4.1 Antiderivatives and Indefinite Integration

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1 Verify that the statement by showing that the derivative of the right side equals the integrand of the left side.

2.

$$\int \left(8x^3 + \frac{1}{2x^2}\right) dx = 2x^4 - \frac{1}{2}x^{-1} + C \quad (1)$$
$$\frac{d}{dx}\left(2x^4 - \frac{1}{2x} + C\right) = \frac{d}{dx}\left(2x^4 - \frac{1}{2}x^{-1} + C\right) \quad (2)$$
$$= 8x^3 + \frac{1}{2}x^{-2} \quad (3)$$
$$= 8x^3 + \frac{1}{2x^2} \quad (4)$$

2 Find the general solution of the differential equation and check the result by differentiation.

6.

$$\frac{dr}{d\theta} = \pi \quad (1)$$
$$r = \pi\theta + C \because \frac{d}{d\theta}[\pi\theta + C] = \pi \quad (2)$$
$$(3)$$

8.

$$\frac{dy}{dx} = 2x^{-3} \quad (1)$$
$$y = \frac{2x^{-2}}{-2} + C \quad (2)$$
$$= \frac{-1}{x^2} + C \because \frac{d}{dx}\left(\frac{-1}{x^2} + C\right) = 2x^{-3} \quad (3)$$

3 Complete the table.

Original integral	Rewrite	Integrate	Simplify
10. $\int \frac{1}{x\sqrt{x}} dx$	$\frac{1}{4}\int x^{-2} dx$	$\frac{1}{4}\frac{x^{-1}}{-1} + C$	$-\frac{1}{4x} + C$
14. $\int \frac{1}{(3x)^2} dx$	$\frac{1}{9}\int x^{-2} dx$	$\frac{1}{9}\left(\frac{x^{-1}}{-1}\right) + C$	$\frac{-1}{9x} + C$

4 Find the indefinite integral and check the result by differentiation.

16.

$$\int (13 - x)dx \tag{1}$$
$$= 13x - \frac{x^2}{2} + C \because \frac{d}{dx}\left(13x - \frac{x^2}{2} + C\right) = 13 - x \tag{2}$$

20.

$$\int (x^3 - 10x - 3)dx \tag{1}$$
$$= \frac{x^4}{4} - 5x^2 - 3x + C \because \frac{d}{dx}\left(\frac{x^4}{4} - 5x^2 - 3x + C\right) = x^3 - 10x - 3 \tag{2}$$

24.

$$\int (\sqrt[4]{x^3} + 1)dx \tag{1}$$
$$= \int (x^{3/4} + 1)dx \tag{2}$$
$$= \frac{4}{7}x^{7/4} + x + C \because \frac{d}{dx}\left(\frac{4}{7}x^{7/4} + x + C\right) = x^{3/4} + 1 = \sqrt[4]{x^3} + 1 \tag{3}$$

28.

$$\int \frac{x^2 + 2x - 3}{x^4}dx \tag{1}$$
$$= \int (x^{-2} + 2x^{-3} - 3x^{-4})dx \tag{2}$$
$$= \frac{x^{-1}}{-1} + \frac{2x^{-2}}{-2} - \frac{3x^{-3}}{-3} + C \tag{3}$$
$$= \frac{-1}{x} - \frac{1}{x^2} + \frac{1}{x^3} + C \because \frac{d}{dx}\left(\frac{-}{1}x - \frac{1}{x^2} + \frac{1}{x^3} + C\right) = x^{-2} + 2x^{-3} - 3x^{-4} = \frac{x^2 + 2x - 3}{x^4} \tag{4}$$

32.

$$\int (1 + 3t)t^2 dt \tag{1}$$
$$= \int (t^2 + 3t^3)dt \tag{2}$$
$$= \frac{1}{3}t^3 + \frac{3}{4}t^4 + C \because \frac{d}{dt}\left(\frac{1}{3}t^3 + \frac{3}{4}t^4 + C\right) = t^2 + 3t^3 = (1 + 3t)t^2 \tag{3}$$

36.

$$\int (t^2 - \cos t) dt \tag{1}$$
$$= \frac{t^3}{3} - \sin t + C \because \frac{d}{dt}\left(\frac{t^3}{3} - \sin t + C\right) = t^2 - \cos t \tag{2}$$

40.

