables that appear in the equations. Connor and Taylor (NF 1977, PPCF 1988) formalized this analysis to obtain the scaling of a confinement time in several different limits. To illustrate the technique of scale invariance, a simple example for the limiting case of electrostatic, collisionless plasmas will be given.

In the electrostatic, collisionless limit, the plasma is assumed to be described by the Vlasov equations for the distribution functions f_j of electrons ($j = e$) and ions ($j = i$)

$$\frac{\partial f_j}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla} f_j + \frac{e_j}{m_j}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f_j}{\partial \boldsymbol{v}} = 0, \qquad j = i, e$$

Where the magnetic field $\boldsymbol{B}$ is fixed and the electrostatic field $\boldsymbol{E}$ is determined by the quasi-neutrality condition

$$\sum_j e_j \int d^3v\ f_j = 0.$$

[1] In SI units the number of dimensions is four (mass, length, time and electric current). In this case to the eleven dimensional parameters describing tokamak plasma behavior one need to add the vacuum permittivity ε_0 or the vacuum permeability μ_O.

The quantities e_j and m_j are the charge and mass of each component of the plasma, and $\boldsymbol{v}$ is the particle velocity. The principle of scale invariance implies that scale transformation of the form

$$f_j \to C_f f_j, \qquad \boldsymbol{v} \to C_v \boldsymbol{v}, \qquad \boldsymbol{x} \to C_x \boldsymbol{x}, \qquad t \to C_t t, \qquad \boldsymbol{E} \to C_E \boldsymbol{E}, \qquad \boldsymbol{B} \to C_B \boldsymbol{B}$$

that leaves Vlasov equation invariant are dimensionally correct. There are three independent transformations of this type that may expressed as

$A_1: f_j \to C_1 f_j$

$A_2: \boldsymbol{v} \to C_2 \boldsymbol{v}, \qquad \boldsymbol{B} \to C_2 \boldsymbol{B}, \qquad t \to C_2^{-1} t, \qquad \boldsymbol{E} \to C_2^2 \boldsymbol{E}$

$A_3: \boldsymbol{x} \to C_3 \boldsymbol{x}, \qquad \boldsymbol{B} \to C_3^{-1} \boldsymbol{B}, \qquad t \to C_3 t, \qquad \boldsymbol{E} \to C_3^{-1} \boldsymbol{E}$

Demonstrate the last three transformations. To do so make dimensionless each variable taking into account a reference distribution function $\mathrm{f_0}$***, reference velocity*** U***, reference length*** L***, reference electric current*** A***, and a reference <u>constant</u> mass*** M***. For example for velocity and time the expressions are (star denotes a dimensionless quantity)***

$$\mathrm{v} = \mathrm{v_* U}, \qquad \mathrm{t} = \mathrm{t_*} \frac{\mathrm{L}}{\mathrm{U}}, \qquad \dots$$

Attention for E ***and*** B ***consider normalized quantities:*** $\frac{\mathrm{e}}{\mathrm{m}}\mathrm{E}$ ***and*** $\frac{\mathrm{e}}{\mathrm{m}}\mathrm{B}$***.***

Now the scale invariance principle implies that any variable derived from the Vlasov equation, such as a confinement time, must transform appropriately under A_1 through A_3. Consider a power law scaling for the confinement time of the form

$$\tau \sim \mathrm{n^p\, T^q\, B^r\, a^s}$$

Where the density n and temperature T are taken to be characteristic values of these quantities, for example the value at the plasma mid-radius. Such characteristic values are sufficient since the profiles of these quantities are determined by the equations under investigation. The quantity a is again a measure of the plasma size. It follows, recalling the definitions

$$n = \int d^3v\ f$$

$$nT = \int d^3v\ \frac{mv^2}{3} f$$

that the four indices p, q, r and s are constrained by the three transformations to satisfy the relations

$$p = 0, \qquad 2q + r = -1, \qquad s - r = 1.$$

Therefore, only one index (q) is independent and

$$B\tau \sim \left(\frac{\mathrm{T}}{\mathrm{B^2 a^2}}\right)^{\mathrm{q}}.$$

From a dimensional analysis prove this last relation; remember that τ ***has the time dimension*** $\mathrm{LU^{-1}}$ ***and that the mass is assumed to be constant.***

