# 5.1 The Natural Logarithmic Function: Differentiation

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## 1 Use a graphing utility to evaluate the logarithm by (a) using the natural logarithm key and (b) using the integration capabilities to evaluate the integral $\int_1^x (\frac{1}{t})dt$

(a)

$$\ln 8.3 \approx 2.1163 \tag{1}$$

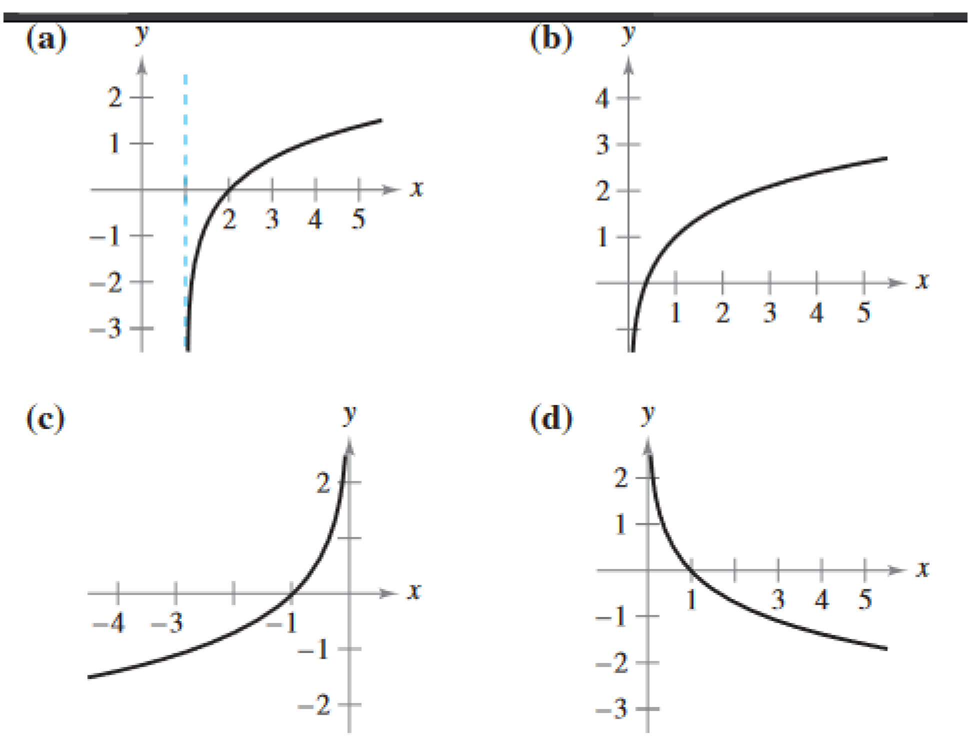
$$\int_1^{8.3} \frac{1}{t} dt \approx 2.1163 \tag{2}$$

(b)

$$\ln 0.6 \approx -0.5108 \tag{1}$$

$$\int_1^{0.6} \frac{1}{t} dt \approx -0.5108 \tag{2}$$

## 2 Match the function with its graph.




8.

$$f(x) = -\ln x \tag{1}$$

Matches (d) since the graph reflects the x-axis

10.

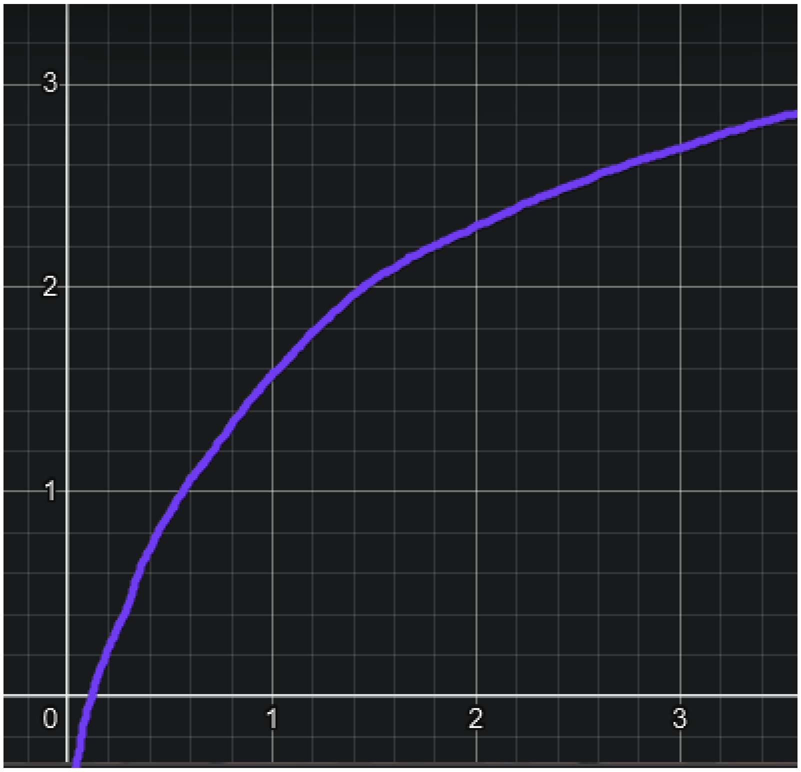
$$f(x) = -\ln(-x) \tag{1}$$

Matches (c) since the graph reflects both axes.

