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REVIEW ARTICLE

INVARIANCE PRINCIPLES AND PLASMA CONFINEMENT

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Abstract—If the equations that describe the observed anomalous transport in magnetic confinement systems are invariant under a scale transformation then any confinement time calculated from them must exhibit the same invariance, no matter how intractable the calculation. This principle places constraints on the form of the confinement time scaling which are characteristic of the plasma model represented by the equations. These constraints reduce the number of parameters which have to be investigated empirically and can serve to indicate which plasma models could describe the observed losses. By extending this argument to specific models of turbulent transport in which the magnetic configuration and the physical mechanism for transport are identified, one can closely define the form of local transport coefficients—in certain cases completely determining them. This paper provides a review of these ideas, shows how they provide a theoretical framework for discussing existing empirical confinement scalings and illustrates their use in classifying and determining the form of local transport coefficients.

I. INTRODUCTION

THE TOPIC of transport in magnetic confinement systems presents a challenging problem in plasma physics whose solution would help to provide a realistic estimate of the parameters of a potential fusion reactor. If transport in such systems were due solely to Coulomb collisions then we would have a predictive theory (classical, or neo-classical theory) available but there is strong evidence that transport is generally anomalously rapid—by up to two orders of magnitude in the case of electrons in Tokamaks, for instance. The task of providing a theory of this anomalous transport is impeded by the diversity of plasma phenomena that could be involved, making it even difficult to decide upon the responsible mechanism. If this hurdle can be overcome, there still remains the problem of actually calculating the anomalous confinement time, say from the non-linearly saturated state of some plasma physics instability. Thus, not surprisingly, much reliance is placed on empirical scaling laws.

$$\tau = F(n, T, B, a \ldots) \tag{1}$$

for the energy confinement time τ as a function of characteristic values of plasma parameters such as density n, temperature T, magnetic field B, linear dimension a, etc. These are derived from existing experiments and then extrapolated, with uncertain validity, to reactor scale parameters.

A more general, plasma physics based, framework within which to discuss both empirical and theoretical approaches would be desirable. A set of related techniques—dimensional analysis, similarity ideas and invariance principles—offers this possibility and has been long familiar in fluid dynamics (LAMB, 1932; LANDAU and LIFSHITZ, 1959).

For example, there is the well-known demonstration using dimensional analysis that the drag force F on a body of characteristic size a, moving with velocity u through a fluid of density ρ and viscosity η described by the Navier Stokes equation, must be of the form

$$F = \rho u^2 a^2 f(R) \tag{2}$$

where the Reynolds number $R = \rho ua/\eta$. The function f depends, of course, on details of the shape of the body but the problem has been reduced to solving for this function of a single parameter. Clearly this result leads to the concept of similarity; i.e. if one knows the value of F, theoretically or empirically, for a particular case, then one knows the results for a whole family of situations with different parameters but the same value of R as that case.

As discussed by LAMB (1932) dimensional analysis is related to the invariance properties of the Navier Stokes equation under scale transformations. To demonstrate this we consider the Navier Stokes equation for the velocity $\mathbf{v}$ of an incompressible fluid (LAMB, 1932; LANDAU and LIFSHITZ, 1959).

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla p + \eta\Delta\mathbf{v} \tag{3}$$

where p is the pressure, and seek the scale transformations that leave this equation invariant. There are three:

$$\begin{array}{llll} \mathbf{v} \to \alpha\mathbf{v} & t \to \alpha^{-1}t & p \to \alpha^2 p & \eta \to \alpha\eta \\ \mathbf{x} \to \beta\mathbf{x} & t \to \beta t & \eta \to \beta\eta & \\ \rho \to \gamma\rho & p \to \gamma p & \eta \to \gamma\eta. & \end{array} \tag{4}$$

Now a force F must transform as $\rho v x^3/t$ so that if we seek a power law scaling in terms of the available quantities

$$F \sim \rho^p u^q a^r \eta^s \tag{5}$$

F must scale appropriately under the three transformations (4). This imposes three constraints on the indices p, q, r and s such that

$$F \sim \rho u^2 a^2 (\rho a u/\eta)^p. \tag{6}$$

Had we sought a more general expression than the power law (5) we would have recovered the result (2).

By making an additional assumption concerning the physics we can constrain the form of $f(R)$—indeed determine it up to a constant multiplier. Thus, considering the limit $R \ll 1$ we can ignore the inertial terms in equation (3) and deduce from dimensional analysis or scale invariance that

$$f(R) = c/R \tag{7}$$

where c is a constant depending only upon the shape of the body. In fact a detailed calculation is needed to obtain c which takes the value 6π for a spherical body (LANDAU and LIFSHITZ, 1959).

In the opposite limit $R > R_c$, where $R_c \gg 1$ is a critical value of R, one expects the flow to become turbulent and then detailed calculations are completely intractable. However, if one assumes that the effects of turbulence can be represented by a turbulent viscosity η_T which is itself independent of classical viscosity, dimensional analysis implies (LANDAU and LIFSHITZ, 1959)

$$\eta_T \sim \rho l u \sim \eta R \tag{8}$$

where u is a typical flow velocity having a gradient length l. Since turbulence only exists for $R > R_c$ an order of magnitude estimate is $\eta_T \sim \eta(R/R_c)$ (LANDAU and LIFSHITZ, 1959).

The idea of constraints arising from invariance properties of equations, the tighter constraints imposed by simplifying those equations and the concept of turbulent transport coefficients, all illustrated above for a viscous fluid, will be recurrent themes in this review of invariance principles and plasma confinement.

The equations of plasma physics are, of course, considerably more complex than the Navier Stokes equation. This is reflected in the larger number of dimensionless parameters available. In particular we have ρ/a', λ/a and λ_D/a where ρ is a Larmor radius, λ is the collisional mean free path, λ_D is the Debye length and a is a macroscopic length characterising the confinement system, and also the quantity β which is the ratio of plasma pressure to magnetic field pressure. In addition to these basic parameters which govern the physical processes, we have constants such as the electron–ion mass ratio and various geometrical ratios peculiar to particular confinement systems.

For such a complicated system of equations the invariance approach offers a more powerful and less heuristic technique than conventional dimensional analysis. However, insofar as it might be considered to be the systematic way to perform dimensional analysis, the distinction can become a matter of semantics. In Section II we describe an invariance principle which allows one to constrain the form of confinement time scaling laws (CONNOR and TAYLOR, 1977; LACINA, 1971). Since these constraints are characteristic of the particular plasma physics model assumed to be responsible for the anomalous transport, this provides a classification of scaling laws, theoretical or empirical, according to these models. We describe the relevant constraints for a number of conventional plasma models: electrostatic Vlasov, electrostatic Fokker–Planck, high-β Vlasov and high-β Fokker–Planck, all with and without the quasi-neutrality assumption. In addition the simpler, but popular, ideal and resistive MHD models are considered. These results give rise to the concept of similarity and we describe some consequences for the performance of families of similar confinement devices. An interpretation of the constraints in terms of dimensionless parameters (KADOMTSEV, 1975) is discussed, but the invariance approach is capable of extension to situations where dimensional analysis would require considerable physical intuition to obtain similar results. Thus illustrations are given of how it may be possible to gain information on scaling of confinement with dimensionless numbers such as the geometrical ratios characterizing a magnetic configuration or quantities such as atomic number and charge.

In Section III we demonstrate a more powerful extension of the invariance technique to discuss turbulent transport processes (CONNOR and TAYLOR, 1984). The simplification of a set of equations resulting from the assumption of transport due to local turbulence associated with a microscopic scale length such as the Larmor radius ρ, rather than a macroscopic scale a, enables the invariance principle to provide extra constraints on the form of a local thermal diffusivity χ. For particularly simple models such as resistive MHD one may be able to completely determine the form of χ in this way, but for more complicated situations such as drift wave turbulence this may not be possible. However, valuable information can always be obtained and the constraints serve at least to classify possible theoretical and empirical forms for χ. A number of illustrations of this technique drawn from the processes thought to be responsible for anomalous transport in toroidal devices are given. The relation of the results to explicit analytic and numerical treatments of the same problems are discussed.

Finally, in Section IV, we summarize our conclusions on the value of this approach to the description of confinement time scalings and discuss its relation to dimensional analysis.

II. THE INVARIANCE PRINCIPLE AND ITS IMPLICATIONS FOR GLOBAL CONFINEMENT

In this section we describe the Invariance Principle and its implications for empirical and theoretical scaling laws for global confinement time (CONNOR and TAYLOR, 1977).

1. *The invariance principle*

A statement of the Invariance Principle is: "If the confinement of plasma is described by the equations of some particular plasma model then a confinement time calculated from that model must reflect any invariance properties of those equations, no matter how complex the calculation."

This principle constrains the possible forms of the confinement time scalings in a manner which is characteristic of the particular model involved. Before listing the consequences for the various conventional plasma models we illustrate this for a particularly simple case. The example chosen is a model, which we label A, where anomalous transport is imagined to be completely described by the electrostatic Vlasov equations for both electrons and ions, supplemented by the charge neutrality condition. Thus we ignore effects of finite β, Coulomb collisions and fluctuations on the scale of the Debye length. Model A encompasses drift wave models such as transport due to the collisionless trapped particle modes and the ion temperature gradient mode.

The plasma is described by the Vlasov equation for the distribution functions f_j

$$\frac{\partial f_j}{\partial t} + \mathbf{v}\cdot\nabla f_j + \frac{e_j}{m_j}(\mathbf{E} + \mathbf{v} \times \mathbf{B})\frac{\partial f_j}{\partial \mathbf{v}} = 0 \qquad j = i, e \tag{9}$$

where $\mathbf{B}$ is a fixed magnetic field and the electrostatic field $\mathbf{E}$ is determined by the quasi-neutrality condition

$$\sum_j e_j \int d^3v f_j = 0. \tag{10}$$

We seek scale transformations

$$f_j \to \alpha f_j,\ \mathbf{v} \to \beta\mathbf{v},\ \mathbf{x} \to \gamma\mathbf{x},\ t \to \delta t,\ \mathbf{E} \to \mu\mathbf{E},\ \mathbf{B} \to \nu\mathbf{B}, \tag{11}$$

isotropic in space, which leave equations (9) and (10) invariant. There are three, and only three, independent ones which may be expressed as

$$\begin{array}{lllll} A_1 & f_j \to \alpha f_j & & & \\ A_2 & \mathbf{v} \to \beta\mathbf{v}, & \mathbf{B} \to \beta\mathbf{B}, & t \to \beta^{-1}t, & \mathbf{E} \to \beta^2\mathbf{E} \\ A_3 & \mathbf{x} \to \gamma\mathbf{x}, & \mathbf{B} \to \gamma^{-1}\mathbf{B}, & t \to \gamma t, & \mathbf{E} \to \gamma^{-1}\mathbf{E}. \end{array} \tag{12}$$

The particular specification A_1–A_3 is arbitrary to the extent that any three independent products of these transformations would also leave equations (9) and (10) invariant. Now the invariance principle implies that any confinement time calculated from equations (9) and (10) must transform appropriately under A_1–A_3. We consider a power law scaling for the confinement time

$$\tau \sim n^p T^q B^r a^s \tag{13}$$

where n, T, etc. are taken to be characteristic values, say at the plasma centre—such characteristic values are sufficient since their complete profiles are determined by the equations we are investigating! It follows, recalling $n = \int d^3v f$ and $nT = \int d^3v(mv^2/3)f$, that the four indices p, q, r and s are constrained by the three transformations to satisfy the relations

$$p = 0,\ 2q + r = -1,\ s - r = 1 \tag{14}$$

i.e there is only one independent index

$$\tau \sim \frac{1}{B}\left(\frac{T}{a^2B^2}\right)^q. \tag{15}$$

By considering a more general representation of τ as a Taylor series

$$\tau = \sum_{p,q,r,s} c_{pqrs} n^p T^q B^r a^s \tag{16}$$

it is clear that the transformations A_1–A_3 constrain τ to the form

$$B\tau = F(T/a^2B^2). \tag{17}$$

It should be stressed that we have assumed homogeneous boundary conditions at the plasma edge, i.e. effects due to atomic physics etc. at the periphery do not affect the scaling properties.

