Notes - 3.9 Differentials

Juan J. Moreno Santos

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1 Warm-up

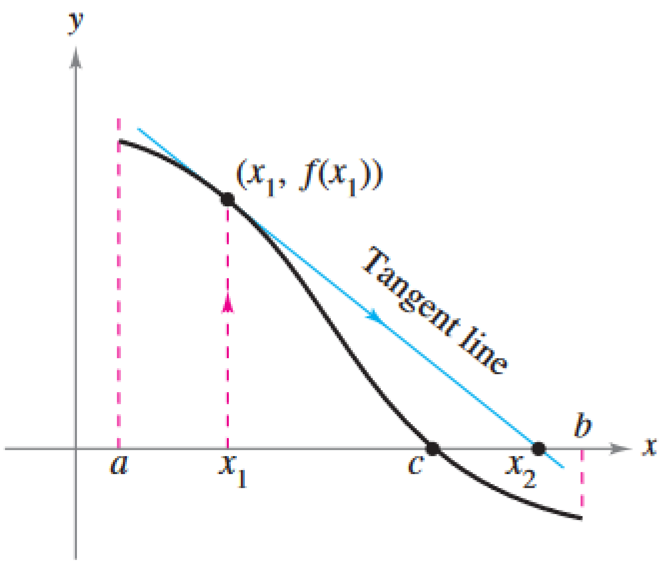
1. If $f(x) = x^5 - 1$, f^{-1} is:

$$f^{-1}(x) = \sqrt[5]{x+1} \tag{1}$$

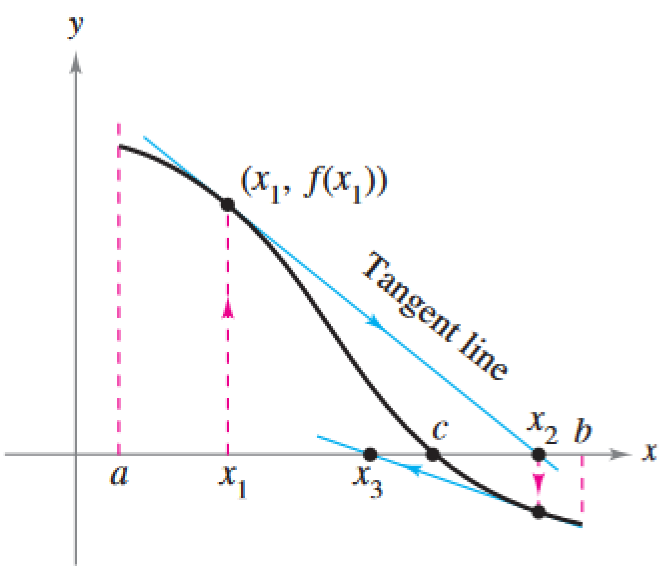
2. $\frac{d}{dx}(x^2 - 3xy + y^2 = -1)$ at $(1, 1)$:

$$f(x) = x^2 - 3xy + y^3 = -2 \tag{1}$$
$$\frac{dy}{dx} = 3xdx - 3ydx + (-3xdy) + 2ydy = 0 \tag{2}$$
$$2ydy - 3xdy = 3ydx - 2xdx \tag{3}$$
$$dy(2y - 3x) = dx(3y - 2x) \tag{4}$$
$$\frac{dy}{dx} = \frac{3y - 2x}{2y - 3x} = \frac{3(1) - 2(1)}{2(1) - 3(1)} = -1 \tag{5}$$

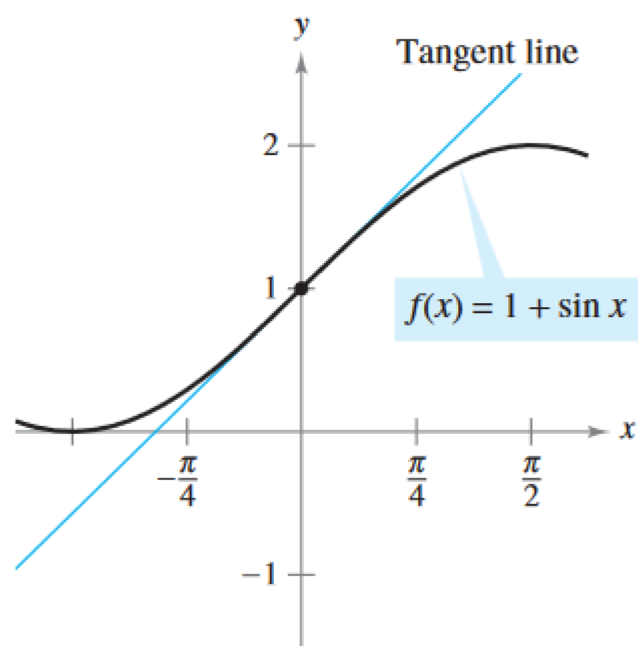
- Understand the concept of a tangent line approximation.
- Compare the value of the differential, *dy*, with the actual change in *y*, Δy.
- Estimate a propagated error using a differential.
- Find the differential of a function using differentiation formulas.