Note that this exercise determines the relationships among the dimensional parameters that must hold on both sides of the equation, but still requires the addition of dimensional constants of nature (m_i, e_j) to yield dimensionless parameters. For the left-hand side of this last equation the cyclotron frequency $\Omega = e_j B / m_j$ provides a convenient normalization, since it has the correct units and is proportional to B. The quantity in the parenthesis on the right-hand side can be recognized as the square of the gyroradius $\rho =$

$\sqrt{Tm_j}/(e_j B)$ normalized to the physical size of the plasma a. Writing this result in a fully dimensionless form gives

$$\Omega\tau = F(\rho_*),$$

where $\rho_* = \rho/a$.

Demonstrate this relation.

The above result may be extended (see Connor and Taylor NF 1977 and PPCF 1988) to include the effects of collisions by the addition of a Fokker–Planck operator into the Vlasov equations. Finite β effects may be included by introducing a self-consistent magnetic field that satisfies the Maxwell equations. The dimensionless confinement time then has the more general form

$$\Omega\tau = F(\rho_*, \nu_*, \beta, \{r_i\}),$$

where ν_* is a normalized collision frequency and the $\{r_i\}$ are ratios of quantities with equal dimensions.

From the present analysis and previous section "Applying Buckingham Π theorem to plasmas" we find that the following set of dimensionless variables can be use:

Dimensionless parameter	Definition in dimensional parameters	Description
$\boldsymbol{\rho_*}$	$\lambda_\rho \frac{\sqrt{M\hat{T}}}{\varepsilon R B}$	Gyroradius normalized to minor radius
$\boldsymbol{\beta}$	$\lambda_\beta \frac{\hat{n}\hat{T}}{B^2}$	Energy density normalized to magnetic energy density
$\boldsymbol{\nu_*}$	$\lambda_\nu Z^4 q R \varepsilon^{-3/2} \frac{\hat{n}}{\hat{T}^2}$	Collisionality, ratio of the particle collision frequency to the banana orbit frequency
$\boldsymbol{q}$		Safety factor (you can use q_{95})
$\boldsymbol{\varepsilon}$	$\frac{a}{R}$	Inverse aspect ratio
$\boldsymbol{\kappa}$		Elongation

Here we use:

- $\hat{n}$ is density in m^{-3}
- $\hat{T}$ is the thermal energy, expressed in eV
- $\lambda_\rho = \sqrt{m_p/e}$
- $\lambda_\beta = e\mu_0$
- $\lambda_\nu = \sqrt{\pi}\, e^2\ \ln\Lambda/(12\pi^2\varepsilon_0^2)$, with $\ln\Lambda \approx 10$ the Coulomb logarithm
- $M = 2,\ Z = 1$ (deuterium) and $T_e/T_i = 1$ for simplicity

Now the energy confinement time in dimensionless, also called 'physics' variables writes

$$\Omega\tau \propto\ \rho_*^{\alpha_\rho}\ \nu_*^{\alpha_\nu}\ \beta^{\alpha_\beta}\ q^{\alpha_q}\ \varepsilon^{\alpha_\varepsilon}\ \kappa^{\alpha_\kappa}$$

The confinement time τ_E scales with the four dimensional variables $(\hat{n}, \hat{T}, a, B)$, see section 2. Applying Buckingham Π theorem to plasmas [Luce et al. PPCF 2008]Also $\omega_c \tau_E$ (with $\omega_c = eB/Mm_p$, the ion cyclotron frequency) is expected to scale with (ρ_*, ν_*, β), we shall call this transformation the Kadomtsev transformation.

Because the number of variables reduces from four to three, the power exponent of the one variable has to vanish in the transformation. For instance:

$$\tau_E = \mathrm{G}(\hat{n}, \hat{T}, a, B) \rightarrow \omega_c \tau_E = H(\rho_*, \nu_*, \beta, B^0)$$

The constraint of vanishing B-exponent is referred as the Kadomtsev's constraint.

