4.4 The Fundamental Theorem of Calculus

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1 Use a graphing utility to graph the integrand. Use the graph to determine whether the definite integral is positive, negative, or zero.

2.

$$f(x) = \cos x \tag{1}$$
$$\int_0^\pi \cos x dx = 0 \tag{2}$$

4.

$$f(x) = x\sqrt{2-x} \tag{1}$$
$$\int_{-2}^{2} x\sqrt{2-x} dx = negative \tag{2}$$

2 Evaluate the definite integral of the algebraic function. Use a graphing utility to verify your result.

6.

$$\int_4^9 5dv \tag{1}$$
$$= (5v)_4^9 \tag{2}$$
$$= 5(9) - 5(4) = 25 \tag{3}$$

10.

$$\int_1^7 (6x^2 + 2x - 3)dx \tag{1}$$
$$= (2x^3 + x^2 - 3x)_1^7 \tag{2}$$
$$= (2(7)^3 + (7)^2 - 3(7)) - (2 + 1 - 3) = 714 \tag{3}$$

14.

$$\int_{-2}^{-1} \left(u - \frac{1}{u^2}\right) du \tag{1}$$

$$= \left(\frac{u^2}{2} + \frac{1}{u}\right)_{-} 2^{-}1 \tag{2}$$

$$= \left(\frac{1}{2} - 1\right) - \left(2 - \frac{1}{2}\right) = -2 \tag{3}$$

18.

$$\int_1^8 \sqrt{\frac{2}{x}} dx \tag{1}$$

$$= \sqrt{2} \int_1^8 x^{-1/2} dx \tag{2}$$

$$= (\sqrt{2}(2) x^{1/2})_1^8 \tag{3}$$

$$= (2\sqrt{2x})_1^8 \tag{4}$$

$$= 8 - 2\sqrt{2} \tag{5}$$

22.

$$\int_{-8}^{-1} \frac{x - x^2}{2\sqrt[3]{x}} dx \tag{1}$$

$$= \frac{1}{2} \int_{-8}^{-1} (x^{2/3} - x^{5/3}) dx \tag{2}$$

$$= \frac{1}{2} \left(\frac{3}{5} x^{5/3} - \frac{3}{8} x^{8/3}\right)_{-8}^{-1} \tag{3}$$

$$= -\frac{1}{80}(39) + \frac{32}{80}(144) \tag{4}$$

$$= \frac{4569}{80} \tag{5}$$

26.

$$\int_0^4 |x^2 - 4x + 3| dx \tag{1}$$

$$= \int_0^1 (x^2 - 4x + 3) dx - \int_1^3 (x^2 - 4x + 3) dx + \int_3^4 (x^2 - 4x + 3) dx \tag{2}$$

$$= \left(\frac{x^3}{3} - 2x^2 + 3x\right)_0^1 - \left(\frac{x^3}{3} - 2x^2 + 3x\right)_1^3 + \left(\frac{x^3}{3} - 2x^2 + 3x\right)_3^4 \tag{3}$$

$$= \left(\frac{1}{3} - 2 + 3\right) - (9 - 18 + 9) + \left(\frac{1}{3} - 2 + 3\right) + \left(\frac{64}{3} - 32 + 12\right) - (9 - 18 + 9) \tag{4}$$

$$= \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4 \tag{5}$$

3 Evaluate the definite integral of the trigonometric function. Use a graphing utility to verify your result.

28.

$$\int_0^{\pi} (2 + \cos x)dx \tag{1}$$
$$= (2x + \sin x)_0^{\pi} \tag{2}$$
$$= (2\pi + 0) - 0 = 2\pi \tag{3}$$

32.

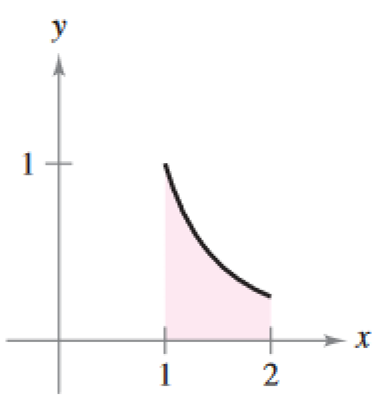
$$\int_{\pi/4}^{\pi/2} (2 - \csc^2 x)dx \tag{1}$$
$$= (2x + \cot x)_{\pi/4}^{\pi/2} \tag{2}$$
$$= (\pi + 0) - \left(\frac{\pi}{2} + 1\right) \tag{3}$$
$$= \frac{\pi}{2} - 1 \tag{4}$$
$$= \frac{\pi - 2}{2} \tag{5}$$