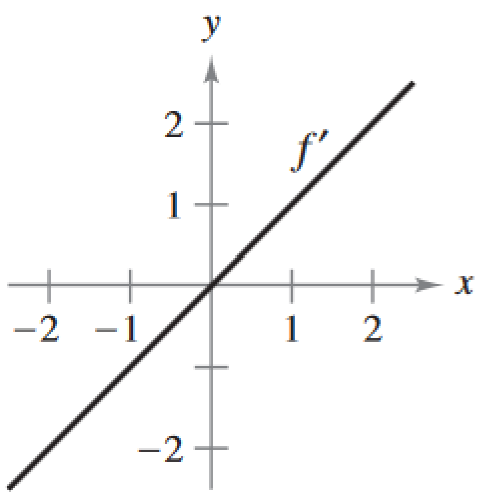
$$\int \sec y(\tan y - \sec y) dy \tag{1}$$
$$= \int (\sec \tan y - \sec^2 y) dy \tag{2}$$
$$= \sec y - \tan y + C \because \frac{d}{dy}(\sec y - \tan y + C) = \sec y \tan y - \sec^2 y = \sec y(\tan y - \sec y) \tag{3}$$

44.

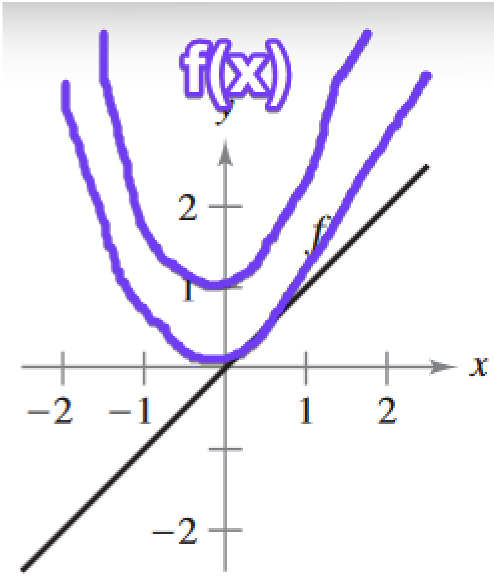
$$\int \frac{\sin x}{1 - \sin^2 x} dx \tag{1}$$
$$= \int \frac{\sin x}{\cos^2 x} dx \tag{2}$$
$$= \int \tan x \sec x dx \tag{3}$$
$$= \sec x + C \tag{4}$$

5 The graph of the derivative of a function is given. Sketch the graphs of two functions that have the given derivative.

46.



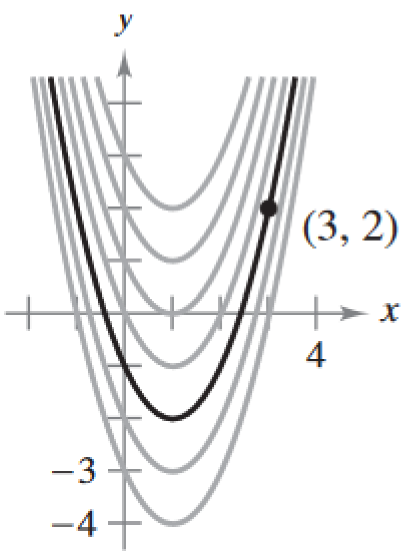
$$f'(x) = x \tag{1}$$
$$f(x) = \frac{x^2}{2} + C \therefore f(x) = \frac{x^2}{2} + 1 \; \frac{x^2}{2} \tag{2}$$



6 Find the equation of y. given the derivative and the indicated point on the curve.

50.

