Homework 3

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Question 1

Suppose that among the 232 students registered in CS 361, there are 180 students that have taken both a calculus and a linear algebra class in the past, and there are 4 students that have taken neither.

(a) How many students have taken at least one of those two math classes in the past?

$$232 - 4 = 228 \text{ students}$$

(b) Now suppose furthermore that the number of students that have not taken linear algebra is 3 times the number of students that have not taken calculus. How many students have taken a linear algebra class in the past?

Let's define x as the students who have not taken linear algebra and y as the number of students that have not taken calculus. $232 - 180 - 4 = 48$. There are 48 students who have only taken one of the two math classes. We can get a system of equations since $x + y = 48$ according to our own logic, and $3(y + 4) = x + 4$ according to the information given about the ratio between $y + 4$ and $x + 4$. This simplifies down to:

$$\begin{cases} x - 3y = 8 \\ x + \ \ y = 48 \end{cases}$$

We can solve this system to get $x = 38$ and $y = 10$. This means that 38 students have taken calculus and not taken linear algebra, but we can't forget about the 4 students that have taken neither. Therefore, the number of students that have not taken linear algebra is $38 + 4 = 42$.

Question 2 (Textbook 3.19)

You shuffle a standard deck of cards, then draw four cards. Express your answers as products of fractions or using choose notation. There's no need to compute numerical answers.

(a) What is the probability all four are the same suit?

$$\frac{4 * \binom{13}{4}}{\binom{54}{4}}$$

(b) What is the probability all four are red?

$$\frac{\binom{26}{4}}{\binom{54}{4}}$$

(c) What is the probability each has a different suit?

$$\frac{39}{51} * \frac{26}{50} * \frac{13}{49}$$

Question 3 (Textbook 3.27)

(Note that each part asks for a single probability. Express your answers using choose and/or summation notation. There's no need to compute numerical answers.)

Magic the Gathering is a popular card game. Cards can be land cards, or other cards. We consider a game with two players. Each player has a deck of 40 cards. Each player shuffles their deck, then deals seven cards, called their hand.

(a) Assume that player one has 10 land cards in their deck and player two has 20. With what probability will *each* player have four lands in their hand?

$$\frac{\binom{10}{4} * \binom{30}{3}}{\binom{40}{7}} * \frac{\binom{20}{4} * \binom{20}{3}}{\binom{40}{7}}$$

(b) Assume that player one has 10 land cards in their deck and player two has 20. With what probability will player one have two lands and player two have three lands in hand?

$$\frac{\binom{10}{2} * \binom{30}{5}}{\binom{40}{7}} * \frac{\binom{20}{3} * \binom{20}{4}}{\binom{40}{7}}$$

(c) Assume that player one has 10 land cards in their deck and player two has 20. With what probability will player two have more lands in hand than player one?

Let's say that the sets X_n are defined where $1 <= n <= 7$ represent the ways of player two getting n land cards. We could calculate $|X_n| = \frac{\binom{20}{7-n} * \binom{20}{n}}{\binom{40}{7}}$

Let's say that the sets Y_n are defined where $0 <= n <= 6$ represents the ways of player one getting n land cards. We could calculate $|Y_n| = \frac{\binom{30}{7-n} * \binom{10}{n}}{\binom{40}{7}}$.

The probability of player two having more lands than player one would look like this:

$$Y_0(\sum_{n=1}^{7}|X_n|) + Y_1(\sum_{n=2}^{7}|X_n|) + Y_2(\sum_{n=3}^{7}|X_n|) + Y_3(\sum_{n=4}^{7}|X_n|) + Y_4(\sum_{n=5}^{7}|X_n|) + Y_5(\sum_{n=6}^{7}|X_n|) + Y_6(\sum_{n=7}^{7}|X_n|)$$