2. *Characteristic constraints*

The same procedure can be applied to more complex models than model A. We list in Table 1 the constraints appropriate to all the conventional plasma models which we describe below.

TABLE 1.—CONFINEMENT TIME SCALING LAWS FOR CONVENTIONAL PLASMA MODELS

	Plasma model	Scaling law for $B\tau$	Constraints on power law scaling (24)	Free indices	Dimensionless arguments of F in equations (43 and 44)
A	Quasi-neutral electrostatic Vlasov	$F\left(\frac{T}{a^2B^2}\right)$	$p = 0, r = s = -2q$	1	$F\left(\frac{\rho_i}{a}\right)$
B	Quasi-neutral electrostatic Fokker–Planck	$F\left(\frac{T}{a^2B^2}, \frac{na^2}{B^4a^5}\right)$	$3p + 2q + s = 0$ $4p + 2q + r = 0$	2	$F\left(\frac{\rho_i}{a}, \frac{\lambda_e}{a}\right)$
C	Quasi-neutral high-β Vlasov	$F\left(na^2, \frac{T}{a^2B^2}\right)$	$2p - 2q - s = 0$ $r + 2q = 0$	2	$F\left(\frac{\rho_i}{a}, \beta\right)$
D	Quasi-neutral high-β Fokker–Planck	$F(na^2, Ta^{1/2}, Ba^{5/4})$	$2p + q/2 + 5r/4 - s = 0$	3	$F\left(\frac{\rho_i}{a}, \frac{\lambda_e}{a}, \beta\right)$
E	Electrostatic Vlasov	$F\left(\frac{n}{B^2}, \frac{T}{a^2B^2}\right)$	$2p + 2q + r = 0$ $2q + s = 0$	2	$F\left(\frac{\rho_i}{a}, \frac{\lambda_D}{a}\right)$
F	Electrostatic Fokker–Planck	$F\left(\frac{n}{B^2}, \frac{T}{a^2B^2}, Ba^{3/2}\right)$	$6p + 2q + 3r - 2s = 0$	3	$F\left(\frac{\rho_i}{a}, \frac{\lambda_e}{a}, \frac{\lambda_D}{a}\right)$
G	High-β Vlasov	$F\left(\frac{n}{B^2}, \frac{T}{a^2B^2}, aB\right)$	$2p + r - s = 0$	3	$F\left(\frac{\rho_i}{a}, \beta, \frac{\lambda_D}{a}\right)$
H	High-β Fokker–Planck	$F(n, T, B, a)$	—	4	$F\left(\frac{\rho_i}{a}, \frac{\lambda_e}{a}, \beta, \frac{\lambda_D}{a}\right)$
I	Ideal MHD	$(na^2)^{1/2}F\left(\frac{nT}{B^2}\right)$	$p = q + 1/2$ $r + 2q = 0$ $s = 1$	1	$F(\beta)$
J	Resistive MHD	$(na^2)^{1/2}F\left(\frac{nT}{B^2}, Ta^{1/2}\right)$	$p - q + 2s - 5/2 = 0$ $2p + r - 1 = 0$	2	$F(\beta, S)$

Model B—The quasi-neutral electrostatic Fokker–Planck model. In this model the Vlasov equation (9) is replaced by the Fokker–Planck equation with the Landau collision integral for Coulomb collisions. [This differs from the classification of LACINA (1971) in which collisions with neutrals were also included.] Such a model encompasses collisional drift waves and dissipative trapped particle drift waves. Taken together with model A it describes most of the phenomena employed in the earlier Six Regime (DÜCHS *et al.*, 1977) and present (ROMANELLI *et al.*, 1986; WALTZ *et al.*, 1987) drift wave models of anomalous transport in Tokamaks.

Model C—The quasi-neutral high-β Vlasov model. Here the Vlasov model A is modified to include a self-consistent magnetic field **B** satisfying Maxwell's equation

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \mathbf{j} = \sum_j e_j \int d^3v \mathbf{v} f_j. \tag{18}$$

Such a model describes finite-β collisionless drift-Alfvén waves and includes the physics underlying the transport coefficients of OHKAWA (1978), PARAIL and POGUTSE (1980) and KADOMTSEV and POGUTSE (1985).

Model D—The quasi-neutral high-β Fokker–Planck model. This model, which combines models B and C and includes, for example, the two-fluid equations of BRAGINSKII (1965), is often thought to be adequate to describe anomalous transport in toroidal systems.

Models E, F, G and H replace the charge neutrality condition equation (10) by the more general Poisson equation

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0, \rho = \sum_j e_j \int d^3v f_j \tag{19}$$

in models A, B, C and D respectively. Such models may be more relevant to transport processes in mirror machines where higher frequency, shorter wavelength velocity space instabilities occur (KADOMTSEV, 1961).

These choices exhaust all conventional plasma physics phenomena but, because of traditional theoretical interest, we include two special fluid models—Model I is ideal and Model J is resistive MHD. The equations of these models comprise a continuity equation relating the density $\rho = m_i n$ and the macroscopic velocity $\mathbf{v}$

$$\frac{\partial \rho}{\partial \tau} + \nabla \cdot \rho \mathbf{v} = 0 \tag{20}$$

a momentum equation

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v}\right) + \nabla p - \mathbf{j} \times \mathbf{B} = 0 \tag{21}$$

and an energy equation for the pressure p, which is related to the temperature T through $p = nT$, such as

$$\frac{\rho^{\gamma}}{\gamma - 1}\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla\right)(p\rho^{-\gamma}) = \eta |\mathbf{j}|^2 \tag{22}$$

with the Spitzer resistivity $\eta \sim T^{-3/2}$ and γ the ratio of specific heats. The models are completed by the Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j} \tag{23}$$

and Maxwell's equations for the electromagnetic field. These equations are appropriate to describe ideal and resistive ballooning modes in Tokamaks and resistive interchange modes in pinches.

In Table 1 we list the constraints on the form of confinement time scaling laws and on the indices for supposed power laws

$$B\tau = n^p T^q B^r a^s \tag{24}$$

for models A–J. The last column of Table 1, involving dimensionless quantities, will be discussed in Subsection 3(d).

3. *Applications of the constraints*

(a) *Theoretical models*—Any theoretical calculation must, of course, conform to the appropriate constraints. Apart from a trivial application as a check on such calculations this also provides a way of classifying the theories. Thus, for example, the complicated and varied results involved in the confinement scenarios based on drift wave transport discussed by FURTH (1977) are classified quite simply in Table 2 in terms of the values of the two independent indices pertaining to the quasi-neutral electrostatic Fokker–Planck model (model B of Table 1). Alternatively one learns that theory has only to supply the values of a limited number of indices!

TABLE 2.—CONFINEMENT DUE TO DRIFT WAVE TRANSPORT MODELS OF FURTH (1977)

$$B\tau \sim \left(\frac{T}{a^2B^2}\right)^p \left(\frac{na^2}{B^4a^5}\right)^q$$

Model	p	q
Bohm	-1	0
Dissipative trapped ion	$-7/2$	1
Collisionless trapped electron	$-3/2$	0
Dissipative trapped electron	$-7/2$	1
Pseudo-classical	$1/2$	-1

(b) *Empirical scaling laws*—An examination of empirical scaling laws in the context of these constraints is a more significant application of the invariance principle. Thus, by examining which, if any, model is consistent with an empirical law we can, in principle, identify the nature of the plasma physics responsible for the anomalous transport. Such a preliminary screening of data would guide the choice of a theoretical model to be investigated in more detail.

Table 1 lists the constraints to compare with empirical laws. However, one must note that for Ohmically heated pinches and Tokamaks the temperature is not an independent quantity but is determined by the energy balance condition

$$\frac{nT}{\tau} \sim \eta\frac{I^2}{a^4} \tag{25}$$

where $\eta \sim Z/T^{3/2}$ is the Spitzer resistivity, with Z the effective charge of the plasma, and I is the total current. Combining equation (18) with equation (25) yields

$$T \sim \left(\frac{B^2\tau}{na^2}\right)^{2/5}. \tag{26}$$

As a result of this constraint the power law scaling (24) now takes the form

$$B\tau \sim n^x B^y a^z \tag{27}$$

and the results in Table 1 are modified.

Table 3 shows the effect for the more customary models A, B, C, D, I and J. Since the ohmic heating condition itself involves the physics of the quasi-neutral high-β Fokker–Planck model, the constraints are less sensitive to the choice of model.

TABLE 3.—SCALING LAWS FOR OHMICALLY HEATED PLASMAS

	Plasma model	Scaling law for $B\tau$	Constraints on power law scaling (27)	Free indices
A	Quasi-neutral electrostatic Vlasov	$B\tau = F(na^7B^4)$	$z = 7x, y = 4x$	1
B	Quasi-neutral electrostatic Fokker–Planck	$B\tau = F(na^2, a^5B^4)$	$8x + 5y - 4z = 0$	2
C	Quasi-neutral high-β Vlasov	$B\tau = F(na^2, a^5B^4)$	$8x + 5y - 4z = 0$	2
D	Quasi-neutral high-β Fokker–Planck	$B\tau = F(na^2, a^5B^4)$	$8x + 5y - 4z = 0$	2
I	Ideal MHD	$B = n^{1/2}aF\left(\frac{n^2a^4}{a^5B^4}\right)$	$4z + 2x = 5, y + 2x = 1$	1
J	Resistive MHD	$B\tau = n^{1/2}aF\left(\frac{n^2a^4}{a^5B^4}\right)$	$4z + 2x = 5, y + 2x = 1$	1

In Table 4 we show some contemporary empirical laws for Tokamak confinement which can be compared with the constraints on the indices given in Tables 1 and 3.

TABLE 4.—EMPIRICAL LAWS FOR TOKAMAK CONFINEMENT

Law		Indices of Table 1 or Table 3 (ohmic)
Name	Form of τ	
Neo-Alcator (ohmic)	$na^{1.04}R^{2.04}q$	$x = 1 \quad y = 1 \quad z = 3.08$
Merezhkin–Mukhovatov	$nT^{-0.5}a^{0.25}R^{2.75}q$	$p = 1 \quad q = -\frac{1}{2} \quad r = 1 \quad s = 3$
Goldston	$n^{-1}T^{-1}B^2a^{1.26}R^{0.5}q^{-2}$	$p = -1 \quad q = -1 \quad r = 3 \quad s = 1.76$

It is evident from comparing the Neo-Alcator law (LIEWER, 1985) for ohmic Tokamaks with Table 3 that it is at least consistent with the three conventional models B, C and D which cannot be differentiated on account of the ohmic constraint. The MEREZHKIN-MUKHOVATOV (1981) law accurately satisfies the constraint appropriate to the quasi-neutral high-β Vlasov Model C of Table 1. On the other hand the GOLDSTON (1984) law, derived from experiments with substantial auxiliary heating, obeys the constraints of the quasi-neutral high-β Fokker–Planck model reasonably well and, surprisingly, equally well satisfies those of the simpler resistive MHD model J. This type of preliminary screening of the empirical data can be a valuable guide to a choice of theoretical models for transport.