$$\frac{dy}{dx} = 2(x-1)$$



$$= 2x - 2,\ (3,2) \tag{3}$$

$$y = \int 2(x-1)dx \tag{4}$$

$$= x^2 - 2x + C \tag{5}$$

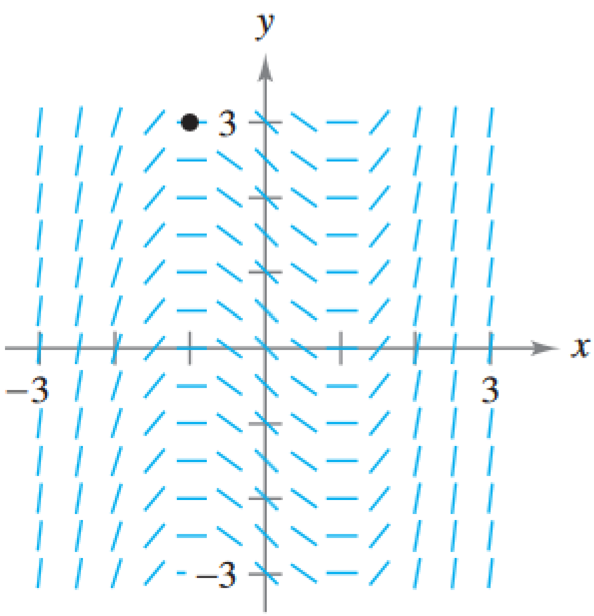
$$2 = (3)^2 - 2(3) + C \Rightarrow C = -1 \tag{6}$$

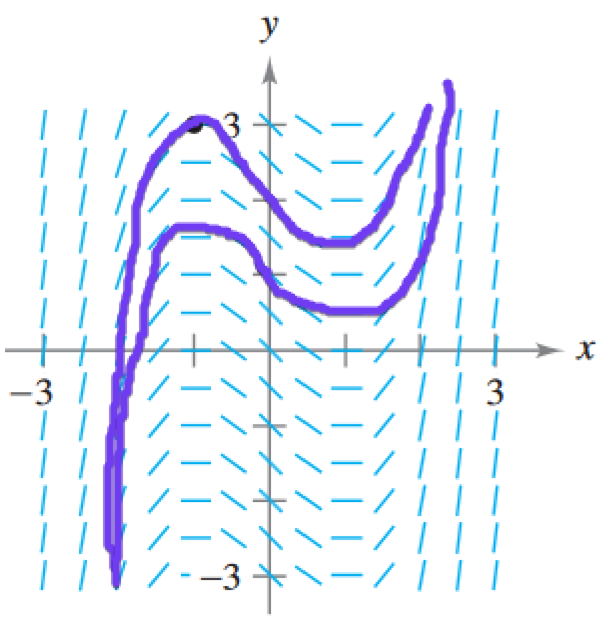
$$y = x^2 - 2x - 1 \tag{7}$$

7 **A differential equation, a point, and a slope field are given. A slope field (or direction field) consists of line segments with slopes given by the differential equation. These line segments give a visual perspective of the slopes of the solutions of the differential equation. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the indicated point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a).**

52.

$$\frac{dy}{dx} = x^2 - 1, \ (-1, 3)$$





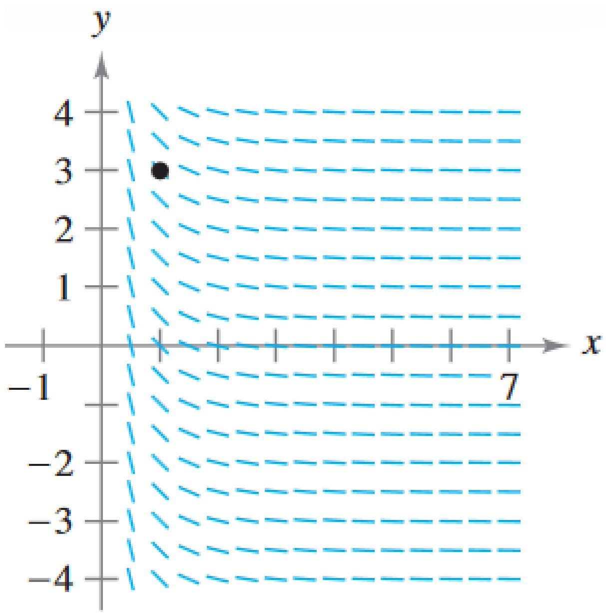
(a)

