Homework #4

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Question 1

Textbook Problem 4.2
Define a random variable X by the following procedure. Draw a card from a standard deck of playing cards. If the card is knave, queen, or king, then $X = 11$. If the card is an ace, then $X = 1$; otherwise, X is the number of the card (i.e. two through ten). Now define a second random variable Y by the following procedure. When you evaluate X, you look at the color of the card. If the card is red, then $Y = X - 1$; otherwise, $Y = X + 1$.

(a) What is $P(\{X \leq 2\})$?

X can only be an Ace or 2 to fit this condition, which is two out of the 13 options, so $\boxed{\frac{2}{13}}$

(b) What is $P(\{X \geq 10\})$?

X can only be a 10, Jack, Queen, or King, so we get $\boxed{\frac{4}{13}}$

(c) What is $P(\{X \geq Y\})$?

$X \geq Y$ only when $Y = X - 1$, which is any time X is red, so we have a probability of $\boxed{\frac{1}{2}}$

(d) What is the probability distribution of $Y - X$?
The two possible options for $Y - X$ are -1 and 1, depending on if $Y = X - 1$ or $Y = X + 1$. Since X has a probability of 1/2 of being red or not red, both options occur with a probability of 1/2.

(e) What is $P(\{Y \geq 12\})$?
The maximum value of Y is 12, which can only be achieved if $X = 11$ and is not red (ie black). There are 3 possible face cards out of the 13 different cards that would give $X = 11$, and since only half are black and these events are independent, we get a final probability of $\left(\frac{3}{13}\right)\left(\frac{1}{2}\right) = \boxed{\frac{3}{26}}$

Question 2

Textbook problem 4.6. Express your answers using choose and/or summation notation. There's no need to compute numerical answers.
Magic the Gathering is a popular card game. Cards can be land cards, or other cards. We consider a game with two players. Each player has a deck of 40 cards. Each player shuffles their deck, then deals seven cards, called their hand. The rest of each player's deck is called their library. Assume that player one has 10 land cards in their deck and player two has 20. Write L_1 for the number of lands in player one's hand and L_2 for the number of lands in player two's hand. Write L_t for the number of lands in the top 10 cards of player one's library.

(a) Write $S = L_1 + L_2$. What is $P(\{S = 0\})$?
$S = 0$ only occurs when both Player one and two have 0 land cards in their hand. For player one, this means we choose 7 cards from their 30 not land cards out of all possible combinations of 7 from their 40 card deck. This gives a probability of $\frac{\binom{30}{7}}{\binom{40}{7}}$. The same logic applies for Player 2 but with 20 non land cards, giving a probability of $\frac{\binom{20}{7}}{\binom{40}{7}}$. Since both events are independent, the event they both happen (ie their joint probability) is their product, giving a final probability of $\boxed{\frac{\binom{30}{7}\binom{20}{7}}{\binom{40}{7}^2}}$

(b) Write $D = L_1 - L_2$. What is $P(\{D = 0\})$?
$D = 0$ only when $L_1 = L_2$. This can be split up into 7 cases, when both have 0 land in their hand, 1, 2, 3, ..., 7 lands in their hand. The probability of having i lands in your hand when you have k lands in your deck is $\frac{\binom{k}{i}\binom{40-k}{7-i}}{\binom{40}{7}}$. Combining this into a summation for all 7 cases and considering both players, we get:

$$\boxed{\sum_{i=0}^{7} \frac{\binom{10}{i}\binom{30}{7-i}\binom{20}{i}\binom{20}{7-i}}{\binom{40}{7}^2}}$$

(c) What is the probability distribution for L_1?
We can follow the same logic as above. There are 8 possibilities for $L_1 : L_1 = \{0, 1, \ldots, 7\}$. For each possibility i, we can write its probability as $\frac{\binom{10}{i}\binom{30}{7-i}}{\binom{40}{7}}$

(d) Write out the probability distribution for $P(L_1|L_t = 10)$
There are only 10 lands in Player one's deck, and we know by the L_t condition that all 10 are still in the deck after player one draws his hand. Thus, the only option for player one's hand is $L_1 = 0$ with probability 1.

(e) Write out the probability distribution for $P(L_1|L_t = 5)$
Here we know 5 lands are still in the deck, so there are 5 available to choose from for our hand. This gives 6 possibilities: $L_1 = \{0, 1, \ldots, 5\}$. When considering their probabilities, we can view the top 10 cards of the deck as part of the hand, just separating the lands in the actual hand and the top of the deck.

