Notes - 4.2 Area

Juan J. Moreno Santos

November 2023

1 Warm-up 11/14/2023

1. If $x + 7y = 29$ is an equation of the line normal (perpendicular) to the graph of f(x) at the point (1, 4), then $f'(x) =$
Find the slope of the line by putting the equation in slope-intercept form:

$$7y = -x + 29 \tag{1}$$
$$y = -\frac{1}{7} + b \tag{2}$$

We don't consider b in this problem.
Now, the reciprocal of $-\frac{1}{7}$ is

$$7 \tag{3}$$

Therefore, $f'(x) = 7$.

Find $\frac{d}{dx}(y^2 - 2xy = 16)$ by implicit differentiation:

$$2ydy - (xdy + ydx) = 0 \tag{1}$$
$$2ydy - 2xdy = 2ydx \tag{2}$$
$$(2y - 2x)dy = 2ydx \tag{3}$$
$$\frac{dy}{dx} = \frac{2y}{2y - 2x} \tag{4}$$
$$= \frac{2y}{2(y - x)} \tag{5}$$
$$= \frac{y}{y - x} \tag{6}$$

2 4.2 Area

1. Use sigma notation (summation) to write and evaluate a summation
2. Understand the concept of Area
3. Approximate the area of a plane region
4. Find the area of a plane region using limits

2.1 Sigma notation

The sum of n terms $a_1, a_2, a_3, \cdots a_n$ is written as

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

Where

1. i is the index of summation
2. a_i is the ith term of the sum
3. n and 1 are the upper and lower bounds of summation, respectively

2.2 Examples of Sigma Notation

$$\sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6 \tag{1}$$

$$\sum_{i=0}^{5} (i+1) = 1 + 2 + 3 + 4 + 5 + 6 \tag{2}$$

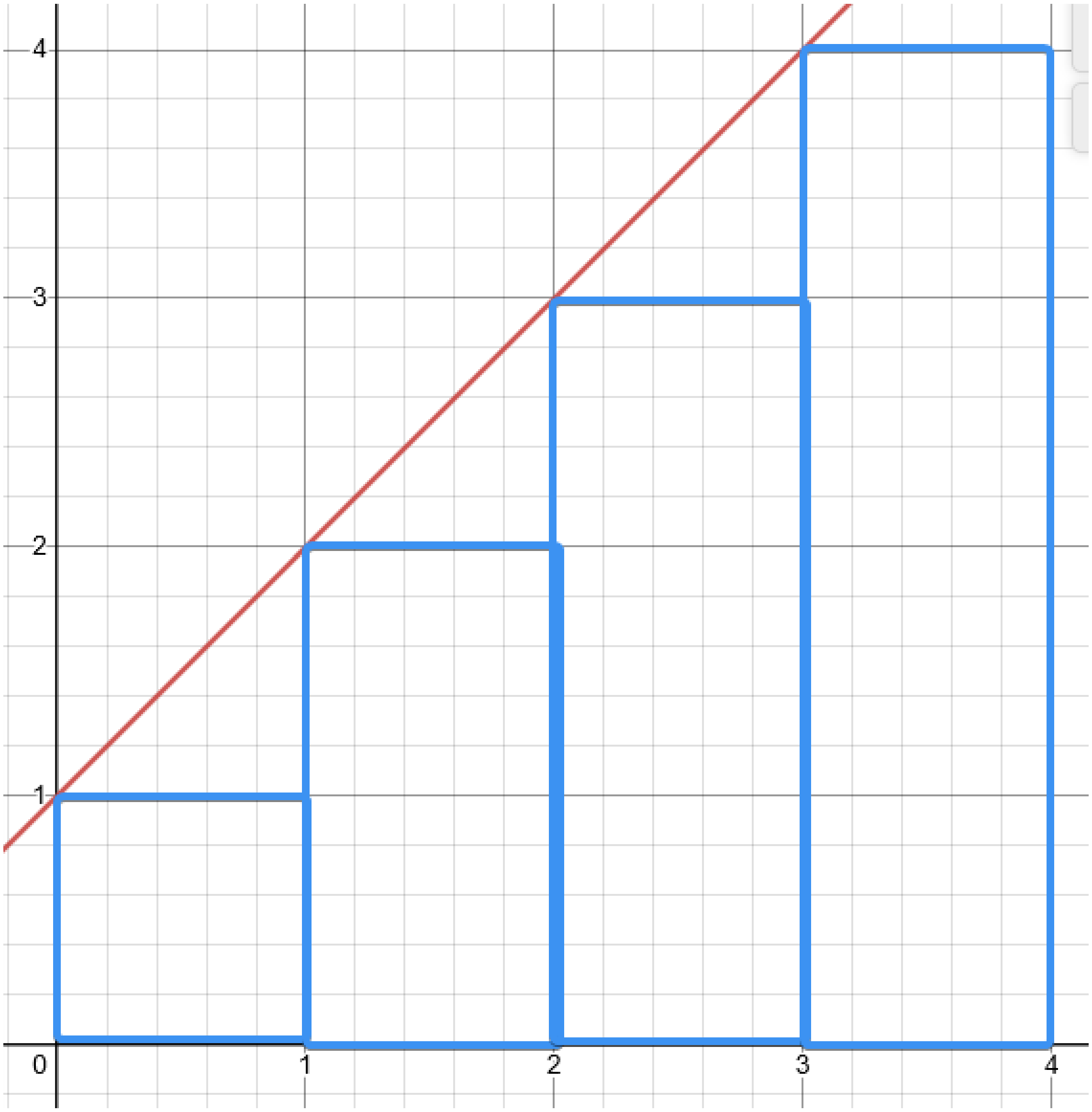
$$\sum_{j=3}^{7} j^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2 \tag{3}$$

