Tokamaks and dimensional analysis

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An attempt is made to establish certain general relationships between the parameters of the magnet system of a tokamak and the parameters of the plasma confined in it on the basis of dimensional analysis.

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As a result of the advances achieved in research on controlled thermonuclear reactions in tokamaks, much hope is now pinned on these systems, and there are plans to construct larger-scale tokamaks. Since it is therefore necessary to predict the plasma properties in these larger systems, the experimental data must in some cases be extrapolated over orders of magnitude. If we completely understood all the events which occur in the plasma, there would be no difficulty in such an extrapolation, since all the properties could be calculated theoretically. However, at present we do not have such a clear understanding, nor do we have a complete theory for the behavior of the hot plasma in a toroidal device which agrees with experiment (because of the complexity of the collective nonlinear processes which occur in the plasma). It therefore becomes necessary to rely to a large extent on empirical relations, established on the basis of only a limited range of parameters. Under these circumstances it would seem pertinent to approach the tokamak plasma from a more general point of view, by adopting the familiar approach based on dimensional analysis.

Simply speaking, a tokamak is a toroidal chamber, with a minor radius a and a major radius R, which is placed inside coils which produce a strong longitudinal magnetic field B_z. The chamber is filled with quasineutral hydrogen or deuterium plasma with a particle density n, and a current I produced by an azimuthal (more precisely, poloidal) magnetic field $B_\theta = 2I/ca$ flows through the plasma. A plasma in a toroidal system is usually at local thermodynamic equilibrium, so that the particle velocity distribution at any point is approximately Maxwellian, with electron and ion temperatures T_e and T_i and velocities v_e and v_i (in certain situations, electron beams appear, but we ignore these cases for the present).

We now idealize the situation somewhat; specifically, we assume that the tokamak chamber is filled by a fully ionized hydrogen plasma which is pure (i.e., in which there are no impurities), and we assume certain standard ideal conditions at the walls (e.g., we assume that all charged particles incident on the wall are absorbed there[1)] or that these charged particles return to the plasma with zero energy). We thus neglect the complicated (but inevitable) phenomena associated with the presence of neutral particles and impurities. In other words, we are asking how a fully ionized pure plasma should behave in a tokamak. Since all quantum effects (e.g., ionization and radiation) are ruled out, the behavior of such a plasma is governed completely by the classical laws of motion of its constituent charged particles and by the Maxwell equations for the electromagnetic field. We assume that the plasma is always at rest, i.e., that there are no velocities in the plasma other than the slow diffusion toward the walls and the slight relative motion of the electrons and ions which produces the electric current. Then the velocities v_e and v_i can be treated as derived quantities, so that there are only the following 11 basic quantities:

$$a,\ R,\ B_z,\ B_\theta,\ c,\ e,\ m,\ M,\ T_e,\ T_i,\ n, \tag{1}$$

where e and m are the charge and the mass of the electron, M is the ion mass, and c is the speed of light. The last six quantities are purely plasma properties, while the first four are governed by the external conditions. Here T_e, T_i, and n can be understood as the average values over the cross section, since the profiles of these functions are governed by the same parameters in (1).

Using the 11 quantities in (1) we can construct eight dimensionless combinations which must characterize the plasma properties in a form which does not depend on the system of units chosen. We of course would like to choose these dimensionless quantities in such a manner that they have some physical meaning. We choose the following dimensionless parameters, which we immediately classify into three groups:

$$\text{I. } \nu=qR/\lambda_e,\quad K=\rho_\theta/a,\quad \beta_\theta=\frac{8\pi n(T_e+T_i)}{B_\theta^2}, \tag{2}$$

$$\text{II. } A=R/a,\quad \theta=T_e/T_i,\quad q=aB_z/RB_\theta, \tag{3}$$

$$\text{III. } \mu=m/M,\quad N=\frac{4\pi}{3}r_d^3 n, \tag{4}$$

where r_d is the Debye length, N is the Debye number (i.e., the number of particles within a Debye sphere), ν is the electron collision parameter, $\lambda_e \sim Nr_d$ is the electron mean free path, $\rho_\theta = Mcc_s/eB_\theta$ is the mean ion gyroradius at the electron temperature in the poloidal field, $c_s = \sqrt{T_e/M}$ is the sound velocity, and q is the safety factor with respect to a helical perturbation mode.

The first group, (2), includes the quantities in whose interrelations we are primarily interested. Group (3) contains quantities which vary over a restricted range (under the conditions of most practical interest all these quantities lie between 1 and 6). The third group, (4), contains one very small parameter, $\mu = m/M$, which varies over a restricted range, and one very large parameter, N. For our purposes the parameter N is a "night clerk": Under ordinary, quiet conditions, without runaway electrons, it probably has no important bearing on our primary concern, plasma confinement (in other words, the quantity $N^{-1} \to 0$ drops out of the group of main parameters). The parameter N can be important only in unusual situations, e.g., when there is an anomalous resistance due to the driving of oscillations by a beam of runaway electrons. (We note that a parameter important in this connection is u/c_s, where u is the directed or current velocity of the electrons, and c_s is the sound velocity; in terms of our dimensionless parameters this ratio can be written $u/c_s \sim K/\beta_\theta$.)