For each i, we can do the same thing as above to determine its probability, but adjust for 5 lands, giving a probability of:

$$\frac{\binom{5}{i}\binom{30}{7-i}\binom{10-i}{5}}{\binom{40}{7}\binom{33}{10}}$$

Notice that this is the same as before, but with an additional term in the numerator and denominator for choosing 5 lands of the remaining $10 - i$ lands for the top of the deck, and choosing 10 cards of the random 33 left for all possibilities of the top of the deck.

Question 3

	x=1	x=2	x=3	x=5
y=1	0.05	0.1	0.05	0.1
y=2	0.05	0.09	0.01	0.05
y=4	0.1	0.21	0.04	0.15

Determine the value of $E[X], E[Y], E[(3X-2Y)^2]$
$E[X] = \sum_{x\in D} xP(X=x) = 1(.05+.05+.1)+2(.1+.09+.21)+3(.05+.01+.04)+5(.1+.05+.15) = 1(.2)+2(.4)+3(.1)+5(.3) = \boxed{2.8}$
$E[Y] = \sum_{y\in D} yP(Y=y) = 1(.05+.1+.05+.1)+2(.05+.09+.01+.05)+4(.1+.21+.04+.15) = 1(.3)+2(.2)+4(.5) = \boxed{2.7}$
$E[(3X-2Y)^2] = E[9X^2-12XY+4Y^2] = 9E[X^2]-12E[XY]+4[Y^2]$

$$E[X^2] = 1^2(.05+.05+.1)+2^2(.1+.09+.21)+3^2(.05+.01+.04)+5^2(.1+.05+.15) = 10.2$$

$$E[Y^2] = 1^2(.05+.1+.05+.1)+2^2(.05+.09+.01+.05)+4^2(.1+.21+.04+.15) = 9.1$$

$$E[XY] = 1(1)(.05)+1(2)(.1)+(1)(3)(.05)+(1)(5)(.1)+(2)(1)(.05)+$$

$$(2)(2)(.09)+(2)(3)(.01)+(2)(5)(.05)+4(1)(.1)+4(2)(.21)+4(3)(.04)+4(5)(.15) = 7.48$$

$$9(10.2)-12(7.48)+4(9.1) =$$

Question 4

A student invests on three risky projects. He will spend \$100 on each project. Let S1, S2 and S3 be the events that the first, second and third project respectively return some money (not the net return counting the expenditure). Suppose all events are mutually independent and we use p_1, p_2, p_3 to represent the corresponding probability that the project is successful. $p_1 = P(S1) = 0.2$ and $S1$ returns \$1000, $p_2 = P(S2) = 0.3$ and $S2$ returns \$500, $p_3 = P(S3) = 0.6$ and $S3$ returns \$100. Let W be the random variable that represents the student's net returns in dollars.

(a) Calculate the possible values for W and the corresponding probabilities.
There are three projects with binary values - 1 if they succeed, 0 if they don't, so we can represent each possibility as the string $s_1 s_2 s_3$. Note that for W, we have spend a total of \$300 regardless of project success. Also note that all events are mutually independent, so $P(S_i \cap S_j) = P(S_i)P(S_j)$

Project Successes	**W**	**Probability**
000	-300	$P(S1^C \cap S2^C \cap S3^C) = (.8)(.7)(.4) = .224$
001	100 - 300 = -200	$P(S1^C \cap S2^C \cap S3) = (.8)(.7)(.6) = .336$
010	500 - 300 = 200	$P(S1^C \cap S2 \cap S3^C) = (.8)(.3)(.4) = .096$
011	500 + 100 - 300 = 300	$P(S1^C \cap S2 \cap S3) = (.8)(.3)(.6) = .144$
100	1000 - 300 = 700	$P(S1 \cap S2^C \cap S3^C) = (.2)(.7)(.4) = .056$
101	1000 + 100 - 300 = 800	$P(S1 \cap S2^C \cap S3) = (.2)(.7)(.6) = .084$
110	1000 + 500 - 300 = 1200	$P(S1 \cap S2 \cap S3^C) = (.2)(.3)(.4) = .024$
111	1000 + 500 + 100 - 300 = 1300	$P(S1 \cap S2 \cap S3) = (.2)(.3)(.6) = .036$

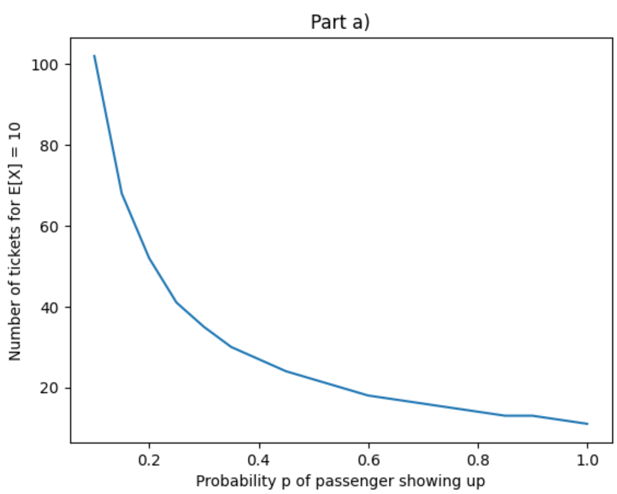
(b) What is the expected value of the winnings, $E[W]$?