Inverting the equations in the table, the following relationships for the Kadomtsev transformation are found:

$$\hat{\mathrm{n}} = (\mathrm{B}^8 \rho_*^2 \nu_*^2 \beta^3)^{\frac{1}{5}} \left(\lambda_\rho^2 \lambda_\nu^2 \lambda_\beta^3 \frac{\mathrm{q}^2 \mathrm{M}}{\varepsilon^5} \right)^{-\frac{1}{5}}$$

$$\hat{\mathrm{T}} = \left(\frac{\mathrm{B}\beta}{\rho_* \nu_*} \right)^{\frac{2}{5}} \left(\frac{\lambda_\rho \lambda_\nu}{\lambda_\beta} \frac{\mathrm{q} \mathrm{M}^{\frac{1}{2}}}{\varepsilon^{\frac{5}{2}}} \right)^{\frac{2}{5}}$$

$$\mathrm{R} = \left(\frac{\beta}{\rho_*^6 \nu_* \mathrm{B}^4} \right)^{\frac{1}{5}} \left(\frac{\lambda_\rho^6 \lambda_\nu}{\lambda_\beta} \frac{\mathrm{q} \mathrm{M}^3}{\varepsilon^{\frac{15}{2}}} \right)^{\frac{1}{5}}$$

The expression of the energy confinement time in dimensionless variables can be rewrite as:

$$\omega_c \tau_E = \frac{\mathrm{e}}{\mathrm{m}} \frac{\left(\mathrm{C}\, \mathrm{C}_\mathrm{I}^{\alpha_\mathrm{I}} \mathrm{C}_\mathrm{tr}^{\alpha_\mathrm{P}} \right)^{\frac{1}{1+\alpha_\mathrm{P}}}}{\lambda_\rho^{\chi_\rho} \lambda_\nu^{\chi_\nu} \lambda_\beta^{\chi_\beta}} \mathrm{M}^{\chi_\mathrm{M}} \mathrm{k}^{\chi_\mathrm{k}} \varepsilon^{\chi_\varepsilon} \mathrm{q}^{\chi_\mathrm{q}} \mathrm{B}^{\chi_\mathrm{B}} \rho_*^{\chi_\rho} \nu_*^{\chi_\nu} \beta_\beta^{\chi_\beta}$$

Where $\mathrm{C}_\mathrm{I} = 2\pi \mathrm{F} 10^{-6}/\mu_0$ (F=1 for circular plasma, F=k for diverted plasma) and $\mathrm{C}_\mathrm{tr} = 6\pi^2 \times 10^{-3}\mathrm{e}$.
It is possible to compute the exponents of each variable as:

$$\chi_\mathrm{M} = \frac{5\alpha_\mathrm{M} + 3\alpha_\mathrm{R} + 3\alpha_\mathrm{I} + 4\alpha_\mathrm{P} - \alpha_\mathrm{n} - 5}{5(1+\alpha_\mathrm{P})}$$

$$\chi_\mathrm{k} = \frac{\alpha_\mathrm{k} + \alpha_\mathrm{P}}{(1+\alpha_\mathrm{P})}$$

$$\chi_\varepsilon = \frac{2\alpha_\varepsilon - 3\alpha_\mathrm{R} + \alpha_\mathrm{I} - 5\alpha_\mathrm{P} + 2\alpha_\mathrm{n}}{2(1+\alpha_\mathrm{P})}$$

$$\chi_\mathrm{q} = \frac{\alpha_\mathrm{R} - 4\alpha_\mathrm{I} + 3\alpha_\mathrm{P} - 2\alpha_\mathrm{n}}{5(1+\alpha_\mathrm{P})}$$

$$\chi_\rho = \frac{2(-3\alpha_\mathrm{R} - 3\alpha_\mathrm{I} - 9\alpha_\mathrm{P} + \alpha_\mathrm{n}}{5(1+\alpha_\mathrm{P})}$$

$$\chi_\nu = \frac{-\alpha_R - \alpha_I - 3\alpha_P + 2\alpha_n}{5(1+\alpha_P)}$$

$$\chi_\beta = \frac{\alpha_R + \alpha_I + 8\alpha_P + 3\alpha_n}{5(1+\alpha_P)}$$

$$\chi_B = \frac{5\alpha_B - 4\alpha_R + \alpha_I + 3\alpha_P + 8\alpha_n + 5}{5(1+\alpha_P)}$$