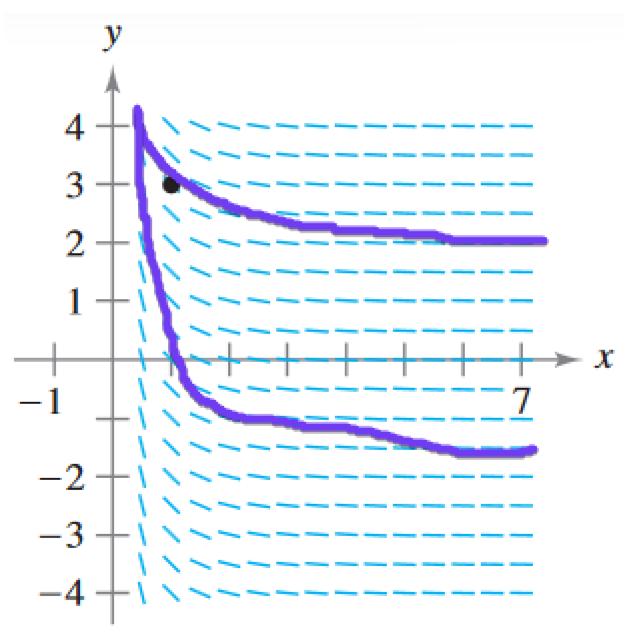
(b)

$$\frac{dy}{dx} = x^2 - 1,\ (-1, 3) \tag{1}$$
$$y = \frac{x^3}{3} - x + C \tag{2}$$
$$3 = \frac{(-1)^3}{3} - (-1) + C \tag{3}$$
$$C = \frac{7}{3} \tag{4}$$
$$y = \frac{x^3}{3} - x + \frac{7}{3} \tag{5}$$

54.

$$\frac{dy}{dx} = -\frac{1}{x^2},\ x > 0,\ (1, 3)$$





(a)

(b)

$$y = \int -\frac{1}{x^2} dx \tag{1}$$
$$= \int -x^{-2} dx \tag{2}$$
$$= \frac{-x^{-1}}{-1} + C \tag{3}$$
$$= \frac{1}{x} + C \tag{4}$$
$$3 = \frac{1}{1} + C \Rightarrow C = 2 \tag{5}$$
$$y = \frac{1}{x} + 2 \tag{6}$$

8 (a) Use a graphing utility to graph a slope field for the differential equation, (b) use integration and the given point to find the particular solution of the differential equation, and (c) graph the solution and the slope field in the same viewing window.

56.

$$\frac{dy}{dx} = 2\sqrt{x}, \ (4, 12) \tag{1}$$
$$y = \int 2x^{1/2} dx \tag{2}$$
$$= \frac{4}{3}x^{3/2} + C \tag{3}$$
$$12 = \frac{4}{3}(4)^{3/2} + C \tag{4}$$
$$= \frac{4}{3}(8) + C \tag{5}$$
$$= \frac{32}{3} + C \Rightarrow C = \frac{4}{3} \tag{6}$$
$$y = \frac{4}{3}x^{3/2} + \frac{4}{3} \tag{7}$$

9 Solve the differential equation.

58.

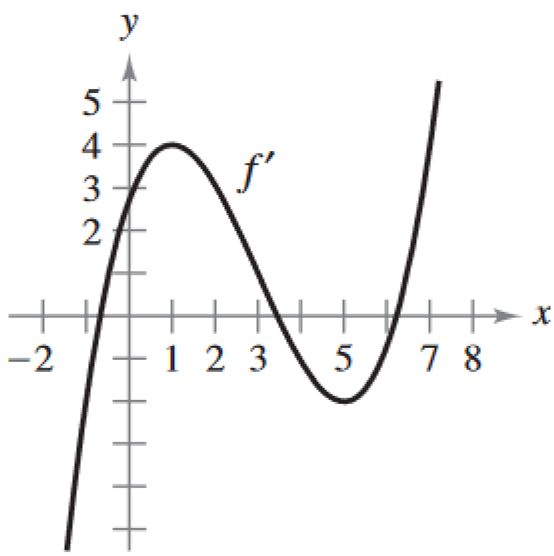
$$g'(x) = 6x^2, \ g(0) = -1 \tag{1}$$
$$g(x) = \int 6x^2 dx = 2x^3 + C \tag{2}$$
$$g(0) = -1 = 2(0)^3 + C \Rightarrow C = -1 \tag{3}$$
$$g(x) = 2x^3 - 1 \tag{4}$$