(a)



(b)
The *x*-intercept of the tangent line approximates the zero of *f*.
Figure 3.60



The tangent line approximation of *f* at the point (0, 1)

We are familiar with the point-slope form of a function:

$$(y - y_1) = m(x - x_1)$$

The euqation for the tangent line at the point $(c, f(c))$ is given by:

$$y - f(c) = f'(c)(x - c)$$

$$y = f(c) + f'(c)(x - c)$$

It has the same format as the point-slope form of a function.

EXAMPLE 1 Using a Tangent Line Approximation

Find the tangent line approximation of

$$f(x) = 1 + \sin x$$

at the point (0, 1). Then use a table to compare the y-values of the linear function with those of $f(x)$ on an open interval containing $x = 0$.

Solution The derivative of f is

$$f'(x) = \cos x. \qquad \text{First derivative}$$

So, the equation of the tangent line to the graph of f at the point (0, 1) is

$$y - f(0) = f'(0)(x - 0)$$

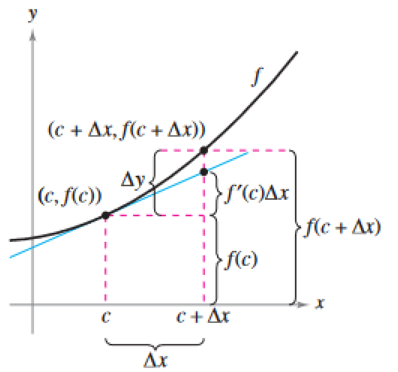
$$y - 1 = (1)(x - 0)$$

$$y = 1 + x. \qquad \text{Tangent line approximation}$$

The table compares the values of y given by this linear approximation with the values of $f(x)$ near $x = 0$. Notice that the closer x is to 0, the better the approximation is. This conclusion is reinforced by the graph shown in Figure 3.65.

x	-0.5	-0.1	-0.01	0	0.01	0.1	0.5
$f(x) = 1 + \sin x$	0.521	0.9002	0.9900002	1	1.0099998	1.0998	1.479
$y = 1 + x$	0.5	0.9	0.99	1	1.01	1.1	1.5

Differentials



When Δx is small, $\Delta y = f(c + \Delta x) - f(c)$ is approximated by $f'(c)\Delta x$.
Figure 3.66

When the tangent line to the graph of f at the point $(c, f(c))$

$$y = f(c) + f'(c)(x - c) \qquad \text{Tangent line at } (c, f(c))$$

is used as an approximation of the graph of f, the quantity $x - c$ is called the change in x, and is denoted by Δx, as shown in Figure 3.66. When Δx is small, the change in y (denoted by Δy) can be approximated as shown.

$$\Delta y = f(c + \Delta x) - f(c) \qquad \text{Actual change in y}$$
$$\approx f'(c)\Delta x \qquad \text{Approximate change in y}$$

For such an approximation, the quantity Δx is traditionally denoted by dx, and is called the **differential of x.** The expression $f'(x)\,dx$ is denoted by dy, and is called the **differential of y.**

DEFINITION OF DIFFERENTIALS

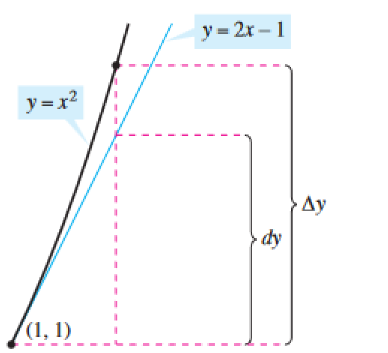
Let $y = f(x)$ represent a function that is differentiable on an open interval containing x. The **differential of x** (denoted by dx) is any nonzero real number. The **differential of y** (denoted by dy) is

$$dy = f'(x)\,dx.$$

In many types of applications, the differential of y can be used as an approximation of the change in y. That is,

$$\Delta y \approx dy \qquad \text{or} \qquad \Delta y \approx f'(x)dx.$$

EXAMPLE 2 Comparing Δy and dy



The change in y, Δy, is approximated by the differential of y, dy.
Figure 3.67

Let $y = x^2$. Find dy when $x = 1$ and $dx = 0.01$. Compare this value with Δy for $x = 1$ and $\Delta x = 0.01$.