An alternative role of the constraints is in reducing the number of parameters it is necessary to vary in order to establish an empirical law. Thus if we believe, for example, that confinement is indeed described by the quasi-neutral high-β Fokker–Planck model then we can establish the nature of the function $F(na^2, Ta^{1/2}, Ba^{5/4})$ by varying just n, T and B—the variation with a is then completely determined.

(c) *Similarity and similar devices*—As we noted in the Introduction viscous fluid flows with the same Reynolds number R may be regarded as similar (LAMB, 1932; LANDAU and LIFSHITZ, 1959)—i.e. if we know the properties of one flow then we can infer those for all other flows with the same value of R. Likewise if we believe the confinement properties of a device are controlled by the physics of some particular model, then it may be possible to define a family of similar devices whose confinement properties we can confidently predict (LACINA, 1971; KADOMTSEV, 1975; CONNOR and TAYLOR, 1979). Suppose, for example, that confinement is describable by the quasi-neutral high-β Fokker–Planck model

$$B\tau = F(na^2, Ta^{1/2}, Ba^{5/4}). \tag{28}$$

Then one can define a one-parameter family of similar devices (i.e. such that the arguments of F are held fixed) by scaling

$$n \sim a^{-2},\ T \sim a^{-1/2},\ B \sim a^{-5/4} \tag{29}$$

for which

$$\tau \sim B^{-1} \sim a^{5/4}. \tag{30}$$

Note that β is invariant but that the "fusion product" $f = nT\tau$ scales as

$$f \sim B \sim a^{-5/4} \tag{31}$$

suggesting the advantages of compact, high-field devices. Although these ideas were explored by LACINA (1971), his results differ somewhat since his collisional model included collisions with neutrals and therefore did not scale as the Coulomb collision frequency $\nu \sim nT^{-3/2}$.

For simpler models we can define two-parameter families of devices. In the quasi-neutral high-β Vlasov case

$$n \sim a^{-2}, T \sim a^2B^2 \tag{32}$$

for which

$$\tau \sim B^{-1}, f \sim B \tag{33}$$

pointing to the advantages of high-field devices.

For the quasi-neutral electrostatic Fokker–Planck model the family is defined by

$$n \sim a^3B^4, T \sim a^2B^2 \tag{34}$$

implying

$$\tau \sim B^{-1}, f \sim (aB)^5 \tag{35}$$

suggesting the merits of large, high-field devices (in this case β is not invariant: $\beta \sim a^5B^4$).

One could, of course, employ control parameters other than a and B, say n and, for non-ohmically heated situations, heating power P where

$$P \sim \frac{nTa^3}{\tau}. \tag{36}$$

Then for the quasi-neutral high-β Vlasov model

$$f \sim P^{1/3}a^{-1} \tag{37}$$

and for the quasi-neutral electrostatic Fokker–Planck model

$$f \sim (P/a)^{5/7} \tag{38}$$

both showing the advantages of intense heating in compact devices.

Although the above families of similar devices indicate advantageous directions for designing experiments, these are not necessarily optimal—true optima will require scaling between families, which requires some knowledge of the function F over a range of its various arguments. In this way we can extend the concept of similarity from one set of values of these arguments to a range of their values (CONNOR and TAYLOR, 1979). Consider as an example ohmically heated devices for which

$$\tau = B^{-1}F(na^2, B^4a^5). \tag{39}$$

If we know the performance $\tau_0(n, B) \equiv \tau(n, B, a_0)$ of a parent device of given size a_0, but over a range of n and B, then these observations have a universal nature and can be applied to similar machines of a different size a_1. Thus $\tau_1(n, B) \equiv \tau(n, B, a_1)$ is given by

$$\tau_1(n, B) = \left(\frac{a_1}{a_0}\right)^{5/4} \tau_0\left(\frac{na_1^2}{a_0^2}, B\left(\frac{a_1}{a_0}\right)^{5/4}\right) \tag{40}$$

provided the new arguments of τ_0 belong to the range of observation on the parent machine.

(d) *Dimensional analysis*—The intimate relationship between scale invariance and dimensional analysis leads to an interpretation of the constraints in terms of dimensionless parameters as discussed by KADOMTSEV (1975). The physics of a pure, fully ionised plasma can be described by the four basic parameters

$$\frac{\rho_i}{a}, \frac{\lambda_e}{a}, \beta, \frac{\lambda_D}{a} \tag{41}$$

where ρ_i is the ion Larmor radius, λ_e the electron mean free path and λ_D the electron Debye length, together with geometrical ratios, such as

$$q, a/R \tag{42}$$

in a toroidal confinement system, and the ratios T_e/T_i and m_e/m_i.

The four basic parameters (41) can be related to the four control parameters n, T, B and a so that the confinement time can be written

$$B\tau = \frac{m_i}{e} F\left(\frac{\rho_i}{a}, \frac{\lambda_e}{a}, \beta, \frac{\lambda_D}{a}\right). \tag{43}$$

This provides an interpretation of our previous results. The most general model H contains four independent variables corresponding to all four parameters in equation (43). For the models A–D we assume charge neutrality and the dependence on one variable, λ_D/a, is therefore removed from equation (43). For the Vlasov models a second, λ_e/a, is removed leaving a function of two variables. Alternatively, the electrostatic assumption removes β leaving a function of two different variables. Finally, removing λ_e/a and β we restrict F to be a function of just one variable ρ_i/a. This is summarized in Table 1.

However, this correspondence of physically motivated dimensionless parameters with the individual model A–H is not unique. Since we can express the ion Larmor radius in terms of β and the collisionless skin depth $d = c/\omega_{pe}$ ($\rho_i^2 = \beta(m_i/m_e)d^2$) the scaling law for the quasi-neutral high-β Vlasov model could be related to the physics of d/a and β (OHKAWA, 1978; PARAIL and POGUTSE, 1980; KADOMTSEV and POGUTSE, 1985) rather than ρ_i/a and β. Indeed, one could regard the electrostatic models as depending on β if one naively held d, instead of ρ_i, fixed.

In the case of the ideal and resistive MHD models the only available parameters are β and the Lundquist number S, the ratio of resistive diffusion time $\tau_R = \mu_0 a^2/\eta$ to Alfvén transit time $\tau_A = a(\mu_0\rho)^{1/2}/B$. Thus we can write

$$B\tau = (\mu_0 m_i n)^{1/2} a F(\beta, S) \tag{44}$$

so that confinement is a function of two variables. The ideal MHD limit removes S, leaving a function of only one variable.

Since one expects the confinement properties to depend on the values of physically significant parameters such as ρ_i/a, λ_e/a and β it is important to establish empirical laws in relevant ranges of these quantities and preferably by varying one at a time, holding the others fixed. As emphasized by KADOMTSEV (1975) some, such as ρ_i/a and λ_e/a, may not vary appreciably in moving from present experiments to reactor parameters and large extrapolations of empirical laws may not be necessary.

(e) *Scaling with geometrical ratios and dimensionless numbers*—The intuitive discussion of dimensional analysis given above recovers the results of the invariance approach that we have obtained so far. However, the invariance approach, which might be regarded as the systematic approach to dimensional analysis, has the potential to extract more information from a set of equations. In this way it is possible to determine features of scaling of confinement with dimensionless quantities such as aspect ratio and ionic charge.

As an example of scaling with geometrical ratios, we introduce a large aspect ratio Tokamak ordering $\varepsilon = a/R \sim B_p/B_T \sim p/B_T^2 \ll 1$, where a and R are the minor and major radii and B_p and B_T are the poloidal and toroidal magnetic fields, into the equations (20)—(23) of resistive MHD to obtain the reduced equations of STRAUSS

(1983). In a co-ordinate system r, θ, ζ, where ζ is a toroidal angle and r and θ are polar co-ordinates in the poloidal plane, so that the major radius $R = R_0 + r\cos\theta$, the poloidal field and fluid velocity can be expressed in terms of stream functions φ and ψ

$$\mathbf{B}_p = \frac{\nabla\psi \times \mathbf{e}}{R_0}, \mathbf{v}_p = \frac{\mathbf{e} \times \nabla\varphi}{B_0} \tag{45}$$

where B_0 is the toroidal field at $R = R_0$ and $\mathbf{e}$ is a unit toroidal vector.

The flux function ψ evolves through the induction equation

$$\frac{\partial\psi}{\partial t} = -\frac{R_0}{B_0}\mathbf{B}\cdot\nabla\varphi + \frac{\eta\nabla_\perp^2\psi}{\mu_0} \tag{46}$$

where

$$\mathbf{B}\cdot\nabla \equiv \frac{B_0}{R_0}\frac{\partial}{\partial\zeta} + \mathbf{B}_p\cdot\nabla. \tag{47}$$

The velocity stream function φ satisfies the vorticity equation

$$\frac{\rho R_0}{B_0}\frac{\mathrm{d}}{\mathrm{d}t}\nabla_\perp^2\varphi = -\frac{(\mathbf{B}\cdot\nabla)\nabla_\perp^2\psi}{\mu_0} - \frac{\mathbf{e}\cdot\nabla \times (R^2\nabla p)}{R_0} \tag{48}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v}_p\cdot\nabla. \tag{49}$$

In the large aspect ratio ordering, $\nabla\cdot\mathbf{v}$ is small and the set of reduced equations is closed by the equation for convection of pressure

$$\frac{\mathrm{d}p}{\mathrm{d}t} = 0. \tag{50}$$

Because of the large aspect ratio geometric limit these equations allow independent scalings of a and R and of B_p and B_0 which provide information on scaling with geometric ratios. In fact four independent invariant transformations exist and we find

$$\tau = \frac{a^2}{\eta}F\left(\frac{p}{B_0^2}\frac{R}{a}, \frac{\rho^{1/2}R_0\eta}{B_0a^2}, q\right) \tag{51}$$

or, with the specific definition of the Lundquist number $S = \mu_0^{1/2}a^2B_0/\eta\rho^{1/2}Rq$ (i.e. the ratio of resistive diffusion time $\tau_R = \mu_0 a^2/\eta$ to poloidal Alfvén transit time $\tau_{A\mathrm{p}} = (\mu_0\rho)^{1/2}Rq/B_0$) the dimensionless form

$$\tau = \tau_R F(\beta/\varepsilon, S, q). \tag{52}$$

Clearly we have deduced geometrical information which could not have been obtained from the full set of resistive equations (20)–(23); these would only have yielded

$$\tau = \tau_R F(\beta, S, q, \varepsilon). \tag{53}$$

The resistive MHD model supplemented by a radiative energy loss term also furnishes an example of scaling with dimensionless numbers such as atomic mass A and ionic charge Z. The loss of energy by Bremsstrahlung or impurity radiation can be represented by adding a general radiation loss term

$$P_{\text{rad}} \sim n^k T^l Z^m \tag{54}$$

to the energy balance equation (22). We also recall that the resistivity $\eta \sim Z/T^{3/2}$ in Ohm's law (23) and that the mass density $\rho = An$. With these modifications the resistive MHD equations (without geometrical simplifications) admit just three independent scale transformations. The general case (54) is discussed by CONNOR and TAYLOR (1977) but for the special case of Bremsstrahlung, when $k = 2$, $l = 1/2$ and $m = 1$ (ROSE and CLARK, 1961), we obtain

$$B\tau = n^{1/2} a A^{1/2} F\left(\frac{nT}{B^2}, \frac{Ta^{1/2}}{Z^{1/2}A^{1/4}}, aB\right) \tag{55}$$

which might tax intuition.