34.

$$\int_{-\pi/2}^{\pi/2} (2t + \cos t)dt \tag{1}$$
$$= (t^2 + \sin t)_{-\pi/2}^{\pi/2} \tag{2}$$
$$= \left(\frac{\pi^2}{4} + 1\right) - \left(\frac{\pi^2}{4} - 1\right) \tag{3}$$
$$= 2 \tag{4}$$

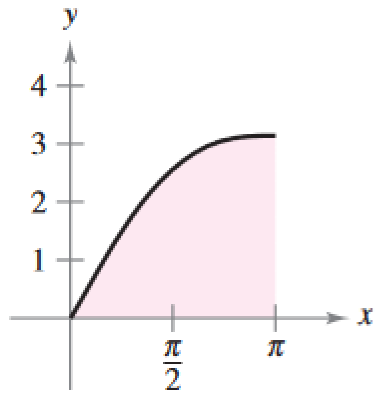
4 Determine the area of the given region.

36. $y = \frac{1}{x^2}$



$$
\begin{aligned}
A &= \int_1^2 \frac{1}{x^2} dx && (1) \\
&= \left(-\frac{1}{x}\right)_1^2 && (2) \\
&= \frac{1}{2} && (3)
\end{aligned}
$$

38.$y = x + \sin x$



$$
\begin{aligned}
A &= \int_0^{\pi} (x + \sin x) dx && (1) \\
&= \left(\frac{x^2}{2} - \cos x\right)_0^{/pi} && (2) \\
&= \frac{\pi^2}{2} + 2 && (3) \\
&= \frac{\pi^2 + 4}{2} && (4)
\end{aligned}
$$

5 Find the area of the region bounded by the graphs of the equations.

40.

$$
\begin{aligned}
y &= x^3 + x,\ x = 2,\ y = 0 && (1) \\
A &= \int_0^2 (x^3 + x) dx && (2) \\
&= \left(\frac{x^4}{4} + \frac{x^2}{2}\right)_0^2 && (3) \\
&= 4 + 2 = 6 && (4)
\end{aligned}
$$

42.

$$y = (3-x)\sqrt{x},\ y = 0 \tag{1}$$
$$A = \int_0^3 (3-x)\sqrt{x}dx \tag{2}$$
$$= \int_0^3 (3x^{1/2} - x^{3/2})dx \tag{3}$$
$$= \left(\frac{3x^{3/2}}{\frac{3}{2}} - \frac{x^{5/2}}{\frac{5}{2}}\right)_0^3 \tag{4}$$
$$= \left(2x^{3/2} - \frac{2}{5}x^{5/2}\right)_0^3 \tag{5}$$
$$= 2 \cdot 3 \cdot \sqrt{3} - \frac{2}{5}9\sqrt{3} \tag{6}$$
$$= \frac{12}{5}\sqrt{3} \tag{7}$$

6 Find the value(s) of c guaranteed by the Mean Value Theorem for Integrals for the function over the given interval.

46.

$$f(x) = \frac{9}{x^3},\ [1,3] \tag{1}$$
$$\int_1^3 \frac{9}{x^3}dx = \left(-\frac{9}{2x^2}\right)_1^3 \tag{2}$$
$$= -\frac{1}{2} + \frac{9}{2} = 4 \tag{3}$$
$$f(c)(3-1) = 4 \tag{4}$$
$$\frac{9}{c^3} = 2 \tag{5}$$
$$c^3 = \frac{9}{2} \tag{6}$$
$$c = \sqrt[3]{\frac{9}{2}} \approx 1.6510 \tag{7}$$

50.

$$f(x) = \cos x,\ (-\frac{\pi}{3}, \frac{\pi}{3}) \tag{1}$$
$$\int_{-\pi/3}^{\pi/3} \cos x dx = (\sin x)_{-\pi/3}^{\pi/3} = \sqrt{3} \tag{2}$$
$$f(c)\left(\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right) = \sqrt{3} \tag{3}$$
$$\cos c = \frac{3\sqrt{3}}{2\pi} \tag{4}$$
$$c \approx \pm 0.5971 \tag{5}$$

7 Find the average value of the function over the given interval and all values of x in the interval for which the function equals its average value.