62.

$$f''(x) = x^2,\ f'(0) = 8,\ f(0) = 4 \tag{1}$$
$$f'(x) = \int x^2 dx = \frac{1}{3}x^3 + C_1 \tag{2}$$
$$f'(0) = 0 + C_1 = 8 \Rightarrow C_1 = 8 \tag{3}$$
$$f'(x) = \frac{1}{3}x^3 + 8 \tag{4}$$
$$f(x) = \int \left(\frac{1}{3}x^3 + 8\right) dx = \frac{1}{12}x^4 + 8x + C_2 \tag{5}$$
$$f(0) = 0 + 0 + C_2 = 4 \Rightarrow C_2 = 4 \tag{6}$$
$$f(x)\frac{1}{12}x^4 + 8x + 4 \tag{7}$$

10 Capstone

70. Use the graph of f' shown in the figure to answer the following, given that $f(0) = -4$.



(a) Approximate the slope of f at $x = 4$.

$$f'(4) \approx -1 \tag{1}$$

(b) Is it possible that $f(2) = -1$? Explain.
No because the tangent lines' slopes on [0, 2] are greater than 2, and f would have to increase more than 4 on [0, 4].

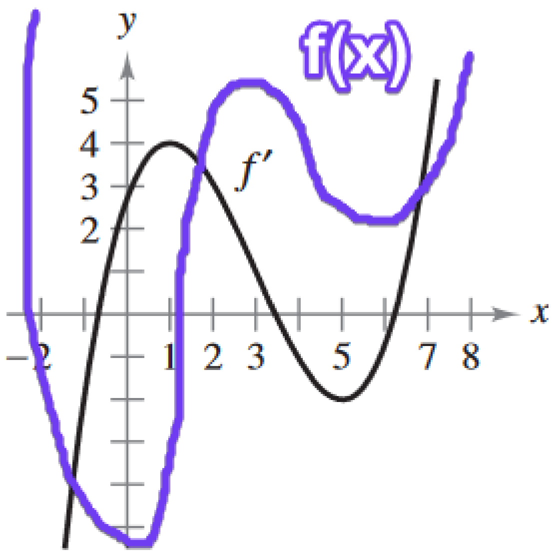
(c) Is $f(5) - f(4) > 0$? Explain. No because f decreases on [4, 5].

(d) Approximate the value of x where f is maximum. Explain.
f is maximum at $x = 3.5$ because $f'(3.5) = 0$.

(e) Approximate any intervals in which the graph of is concave upward and any intervals in which it is concave downward. Approximate the coordinates of any points of inflection.
f is concave upward when f' increases on $(-\infty, 1$ and $(5, \infty)$. f is concave downward on (1, 5). There are points of inflection at $x = 1, 5$.

(f) Approximate the coordinate of the minimum of $f''(x)$.
$f''(x)$ is minimum at $x = 3$.

(g) Sketch an approximate graph of f.



11 Vertical Motion. Use $a(t) = -32$ feet per second per second as the acceleration due to gravity. (Neglect air resistance.)

74. A balloon, rising vertically with a velocity of 16 feet per second, releases a sandbag at the instant it is 64 feet above the ground.

(a) How many seconds after its release will the bag strike the ground?

$$\begin{align}
v_0 &= 16\text{ft/sec} \tag{1}\\
s_0 &= 64\text{ft} \tag{2}\\
s(t) &= -16t^2 + 16t + 64 = 0 \tag{3}\\
-16(t^2 - t - 4) &= 0 \tag{4}\\
t &= \frac{1+\sqrt{17}}{2} \approx 2.562\text{seconds} \tag{5}\\
& \tag{6}
\end{align}$$

(b) At what velocitiy will it hit the ground?

$$\begin{align}
v(t) = s'(t) &= -32t + 16 \tag{1}\\
v\left(\frac{1+\sqrt{17}}{2}\right) &= -32\left(\frac{1+\sqrt{17}}{2}\right) + 16 \tag{2}\\
&= -16\sqrt{17} \approx -65.970\text{ft/sec} \tag{3}
\end{align}$$

12 Vertical Motion. Use $a(t) = -9.8$ meters per second per second as the acceleration due to gravity. (Neglect air resistance.)