You could also write it out like the following:

$$\frac{\binom{30}{7}}{\binom{40}{7}} * (\frac{\binom{20}{6} * \binom{20}{1}}{\binom{40}{7}} + \frac{\binom{20}{5} * \binom{20}{2}}{\binom{40}{7}} + \frac{\binom{20}{4} * \binom{20}{3}}{\binom{40}{7}} + \frac{\binom{20}{3} * \binom{20}{4}}{\binom{40}{7}} + \frac{\binom{20}{2} * \binom{20}{5}}{\binom{40}{7}} + \frac{\binom{20}{1} * \binom{20}{6}}{\binom{40}{7}} + \frac{\binom{20}{7}}{\binom{40}{7}})$$
$$+ \frac{\binom{30}{6} * \binom{10}{1}}{\binom{40}{7}} * (\frac{\binom{20}{5} * \binom{20}{2}}{\binom{40}{7}} + \frac{\binom{20}{4} * \binom{20}{3}}{\binom{40}{7}} + \frac{\binom{20}{3} * \binom{20}{4}}{\binom{40}{7}} + \frac{\binom{20}{2} * \binom{20}{5}}{\binom{40}{7}} + \frac{\binom{20}{1} * \binom{20}{6}}{\binom{40}{7}} + \frac{\binom{20}{7}}{\binom{40}{7}})$$
$$+ \frac{\binom{30}{5} * \binom{10}{2}}{\binom{40}{7}} * (\frac{\binom{20}{4} * \binom{20}{3}}{\binom{40}{7}} + \frac{\binom{20}{3} * \binom{20}{4}}{\binom{40}{7}} + \frac{\binom{20}{2} * \binom{20}{5}}{\binom{40}{7}} + \frac{\binom{20}{1} * \binom{20}{6}}{\binom{40}{7}} + \frac{\binom{20}{7}}{\binom{40}{7}})$$
$$+ \frac{\binom{30}{4} * \binom{10}{3}}{\binom{40}{7}} * (\frac{\binom{20}{3} * \binom{20}{4}}{\binom{40}{7}} + \frac{\binom{20}{2} * \binom{20}{5}}{\binom{40}{7}} + \frac{\binom{20}{1} * \binom{20}{6}}{\binom{40}{7}} + \frac{\binom{20}{7}}{\binom{40}{7}})$$
$$+ \frac{\binom{30}{3} * \binom{10}{4}}{\binom{40}{7}} * (\frac{\binom{20}{2} * \binom{20}{5}}{\binom{40}{7}} + \frac{\binom{20}{1} * \binom{20}{6}}{\binom{40}{7}} + \frac{\binom{20}{7}}{\binom{40}{7}})$$
$$+ \frac{\binom{30}{2} * \binom{10}{5}}{\binom{40}{7}} * (\frac{\binom{20}{1} * \binom{20}{6}}{\binom{40}{7}} + \frac{\binom{20}{7}}{\binom{40}{7}})$$
$$+ \frac{\binom{30}{1} * \binom{10}{6}}{\binom{40}{7}} * (\frac{\binom{20}{7}}{\binom{40}{7}})$$

Question 4

Suppose Sarah a student likes a local restaurant, she always orders from there one dish on both Fri and Sat. On Fri, the restaurant offers 7 different dishes on the menu among which the first five are \$10 each and the next two are \$15 each. On Sat, the restaurant also offers 7 dishes where the first \$10 item from Friday item is replaced with a \$20 item, and the remaining six dishes are the same as those on Fri. Suppose Sarah orders independently each time regardless of the day and she doesn't order on other days.

(a) Write down the sample space of unique pair of orders if Sarah orders one dish both on Fri and Sat.

In total, there are 8 different dishes available to Sarah across the two days: 7 dishes on the first day and then 1 new dish on the next day and 6 of the previous dishes. So the sample space would include all pairs of these 8 unique dishes, which can be counted as having $\binom{8}{2}$ different combos. This assumes that ordering the a pair of dishes across the two days is the same as ordering the two dishes in reverse order. Additionally, we also have to include in this sample space the 6 additional cases where she gets the same dish both days. There are 6 dishes repeated, so the total number of unique pairs of order would be

$$\binom{8}{2} + 6 = 34$$

(b) Now let E_1 be the event that Sarah orders a \$10 dish in the week, E_2 be the event that Sarah orders a \$15 dish in the week, and E_3 be the event that Sarah orders a \$20 dish in the week.

(i) Are E_1 and E_2 independent? Justify your answer with calculations.

