Notes - 4.4 The Fundamental Theorem of Calculus

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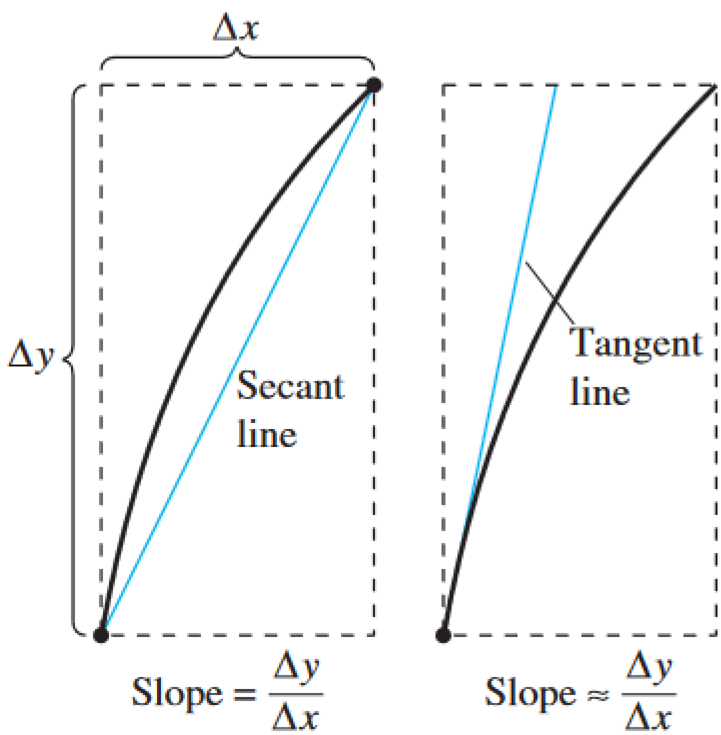
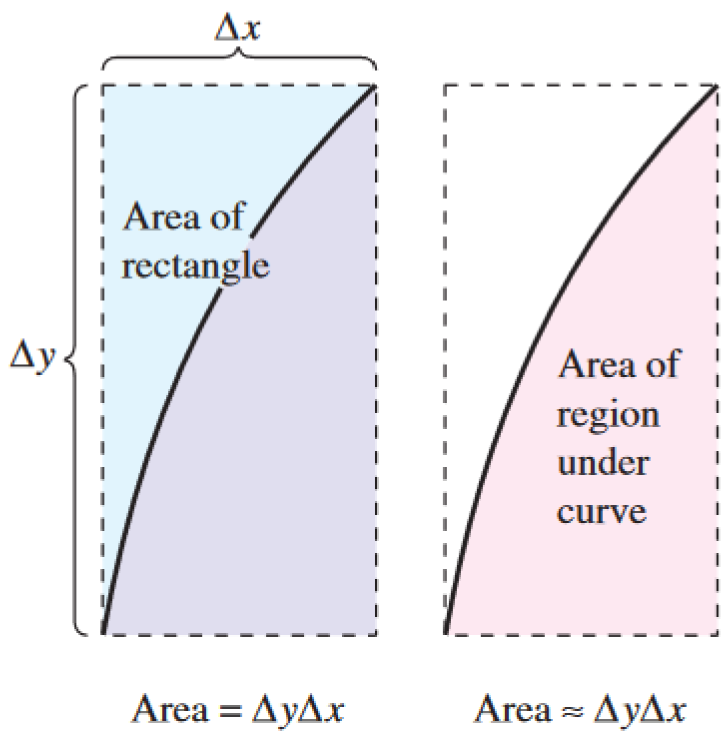
1. Evaluate a definite integral using the Fundamental Theorem of Calculus.
2. Understand and use the Mean Value Theorem for Integrals.
3. Find the average value of a function over a closed interval.
4. Understand and use the Second Fundamental Theorem of Calculus.
5. Understand and use the Net Change Theorem.

The theorem states that differentiation and (definite) integration are inverse operations.