With α_x the coefficients obtained with the engineering scaling law.

Check if the Kadomtsev constraint ($\chi_B = 0$) is satisfied for the IPB98(y,2), ITER96 scaling and your scaling. Then, compute all the other dimensionless coefficients using the Kadomtsev transformations.

Compare to Chapter 2 ITER Physics Basis, Nuclear Fusion, Vol. 39, No. 12, 1999.

In addition, you can compare to [Sips, Nuclear Fusion, Vol. 58, 2018] using a different way of estimating the dimensionless parameters (see Eqs. (2) to (4) in that article).

IV. BONUS: computation of plasma energy using diamagnetic flux measurements

From the system of equation ($\mathbf{1}$), we can consider the simplified set of equations:

$$\begin{cases} \nabla P = \boldsymbol{J} \times \boldsymbol{B} \quad momentum\ eq. \\ \mu_0 \mathbf{J} = \nabla \times \boldsymbol{B} \quad Ampere's\ law \\ \nabla \cdot \boldsymbol{B} = 0 \quad mag. divergence \end{cases} \tag{2}$$

Force balance equation from the momentum equation using Ampere's law can be written:

$$\nabla P = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$$

Using the tensor relation:

$$\boldsymbol{A} \times (\nabla \times \boldsymbol{B}) = (\nabla \boldsymbol{B}) \cdot \boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$$

We can write:

$$\nabla P + \frac{1}{\mu_0}(\nabla \boldsymbol{B}) \cdot \boldsymbol{B} = \frac{1}{\mu_0}(\boldsymbol{B} \cdot \nabla)\boldsymbol{B}$$
$$\nabla P + \frac{\nabla B^2}{2\mu_0} = \frac{1}{\mu_0}(\boldsymbol{B} \cdot \nabla)\boldsymbol{B}$$

With $B^2 = \|\boldsymbol{B}\|^2$, then the right hand side of this last equation can be expressed as:

$$(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} = \nabla \cdot (\boldsymbol{B}\boldsymbol{B}) - (\nabla \cdot \boldsymbol{B}) \cdot \boldsymbol{B}$$

Using the constraint $\nabla \cdot \boldsymbol{B} = 0$, the force balance equation writes:

$$\nabla \cdot \left(P + \frac{B^2}{2\mu_0} - \frac{1}{\mu_0}\boldsymbol{BB} \right) = 0$$

This expression can be summarized as:

$$\nabla \cdot \boldsymbol{T} = 0 \tag{3}$$

With the second order tensor,

$$\boldsymbol{T} = \left(P + \frac{B^2}{2\mu_0} \right)\boldsymbol{I} - \frac{1}{\mu_0}\boldsymbol{BB}.$$

Demonstrate expression (3).

Let us consider a vector $\boldsymbol{r}$, we can obtain the following expression from Eq. (3), here double columns means the tensor trace:

$$\nabla \cdot (\boldsymbol{T} \cdot \boldsymbol{r}) = \boldsymbol{T}{:}\nabla\boldsymbol{r} = \boldsymbol{T}{:}\boldsymbol{I} \tag{4}$$