52.

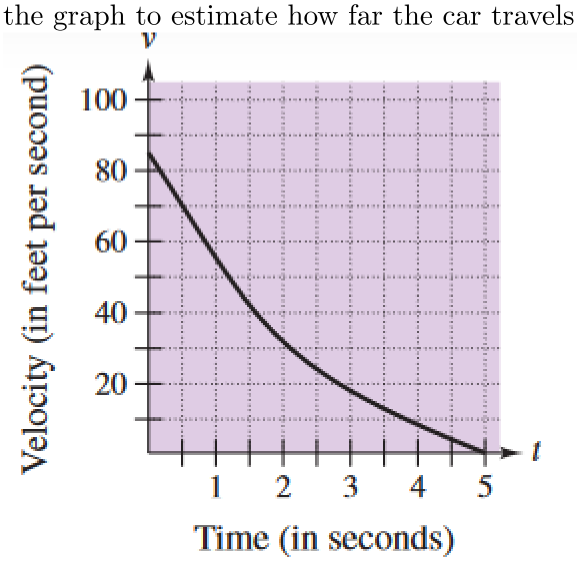
$$f(x) = \frac{4(x^2+1)}{x^2},\ [1,3] \tag{1}$$
$$\frac{1}{3-1}\int_1^3 \frac{4(x^2+1)}{x^2}dx = 2\int_1^3 (1+x^{-2})dx \tag{2}$$
$$= 2\left(x - \frac{1}{x}\right)_1^3 \tag{3}$$
$$= 2\left(3 - \frac{1}{3}\right) = \frac{16}{3} \tag{4}$$
$$\frac{4(x^2+1)}{x^2} = \frac{16}{3} \Rightarrow x = \sqrt{3},\ [1,3] \tag{5}$$

56.

$$f(x) = \cos x,\ [0, \frac{\pi}{2}] \tag{1}$$
$$\frac{1}{(\frac{\pi}{2}) - 0}\int_0^{\pi/2} \cos x dx \tag{2}$$
$$= \left(\frac{2}{\pi}\sin x\right)_0^{\pi/2} \tag{3}$$
$$= \frac{2}{\pi} \tag{4}$$
$$\cos x = \frac{2}{\pi} \tag{5}$$
$$x \approx 0.881 \tag{6}$$

8 Velocity

58.The graph shows the velocity, in feet per second, of a decelerating car after the driver applies the brakes. Use the graph to estimate how far the car travels before it comes to a stop.



The traveled distance is $\int_0^5 v(t)dt$. The area under the curve at $0 \le t \le 5$ is about $29(5) = 145\text{ft}^2$.

9 Average sales

64. A company fits a model to the monthly sales data for a seasonal product. The model is $S(t) = \frac{t}{4} + 1.8 + 0.5\sin\left(\frac{\pi t}{6}\right),\ 0 \le t \le 24$ where S is sales (in thousands) and t is time in months.

(a) Use a graphing utility to graph $f(t) = 0.5\sin\left(\frac{\pi t}{6}\right)$ for $0 \le t \le 24$. Use the graph to explain why the average value of $f(t)$ is 0 over the interval.
The area above the x-axis is the same as the area below the x-axis. Therefore, the average value is 0.

(b) Use a graphing utility to graph $S(t)$ and the line $g(t) = \frac{t}{4} + 1.8$ in the same viewing window. Use the graph and the result of part (a) to explain why g is called the *trend line*. g is called the *trend line* because the average value of S approaches this line.

10 Capstone

66. The graph of f is shown in the figure. The shaded region A has an area of 1.5, and $\int_0^6 f(x)dx = 3.5$. Use this information to fill in the blanks.

(a) $\int_0^2 f(x)dx = -1.5$

(b) $\int_2^6 f(x)dx = 5$

(c) $\int_0^6 |f(x)|dx = 6.5$

(d) $\int_0^2 -2f(x)dx = 3$

(e) $\int_0^6 (2 + f(x))dx = 15.5$

(f) The average value of f over the interval [0. 6] is 0.5833.