76. The Grand Canyon is 1800 meters deep at its deepest point. A rock is dropped from the rim above this point. Write the height of the rock as a function of the time t in seconds. How long will it take the rock to hit the canyon

floor?

$$f(t) = 0 = -4.9t^2 + 1800 \tag{1}$$
$$4.9t^2 = 1800 \tag{2}$$
$$t^2 = \frac{1800}{4.9} \Rightarrow t \approx 9.2\text{seconds} \tag{3}$$

13 Rectilinear Motion. Consider a particle moving along the x-axis where $x(t)$ is the position of the particle at time t, $x'(t)$ is its velocity, and $x''(t)$ is its acceleration.

82.

(a) Find the velocity and acceleration of the particle.

$$x(t) = (t-1)(t-3)^2,\ 0 \le t \le 5 \tag{4}$$
$$= t^3 - 7t^2 + 15 - 9 \tag{5}$$
$$v(t) = x'(t) = 3t^2 - 14t + 15 = (3t-5)(t-3) \tag{6}$$
$$a(t) = v'(t) = 6t - 14 \tag{7}$$

(b) Find the open intervals on which the particle is moving to the right.

$$v(y) > 0 \text{ when} 0 < t < \frac{5}{3} \text{ and } 3 < t < 5 \tag{8}$$
$$\tag{9}$$

(c) Find the velocity of the particle when the acceleration is 0.

$$a(t) = 6t - 14 = 0 \text{ when } t = \frac{7}{3} \tag{10}$$
$$v(\frac{7}{3}) = \left(3\left(\frac{7}{3}\right) - 5\right)\left(\frac{7}{3} - 3\right) = 2\left(-\frac{2}{3}\right) = -\frac{4}{3} \tag{11}$$

14 Rectilinear Motion. Consider a particle moving along the x-axis where $x(t)$ is the position of the particle at time t, $x'(t)$ is its velocity, and $x''(t)$ is its acceleration.

84. A particle, initially at rest, moves along the x-axis such that its acceleration at time $t > 0$ is given by $a(t) = \cos t$. At the time $t = 0$, its position $x = 3$

(a) Find the velocity and position functions for the particle.

$$a(t) = \cos t \tag{1}$$
$$v(t) = \int a(t)dt \tag{2}$$
$$= \int \cos t dt \tag{3}$$
$$= \sin t + C_1 = \sin t \tag{4}$$
$$f(t) = \int v(t)dt = \int \sin t dt = -\cos + C_2 \tag{5}$$
$$f(0) = 3 = -\cos(0) + C_2 = -1 + C_2 \Rightarrow C_2 = 4 \tag{6}$$
$$f(t) = -\cos t + 4 \tag{7}$$

(b) Find the values of t for which the particle is at rest.

$$v(t) = 0 = \sin t,\ t = k\pi,\ k = 0, 1, 2, ... \tag{1}$$

15 Deceleration

86. A car traveling at 45 miles per hour is brought to a stop, at constant deceleration, 132 feet from where the brakes are applied.