Solution Because $y = f(x) = x^2$ you have $f'(x) = 2x$ and the differential dy is given by

$$dy = f'(x)\,dx = f'(1)(0.01) = 2(0.01) = 0.02. \qquad \text{Differential of y}$$

Now, using $\Delta x = 0.01$, the change in y is

$$\Delta y = f(x + \Delta x) - f(x) = f(1.01) - f(1) = (1.01)^2 - 1^2 = 0.0201.$$

Figure 3.67 shows the geometric comparison of dy and Δy. Try comparing other values of dy and Δy. You will see that the values become closer to each other as dx (or Δx) approaches 0. ■

In Example 2, the tangent line to the graph of $f(x) = x^2$ at $x = 1$ is

$$y = 2x - 1 \qquad \text{or} \qquad g(x) = 2x - 1. \qquad \text{Tangent line to the graph of } f \text{ at } x = 1.$$

For x-values near 1, this line is close to the graph of f, as shown in Figure 3.67. For instance,

$$f(1.01) = 1.01^2 = 1.0201 \qquad \text{and} \qquad g(1.01) = 2(1.01) - 1 = 1.02.$$

In other

words,

$$\begin{aligned} y &= x^2 \tag{1}\\ dy &= f'(x)dx = 2xdx = 2(1)(0.01) \tag{2}\\ &= 0.02 \tag{3}\\ & \tag{4} \end{aligned}$$

For the tangent line:

$$\begin{aligned} y &= f(c) + f'(c)(x-c) \tag{5}\\ &= 1 - 2(x-1) \tag{6}\\ &= 2x - 1 \tag{7} \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta y &= f(x+\Delta x) - f(x) \tag{8}\\ &= f(1+0.1) - f(1) \tag{9}\\ &= (1.01)^2 - (1)^2 \tag{10}\\ &= 0.201 \tag{11} \end{aligned}$$

Comparing to equation 4, we can confirm that $\Delta y \approx dy$

3 Error propagation

$$\underbrace{f(x + \overbrace{\Delta x}^{\text{Measurement error}})}_{\text{Exact value}} - \underbrace{f(x)}_{\text{Measured value}} = \overbrace{\Delta y}^{\text{Propagated error}}$$

EXAMPLE 3 Estimation of Error

The measured radius of a ball bearing is 0.7 inch, as shown in Figure 3.68. If the measurement is correct to within 0.01 inch, estimate the propagated error in the volume V of the ball bearing.

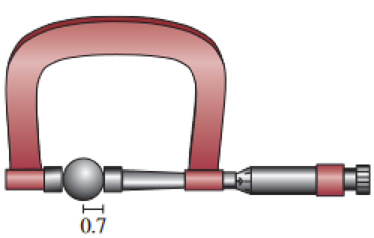
Solution The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere. So, you can write

$r = 0.7$ Measured radius

and

$-0.01 \le \Delta r \le 0.01.$ Possible error

To approximate the propagated error in the volume, differentiate V to obtain $dV/dr = 4\pi r^2$ and write



Ball bearing with measured radius that is correct to within 0.01 inch.
Figure 3.68

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \tag{1}\\ dV &= 4\pi r^2 dr \tag{2}\\ &= 4\pi(0.7)^2(\pm 0.1) \tag{3}\\ &= \pm 0.0.6158 in^3 \tag{4}\\ \Delta y &= f(x+\Delta x) - f(x) \tag{5} \end{aligned}$$

Diving the propagated error by the percentage yields the **relative error**, and converting this decimal gives the **percent error**:

Would you say that the propagated error in Example 3 is large or small? The answer is best given in *relative* terms by comparing dV with V. The ratio

$$\frac{dV}{V} = \frac{4\pi r^2\,dr}{\frac{4}{3}\pi r^3} \qquad \text{Ratio of } dV \text{ to } V$$

$$= \frac{3\,dr}{r} \qquad \text{Simplify.}$$

$$\approx \frac{3}{0.7}(\pm 0.01) \qquad \text{Substitute for } dr \text{ and } r.$$

$$\approx \pm 0.0429$$

is called the **relative error.** The corresponding **percent error** is approximately 4.29%