Considerer the identity

$$\int_V \nabla \cdot \boldsymbol{A}\, dV = \int_S \boldsymbol{A} \cdot d\boldsymbol{S},$$

with V a volume enclosed by a surface S and $d\boldsymbol{S} = \boldsymbol{n}dS$, where $\boldsymbol{n}$ is the unit normal outward from V. From this last identity, equations (3) and (4) yield the integral forms (from volume integrals we can transform to surface integrals):

$$\int_S \left[\left(P + \frac{B^2}{2\mu_0} \right) d\boldsymbol{S} - \frac{1}{\mu_0}\boldsymbol{BB} \cdot d\boldsymbol{S} \right] = 0 \tag{5}$$

$$\int_V \left(3P + \frac{B^2}{2\mu_0} \right) dV = \int_S \left[\left(P + \frac{B^2}{2\mu_0} \right) \boldsymbol{r} \cdot d\boldsymbol{S} - \frac{1}{\mu_0}(\boldsymbol{B} \cdot \boldsymbol{r})(\boldsymbol{B} \cdot d\boldsymbol{S}) \right] \tag{6}$$

Demonstrate integrals (5) and (6).

In cylindrical coordinates: R, ϕ, Z we have $\boldsymbol{r} = R\boldsymbol{e}_R + Z\boldsymbol{e}_Z$. We introduce the notation $B_p$for the magnetic field so that

$$\boldsymbol{B} = \boldsymbol{B}_p + B_\phi \boldsymbol{e}_\phi.$$

The toroidal magnetic field outside the plasma depends only on R,

$$B_{\phi 0} = \frac{R_0 B_0}{R} \tag{7}$$

With R_0, B_0 being a reference radius and toroidal magnetic field. Let us notice two relations:

$$\int_{S_n} R^\alpha \boldsymbol{r}\cdot\boldsymbol{n}\, dS_n = (\alpha+3)\int_V R^\alpha\, dV \tag{8}$$

$$\int_{S_n} R^\alpha \boldsymbol{e_R}\cdot\boldsymbol{n}\, dS_n = 2\pi(\alpha+1)\int_V R^\alpha\, dS_\phi \tag{9}$$

These relations can be demonstrated taking into account the follow identities:

$$dV = 2\pi R dS_\phi$$

$$\int_V \nabla\cdot\boldsymbol{A}\, dV = \int_S \boldsymbol{A}\cdot\, d\boldsymbol{S}$$

With V a volume enclosed by a surface S and $d\boldsymbol{S} = \boldsymbol{n}dS$, where $\boldsymbol{n}$ is the unit normal outward from V and S_ϕ the transverse cross section. Also, take into account the identities for a vector $\boldsymbol{r} = x\boldsymbol{e_x} + y\boldsymbol{e_y} + z\boldsymbol{e_z}$ being the radius of magnitude r, from the origin to the point x, y, z. Then

$$\nabla\cdot\boldsymbol{r} = 3$$

$$\nabla r = \frac{\boldsymbol{r}}{r}$$

Demonstrate at least the identity (8).

Using the unit vector relations,

$$\frac{\partial \mathrm{e}_\phi}{\partial\phi} = -\mathrm{e}_\mathrm{R}, \qquad \frac{\partial \mathrm{e}_\mathrm{R}}{\partial\phi} = \mathrm{e}_\phi$$

We can write:

$$\frac{\partial \mathrm{B}}{\partial\phi} = -\mathrm{B}_\phi \mathrm{e}_\phi + \mathrm{B}_\mathrm{p}\cdot\mathrm{e}_\mathrm{R}\mathrm{e}_\phi$$

From equations (7), (8) and (9) we have

$$\int_{S_n} B_{\phi 0}^2\, \boldsymbol{r}\cdot\boldsymbol{n}\, dS_n = \int_V B_{\phi 0}^2\, dV$$

$$\int_{S_n} B_{\phi 0}^2\, \boldsymbol{e_R}\cdot\boldsymbol{n}\, dS_n = -2\pi\int_V B_{\phi 0}^2\, dS_\phi$$