$$\sum_{k=1}^{n} \frac{1}{n}(k^2+1) = \frac{1}{n}(1^2+1) + \frac{1}{n}(2^2+1) + \cdots + \frac{1}{n}(n^2+1) \tag{4}$$

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \tag{5}$$

Notice how (a) and (b) are the same sequence, but notated differently.

2.3 Approximating the area under a function using summation



Consider that $n = 4$ and f is in the $[0, 4]$ interval. The sum of the rectangles without sigma notation is written as:

$$h_1 \cdot b + h_2 + \cdot b + h_3 \cdot b + h_4 \cdot b$$

Which is the same as:

$$f(x_1)b + f(x_2)b + f(x_3)b + f(x4)b$$

The iterations of x represent Δx In Sigma Notation, this is written as:

$$\sum_{i=1}^{n} f(x_1)\Delta x = \sum_{i=0}^{3}(i+1)1 = 10$$

2.4 Properties of summation

$$\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i \tag{1}$$

$$\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i \tag{2}$$

$$\tag{3}$$

Considering that

$$\Delta x = \frac{b-a}{n} = \frac{4-0}{n} = \frac{4}{n}$$

and

$$x_i = \frac{4}{n} i$$

Therefore,

$$\lim_{n\to\infty} \sum_{i=0}^{n} (i+1)\Delta x = \lim_{\Delta x\to 0} \sum_{i=0}^{n} (i+1)\Delta x \tag{1}$$

$$\lim_{n\to\infty} \sum_{i=0}^{n} (i+1)\Delta x = \Delta x (\sum x_i + \sum 1) \tag{2}$$

$$= \frac{4}{n} (\sum \frac{4i}{n} + \sum 1) \tag{3}$$

$$= \frac{4}{n} \left(\frac{4}{n} \left(\frac{n(n+1)}{2} \right) + n \right) \tag{4}$$

$$= \frac{4}{n} \left(\frac{4n(n+1)}{2n} + \frac{n2n}{2n} \right) \tag{5}$$

$$= \frac{4}{n} \left(\frac{4n^2 + 4n + 2n^2}{2n} \right) \tag{6}$$

Taking the limit,

$$\lim_{h\to\infty} \frac{24n^2 + 16n}{2n^2} = 12 \tag{1}$$

2.5 Summation formulas

$$\sum_{i=1}^{n} c = cn \tag{1}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \tag{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \tag{3}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} \tag{4}$$

2.6 Example - Evaluating a sum

Evaluate $\sum_{i=1}^{n} \frac{i+1}{n}$ for $n = 10$, 100, 1000, and 10000.

$$\sum_{i=1}^{n} \frac{i+1}{n^2} = \frac{1}{n^2} \sum_{i=1}^{n} (i+1) \tag{1}$$

$$= \frac{1}{n^2} \left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \right) \tag{2}$$

$$= \frac{1}{n^2} \left(\frac{n(n+1)}{2} + n \right) \tag{3}$$

$$= \frac{1}{n^2} \left(\frac{n^2 + 3n}{2} \right) \tag{4}$$

$$= \frac{n+3}{2n} \tag{5}$$

Steps:

(a) Factor $\frac{1}{n^2}$ out of the sum.