We assume the usual conditions in a tokamak, with the plasma Joule-heated by the current flowing through it. Under these conditions the electron and ion temperatures are not independent but are governed by all the other plasma properties. Consequently, two of the dimensionless parameters in (2)-(4) must be functions of the others. As the dependent quantities we naturally choose β_θ and θ, so that we have, in particular

$$\beta_\theta=\beta_\theta(\nu, K, q, \mu, A). \quad (5)$$

Here we have omitted the dependence on N; furthermore, we can suppress the dependence on the last two quantities in the argument in (5) for the case of a hydrogen plasma and for a fixed aspect ratio A. As for the dependence on q, it is clearly of practical interest but it is not of much interest for our purposes, i.e., for an ambitious extrapolation, since q is usually restricted to a limited range. Accordingly, the main functional dependence can be written as $\beta_\theta = \beta_\theta(\nu, K)$.

At this point of course we cannot determine which of the quantities ν and K or which combination of these quantities is the main parameter. In particular, we do not rule out the possibility that the "electron magnetization parameter" $\Omega_e\tau_e$, where $\Omega_e = eB_z/mc$ and $\tau_e = \lambda_e\sqrt{m/T_e}$, plays a very important role. This parameter can be expressed in terms of our dimensionless quantities by the rather complicated expression $\Omega_e\tau_e \sim qA/\nu K\sqrt{\mu}$; we see that it always large.

However, even from (5) we can draw a very interesting conclusion: Since we have assumed that one of the parameters – specifically, N – is unimportant, one of the quantities in (1) is free if the dimensionless parameters in (2) and (3) and μ are specified. Let us choose the magnetic field B as this free parameter. Then by using (2) and (3) we can easily find the following scaling conditions, under which the dimensionless parameters remain constant:

$$a\sim B^{-4/5}; \quad n\sim B^{8/5}; \quad T\sim B^{2/5}. \quad (6)$$

Accordingly, there is a single-parameter family of tokamaks which are similar in all respects except in the characteristics of those fine-scale, high-frequency processes which develop in the case of two-stream instabilities and in which the Debye number N is important.

We note that, according to (6), we could choose the quantity $aB^{4/5}$ as the quantitative characteristic of the family of similar tokamaks, and we could approximate it by the current $I \sim aB$. In other words, the governing parameter for the tokamaks should be identified as the product aB or the current $I\sim aB$ and not the longitudinal field by itself. Furthermore, according to (6), the energy of the stabilizing field, $W \sim a^3B^2 \sim B^{-2/5}$, falls off with increasing B but not very rapidly for similar tokamaks; the parameter $n\tau_E$ [see (7) below] also increases slowly with increasing B, in proportion to $B^{2/5}$.

We can pursue this discussion by making a few plausible assumptions. We introduce a, τ_e, and m as the scale values of the length, time, and mass, respectively. Using these quantities and the dimensionless parameters in (2)-(4) we can construct a quantity with any dimensionality. For the problem of plasma confinement the most interesting quantity is the energy confinement time τ_E. If we assume that the energy loss is diffusive in nature, i.e., that $\tau_E \sim a^2$, then it would be natural to write τ_E as

$$\tau_E=\alpha\tau_e/K^2, \quad (7)$$

where the quantity α may depend on the dimensionless parameters. Artsimovich[1] has suggested that energy is carried away through the "electron channel," so that τ_E is governed by the electron thermal conductivity. Artsimovich also assumed that the electron thermal conductivity χ_e is governed by the "pseudoclassical" relation $\chi_e \cong 5\mu\rho_\theta^2/\tau_e$, which corresponds to the assumption α = const. From the balance of the Joule heating and the energy loss we find $\beta_\theta \sim \sqrt{\alpha}$ = const. The quantity β_θ is in fact a rather sluggish function of the dimensionless parameters ν, K, and q, changing by no more than a factor of a few units as these parameters are varied over broad ranges.

On the other hand, β_θ is not strictly constant, and we would like to know just how β_θ behaves as the temperature is raised and as $\nu \sim T^{-2}$ is reduced.

Neoclassical theory[2] predicts that if $\nu < A^{-3/2}$ the diffusion is "banana" diffusion, and the transport coefficients will fall off with ν. The theory for the trapped-particle instability,[3] on the other hand, predicts increasing transport coefficients in this region. Okabayachi et al.[4] have suggested that as we move into the highly collisionless region, $\nu \to 0$, we may find diffusion of the Bohm type, i.e., we may find that the parameter $\Omega_e\tau_e \sim 1/\nu K$ becomes influential. Dimensional analysis of course cannot rule out either of these possibilities, but it does tell us something about the possible scaling laws.