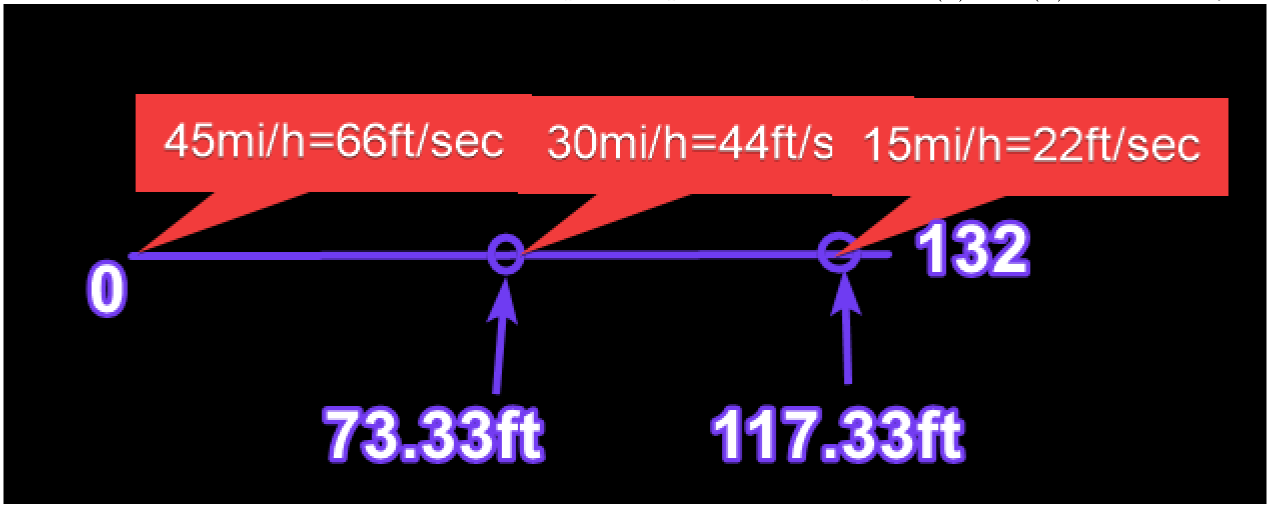
(a) How far has the car moved when its speed has been reduced to 30 miles per hour?

$$\begin{aligned}
v(0) &= 45\text{mi/h} = 66\text{ft/sec} && (1)\\
30\text{mi/h} &= 44\text{ft/sec} && (2)\\
15\text{mi/h} &= 22\text{ft/sec} && (3)\\
a(t) &= -a && (4)\\
v(t) &= -at + 66 && (5)\\
s(t) &= -\frac{a}{2}t^2 + 66t && (6)\\
v(t) &= 0 \text{ after } 132 \text{ ft} && (7)\\
-at + 66 &= 0 \text{ when } t = \frac{66}{a} && (8)\\
s\left(\frac{66}{a}\right) &= -\frac{a}{2}\left(\frac{66}{a}\right)^2 + 66\left(\frac{66}{a}\right) && (9)\\
&= 132 \text{ when } a = \frac{33}{2} = 16.5 && (10)\\
a(t) &= -16.5 && (11)\\
v(t) &= -16.5t + 66 && (12)\\
s(t) &= -8.25t^2 + 66t - 16.5t + 66 && = 44 \quad (13)\\
t &= \frac{22}{16.5} \approx 1.333 && (14)\\
s\left(\frac{22}{16.5}\right) &\approx 73.33 && (15)
\end{aligned}$$

(b) How far has the car moved when its speed has been reduced to 15 miles per hour?

$$\begin{aligned}
-16.5t + 66 &= 22 && (1)\\
t &= \frac{44}{16.5} \approx 2.667 && (2)\\
s\left(\frac{44}{16.5}\right) &\approx 117.33 && (3)
\end{aligned}$$

(c) Draw the real number line from 0 to 132, and plot the points found in parts (a) and (b). What can you conclude?



Less distance is needed to reach the next reduction of speed every time.

16 Determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

90. Each antiderivative of an nth-degree polynomial function is an $(n+1)$th-degree polynomial function. True.

91. If $p(x)$ is a polynomial function, then p has exactly one antiderivative whose graph contains the origin.
True.

92. If $F(x)$ and $G(x)$ are antiderivatives of $f(x)$, then $F(x) = G(x) + C$.
True.

93. If $f'(x) = g(x)$, then $\int g(x)dx = f(x)_C$.
True.

94. $\int f(x)g(x)dx = \int f(x)dx \int g(x)dx$
False.

$$\int x \cdot x dx \neq \int x dx \cdot \int x dx \because \frac{x^3}{3} + C \neq \left(\frac{x^2}{2} + C_1\right)\left(\frac{x^2}{2} + C_2\right)$$

95. The antiderivative of $f(x)$ is unique.
False. It has an infinite number of integrals, each being different by a constant.