(b) Write as two sum

(c) Apply summation formulas 1 and 2

(d) Simplify the expression

Taking the limit:

$$\lim_{n\to\infty} \frac{n+3}{2n} = \lim_{n\to\infty} \left(\frac{n}{2n} + \frac{3}{2n}\right) = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{3}{2n}\right) = \frac{1}{2} + 0 = \frac{1}{2}.$$

3 Warm-up 11/16/2023

1. If $f(x) = x\sqrt{x}$ then $f'(1) =$

$$\begin{aligned} & \frac{d}{dx}(x\sqrt{x}) && (1)\\ &= \frac{d}{dx}(x \cdot x^{1/2} = x^{3/2}) && (2)\\ &= \frac{3}{2}x^{1/2} && (3)\\ f'(1) &= \frac{3}{2}(1)^{1/2} = \frac{3}{2} && (4) \end{aligned}$$

2. If the line y=3x-5 is tangent to the graph of $y = f(x)$ at the point (4, 7) then $\lim x \to 0 \frac{f(4+x)-f(4)}{x}$ is:
Remember that this expression is the exact limit definition of the derivative at a certain point (h is changed to x in this case). The slope of the tangent line is 3 because the equation is in the form $y = mx + b$.

4 Limits of the lower and upper sums

Let f be continouous and nonnegative on the interval [a, b]. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

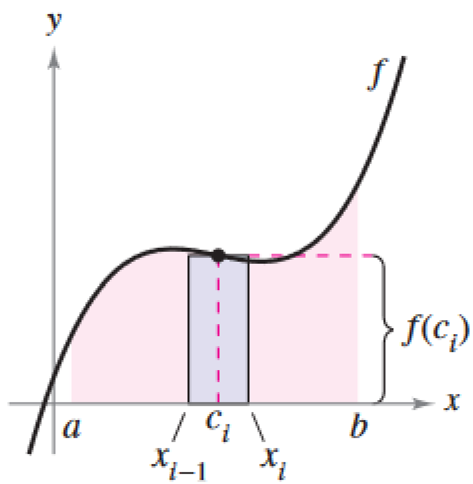
$$\begin{aligned} \lim_{n\to\infty} s(n) &= \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\Delta x && (1)\\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\Delta x && (2)\\ &= \lim_{n\to\infty} S(n) && (3) \end{aligned}$$

where $\Delta x = \frac{b-a}{n}$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.

5 Definition of the are of a region in the plane

Let f be continuous and nonnegative on the interval [a, b]. The are of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx$$



The width of the ith subinterval is $\Delta x = x_i - x_{i-1}$.

6 The fundamental theorem of Calculus

Remember that

$$\frac{d}{dx}(F) = f \tag{1}$$
$$\frac{d}{dx}(f) = f' \tag{2}$$

Therefore,

$$\int_a^b f(x)dx = F(b) - f(a)$$

6.1 Applying the fundamental theorem of Calculus

1. Consider the previous function $f(x) = x + 1$. By that, $F(x) = \frac{1}{2}x^2$

$$\int_0^4 x + 1dx \tag{1}$$
$$= \frac{1}{2}x^2 + x + C|_0^4 \tag{2}$$
$$= \frac{1}{4}^2 + (4) + C - (\frac{1}{2}(0)^2 + 0 + C) \tag{3}$$
$$= 12u^2 \tag{4}$$

2.

$$\int_0^\pi \sin x dx \tag{1}$$
$$= -\cos x|_0^\pi \tag{2}$$
$$= -(-1) - (-1) \tag{3}$$
$$= 2 \tag{4}$$

7 P-set 76

Use the Midpoint Rule

$$\text{Area} \approx \sum_{i=1}^{n} f\left(\frac{x_1 + x_{i-1}}{2}\right)\Delta x$$

with $n = 4$ to approximate the are of the region bounded by the graph of the function and the x-axis over the given interval

$$f(x) = \sin x, \quad \left[0,\ \frac{\pi}{2}\right], \quad n = 4$$

Let

$$c_i = \frac{x_i + x_i - 1}{2}$$

$$\Delta x = \frac{\pi}{8}, \quad c_1 = \frac{\pi}{16}, \quad c_2 = \frac{3\pi}{16}, \quad c_3 = \frac{5\pi}{16}, \quad c_4 = \frac{7\pi}{16} \tag{1}$$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_1)\Delta x = \sum_{i=1}^{4} \sin\left(\frac{(2n-1)\pi}{16}\right)\frac{\pi}{8} \tag{2}$$

$$\approx 1.006 \tag{3}$$

8 Solving differential equations

8.1 Separate

$$\frac{dy}{dx} = 12x^3 \tag{1}$$

$$dy = 12x^3 dx \tag{2}$$

8.2 Integrate

$$\int dy = \int 12x^3 dx \tag{1}$$

$$y + C = 3x^4 + C \tag{2}$$

$$y = 3x^4 + C \tag{3}$$