For example, let us make the plausible assumption that the coefficient α in (7) is governed by the parameters ν and K alone, which are measures of the plasma-confinement properties along and across the magnetic field; specifically, we are assuming that α is independent of β_θ. If α is independent of β_θ, then from the standpoint of scaling this quantity we can speak of a two-parameter family of similar systems. Holding all parameters besides N and β_θ fixed, we find the following scaling conditions (here the free quantities are n and B):

$$a\sim n^{1/4}/B^{4/5}; \quad T\sim(n/B)^{2/3}; \quad \beta_\theta\sim n^{5/3}/B^{8/3}. \quad (8)$$

Accordingly, if we wish to reproduce the same parameters ν, K, and $\alpha = \alpha(\nu, K)$ in the model apparatus, we can choose the parameters of this apparatus from (8), but we must be able to vary β_θ, e.g., by means of additional heating methods.

If we had chosen the current I and the field B as the governing quantities, we would have obtained $T\sim I^2$, $n\sim I^3B$, and $\beta_\theta \sim I^4/B$ instead of (8). As we see, β_θ should increase rapidly with I. We can say that a small apparatus with Joule heating models a large apparatus with additional heating.

Of course we are not ruling out the possibility that α can be a very weak function of one of the parameters, e.g., K, in which case the entire problem would reduce to one of determining the function $\alpha(\nu)$, i.e., a function of a single parameter, ν. However, at this point we have no experimental basis for this assertion, so that on the basis of dimensionality theory alone we cannot rule out a dependence on K. In the TM-3, for example, at low collision frequencies the confinement time is proportional to the quantity[5] τ_B/ν, where τ_B is the Bohm diffusion time, leading to a dependence $\alpha \sim K$; from the relation $\beta_\theta^2 \sim K$ follows the empirical relation $T_e n^{4/3} \sim I^2$, which has been well-established[6] in the TM-3 experiments.

On the basis of dimensional analysis we can make another assertion. We note that with fixed values of the magnetic field, q, and A, we have the dimensionless parameters $\nu \sim nR/T^2$ and $K \sim \sqrt{T}/I$. We thus see that if there are simultaneous increases in the size of the apparatus, the current, and the temperature these parameters do not change as rapidly as when only one of these quantities is changed. For example, if the dimension R is increased by an order of magnitude while the density is held constant, the temperature can be tripled without a change in ν, and the next tripling of the temperature reduces ν by only an order of magnitude. In precisely the same manner, an increase in the current I accompanied by an increase in T does not reduce K as much as a change in I alone. Accordingly, as we move to larger systems and simultaneously increase the temperature and the dimensions, the extrapolation may not be as ambitious as it seems at first glance. For example, the T-10 system with $n = 2 \cdot 10^{13}$, $T_e = 3$ keV, $a = 0.4$ m, $R = 1.5$ m, and $I = 0.8$ MA, and the T-20, with the proposed properties $n = 5 \cdot 10^{13}$, $T = 10$ keV, $a = 2$ m, $R = 5$ m, and $I = 5$ MA have the same values of ν, and the values of K differ by a factor of only 3.

Finally, we note that among the basic dimensionless parameters there are several very large ones, e.g., N, $\Omega_e\tau_e$, and a/ρ_θ. We do not rule out the possibility that under certain conditions even small effects (e.g., an interaction with the walls or slight perturbations of the magnetic field) may combine with these large parameters to lead to important changes in the plasma.

In other words, with large parameters like these we should not be surprised if the plasma reacts strongly to small external perturbations, although a detailed explanation of such a reaction would require a definite physical picture of the events.

[1)] If the density is to be maintained at a constant level, there must thus be some volume source of particles.

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Gas breakdown in a tokamak

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In an analysis of the initial stage of the discharge in a tokamak it is assumed that all the electrons formed during ionization of the neutral gas experience nearly continuous acceleration. The time over which a given electron density is reached is calculated as a function of the neutral gas density and the parameters of the helical electric field. The mechanism for plasma loss to the walls is analyzed; the time scale of this loss is governed by a peculiar inertial flow of the plasma and depends on whether the chamber walls are metal or dielectric. A condition for gas breakdown in a toroidal system is formulated. This theory explains why the initial gas density strongly affects the possibility of breakdown.

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1. INTRODUCTION

There has been remarkably little study of the initial stage of the discharge in toroidal systems of the tokamak type, although the discharge-formation conditions can obviously have a strong influence on both the energy balance of the entire system and the subsequent behavior of the plasma filament (through a degradation of the vacuum conditions, the appearance of impurities, etc., if the plasma filament interacts strongly with the wall at the time of its formation).

Only very recently have experiments been reported on the discharge formation in modern tokamaks (see, e.g.,