## 3 Sketch the graph of the function and state its domain.

12.$f(x) = -2\ln x$
Domain: $x > 0$


16.$g(x) = 2 + \ln x$
Domain: $x > 0$




## 4 In Exercises 19 and 20, use the properties of logarithms to approximate the indicated logarithms, given that $\ln 2 \approx 0.6931$ and $\ln 3 \approx 1.0986$.

20.

(a)

$$\begin{aligned}
&\ln 0.25 && (1)\\
&= \ln \frac{1}{4} && (2)\\
&= \ln 1 - \ln 2 \ln 2 \approx -1.3682 && (3)
\end{aligned}$$

(b)

$$\begin{aligned}
&\ln 24 && (1)\\
&= 3\ln 2 + \ln 3 \approx 3.1779 && (2)
\end{aligned}$$

(c)

$$\begin{aligned}
&\ln \sqrt[3]{12} && (1)\\
&= \frac{1}{3}(2\ln 2 + \ln 3) \approx 0.8283 && (2)
\end{aligned}$$

(d)

$$\begin{aligned}
&\ln \frac{1}{72} && (1)\\
&= \ln 1 - (3\ln 2 + 2\ln 3) \approx -4.2765 && (2)
\end{aligned}$$

## 5 Use the properties of logarithms to expand the logarithmic expression.

22.

$$\ln\sqrt{x^5} \tag{1}$$
$$= \ln x^{\frac{5}{2}} = \frac{5}{2}\ln x \tag{2}$$

26.

$$\ln\sqrt{a-1} \tag{1}$$
$$= \ln(a-1)^{\frac{1}{2}} = \left(\frac{1}{2}\right)\ln(a-1) \tag{2}$$

28.

$$\ln 3e^2 \tag{1}$$
$$= \ln 3 + 2\ln e = 2 + \ln 3 \tag{2}$$

## 6 Write the expression as a logarithm of a single quantity.

32.

$$3\ln x + 2\ln y - 4\ln z \tag{1}$$
$$= \ln x^3 + \ln y^2 - \ln z^4 \tag{2}$$
$$= \ln\frac{x^3y^2}{z^4} \tag{3}$$

34.

$$2(\ln x - \ln(x+1) - \ln(x-1)) \tag{1}$$
$$= 2\ln\frac{x}{(x+1)(x-1)} \tag{2}$$
$$= \ln\left(\frac{x}{x^2-1}\right)^2 \tag{3}$$

## 7 (a) Verify that $f = g$ by using a graphing utility to graph $f$ and $g$ in the same viewing window and (b) verify that $f = g$ algebraically.

38.

$$f(x) = \ln\sqrt{x(x^2+1)} \tag{1}$$
$$= \frac{1}{2}\ln(x(x^2+1)) \tag{2}$$
$$= \frac{1}{2}(\ln x + \ln(x^2+1)) = g(x) \tag{3}$$

## 8 Find the limit.

42.

$$\lim_{x\to 5^+} \ln \frac{x}{\sqrt{x-4}} \tag{1}$$
$$= \ln 5 \approx 1.61 \tag{2}$$

## 9 Find an equation of the tangent line to the graph of the logarithmic function at the point (1, 0).

44.

$$y = \ln x^{\frac{3}{2}} = \frac{3}{2}\ln x \tag{1}$$
$$y' = \frac{3}{2x} \tag{2}$$

The slope at (1, 0) is $\frac{3}{2}$, and the tangent line is:

$$y - 0 = \frac{3}{2}(x-1) \tag{1}$$
$$y = \frac{3}{2}x - \frac{3}{2} \tag{2}$$

## 10 Find the derivative of the function.

48.

$$f(x) = \ln(x-1) \tag{1}$$
$$f'(x) = \frac{1}{x-1} \tag{2}$$

52.

$$y = x^2 \ln x \tag{1}$$
$$y' = x^2\left(\frac{1}{x}\right) + 2x\ln x \tag{2}$$
$$= x + 2x\ln x \tag{3}$$
$$= x(1 + 2\ln x) \tag{4}$$

56.