III. LOCAL TURBULENT TRANSPORT COEFFICIENTS

Global confinement properties may well be determined by the interaction of a variety of processes in different parts of a device. For example, conventional descriptions (DÜCHS *et al.*, 1977; ROMANELLI *et al.*, 1986; WALTZ *et al.*, 1987) of Tokamak confinement involve three radial zones: an inner sawteeth dominated region, a "confinement" zone controlled by anomalous transport due to micro-instabilities, such as trapped electron drift waves and ion temperature gradient modes, and an edge region. This edge region may be affected by turbulence due to instabilities specific to the plasma periphery (rippling modes, resistive ballooning modes and collisional drift waves) or by non-plasma physics phenomena such as radiation.

It is therefore more profitable to consider the scaling properties of *local* transport coefficients. These are more likely to be described by a single plasma physics model than are the global confinement properties and a study of their empirical scaling laws within the framework of invariance constraints would be more fruitful. However, local transport coefficients can, in principle, depend on many more parameters. Whereas τ depends only on global dimensions such as a and R (since all profile effects are themselves determined by the governing equations and are accounted for automatically by the invariance principle), a local thermal diffusivity χ can depend, in general, on a number of lengths representing the gradients of the quantities driving the turbulence and the magnetic field geometry, e.g.

$$L_T = \left(\frac{\mathrm{d}}{\mathrm{d}r}\ln T\right)^{-1}, L_n = \left(\frac{\mathrm{d}}{\mathrm{d}r}\ln n\right)^{-1}, L_s = \left(\frac{r}{Rq^2}\frac{\mathrm{d}q}{\mathrm{d}r}\right)^{-1} \equiv \frac{Rq}{s}$$

with $s = (r/q)(\mathrm{d}q/\mathrm{d}r)$ (only if some of these vanish can a *local* theory depend on higher derivatives). These gradient lengths themselves are, of course, determined by the global transport properties and not just the local problem. Fortunately these extra degrees of freedom in the form of χ will be seen to be offset by additional constraints arising from the assumption of a local transport coefficient. These arise because the local turbulence causing the transport has a scale characterized by a microscopic length such as a Larmor radius, the collisionless skin depth or a resistive layer, which is distinct from the macroscopic length a. As a result an extra invariant scaling is possible which further constrains the form of χ. This aids the empirical identification of a plasma physics model for χ and limits the possible forms of theoretical models—indeed in certain cases it completely determines them as demonstrated by CONNOR and TAYLOR (1984) for resistive fluid turbulence. Thus the invariance technique is a powerful tool to complement explicit non-linear turbulence calculations.

In the following subsections we shall describe a number of applications of the invariance technique to local transport coefficients. We begin with the minimal assumption that transport is caused by local turbulence described by the non-linear gyro-kinetic equations and Maxwell's equations and then move towards simpler geometries, such as the large aspect ratio torus, and simpler models, such as resistive MHD. In this way we shall encompass many mechanisms thought to be involved in anomalous transport in Tokamaks and pinches.

1. *Non-linear gyro-kinetic model*

FRIEMAN and CHEN (1982) derived a non-linear gyro-kinetic equation for the fluctuation δf in the distribution function f in response to electrostatic potential and magnetic field fluctuations $\delta\varphi$ and $\delta\mathbf{B}$ respectively, by ordering

$$\frac{\delta f}{f} \sim \frac{\rho\delta\varphi}{T} \sim \frac{\delta\mathbf{B}}{B} \sim \frac{1}{\Omega}\frac{\partial}{\partial t} \sim \rho\nabla_{\parallel} \sim \frac{\rho}{a},\ \rho\nabla_{\perp} \sim 1 \tag{56}$$

for the fluctuations and $(1/\Omega)\,\partial/\partial t \sim (\rho/a)^3$ for macroscopic quantities. Here ρ is a Larmor radius, a is a macroscopic length, Ω a gyro-frequency, while $\nabla_{\parallel}$ and $\nabla_{\perp}$ are gradient operators parallel and perpendicular to the equilibrium field $\mathbf{B}$. Writing

$$\delta f = -e\frac{\delta\varphi}{T}F_M + g \tag{57}$$

they obtained an equation for g which, in the electrostatic Vlasov limit, is

$$g\frac{\partial\bar{g}}{\partial t} + \left(\mathbf{v}_D + \frac{\mathbf{n}\times\nabla\overline{\delta\varphi}}{B}\right)\cdot\nabla\bar{g} + v_{\parallel}\mathbf{n}\cdot\nabla\bar{g} - \frac{e}{T}F_M\frac{\partial\overline{\delta\varphi}}{\partial t} + \frac{1}{B}\mathbf{n}\times\nabla\overline{\delta\varphi}\cdot\nabla F_M = 0 \tag{58}$$

where $\mathbf{n}$ is a unit vector along $\mathbf{B}$, F_M is the equilibrium Maxwellian distribution and the magnetic drift velocity

$$\mathbf{v}_D = (1/\Omega)[\mathbf{n} \times \mu\nabla B + v_{\parallel}^2\mathbf{n}\cdot\nabla\mathbf{n}].$$

This equation has been written in the co-ordinates ($\mathbf{R}$, μ, κ, ζ), where $\mathbf{R} = \mathbf{r} - \mathbf{n} \times \mathbf{v}_{\perp}/\Omega$ is a guiding centre position, $\mu = v_{\perp}^2/2B$, $\kappa = v_{\parallel}^2/2 + \mu B$ and ζ is the gyro-phase, and an overbar represents a gyro-phase average. The potential fluctuation $\delta\varphi$ is determined self-consistently from quasi-neutrality. A turbulent heat flux $\mathbf{q}$ arising from such fluctuations can be calculated from

$$\mathbf{q} = \int \mathrm{d}^3v \frac{mv^2}{2}\left\langle \overline{\delta f}\frac{\mathbf{n} \times \nabla\overline{\delta\varphi}}{B}\right\rangle \tag{59}$$

where the angle brackets represent an average over an ensemble of fluctuations.

HAGAN and FRIEMAN (1986) have applied the scale invariance technique to these equations. In the collisionless limit they conclude that, if the radial heat flux q is expressed in terms of a thermal diffusivity χ (which can itself depend on radial gradients), i.e. $q = -\chi n\, \mathrm{d}T/\mathrm{d}r$, then

$$\chi \sim T^{3/2}/B^2a \tag{60}$$

where a represents a characteristic macroscopic length such as L_T, L_s, etc. and a possible function of geometrical ratios has been suppressed. Alternatively, this could be established through the relation $\chi \sim (\Delta x)^2/\Delta t$, where Δx and Δt are characteristic lengths and times of the turbulent fluctuations whose scaling properties can be derived using the invariance approach (CONNOR and TAYLOR, 1984).

One should note that equation (60) does not correspond to Bohm diffusion $\chi \sim T/B$. This could be obtained by using the Larmor period as a characteristic time Δt, but the gyro-kinetic model averages over that timescale leaving only the thermal transit time implicit in equation (60). [It is also possible to derive the Bohm result by considering $\mathbf{E} \times \mathbf{B}$ motion in a potential $\varphi \sim T/e$ with any scale length $l \ll a$ (KROMMES, 1984).]

Thus, apart from geometrical ratios, χ has been completely determined up to an overall constant! If we estimate a global confinement time from $\tau \sim a^2/\chi$, we find

$$\tau \sim \frac{a^3B^2}{T^{3/2}} \tag{61}$$

—in other words the function $F(T/a^2B^2)$ appropriate to the quasi-neutral electrostatic Vlasov model A has been completely determined and the index in equation (15) is given by $q = -3/2$.

This extra information has arisen from the assumption of a local turbulence model which allows us to express

$$f = F_M + \delta f \tag{62}$$

and scale F_M and δf separately. The existence of such a separation in the non-linearly saturated state is, in turn, a consequence of the existence of a microscopic scale ρ, distinct from a, in the ordering (56) so that both ρ and a can be scaled independently. The continuous scalings of the co-ordinate perpendicular to the magnetic field and within the magnetic surface of a torus are permitted, despite the periodicity in such a direction, because we are considering high mode number turbulence.

An interesting aspect of the work of HAGAN and FRIEMAN (1986) is their concern that the ensemble averaging operation $\langle \overline{\delta f}\, \overline{\delta\varphi} \rangle$ in equation (59) may affect the scaling properties. Therefore they develop the Direct Interaction Approximation equations for the two point correlation function $\langle \overline{\delta f}\, \overline{\delta\varphi} \rangle$. The scaling properties of these equations confirm the above results.

Had we considered geometrical and profile parameters in equation (58) we would have concluded

$$\chi \sim \frac{T^{3/2}}{B^2 L_n} F(r/R, L_T/L_n, L_s/L_n, q, s) \tag{63}$$

but, since the transport processes themselves determine L_n, L_T and L_s, we can deduce

$$\tau \sim \frac{a^3 B^2}{T^{3/2}} F(a/R, q). \tag{64}$$

In practice a/R and q do not vary greatly so that we have a well defined prediction for τ from this model although the radial profiles of temperature, etc. depend on the function F in equation (63).

Including collisions (HAGAN and FRIEDMAN, 1986) and finite β effects but suppressing geometrical and profile parameters, leads to

$$\chi \sim \frac{T^{3/2}}{B^2 a} F\left(\frac{na}{T^2}, \frac{nT}{B^2}\right). \tag{65}$$

This has a simple interpretation in terms of dimensionless parameters:

$$\chi \sim \frac{\rho^2}{a} v_{\text{th}} F(\nu a/v_{\text{th}}, \beta) \tag{66}$$

where v_{th} is a thermal velocity and ν a collision frequency.

It is interesting to note that the conventional drift wave models of transport satisfy the low-β limit of equation (65)—indeed a number of results in the literature can be classified by their power of (na/T^2) as in Table 5.

TABLE 5.—CLASSIFICATION OF DRIFT WAVE TRANSPORT MODELS BY THE INDEX p, WHERE $\chi \sim \frac{T^{3/2}}{B^2 a}\left(\frac{na}{T^2}\right)^p$

Drift wave	Index p	Reference
Collisional	1	1,2,3
	0	1,2
	2/3	4
	1/3	1
Dissipative trapped electron	−1	1,2,5,6
	−4/3	7
	−2	8
Collisionless trapped electron	0	1,2,5
η_i-mode	0	2,5,6,9,10
Dissipative trapped ion	−1	1

Key to References in Table 5
1. DÜCHS *et al.* (1977).
2. WALTZ *et al.* (1987).
3. YOSHIKAWA (1970).
4. TERRY and DIAMOND (1985).
5. ROMANELLI *et al.* (1986).
6. KADOMTSEV and POGUTSE (1970).
7. SIMILON and DIAMOND (1984).
8. CHEN *et al.* (1977).
9. HORTON *et al.* (1981).
10. LEE and DIAMOND (1986).