Therefore equations (6) and the projection of equation (5) on the equatorial plane can be written

$$\int_V \left(3P + \frac{B_p^2 + B_\phi^2 - B_{\phi 0}^2}{2\mu_0}\right) dV = \int_S \left[\frac{B_\tau^2 - B_n^2}{2\mu_0}\boldsymbol{n}\cdot\boldsymbol{r} - \frac{B_n B_\tau}{\mu_0}\boldsymbol{\tau}\cdot\boldsymbol{r}\right] dS_n \tag{10}$$

$$2\pi \int_V \left(P + \frac{B_p^2 + B_{\phi 0}^2 - B_\phi^2}{2\mu_0}\right) dS_\phi = \int_S \left[\frac{B_\tau^2 - B_n^2}{2\mu_0} \boldsymbol{n} \cdot \boldsymbol{e}_R - \frac{B_n B_\tau}{\mu_0} \boldsymbol{\tau} \cdot \boldsymbol{e}_R\right] dS_n \tag{11}$$

We denote by $\boldsymbol{n}$ and $\boldsymbol{\tau}$ the normal and tangential vector. Here B_ϕ is the toroidal field inside the volume considered and B_n, B_τ are the normal and tangential components respectively of the poloidal field on the surface S_n:

$$B_n = \boldsymbol{B}_p \cdot \boldsymbol{n}, \qquad B_\tau = \boldsymbol{B}_p \cdot \boldsymbol{\tau}.$$

The parameters internal inductance l_i, poloidal beta β_I and the diamagnetic parameter μ_I are given by:

$$l_i = \frac{2\pi}{V\mu_0^2 I_p^2} \int_V B_p^2 \, dV$$

$$\beta_I = \frac{4\pi}{V\mu_0 I_p^2} \int_V p \, dV$$

$$\mu_I = \frac{2\pi R_0}{V} \frac{2B_{\phi 0}}{\mu_0^2 I_p^2} \int_V B_{\phi 0}^2 - B_\phi^2 dV$$

We introduce also the mean radius R_T

$$R_T = \frac{\int \left(2\mu_0 p + B_p^2 + B_{\phi 0}^2 - B_\phi^2\right) R \, dS_\phi}{\int \left(2\mu_0 p + B_p^2 + B_{\phi 0}^2 - B_\phi^2\right) dS_\phi}$$

The following notation for the surface integrals are introduced

$$s_1 = C_1 \int_S \left[\frac{B_\tau^2 - B_n^2}{2\mu_0} \boldsymbol{n} \cdot \boldsymbol{r} - \frac{B_n B_\tau}{\mu_0} \boldsymbol{\tau} \cdot \boldsymbol{r}\right] dS_n \tag{12}$$

$$s_2 = C_2 \int_S \left[\frac{B_\tau^2 - B_n^2}{2\mu_0} \boldsymbol{n} \cdot \boldsymbol{e}_R - \frac{B_n B_\tau}{\mu_0} \boldsymbol{\tau} \cdot \boldsymbol{e}_R\right] dS_n \tag{13}$$

Relations (10) and (11) can be rewritten in the compact form

$$3\beta_I + l_i - \mu_I = 2(s_1 + s_2) \tag{14}$$

$$\beta_I + l_i + \mu_I = 2\, s_2 \frac{R_T}{R} \tag{15}$$

Taking into account the relative difference of our arbitrarily introduced radius R and radius R_T we define

$$\delta = \frac{R - R_T}{R}$$

Eliminating l_i from relations (14) and (15) we obtain

$$\beta_I - \mu_I = s_1 + s_2\delta \tag{16}$$

For l_i we have

$$l_i = 2s_2\left(1 - \frac{3}{2}\delta\right) - s_1 - 2\mu_I \tag{17}$$

Combining Eqs. (16) and (17) we can eliminate μ_I and obtain:

$$\beta_I + \frac{l_i}{2} = \frac{s_1}{2} + s_2\left(1 - \frac{\delta}{2}\right) \tag{18}$$

BONUS: demonstrate relation* (18)*.

V. Python code template

Copy from public CEA Partner Zone (from Altair for example) with command:

```
cp -r /home/JM240179/public/FUSION_MASTER/TP_master_fusion_confinement .
```

Alternatively, clone from GitHub:

```
git clone git@github.com:jmoralesFusion/TP_master_fusion_confinement.git
```