$$y = \ln(t(t^2+3)^3) \tag{1}$$
$$= \ln t + 3\ln(t^2+3) \tag{2}$$
$$y' = \frac{1}{t} + \frac{2}{t^2+3}(2t) \tag{3}$$
$$= \frac{1}{t} + \frac{6t}{t^2+3} \tag{4}$$

60.

$$
\begin{aligned}
h(t) &= \frac{\ln t}{t} && (1)\\
h'(t) &= \frac{t\left(\frac{1}{t}\right) - \ln t}{t^2} && (2)\\
&= \frac{1 - \ln t}{t^2} && (3)
\end{aligned}
$$

64.

$$
\begin{aligned}
y &= \ln \sqrt[3]{\frac{x-1}{x+1}} && (1)\\
&= \frac{1}{3}(\ln(x-1) - \ln(x+1)) && (2)\\
y' &= \frac{1}{3}\left(\frac{1}{x-1} - \frac{1}{x+1}\right) && (3)\\
&= \frac{1}{3} \cdot \frac{2}{x^2-1} && (4)\\
&= \frac{2}{3(x^2-1)} && (5)
\end{aligned}
$$

68.

$$
\begin{aligned}
y &= \frac{-\sqrt{x^2+4}}{x} - \frac{1}{4}\ln\left(\frac{2+\sqrt{x^2+4}}{x}\right) && (1)\\
&= \frac{-\sqrt{x^2+4}}{2x^2} - \frac{1}{4}\ln(2+\sqrt[root]{x^2+4}) + \frac{1}{4}\ln x && (2)\\
\frac{dy}{dx} &= \frac{-2x^2(\frac{x}{\sqrt{x^2+4}}) + 4x\sqrt{x^2+4}}{4x^4} - \frac{1}{4}\left(\frac{1}{2+\sqrt{x^2+4}}\right)\left(\frac{x}{\sqrt{x^2+4}}\right) + \frac{1}{4}\ln x && (3)\\
&= \frac{-1}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} - \frac{1}{4} \cdot \frac{(2-\sqrt{x^2+4})}{-x^2}\left(\frac{x}{\sqrt{x^2+4}}\right) + \frac{1}{4x} && (4)\\
&= \frac{-1 + \left(\frac{1}{2}\right)(2-\sqrt{x^2+4})}{2x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x} && (5)\\
&= \frac{-\sqrt{x^2+4}}{4x\sqrt{x^2+4}} + \frac{\sqrt{x^2+4}}{x^3} + \frac{1}{4x} && (6)\\
&= \frac{\sqrt{x^2+4}}{x^3} && (7)
\end{aligned}
$$

72.

$$
\begin{aligned}
y &= \ln|\sec x + \tan x| && (1)\\
\frac{dy}{dx} &= \frac{\sec x \tan + \sec^2 x}{\sec x + \tan x} && (2)\\
&= \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} && (3)\\
&= \sec x && (4)
\end{aligned}
$$

76.

$$g(x) = \int_1^{\ln x} (t^2+3)dt \tag{1}$$
$$g'(x) = ((\ln x)^2+3)\frac{d}{dx}(\ln x) \tag{2}$$
$$= \frac{(\ln x)^2+3}{x} \tag{3}$$

# 11 Find an equation of the tangent line to the graph of at the given point.

78.

$$f(x) = 3x^2 - \ln x,\ (0,4) \tag{1}$$
$$\frac{dy}{dx} = -2x - \frac{1}{\left(\frac{1}{2}\right)x+1}\left(\frac{1}{2}\right) \tag{2}$$
$$= -2x - \frac{1}{x+2} \tag{3}$$
$$\frac{dy}{dx} = -\frac{1}{2} \text{ when } x=0 \tag{4}$$
$$y-4 = -\frac{1}{2}(x-0) \tag{5}$$
$$y = -\frac{1}{2}x+4 \tag{6}$$

82.

$$f(x) = \frac{1}{2}x\ln(x^2),\ (-1,0) \tag{1}$$
$$f'(x) = \frac{1}{2}\ln(x^2) + \frac{1}{2}\left(\frac{2x}{x^2}\right) \tag{2}$$
$$= \frac{1}{2}\ln(x^2)+1 \tag{3}$$
$$f'(-1) = 1 \tag{4}$$
$$y-0 = 1(x+1) \tag{5}$$
$$y = x+1 \tag{6}$$

# 12 Use implicit differentiation to find $\frac{dy}{dx}$

84.

$$\ln(xy)+5x = 30 \tag{1}$$
$$\ln x + \ln y + 5x = 30 \tag{2}$$
$$\frac{1}{x} + \frac{1}{y}\cdot\frac{dy}{dx} + 5 = 0 \tag{3}$$
$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x} - 5 \tag{4}$$
$$\frac{dy}{dx} = -\frac{y}{x} - 5y \tag{5}$$
$$= -\left(\frac{y+5xy}{x}\right) \tag{6}$$