2. *Geometrical simplification in a large aspect ratio torus*

In order to discuss the dependence of χ on geometrical factors it is necessary to introduce a specific geometry. In a large aspect ratio torus with circular concentric surfaces equation (58) takes the form

$$\left[\frac{\partial}{\partial t} + \frac{v_{\parallel}}{Rq}\left(\frac{\partial}{\partial\theta} + sx\frac{\partial}{\partial y}\right) - \frac{m}{eBR}\left(\frac{v_{\perp}^2}{2} + v_{\parallel}^2\right)\left(\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}\right) + \frac{1}{B}\left(\frac{\partial}{\partial x}\overline{\delta\varphi}\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\overline{\delta\varphi}\frac{\partial}{\partial x}\right)\right]\bar{g} - \frac{eF_M}{T}\frac{\partial}{\partial t}\overline{\delta\varphi} + \frac{1}{B}\mathbf{n}\times\nabla\overline{\delta\varphi}\cdot\nabla F_M = 0 \quad (67)$$

where we have introduced co-ordinates x, y and θ with respect to a rational magnetic surface of radius r_0. x is the radial distance from the surface, y is perpendicular to the magnetic field but in the surface and θ is a periodic variable along the magnetic field which allows for ballooning effects. Turbulence is described by this equation together with quasi-neutrality.

Scale invariance then implies

$$\chi = \frac{\rho_i^2 v_{\text{thi}}}{L_n} F(\varepsilon, \varepsilon_n, \eta_i, \eta_e, q, s, T_e/T_i, m_e/m_i) \quad (68)$$

where $\varepsilon_n = L_n/R$, $\eta_j = \mathrm{d}(\ln T_j)/\mathrm{d}(\ln n)$ and $\varepsilon = r_0/R$. The collisionless trapped electron turbulent transport coefficient (DÜCHS *et al.*, 1977; ROMANELLI *et al.*, 1986; WALTZ *et al.*, 1987)

$$\chi^{TE} \sim \varepsilon^{1/2} \frac{\rho_i^2 v_{\text{thi}}}{L_n} \tag{69}$$

is seen to be an example of this model.

In the cylindrical limit $\varepsilon, \varepsilon_n \rightarrow 0$, we can replace the parallel gradient operator $(\partial/\partial\theta + sx\,\partial/\partial y)$ acting on a Fourier mode centred on a surface $x = x_k$ by $s(x - x_k)\,\partial/\partial y$. Then applying scale invariance arguments to the turbulence allows us to condense expression (68) to the form

$$\chi = \frac{\rho_i^2 v_{\text{thi}}}{L_n} F(\varepsilon_n s/q, \eta_i, \eta_e, T_e/T_i, m_e/m_i) \tag{70}$$

where $\varepsilon_n s/q \equiv L_n/L_s$. The calculation of a turbulent diffusivity due to the universal mode by HIRSHMAN and MOLVIG (1979)

$$\chi^{HM} \sim \frac{(T_e/T_i)^4}{(1 + T_e/T_i)^{9/2}} \left(\frac{L_s}{L_n}\right)^{7/2} \frac{\rho_i^2 v_{\text{thi}}}{L_n} \tag{71}$$

is an example of this situation. Another is furnished by the slab version of the η_i-mode but we shall discuss that case later in the context of a fluid model.

The incorporation of electro-magnetic effects into this slab model can be achieved by substituting

$$\delta\varphi \rightarrow \delta\varphi - v_{\parallel}\delta A_{\parallel} \tag{72}$$

where the parallel component of the vector potential $\delta A_{\parallel}$ satisfies Ampere's law

$$\nabla^2 \delta A_{\parallel} = -\mu_0 \sum_j e_j \int \mathrm{d}^3 v\, v_{\parallel} \delta f_j. \tag{73}$$

In general this extension will add the parameter β to the arguments in equation (70), but if we consider short wavelength turbulence such that $k_{\perp}\rho_i \gg 1$ when the ions will take up a Boltzmann distribution,

$$\chi \sim \frac{\rho_e^2 v_{\text{the}} L_s}{L_n^2} F\left(\frac{\beta_e L_s^2}{L_n^2}, \eta_e\right) \tag{74}$$

where we set $T_e = T_i$ for simplicity. Demanding a finite result as $B \rightarrow \infty$ (i.e. $\rho_e, \beta \rightarrow 0$), the β dependence of F is determined and

$$\chi \sim \frac{c^2}{\omega_{pe}^2} \frac{v_{\text{the}}}{L_s} F(\eta_e) \tag{75}$$

reminiscent of the results of OHKAWA (1978) and PARAIL and POGUTSE (1980). In the limit $\eta_e \gg 1$ it can be shown further that $F(\eta_e) \rightarrow$ constant, inconsistent with the result of GUZDAR *et al.* (1986) for the η_e-mode, but agreeing with that of ROZHANSKII

(1981). Related results for the limit of magnetized ions when χ is a function of $v_{\text{the}}/v_A \equiv \sqrt{\beta m_i/m_e}$ appear in CONNOR and TAYLOR (1985).

Returning to electrostatic turbulence we consider the effect of collisions which also introduce a further parameter into the argument of the function F in equation (68). In particular we focus on dissipative trapped electron turbulence, thought to play an important role in Tokamak confinement, but which has spawned a variety of theoretical calculations for χ. The conventional model describes the ions and passing electrons by the collisionless gyro-kinetic equation, equation (67), but trapped electrons are affected by collisions. The turbulence is assumed to satisfy $\partial/\partial t \ll v_{\text{the}}/qR$ so that passing electrons are adiabatic and follow a Boltzmann distribution $\bar{g}_P = 0$. Provided collisions are sufficiently rare, $\nu_{\text{eff}} \equiv \nu/\varepsilon \ll \varepsilon^{1/2} v_{\text{the}}/qR$, (i.e. the effective collision frequency for the small angle scattering of trapped particles is lower than their bounce frequency) the trapped particles satisfy a bounce-averaged Fokker-Planck equation. If we further assume that $\langle\omega_D\rangle$ and $\partial/\partial t \ll \nu_{\text{eff}}$, where $\langle\omega_D\rangle$ is the bounce-averaged magnetic drift, then the trapped electron distribution function g_T satisfies

$$-\frac{\nu_e}{2\varepsilon}\frac{\partial^2}{\partial(k^2)^2}\langle\bar{g}_T\rangle = \frac{eF_M}{T_e}\frac{\partial}{\partial t}\langle\overline{\delta\varphi}\rangle - \frac{1}{B}\mathbf{n}\times\nabla\langle\overline{\delta\varphi}\rangle\cdot\nabla F_M. \tag{76}$$

Here $k^2 = [1 - \lambda B_0(1-\varepsilon)]/2\varepsilon$ is related to the pitch angle λ and the bounce-average operation is defined by

$$\langle A\rangle = (\textstyle\int \mathrm{d}\theta A/|v_\parallel|)/(\textstyle\int \mathrm{d}\theta/|v_\parallel|) \tag{77}$$

where $v_\parallel^2 = 2\kappa[1 - \lambda B_0(1 - \varepsilon\cos\theta)]$ and the integration over θ is between turning points θ_1 and θ_2 (i.e. $v_\parallel(\theta_1) = v_\parallel(\theta_2) = 0$). Because the trapped electrons only occupy a small range $\delta\lambda = 2\varepsilon/B_0$ about $\lambda = 1/B_0$, the Fokker–Planck collision term can be represented by the pitch angle scattering contribution appearing in equation (76).

If we consider the scale invariance properties of equation (67) for the ion distribution $\bar{g}_i$ and equation (76) for the trapped electron distribution $\bar{g}_T$ and the $\varepsilon^{1/2} \ll 1$ limit of the quasi-neutrality equation linking $\bar{g}_i$, $\bar{g}_T$ and $\bar{g}_P$, then (CONNOR and TAYLOR, 1985)

$$\chi = \frac{\rho_i^2 v_{\text{thi}}}{L_n} F(\varepsilon^{3/2} v_{\text{thi}}/\nu_e L_n, \varepsilon_n, q, s, \eta_e, \eta_i, T_e/T_i). \tag{78}$$

Despite the variety of geometrical ratios in equation (78), the crucial n and T dependence only occurs in one of the arguments of F and the B dependence of χ is completely specified. We can again proceed to a "cylindrical" limit $\varepsilon_n \longrightarrow 0$, but holding ε finite, to obtain

$$\chi = \frac{\rho_i^2 v_{\text{thi}}}{L_n} F(\varepsilon^{3/2} v_{\text{thi}}/\nu_e L_n, L_n/L_s) \tag{79}$$

where we suppress dependence on η_e, η_i and T_e/T_i for simplicity. It is now possible to classify a number of turbulence theories in the literature according to the form of the function F. Thus, in the strong turbulence mixing length theory of KADOMTSEV and POGUTSE (1970).

$$F(x, y) \sim x \tag{80}$$

(corresponding to the choice $p = -1$ in Table 5) which differes from the weakly turbulent mode coupling theory of CHEN *et al.* (1977)

$$F(x, y) \sim x^2 \tag{81}$$

($p = -2$ in Table 5). These contrast with the theory of SIMILON and DIAMOND (1984) involving toroidal mode structure when the quantity s appears:

$$F \sim x^{4/3} s^{-7/3} \tag{82}$$

(i.e. $p = 4/3$ in Table 5).

It is clear that scale invariance can provide some valuable information but rarely completely determines the form of χ in these complex gyro-kinetic models.

3. *Ion temperature gradient turbulence*

By employing a simpler fluid description of turbulence we can expect more success in determining the form of thermal diffusivities by scale invariance than in the case of the gyro-kinetic models above. In this subsection we consider the turbulence due to the ion temperature gradient, or η_i, mode. This turbulence is described by the equations of continuity, momentum along the magnetic field and energy balance for the ions and a Boltzmann distribution for the electrons. We consider fluctuations $\tilde{p}_i$ about the mean ion pressure P_{0i}, $\tilde{\varphi}$ the electrostatic potential and $\tilde{v}_\parallel$ the parallel ion velocity. In a cylindrical limit they satisfy the equations (HORTON *et al.*, 1980)

$$\left\{\begin{aligned}(1 - \nabla_\perp^2)\frac{\partial \varphi}{\partial \tau} = &-\frac{\partial \varphi}{\partial y} - \nabla_\parallel v + [\varphi + p, \nabla_\perp^2 \varphi] \\ &-\kappa \nabla_\perp^2 \frac{\partial \varphi}{\partial y} + \left[\frac{\partial p}{\partial x}, \frac{\partial \varphi}{\partial x}\right] + \left[\frac{\partial p}{\partial y}, \frac{\partial \varphi}{\partial y}\right]\end{aligned}\right\} \tag{83}$$

$$\frac{\partial v}{\partial \tau} = -\nabla_\parallel(\varphi + p) - [\varphi, v] \tag{84}$$

$$\frac{\partial p}{\partial \tau} = -[\varphi, p] - \Gamma \nabla_\parallel v - \kappa \frac{\partial \varphi}{\partial y} \tag{85}$$

where

$$\left[[A, B] = \frac{\partial}{\partial x}A\frac{\partial}{\partial y}B - \frac{\partial}{\partial y}A\frac{\partial}{\partial x}B\right],$$
$$\left[\nabla_{\|} = \frac{\partial}{\partial \xi} + \frac{L_n}{L_s}x\frac{\partial}{\partial y}, \ \Gamma = \frac{\gamma T_i}{T_e}, \ \kappa = \frac{T_i}{T_e}(1 + \eta_i)\right]. \quad (86)$$

The equations have been simplified by suitable normalizations: we have normalized distances to $\rho = (m_i T_e)^{1/2}/eB$, times to L_n/C_s where $C_s = (T_e/m_i)^{1/2}$ and the normalized fluctuations in potential, ion pressure and ion parallel velocity are defined by

$$\tilde{\varphi} = \frac{T_e}{e}\frac{\rho}{L_n}\varphi, \ \tilde{p}_i = P_{0i}\frac{\rho}{L_n}p, \ \tilde{v}_{\|} = C_s\frac{\rho}{L_n}v \quad (87)$$

respectively. The advantage of introducing normalized variables is that it exhibits clearly the remaining dimensionless parameters L_n/L_s, Γ and κ. The invariance principle would of course extract this information itself but the use of normalized variables reduces the effort required, allowing concentration on more subtle invariance properties.