$$.224(-300)+.336(-200)+.096(200)+.144(300)+.056(700)+.084(800)+.024(1200)+.036(1300) = \boxed{110}$$

Question 5

Textbook problem 4.23. Do both parts of this problem. For part (b) of this problem, simulate with $N = 10^5$ trials and submit your answer as a graph (plot) of a function of p where $p = 0, 0.1, 0.2 \ldots 0.9, 1$. Submit your code for this problem by pasting it in your pdf, not in a separate file.
An airline company runs a flight that has 10 seats. Each passenger who buys a ticket has a probability p of turning up for the flight. The gender of the passengers is not known until they turn up for a flight, and women buy tickets with the same frequency that men do. The pilot is eccentric, and will not fly unless at least two women turn up.

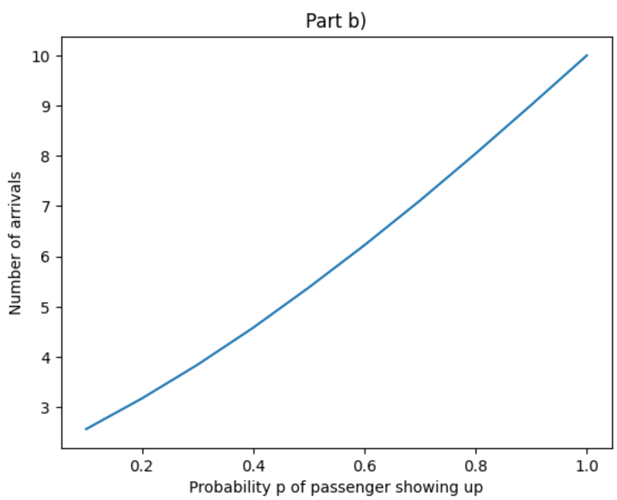
(a) How many tickets should the airline sell to ensure that the expected number of passengers that turn up is greater than 10?
This depends on the probability p, so I created a simulation iterating through different values of p from 0 to 1. For each p, I would increase the number of tickets sold until 10 people arrived, and ran this iteration 10000 times to get an expected number of arrivals for each p and n (number of tickets sold). I then graphed the different values of p and how many tickets needed to be sold n to get an expected amount of more than 10 people to arrive.




```
trials = 10000
p_vals = []
n_vals = []
for p in np.arange(.1, 1.05, .05):
    n = 10
    p_vals.append(p)
    expected_val = 0
    while (expected_val < 10):
        for t in range(trials):
            coming = 0
            for i in range(n):
                if (random.random() <= p):
                    coming += 1
            expected_val += coming
        expected_val /= trials
        n += 1
    print(expected_val)
    print(n)
    n_vals.append(n)
```

```
plt.plot(p_vals, n_vals)
plt.title("Part a)")
plt.xlabel("Probability p of passenger showing up")
plt.ylabel("Number of tickets for E[X] = 10")
plt.show()
```

(b) The airline sells 10 tickets. What is the expected number of passengers on the aircraft, given that it flies? (i.e. that at least two women turn up). Estimate this value with a simulation.
This value also depends on p, so I created a similar simulation iterating through p, and this time running 10^5 iterations of generating 10 passengers with each p probability of arriving and .5 probability of being male or female. Then, for each trial. I would ensure at least two females arrived, then counted how many passengers in total arrived. I took the average of all 10^5 trials to get an expected value of how many passengers showed up for each probability p given at least two women arrived.




```
p_vals = []
e_vals = []
for p in np.arange(0, 1.1, 0.1):
    p_vals.append(p)
    expected_val = []
    for i in range(10**5):
        arrive = [random.random() <= p for _ in range(10)]
        gender = [random.randint(0, 1) for _ in range(10)]
        women_arrived = 0
        for w in range(len(arrive)):
            if (arrive[w] and gender[w] == 1):
                women_arrived += 1
        if (women_arrived >= 2):
            expected_val.append(np.sum(arrive))
    print(np.average(expected_val))
    e_vals.append(np.average(expected_val))
plt.plot(p_vals, e_vals)
plt.title("Part b)")
plt.xlabel("Probability p of passenger showing up")
plt.ylabel("Number of arrivals")
plt.show()
```

Question 6

Textbook Problem 4.14