To check if E_1 and E_2 are independent, we need to verify if:

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

Sarah can choose from 7 dishes on Friday. There are 5 dishes priced at \$10. Therefore, the probability that Sarah orders a \$10 dish on Friday is: $P(E_1) = \frac{5}{7}$

Similarly, there are 2 dishes priced at \$15 on Friday. Hence, the probability that Sarah orders a \$15 dish is: $P(E_2) = \frac{2}{7}$

The events E_1 and E_2 cannot occur simultaneously, as Sarah cannot order both a \$10 dish and a \$15 dish on the same day. Therefore:

$$P(E_1 \cap E_2) = 0$$

For E_1 and E_2 to be independent, we require $P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$. Let's compute the right-hand side:

$$P(E_1) \cdot P(E_2) = \frac{5}{7} \cdot \frac{2}{7} = \frac{10}{49}$$

Since $P(E_1 \cap E_2) = 0 \neq \frac{10}{49}$, the events E_1 and E_2 are not independent.

(ii) Are E_2 and E_3 independent? Justify your answer with calculations.

Now, let's check if E_2 and E_3 are independent. We need to verify if:

$$P(E_2 \cap E_3) = P(E_2) \cdot P(E_3)$$

On Saturday, there is 1 dish priced at \$20, and the remaining 6 dishes are the same as on Friday. Thus, the probability that Sarah orders the \$20 dish on Saturday is$P(E_3) = \frac{1}{7}$

Since Sarah orders independently on Friday and Saturday, the probability that she orders a \$15 dish on Friday and a \$20 dish on Saturday is simply the product of the individual probabilities:

$$P(E_2 \cap E_3) = P(E_2) \cdot P(E_3) = \frac{2}{7} \cdot \frac{1}{7} = \frac{2}{49}$$

Since $P(E_2 \cap E_3) = P(E_2) \cdot P(E_3)$, the events E_2 and E_3 are independent.

Question 5 (Textbook 3.44)

A student takes a multiple choice test. Each question has N answers. If the student knows the answer to a question, the student gives the right answer, and otherwise guesses uniformly and at random. The student knows the answer to 70% of the questions. Write K for the event a student knows the answer to a question and R for the event the student answers the question correctly.

(a) What is $P(K)$?

This is the probability that the student knows the answer to a given question on the test. Since the student knows the answer to 70% of the questions, this would be 0.7

(b) What is $P(R|K)$?

The probability that the student gives the correct answer would be 1 given that the student knows the correct answer.

(c) What is $P(K|R)$, as a function of N?

Using Baye's, we can find that

$$P(K|R) = \frac{P(R|K)P(K)}{P(R)} = \frac{1(0.7)}{P(R)}$$

P(R) can be represented as knowing and not knowing the answer. If the student does not know the answer they would have 1/N probability of getting it right.

$$P(R) = P(R|K) * P(K) + P(R|K^C) * P(K^C) = 1(0.7) + \frac{0.3}{N} = 0.7 + \frac{0.3}{N}$$

The student knows the answer 70% of the time. They don't know the answer 30% of the time. Therefore,

$$P(K|R) = \frac{0.7}{0.7 + \frac{0.3}{N}}$$

(d) What values of N will ensure that $P(K|R) > 99\%$? Setting this equal to 0.99, we get:

$$0.99 = \frac{0.7}{0.7 + \frac{0.3}{N}}$$

$$0.99(0.7 + \frac{0.3}{N}) = 0.7$$

$$0.693 + \frac{0.297}{N} = 0.7$$

$$\frac{0.297}{N} = .007 \implies N > 42$$

Extra Credit

Let A, B and C are events in a sample space while A and B are disjoint events. We know $P(A) = 2P(B)$, $P(C|A) = 2/7$, and $P(C|B) = 4/7$. What is $P(C|(A \cup B))$?

$$P(C|B) = \frac{P(C \cap B)}{P(B)} = \frac{P(C \cap B)}{x} = 4/7$$

$$P(C|A) = \frac{P(C \cap A)}{P(A)} = \frac{P(C \cap A)}{2x} = 2/7$$

$$P(C|(A \cup B) = \frac{P(C \cap (A \cup B))}{P(A \cup B)} = \frac{P(C \cap (A \cup B))}{3x}$$

$P(C \cap (A \cup B)) = P(C \cap A) + P(C \cap B) = \frac{4}{7}x + \frac{2}{7} * 2x = \frac{8}{7}x$
So therefore,

$$P(C|(A \cup B) = \frac{\frac{8}{7}x}{3x} = \frac{8}{21}$$