Clearly a turbulent diffusivity must have the form (CONNOR, 1986*a*)

$$\chi_i = \frac{\rho^2 C_s}{L_n}F(L_n/L_s, \kappa, \Gamma). \quad (88)$$

By considering additional limits we can deduce the form of the function F. Thus we can demonstrate *a posteriori* that the limit $\eta_i \equiv L_n/L_T \gg 1$, already required for the validity of the fluid model of the η_i-mode, is consistent with the approximations $\nabla_\perp^2 \gg 1$, $\varphi \ll p$ and the neglect of the parallel compression in equation (85). It then follows from scale invariance that (CONNOR, 1986*a*)

$$\chi_i = \frac{\rho^2 C_s}{L_n}\left(\frac{L_s}{L_n}\right)^{1/2} F\left(\frac{L_n^2}{L_T L_s}\right). \quad (89)$$

Further progress is possible in the two limits $L_n^2/L_T L_s \ll 1$ and $L_n^2/L_T L_s \gg 1$, allowing the approximations $\partial\varphi/\partial\tau \ll \partial\varphi/\partial y$ and $\partial\varphi/\partial\tau \gg \partial\varphi/\partial y$ respectively (consistency can also be verified *a posteriori*). The extra invariant scale transformations permitted by these simplifications allows a determination of the form of F (CONNOR, 1986*a*)

$$\chi_i \sim \frac{\rho^2 C_s}{L_s}\left(\frac{L_n}{L_T}\right)^{3/2}, \qquad \frac{L_n^2}{L_T L_s} \ll 1$$
$$\chi_i \sim \frac{\rho^2 C_s}{L_T^{3/4} L_s^{1/4}}, \qquad \frac{L_n^2}{L_T L_s} \gg 1. \quad (90)$$

These expressions resemble explicit non-linear calculations (LEE and DIAMOND, 1986) but differ in particular by logarithmic factors depending on viscosity. Attainment of a true steady state for η_i-turbulence requires viscous dissipation which is absent from the model equations (83)–(86) (LEE and DIAMOND, 1986; WAKATANI and HASEGAWA, 1984).

HORTON *et al.* (1981) have considered equations describing the toroidal version of the ion temperature gradient turbulence. These involve extra parameters L_n/R and q, where R is the major radius of the torus and q the safety factor. Invariance arguments indicate

$$\chi_i = \frac{\rho^2 C_s}{(L_n L_T)^{1/2}} F\left(\frac{L_n^2}{RL_T}, q, s\right). \tag{91}$$

This is consistent with the result of HORTON *et al.* (1981), but not with those of ROMANELLI *et al.* (1986) and WALTZ *et al.* (1987), where somewhat arbitrary statements are made about $k_\perp \rho$. In the interesting limit $L_n^2/RL_T \gg 1$, χ_i is further constrained to the form

$$\chi_i = \frac{\rho^2 C_s}{L_T^{3/4} R^{1/4}} F(q, s) \tag{92}$$

independent of L_n.

The expressions obtained in this subsection are all seen to be special cases of the general collisionless gyro-kinetic form (63).

4. *Resistive fluid models*

The above discussion of the η_i-turbulence has demonstrated the extra constraints on the form of χ arising from a simplified fluid description. In this subsection we explore the consequences of a resistive fluid model, which encompasses pressure gradient driven turbulence, resistivity gradient driven turbulence and resistive drift wave turbulence. The former has been suggested as an explanation of degradation in confinement with increasing β in Tokamaks (CARRERAS *et al.*, 1983) and pinches (AN *et al.*, 1985; CARRERAS *et al.*, 1987), and the latter two as explanations of transport at the periphery of Tokamaks (GARCIA *et al.*, 1985; TERRY and DIAMOND, 1985).

(a) *Pressure gradient driven turbulence*—In order to discuss pressure gradient turbulence in a Tokamak we use the reduced MHD equations (45)–(50) and decompose the independent variables into a mean and a fluctuating part

$$\psi = \psi_0 + \tilde{\psi}_1, \varphi = \tilde{\varphi}_1, p = p_0 + \tilde{p}_1 \tag{93}$$

where the fluctuations vary on a scale $a\delta$, with $\delta \sim S^{-1/2}$, in the directions x and y perpendicular to $\mathbf{B}$. We introduce dimensionless variables to exploit the information obtained in Section II.

$$\tilde{p}_1 = p_0\delta\tilde{p},\ \tilde{\psi}_1 = \frac{r^2B_0}{q}\delta^2\tilde{\psi},\ \tilde{\varphi}_1 = \frac{r^2B_0^2}{\mu_0^{1/2}qR\rho^{1/2}}\delta^2\tilde{\varphi},\ t = \frac{\mu_0^{1/2}Rq\rho^{1/2}}{B_0}\tau. \tag{94}$$

Employing an ordering which balances linear and non-linear effects equations (45)–(50) become (CONNOR and TAYLOR, 1984)

$$\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}\tau} = -\nabla_{\parallel}\tilde{\varphi} + \frac{1}{S}\nabla_{\perp}^2\tilde{\psi} \tag{95}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\nabla_{\perp}^2\tilde{\varphi} = -\nabla_{\parallel}\nabla_{\perp}^2\tilde{\psi} - [\tilde{\psi}, \nabla_{\perp}^2\tilde{\psi}] + \frac{\beta^*}{q^2}\left[\sin\theta\frac{\partial\tilde{p}}{\partial x} + \cos\theta\frac{\partial\tilde{p}}{\partial y}\right] \tag{96}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{p} + \kappa\frac{\partial\tilde{\varphi}}{\partial y} = 0 \tag{97}$$

where

$$\left[\nabla_{\parallel} = \frac{\partial}{\partial\theta} + sx\frac{\partial}{\partial y}\right]$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}\tau}A = \frac{\partial}{\partial\tau}A - [\tilde{\psi}, A], \nabla_{\perp}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]$$

$$s = \frac{r}{q}\frac{\mathrm{d}q}{\mathrm{d}r},\ \kappa = \frac{r}{p_0}\frac{\mathrm{d}p_0}{\mathrm{d}r} \tag{98}$$

and we have locally defined values of S and a reduced β

$$S = \frac{\mu_0^{1/2}r^2B_0}{\eta\rho^{1/2}Rq} \qquad \beta^* = \frac{2\mu_0Rq^2}{r}\frac{p_0}{B_0^2}. \tag{99}$$

The invariance properties of these equations enable one to deduce a thermal diffusivity

$$\chi = \frac{\eta}{\mu_0}F(\alpha, s) \tag{100}$$

where

$$\alpha = -(2\mu_0Rq^2/B_0^2)\,\mathrm{d}p/\mathrm{d}r \tag{101}$$

is the parameter governing ideal MHD ballooning stability (CONNOR *et al.*, 1978).

However, we can determine the form of F in the electrostatic approximation $\partial/\partial\tau \ll \nabla_{\perp}^2$ if we assume that the dominant non-linear process is convection of pressure, and ignore gradients of the fluctuations in the poloidal y-direction relative to the radial x-direction. It can be shown, *a posteriori*, that these last two assumptions are valid if $\alpha/s \ll 1$. These approximations allow a systematic expansion of the equations in the radial localization of the fluctuations and from the resulting simplified equations we learn

$$F \sim \alpha/s. \tag{102}$$

Furthermore, we can determine the scaling properties of the corresponding fluctuating magnetic fields $\delta\mathbf{B}$ and, using the Rechester–Rosenbluth formulae (RECHESTER and ROSENBLUTH, 1978), deduce the thermal diffusivity due to the stochastic magnetic fields arising from this pressure gradient driven turbulence. In the collisionless limit the Rechester–Rosenbluth electron thermal diffusivity takes the form

$$\chi_e^{st} \sim v_{the}\left(\frac{\delta B}{B}\right)^2 l_c \tag{103}$$

where l_c is a correlation length of the turbulence and v_{the} is the electron thermal velocity.

Application of the invariance principle implies (CONNOR and TAYLOR, 1984)

$$\chi_e^{st} = \frac{\eta}{\mu_0}\left(\frac{m_i\beta}{m_e}\right)^{1/2} F_1(\alpha, s) \tag{104}$$

or, in the limit $\alpha/s \ll 1$,

$$\chi_e^{st} \sim \frac{\eta}{\mu_0}\left(\frac{m_i\beta}{m_e}\right)^{1/2}\left(\frac{\alpha}{s}\right)^{3/2}. \tag{105}$$

Thus, it has been possible in this model of stochastic field transport to introduce additional physics associated with electron thermal motion while still completely constraining the form of χ_e. This result closely resembles that obtained by CARRERAS *et al.* (1983) in their calculation of transport due to resistive ballooning modes.

Not surprisingly similar results can be obtained for resistive interchange turbulence in a cylindrical pinch. Considering a low β pinch we introduce the parameters

$$\delta = \frac{2\mu_0 r}{B^2}\frac{\mathrm{d}p}{\mathrm{d}r}, \sigma = \frac{B_\theta}{B}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{rB_z}{B_\theta}\right) \tag{106}$$

and find that the analogues of equations (100) and (104) are

$$\chi = \frac{\eta}{\mu_0}F(\delta,\ \sigma), \chi_e^{st} = \frac{\eta}{\mu_0}\left(\frac{m_i\beta}{m_e}\right)^{1/2} F_1(\delta, \sigma). \tag{107}$$

If we again assume that convection of pressure is the dominant non-linear process and make the electrostatic approximation, valid when $\delta \ll 1$, then we find

$$\chi \sim \frac{\eta}{\mu_0}\delta/\sigma^2,\ \chi_e^{st} \sim \frac{\eta}{\mu_0}\left(\frac{m_i\beta}{m_e}\right)^{1/2}(\delta/\sigma^2)^{3/2} \tag{108}$$

where the Suydam criterion (SUYDAM, 1958) is $\delta/\sigma^2 = 1/4$.

It is interesting to note that in the pinch we have not assumed $\partial/\partial x \gg \partial/\partial y$. If this further approximation is introduced an extra invariant transformation is allowed but this does not affect the results (108). It corresponds to an arbitrary characteristic wave-number in the y-direction which remains unspecified, but the transport properties are independent of this parameter.

The expressions (108) are similar to those obtained by AN *et al.* (1985), but they differ from later results (CARRERAS *et al.*, 1987) by a logarithmic factor. As in the case of the η_i-turbulence, dissipation at high wave-numbers is necessary to achieve a steady state and this is absent from the present model. However, if this is provided self-consistently by a turbulent viscosity and thermal diffusivity (CARRERAS *et al.*, 1987) the result remains as a special case of equation (107). A different result $\chi \sim (\eta/\mu_0)(m_i\beta/m_e)^{1/2}\delta^{1/2}$ obtained by HAMEIRI and BHATTACHARJEE (1987) would appear to be related to pressure gradient driven tearing modes, rather than the electrostatic interchanges considered above.