11 Find F as a function of x and evaluate it at $x = 2$, $x = 5$, and $x = 8$.

68.

$$F(x) = \int_2^x (t^3 + 2t - 2)dt \tag{1}$$

$$= \left(\frac{t^4}{4} + t^2 - 2t\right)_2^x \tag{2}$$

$$= \left(\frac{x^4}{4} + x^2 - 2x\right) - (4 + 4 - 4) \tag{3}$$

$$= \frac{x^4}{4} + x^2 - 2x - 4 \tag{4}$$

$$F(2) = 4 + 4 - 4 - 4 = 0 \tag{5}$$

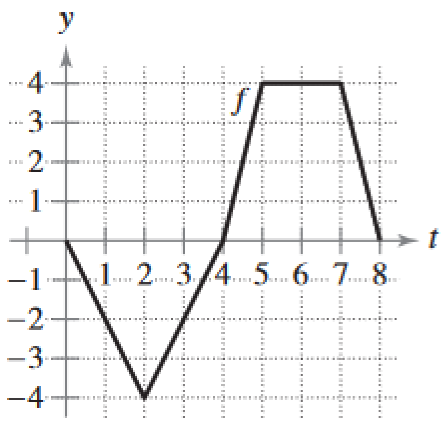
$$F(5) = \frac{625}{4} + 25 - 10 - 4 = 167.25 \tag{6}$$

$$F(8) = \frac{8^4}{4} + 64 - 16 - 4 = 1068 \tag{7}$$

72.

$$F(x) = \int_0^x \sin\theta d\theta = -\cos\theta|_0^x \tag{1}$$
$$= -\cos x + \cos 0 \tag{2}$$
$$= 1 - \cos x \tag{3}$$
$$F(2) = 1 - \cos 2 \approx 1.4161 \tag{4}$$
$$F(5) = 1 - \cos 5 \approx 0.7163 \tag{5}$$
$$F(8) = 1 - \cos 8 \approx 1.1455 \tag{6}$$

74. Let $g(x) = \int_0^x f(t)dt$, where f is the function whose graph is shown in the figure.



(a) Estimate $g(0)$, $g(2)$, $g(4)$, $g(6)$, and $g(8)$

$$g(0) = \int_0^0 f(t)dt = 0 \tag{1}$$
$$g(2) = \int_0^2 f(t)dt = -\frac{1}{2}(2)(4) = -4 \tag{2}$$
$$g(4) = \int_0^4 f(t)dt = -\frac{1}{2}(4)(4) = -8 \tag{3}$$
$$g(6) = \int_0^6 f(t)dt = -8 + 2 + 4 = -2 \tag{4}$$
$$g(8) = \int_0^8 f(t)dt = -2 + 6 = 4 \tag{5}$$

(b) Find the largest open interval on which g is increasing. Find the largest open interval on which g is decreasing.
g is decreasing on (0, 4) and increasing on (4, 8)

(c) Identify any extrema of g.
g is a minimum of -8 at $x = 4$

12 (a) Integrate to find F as a function of x and (b) demonstrate the Second Fundamental Theorem of Calculus by differentiating the result in part (a).

76.

(a)

$$\int_0^x t(t^2+1)dt = \int_0^x (t^3+t)dt \tag{1}$$
$$= \left(\frac{1}{4}t^4 + \frac{1}{2}t^2\right)|_0^x \tag{2}$$
$$= \frac{1}{4}x^4 + \frac{1}{2}x^2 \tag{3}$$
$$= \frac{x^2}{4}(x^2+2) \tag{4}$$

(b)

$$\frac{d}{dx}\left(\frac{1}{4}x^4 + \frac{1}{2}x^2\right) \tag{1}$$
$$= x^3 + x \tag{2}$$
$$= x(x^2+1) \tag{3}$$

13 Use the Second Fundamental Theorem of Calculus to find $F'(x)$

84.

$$F(x) = \int_1^x \sqrt[4]{t}dt \tag{1}$$
$$F'(x) = \sqrt[4]{x} \tag{2}$$

14 Find $F'(x)$.

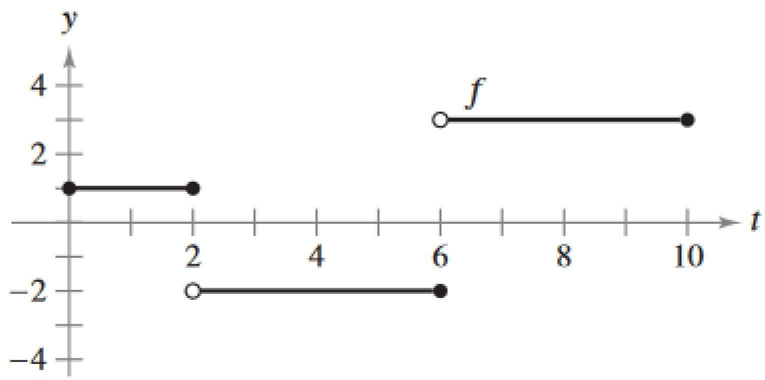
88.