86.

$$4xy + \ln x^2 y = 7 \tag{1}$$
$$4xy + 2\ln x + \ln y = 7 \tag{2}$$
$$4xy' + 4y + \frac{2}{x} + \frac{1}{y}y' = 0 \tag{3}$$
$$\left(4x + \frac{1}{y}\right) y' = -4y - \frac{2}{x} \tag{4}$$
$$y' = \frac{-4y - \frac{2}{x}}{4x + \frac{1}{y}} \tag{5}$$
$$= \frac{-4xy^2 - 2y}{4x^2 y + x} \tag{6}$$

## 13 Locate any relative extrema and inflection points.

92.

$$y = x - \ln x \tag{1}$$
$$\text{Domain: } x > 0 \tag{2}$$
$$y' = 1 - \frac{1}{x} = 0 \text{ when } x = 1 \tag{3}$$
$$y'' = \frac{1}{x^2} > 0 \tag{4}$$

Relative minimum at (1, 1).

96.

$$y = x^2 \ln \frac{x}{4}, \text{ Domain: } x > 0 \tag{1}$$
$$y' = x^2 \left(\frac{1}{x}\right) + 2x \ln \frac{x}{4} = x\left(1 + 2\ln \frac{x}{4}\right) = 0 \text{ when:} \tag{2}$$
$$-1 = 2\ln \frac{x}{4} \Rightarrow \ln \frac{x}{4} = -\frac{1}{2} \Rightarrow x = 4e^{-\frac{1}{2}} \tag{3}$$
$$y'' = 1 + 2\ln \frac{x}{4} + 2x\left(\frac{1}{x}\right) = 3 + 2\ln \frac{x}{4} \tag{4}$$
$$y'' = 0 \text{ when } x = 4e^{-\frac{3}{2}} \tag{5}$$

Relative minimum at $(4e^{-\frac{1}{2}},\ -8e^{-1})$, and point of inflection at $(4e^{-\frac{3}{2}},\ -24e^{-3})$.

## 14 Use logarithmic differentiation to find $\frac{dy}{dx}$.

102.

$$y = \sqrt{x^2(x+1)(x+2)},\ \ x > 0 \tag{1}$$
$$y^2 = x^2(x+1)(x+2) \tag{2}$$
$$2\ln y = 2\ln x + \ln(x+1) + \ln(x+2) \tag{3}$$
$$\frac{2}{y}\cdot\frac{dy}{dx} = \frac{2}{x} + \frac{1}{x+1} + \frac{1}{x+2} \tag{4}$$
$$\frac{dy}{dx} = \frac{y}{2}\left(\frac{2}{x} + \frac{1}{x+1} + \frac{1}{x+2}\right) \tag{5}$$
$$\frac{dy}{dx} = \frac{\sqrt{x^2(x+1)(x+2)}}{2}\left(\frac{2(x+1)(x+2) + x(x+2) + x(x+1)}{x(x+1)(x+2)}\right) \tag{6}$$
$$= \frac{4x^2 + 9x + 4}{2\sqrt{(x+1)(x+2)}} \tag{7}$$

## 15 Determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

111. $\ln(x+25) = \ln x + \ln 25$
False because $\ln x + \ln 25 = \ln(25x) \neq \ln(x+25)$.

112. $\ln xy = \ln x \ln y$
False because the property actually is $\ln xy = \ln x + \ln y$.

113. If $y = \ln \pi$, then $y' = \frac{1}{\pi}$.
False. Since $\pi$ is a constant, $\frac{d}{dx}(\ln \pi) = 0$

114. If $y = \ln e$, then $y' = 1$.
False because if $y = \ln e = 1$, then $y' = 0$.

## 16 Word problems

116. The relationship between the number of decibels $\beta$ and the intensity of a sound $I$ in watts per centimeter squared is $\beta = 10\log_{10}\left(\frac{1}{10^{-16}}\right)$.
Use the properties of logarithms to write the formula in simpler form, and determine the number of decibels of a sound with an intensity of $10^{-10}$ watt per square centimeter.

$$
\begin{aligned}
\beta &= 10\log_{10}\left(\frac{I}{10^{-16}}\right) && (1)\\
&= \frac{10}{\ln 10}(\ln I + 16\ln 10) && (2)\\
&= 160 + 10\log_{10} I && (3)\\
B(10^{-10}) &= \frac{10}{\ln 10}(\ln 10^{-10} + 16\ln 10) && (4)\\
&= \frac{10}{\ln 10}(-10\ln 10 + 16\ln 10) && (5)\\
&= \frac{10}{\ln 10}(6\ln 10) && (6)\\
&= 60 \text{ decibels} && (7)
\end{aligned}
$$