(b) *Resistivity gradient driven turbulence*—As a further example we discuss electrostatic, resistivity gradient driven turbulence in cylindrical geometry with a strong axial magnetic field (GARCIA *et al.*, 1985). This depends on fluctuations in resistivity η; if η is decomposed into a background part η_0 and a fluctuating part $\tilde{\eta}_1$, the latter satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\eta}_1 - \chi_{\parallel}\nabla_{\parallel}^2\tilde{\eta}_1 = \frac{1}{rB}\frac{\partial\tilde{\varphi}_1}{\partial\theta}\frac{\mathrm{d}\eta_0}{\mathrm{d}r} \tag{109}$$

in a pure plasma, since $\eta \sim T^{-3/2}$. Here $\chi_{\parallel}$ is the parallel electron thermal diffusivity which dissipates the $\mathbf{E} \times \mathbf{B}$ convection of the background resistivity η_0. The description of the turbulence is completed by the vorticity equation and the parallel Ohm's law including the perturbation in η.

Introducing similar normalizations to equation (94) we obtain the equations

$$\frac{1}{S}\frac{\mathrm{d}}{\mathrm{d}\tau}\nabla_{\perp}^2\tilde{\varphi} = -\nabla_{\parallel}^2\tilde{\varphi} - J\nabla_{\parallel}\tilde{\eta} \tag{110}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{\eta} - K\nabla_{\parallel}^2\tilde{\eta} = \frac{1}{L}\frac{\partial\tilde{\varphi}}{\partial y} \tag{111}$$

where $\tilde{\eta} = \tilde{\eta}_1/\eta_0$ and

$$K = \frac{\chi_{\parallel}\rho^{1/2}\mu_0^{1/2}}{RqB}, J = \frac{Rq\mu_0 J_z}{B}, \frac{1}{L} = \frac{r}{\eta_0}\frac{\mathrm{d}\eta_0}{\mathrm{d}r} \tag{112}$$

with J_z the axial current.

The invariance properties of these equations imply a diffusion coefficient (CONNOR, 1986*b*).

$$D = \frac{r^2}{s^2} \frac{B}{Rq\rho^{1/2}\mu_0^{1/2}} \frac{1}{S^{5/3}} \left(\frac{J}{L}\right)^{4/3} F(K^{3/2}S^{1/2}Ls^3/J). \tag{113}$$

If we make the customary assumption that $\partial/\partial x \gg \partial/\partial y$, then an extra constraint emerges which implies

$$F \sim K^{1/2}S^{1/6}L^{1/3}s/J^{1/3} \tag{114}$$

or

$$D \sim \eta^{3/2}\chi_{\parallel}^{1/2}\frac{J_z}{sB^2}Rq\rho^{1/2}\frac{\mathrm{d}\ln\eta_0}{\mathrm{d}r}. \tag{115}$$

The consistency condition for $\partial/\partial x \gg \partial/\partial y$ is, *a posteriori*,

$$J \ll S^{1/2}K^{3/2}ls^3. \tag{116}$$

This result (115) differs from a detailed analytic and numerical treatment of resistivity gradient turbulence (GARCIA *et al.*, 1985) because that was expressed in terms of $(m^2)_{\mathrm{rms}}$, a mean value of the square of the poloidal mode number over the turbulent spectrum. The value used resulted from a numerical calculation with specific physical parameters. However $(m^2)_{\mathrm{rms}}$ has scaling properties itself (CONNOR, 1986*b*).

$$(m^2)_{\mathrm{rms}} \sim S^{1/2}J/K^{5/2}ls^3 \tag{117}$$

and, when this is used, the results are reconciled.

In an impure plasma, when $\eta \sim Z_{\mathrm{eff}}/T^{3/2}$, it is possible to have resistivity gradient turbulence excited by a gradient in Z_{eff}. In the simplest model $\tilde{\eta}_1$ is determined by $\mathbf{E} \times \mathbf{B}$ convection of Z_{eff}. In this case $(m^2)_{\mathrm{rms}}$ cannot be controlled by parallel transport as in equation (117) and one must consider $\partial/\partial x \sim \partial/\partial y$, with the result

$$D \sim \eta\left(\frac{Rq}{sB}\right)^2[\rho^{1/2}\eta J_z^2(\mathrm{d}(\ln Z_{\mathrm{eff}})/\mathrm{d}r)^2]^{2/3}. \tag{118}$$

However, if the motion of ions and impurities along the field lines are included, one finds that their mutual friction can play a similar role to that of $\chi_{\parallel}$ in the pure plasma case and can be described by the substitution (RUTHERFORD, 1981)

$$\chi_{\parallel} \rightarrow \chi_Z \sim v_{\mathrm{thi}}^2/Z_{\mathrm{eff}}\nu_i. \tag{119}$$

Provided that the condition equivalent to (116) is satisfied one can then obtain a result analogous to (115) (CONNOR, 1986*b*).

(c) *Resistive drift wave turbulence*—Finally we discuss electrostatic resistive drift waves in cylindrical geometry (YAGI *et al.*, 1987). These are described by combining the vorticity equation with an Ohm's Law containing the Hall term. In an isothermal limit this takes the form

$$\frac{\mathbf{B}\cdot\nabla}{B}\left(\tilde{\varphi}_1 + \frac{T}{en_0}\tilde{n}_1\right) + \eta_0 \tilde{j}_{\|_1} = 0 \tag{120}$$

where the density perturbation $\tilde{n}_1$ is determined by the electron continuity equation

$$\frac{d\tilde{n}_1}{dt} + \frac{1}{n_0 e}\frac{\mathbf{B}\cdot\nabla}{B}\tilde{j}_{\|_1} = \frac{1}{rB}\frac{\partial\tilde{\varphi}_1}{\partial\theta}\frac{dn_0}{dr}. \tag{121}$$

Again, introducing normalizations similar to equation (94), the turbulence is described by the equations

$$\frac{1}{S}\frac{d}{d\tau}\nabla_\perp^2\tilde{\varphi} = \nabla_\|^2(\tilde{\varphi} + \Omega\tilde{n}) \tag{122}$$

$$\frac{d}{d\tau}\tilde{n} - K\frac{\partial\tilde{\varphi}}{\partial y} = \Gamma\frac{1}{d\tau}\nabla_\perp^2\tilde{\varphi} \tag{123}$$

where $\tilde{n} = \tilde{n}_1/n_0$ and

$$\Omega = \frac{TqR\rho^{1/2}\mu_0{}^{1/2}}{er^2B^2}, \Gamma = \frac{\rho^{1/2}}{\mu_0^{1/2}neqR}, K = \frac{r}{n_0}\frac{dn_0}{dr}. \tag{124}$$

Scale invariance implies

$$D = \left(\frac{K}{s}\right)^2 \frac{1}{\Gamma S}\frac{T}{eB}F(s^2 S\Gamma^{3/2}\Omega^{1/2}/K) \tag{125}$$

or, more physically,

$$D = \left(\frac{K}{s}\right)^2 D_{pc} F\left(\frac{K}{s^2}\frac{v_e Rq}{v_{\text{thi}}}\frac{Rq}{r}\right) \tag{126}$$

where $D_{pc} = v_e\rho_e^2(Rq/r)^2$ is the pseudo-classical diffusion coefficient (YOSHIKAWA, 1970; ARTSIMOVICH, 1970). In the limit $K \ll s^2 S\Gamma^{3/2}\Omega^{1/2}$ it can be demonstrated that $\partial/\partial x \gg \partial/\partial y$ and this simplification leads to the determination of F as a constant. This limit resembles results quoted in DÜCHS *et al.* (1977) and WALTZ *et al.* (1987) for collisional drift wave turbulence and corresponds to the choice of index $p = 1$ in Table 5.

IV. DISCUSSION AND CONCLUSIONS

This review has been concerned with the consequences of the Invariance Principle for the scaling of confinement times and turbulent transport coefficients. It has been shown that for any particular model for anomalous transport the principle constrains the form of these scalings in a manner characteristic of that model. These constraints arise from the invariance properties under scale transformations of the equations describing the model. The simpler the equations the more invariant transformations they possess and the tighter the constraints.

This invariance approach is intimately related to dimensional analysis—whether it differs is a matter of semantics. Dimensional analysis is often regarded as a heuristic technique involving a list of significant dimensionless ratios such as β, the collisionality parameter ν_*, etc., thought to characterize a plasma. However, these concepts arose from examining the equations governing plasmas and writing them in a dimensionless form. Such a form is arrived at by suitably normalizing, or scaling, the variables in the equations and the dimensionless ratios emerge as parameters in the equations. The invariance approach offers a systematic procedure to achieve this situation but, more importantly, can extract hidden scaling properties that the usual approach might fail to expose as in Subsections II.3(e) or III.3 for example. The invariance approach could be regarded as the correct approach to dimensional analysis! However, it may well be more efficient to undertake a preliminary attempt at writing the equations in a dimensionless form before applying the invariance technique, as was done in the cases of η_i-turbulence and resistive fluid turbulence.

In Section II we classified the possible mechanisms underlying anomalous transport rather generally. We considered situations in which a particular set of basic equations such as the electrostatic Vlasov equation and quasi-neutrality, or the resistive MHD equations, etc. were adequate to describe the confinement of the plasma as a whole. (This assumed that additional physical processes such as atomic physics phenomena at the plasma boundary did not have a significant influence on the overall confinement properties, say by affecting the boundary conditions of the governing plasma physics equations.) We were then able to identify the characteristic constraints that each such model imposed on the form of the global confinement time regarded as a function of n, T, B and a.

All theoretical calculations of confinement must, of course, conform to the appropriate constraints and this allows a useful classification and concise description of the large number of such calculations in the literature. More constructively, examination of empirical data on the scaling of confinement times against this theoretical framework of constraints provides a technique for associating the observed anomalous transport with one of these models. Conversely, if one were confident that a particular model is appropriate, these constraints limit the number of parameters whose scaling properties need to be determined empirically. The constraints also allow one to construct families of similar confinement systems whose properties can be deduced from the behaviour of one member. These properties generally indicate the advantages of high-field devices for fusion reactors.

If geometrical approximations are introduced into the governing equations, say a large aspect ratio Tokamak limit, it may be possible to discover scale transformations involving separate scalings of lengths in different directions. It is then possible to find constraints on the scaling of confinement with geometrical ratios for a particular plasma model—the reduced resistive MHD model provided an example. Similarly if invariant transformations involving scaling of dimensionless numbers such as charge Z or atomic mass A exist, then the constraints can involve this type of quantity also.

Global confinement time scalings may well result from the interaction of different plasma physics models operating in different spatial regions of a confinement device and will therefore only exhibit the weaker constraints appropriate to a model encompassing all these processes. It is more informative to examine the implications of the invariance principle for the scaling properties of local turbulent transport coefficients—assuming that anomalous transport can indeed be described in such a way—and this is the topic of Section III.

Because it is possible to separate the roles of turbulent fluctuations (which vary on a microscopic scale such as the Larmor radius, collisionless skin depth or resistive layer width, for example) from the background equilibrium quantities in the governing equations, more invariant scale transformations are possible and hence turbulent transport coefficients are more tightly constrained. At the same time, however, a variety of local gradient lengths L_n, L_T, L_s, etc. are introduced which allows considerable freedom in the possible forms of these transport coefficients. In the global problem all these lengths are determined self consistently in terms of the size of the device by the self same equations of the model considered. This has the corollary that if overall confinement is determined by such a local transport coefficient, then the extra constraints imposed by the locality assumption do manifest themselves in the global confinement. Indeed if the collisionless electrostatic gyro-kinetic description is appropriate the confinement scaling is completely determined up to a dependence on a/R and q.