$$F(x) = \int_{-x}^x t^3 dt = \left(\frac{t^4}{4}\right)_{-x}^x = 0 \tag{1}$$
$$F'(x) = 0 \tag{2}$$

92.

$$F(x) = \int_0^{x^2} \sin\theta^2 d\theta \tag{1}$$
$$F'(x) = \sin(x^2)^2(2x) \tag{2}$$
$$= 2x\sin x^4 \tag{3}$$

94. Use the graph of the function f show in the figure and the function called g defined by $g(x) = \int_0^x f(t)dt$.



(a) Complete the table.

x	1	2	3	4	5	6	7	8	9	10
$g(x)$	1	2	0	-2	-4	-6	-3	0	3	6

(b) Where does g have its minimum? Explain.
g is minimum at (6, -6).

(c) Where does g have a maximum? Explain.
g has a relative maximum at (2, 2).

(d) On what interval does g increase at the greatest rate? Explain.
g increases at a rate of 3 on [6, 10].

(e) Identify the zeros of g.
The zeros of g are $x = 3,\ 8$.

15 The velocity function, in feet per second, is given for a particle moving along a straight line. Find (a) the displacement and (b) the total distance that the particle travels over the given interval.

101.

(a)

$$v(t) = \frac{1}{\sqrt{t}},\ 1 \le t \le 4 \tag{1}$$
$$\text{Total Distance} = \text{Displacement} \tag{2}$$
$$\text{Displacement} = \int_1^4 t^{-1/2} dt \tag{3}$$
$$= \left(2t^{1/2}\right)_1^4 \tag{4}$$
$$= 4 - 2 = 2\text{ft to the right} \tag{5}$$

(b) The total distance is 2 feet.

102.

(a)

$$v(t) = \cos t,\ 0 \le t \le 3\pi \quad (1)$$
$$\text{Displacement} = \int_0^{3\pi} \cos t dt \quad (2)$$
$$= (\sin t)_0^{3\pi} = 0\text{ft} \quad (3)$$
$$(4)$$

(b)

$$\text{Total Distance} = \int_0^{\pi/2} \cos t dt - \int_{\pi/2}^{3\pi/2} \cos t dt + \int_{3\pi/2}^{5\pi/2} \cos t dt - \int_{5\pi/2}^{3\pi} \cos t dt \quad (1)$$
$$= (\sin t)_0^{\pi/2} - (\sin t)_{\pi/2}^{3\pi/2} + (\sin t)_{3\pi/2}^{5\pi/2} - (\sin t)_{5\pi/2}^{3\pi} \quad (2)$$
$$= 1 - (-2) + 2 - (-1) = 6 \quad (3)$$

16 Oil Leak

106. At 1:00 P.M., oil begins leaking from a tank at a rate of $4 + 0.75t$ gallons per hour.

(a) How much oil is lost from 1:00 P.M. to 4:00 P.M.?

$$\int_0^3 (4 + 0.75t) dt \quad (1)$$
$$= \left(4t + \frac{0.75}{2} t^2\right)_0^3 \quad (2)$$
$$= \frac{123}{8} = 15.375\text{gal} \quad (3)$$

(b) How much oil is lost from 4:00 P.M. to 7:00 P.M.?

$$\int_3^6 (4 + 0.75t) dt \quad (1)$$
$$= \left(4t + \frac{0.75}{2} t^2\right)_3^6 \quad (2)$$
$$= 22.125\text{gal} \quad (3)$$

(c) Compare your answers from parts (a) and (b). What do you notice? The oil lost in the evening is more because the flow rate increased.

17 Determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

113. If $F'(x) = G'(x)$ on the interval [a, b], then $F(b) - F(a) = G(b) - G(a)$.
True.

114. If f is continuous on [a, b], then f is integrable on [a, b].
True.