In Subsection III.1 we have classified the constraints on the dependence of the thermal diffusivities on local collisionality and β for the conventional gyro-kinetic plasma models and this is a valuable framework against which to compare corresponding empirical scaling laws with a view to identifying the cause of anomalous transport. In order to address the dependence of the transport coefficients on local gradient lengths and geometrical ratios, we considered geometrical approximations—large aspect ratio and cylindrical limits—and examined more closely specified forms of turbulence thought to be relevant to confinement in Tokamaks and pinches. It should be emphasized, however, that the more specific the model the less general is the result deduced from the invariance principle, so a judicious balance must be sought!

In the case of the drift-wave turbulence considered in Subsection III.2, we find there is often considerable arbitrariness in the form of associated thermal diffusivities, but the constraints do specify some dependences and serve to classify the many theoretical calculations available. For simpler models, such as η_i-mode turbulence (Subsection III.3) and resistive fluid turbulence (Subsection III.4), their form can be quite tightly constrained and sometimes completely determined up to an overall constant.

In this role the invariance approach is a valuable addition to the theoretical techniques for obtaining transport coefficients, supplementing analytic and numerical non-linear calculations. It is more robust than many specific analytic non-linear calculations since it relies only on invariance properties of the governing equations, not on the method (and approximations) invoked to solve them. Thus it provides a measure of the generality of such specific results. Insofar as many such calculations are in reality only precise to an overall constant of order unity, the failure of the invariance technique to determine such constants is not particularly to its detriment.

Indeed vigorous discussion of the merits of detailed non-linear calculations yielding apparently precise predictions emphasizes this point (KROMMES, 1986; TERRY and DIAMOND, 1986). On the other hand the technique does assume that the equations considered have a non-trivial, steady state turbulent solution, whereas specific non-linear calculations can demonstrate whether one exists. Thus analytic turbulence calculations of ion temperature gradient and pressure gradient resistive fluid turbulence have indicated the need for additional dissipation in the model equations to achieve a steady state, although this was found to introduce only a weak logarithmic dependence on the extra dissipative processes. This application of the invariance approach should not be confused with heuristic methods like the mixing length estimate (DIAMOND and CARRERAS, 1987). It is, in fact, a rigorous method for obtaining all the constraints on a transport coefficient calculated from a given set of equations. If one is tempted to oversimplify those equations in order to completely determine the form of the transport coefficient, then it is that oversimplification that should be criticized, not the scale invariance technique.

A potentially important role of the invariance technique is in conjunction with numerical computations of turbulence and transport. Because of cost, these can only be carried out at a limited number of parameter values but this might be sufficient to establish "empirically" the functional dependences allowed by the invariance approach. The invariant scaling of the numerically derived mean wavenumber for resistivity gradient turbulence discussed in Subsection III.4(b) provides an example of such a procedure.

Finally we note that the invariance principle can be used to discuss not only turbulent transport coefficients but also characteristics of the turbulence itself—amplitudes, correlation lengths and correlation times (CONNOR and TAYLOR, 1984; CONNOR, 1986*a*; CONNOR, 1986*b*).

In conclusion the constraints imposed by the invariance principle offer a valuable theoretical framework for discussing empirical confinement scaling laws and a useful technique to classify and supplement analytic and numerical calculations of turbulent transport coefficients.

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REFERENCES

AN Z. G., DIAMOND P. H., HAZELTINE R. D., LEBOEUF J. N., ROSENBLUTH M. N., SYDORA R. D., TAJIMA T., CARRERAS B. A., GARCIA L., HENDER T. C., HICKS H. R., HOLMES J. A., LYNCH V. E. and STRAUSS H. R. (1985) *Plasma Physics and Controlled Fusion Research* (Proc. 10th Int. Conf., London, 1984), Vol. 2, p. 231. IAEA Vienna.

ARTSIMOVICH L. A. (1970) *Pis'ma Zh. éksp. teor. Fiz.* **13**, 101 [*Soviet Phys. JETP Lett.* **13**, 70 (1971)].

BRAGINSKII S. I. (1965) *Reviews of Plasma Physics* (edited by M. A. LEONTOVICH), Vol. 1, p. 205. Consultants Bureau, New York.

CARRERAS B. A., DIAMOND P. H., MURAKAMI M., DUNLAP J. L., BELL J. D., HICKS H. R., HOLMES J. A., LAZARUS E. A., PARÉ V. K., SIMILON P., THOMAS C. E. and WIELAND R. M. (1983) *Phys. Rev. Lett.* **50**, 503.

CARRERAS B. A., GARCIA L., CHARLTON L. A., HOLMES J. A., LYNCH V. E. and DIAMOND P. H. (1987) Plasma Physics and Controlled Nuclear Fusion Research (Proc. of 11th Int. Conf., Kyoto, 1986), Vol. 2, p. 615. IAEA Vienna.

CHEN LIU, BERGER R. L., LOMINADZE J. G., ROSENBLUTH M. N. and RUTHERFORD P. H. (1977) *Phys. Rev. Lett.* **39**, 754.

CONNOR J. W. (1986*a*) *Nucl. Fusion* **26**, 193.

CONNOR J. W. (1986*b*) *Nucl. Fusion* **26**, 517.

CONNOR J. W., HASTIE R. J. and TAYLOR J. B. (1978) *Phys. Rev. Lett.* **40**, 396.
CONNOR J. W. and TAYLOR J. B. (1977) *Nucl. Fusion* **17**, 1047.
CONNOR J. W. and TAYLOR J. B. (1979) *Heating in Toroidal Plasmas* (Proc. of Joint Varenna Grenoble Symposium 1978), Vol. 1, p. 1. Pergamon Press.
CONNOR J. W. and TAYLOR J. B. (1984) *Physics Fluids* **27**, 2676.
CONNOR J. W. and TAYLOR J. B. (1985) *Plasma Physics and Controlled Nuclear Fusion Research* (Proc. of 10th Int. Conf., London, 1984), Vol. 2, p. 13. IAEA Vienna.
DIAMOND P. H. and CARRERAS B. A. (1987) *Comments Plasma Phys. Contr. Fusion* **10**, 271.
DÜCHS D. F., POST D. E. and RUTHERFORD P. H. (1977) *Nucl. Fusion* **17**, 565.
FRIEMAN E. A. and CHEN LIU (1982) *Physics Fluids* **25**, 502.
FURTH H. P. (1977) *Nucl. Fusion* **15**, 487.
GARCIA L., DIAMOND P. H., CARRERAS B. A. and CALLEN J. D. (1985) *Physics Fluids* **28**, 2147.
GOLDSTON R. J. (1984) *Plasma Phys. Contr. Fusion* **26**, 87.
GUZDAR P. N., LIU C. S., DONG J. Q. and LEE Y. C. (1986) *Phys. Rev. Lett.* **57**, 2818.
HAGAN W. K. and FRIEMAN E. A. (1986) *Physics Fluids* **29**, 3635.
HAMEIRI E. and BHATTACHARJEE (1987), submitted for publication in *Physics Fluids.*
HIRSHMAN S. P. and MOLVIG K. (1979) *Phys. Rev. Lett.* **42**, 648.
HORTON W., CHOI DUK-IN and TANG W. M. (1981) *Physics Fluids* **24**, 1077.
HORTON W., ESTES R. D. and BISKAMP D. (1980) *Plasma Physics* **22**, 663.
KADOMTSEV B. B. (1961) *Zh. éksp. teor. Fiz.* **40**, 328 [*Soviet Phys. JETP* **13**, 223 (1961)].
KADOMTSEV B. B. (1975) *Fiz. Plasmy* **1**, 531 [*Soviet J. Plasma Phys.* **1**, 295 (1975)].
KADOMTSEV B. B. and POGUTSE O. P. (1970) *Reviews of Plasma Physics* (edited by M. A. LEONTOVICH), Vol. 5, p. 249. Consultants Bureau, New York.
KADOMTSEV B. B. and POGUTSE O. P. (1985) *Plasma Physics and Controlled Nuclear Fusion Research* (Proc. of the 10th Int. Conf., London, 1984), Vol. 2, p. 69. IAEA Vienna.
KROMMES J. A. (1984) *Handbook of Plasma Physics* (edited by R. N. SUDAN and A. A. GALEEV), Chap. 5.5, p. 183, North-Holland Publishing Company, Amsterdam.
KROMMES J. A. (1986) *Physics Fluids* **29**, 2756.
LACINA J. (1971) *Plasma Physics* **13**, 303.
LAMB H. (1932) *Hydrodynamics*, Sixth Edition, Cambridge University Press.
LANDAU L. D. and LIFSHITZ E. M. (1959) *Fluid Mechanics*, Vol. 6 of *Course of Theoretical Physics*. Pergamon Press.
LEE G. S. and DIAMOND P. H. (1986) *Physics Fluids* **29**, 3291.
LIEWER P. C. (1985) *Nucl. Fusion* **25**, 543.
MEREZHKIN V. G. and MUKHOVATOV V. S. (1981) *Pis'ma Zh. éksp. teor. Fiz.* **33**, 463 [*Soviet Phys. JETP Lett.* **33**, 446 (1981)].
OHKAWA T. (1978) *Phys. Lett. A* **67**, 35.
PARAIL V. V. and POGUTSE O. P. (1980) *Pis'ma Zh. éksp. teor. Fiz.* **32**, 408 [*Soviet Phys. JETP Lett.* **32**, 384 (1980)].
RECHESTER A. B. and ROSENBLUTH M. N. (1978) *Phys. Rev. Lett.* **40**, 38.
ROMANELLI F., TANG W. M. and WHITE R. B. (1986) *Nucl. Fusion* **26**, 1515.
ROSE D. J. and CLARKE JR. M. (1961), *Plasmas and Controlled Fusion*, MIT Press, Cambridge, Mass.
ROZHANSKII V. A. (1981) *Pis'ma Zh. éksp. teor. Fiz.* **34**, 60 [*Soviet Phys. JETP Lett.* **34**, 56 (1981)].
RUTHERFORD P. H. (1981) *Physics of Plasmas Close to Thermonuclear Conditions*, Vol. 1, p. 143. Pergamon Press, Oxford.
SIMILON P. L. and DIAMOND P. H. (1984) *Physics Fluids* **27**, 916.
STRAUSS H. R. (1983) *Nucl. Fusion* **23**, 649.
SUYDAM B. R. (1958) *Proc. of Second Int. Conf. on the Peaceful Uses of Atomic Energy* (United Nations, Geneva, 1958), Vol. 31, p. 157.
TERRY P. W. and DIAMOND P. H. (1985) *Physics Fluids* **28**, 1419.
TERRY P. W. and DIAMOND P. H. (1986) *Physics Fluids* **29**, 2758.
WAKATANI M. and HASEGAWA A. (1984) *Physics Fluids* **27**, 611.
WALTZ R. E., DOMINGUEZ R. R., WONG S. K., DIAMOND P. H., LEE G. S., HAHM T. S. and MATTOR N. (1987) *Plasma Physics and Controlled Nuclear Fusion Research* (Proc. 11th Int. Conf., Kyoto, 1986), Vol.1, p. 345. IAEA Vienna.
YAGI M., WAKATANI M. and HASEGAWA A. (1987) *J. Phys. Soc. Japan* **56**, 973.
YOSHIKAWA S. (1970) *Phys. Rev. Lett.* **25**, 353.