(a) Differentiation



(b) Definite integration



1 The fundamental theorem of calculus (again)

If a function f is continuous on the closed interval [a, b] and F is an antiderivative of f on the interval [a, b], then

$$\int_a^b f(x)dx = F(b) - F(a)$$

In other words, every function that is continuous has an integral

1.1 Guidelines for using the fundamental theorem of calculus

1. *Provided you can find* an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the following notation is convenient.

$$\begin{aligned}\int_a^b f(x)\,dx &= F(x)\Big]_a^b \\ &= F(b) - F(a)\end{aligned}$$

   For instance, to evaluate $\int_1^3 x^3\,dx$, you can write

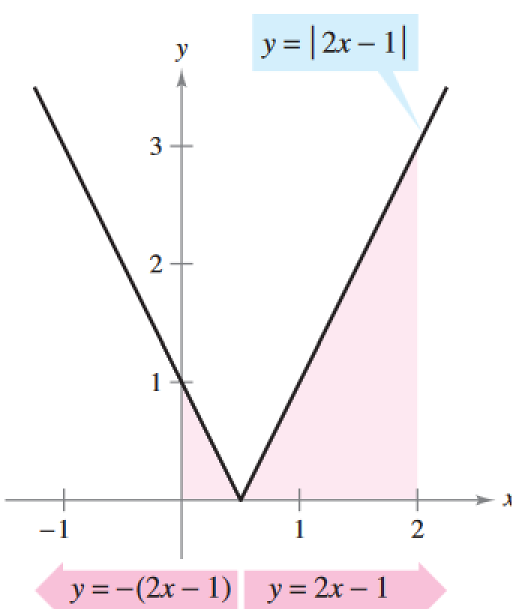
$$\int_1^3 x^3\,dx = \frac{x^4}{4}\Big]_1^3 = \frac{3^4}{4} - \frac{1^4}{4} = \frac{81}{4} - \frac{1}{4} = 20.$$

3. It is not necessary to include a constant of integration C in the antiderivative because

$$\begin{aligned}\int_a^b f(x)\,dx &= \Big[F(x) + C\Big]_a^b \\ &= [F(b) + C] - [F(a) + C] \\ &= F(b) - F(a).\end{aligned}$$

2 Example - A definite integral involving absolute value

Evaluate $\int_0^2 |2x - 1| dx$.



The definite integral of y on $[0, 2]$ is $\frac{5}{2}$.

Solution Using Figure 4.27 and the definition of absolute value, you can rewrite the integrand as shown.

$$|2x - 1| = \begin{cases} -(2x - 1), & x < \frac{1}{2} \\ 2x - 1, & x \geq \frac{1}{2} \end{cases}$$

From this, you can rewrite the integral in two parts.

$$\begin{aligned}\int_0^2 |2x - 1|\,dx &= \int_0^{1/2} -(2x - 1)\,dx + \int_{1/2}^2 (2x - 1)\,dx \\ &= \Big[-x^2 + x\Big]_0^{1/2} + \Big[x^2 - x\Big]_{1/2}^2 \\ &= \left(-\frac{1}{4} + \frac{1}{2}\right) - (0 + 0) + (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right) \\ &= \frac{5}{2}\end{aligned}$$

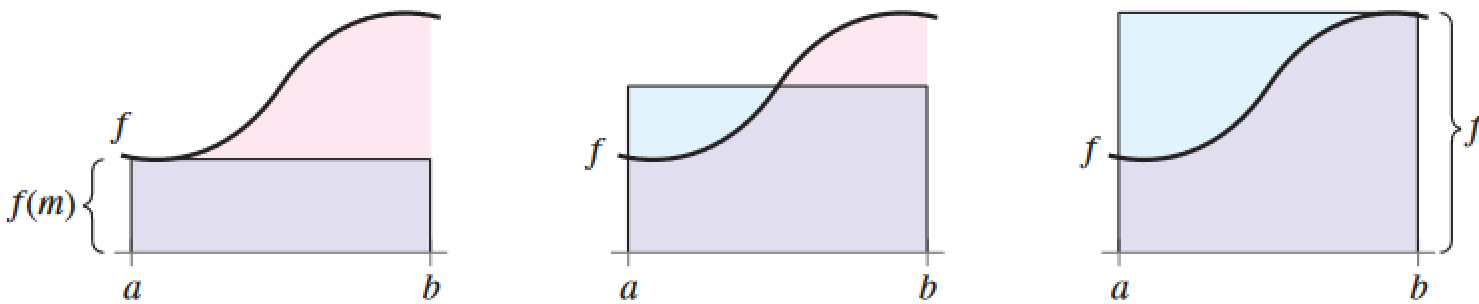
3 Mean value theorem for integrals

If f is continuous on the closed interval [a, b], then there exists a number in the closed interval [a, b] such that

$$\int_a^b f(x)dx = f(c)(b-a)$$

Average height of the function over a certain period of time.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx \qquad \text{or} \qquad f(c)(b-a) = \int_a^b f(x)\,dx.$$



Inscribed rectangle (less than actual area)

$$\int_a^b f(m)\,dx = f(m)(b-a)$$

Mean value rectangle (equal to actual area)

$$\int_a^b f(x)\,dx$$

Circumscribed rectangle (greater than actual area)

$$\int_a^b f(M)\,dx = f(M)(b-a$$

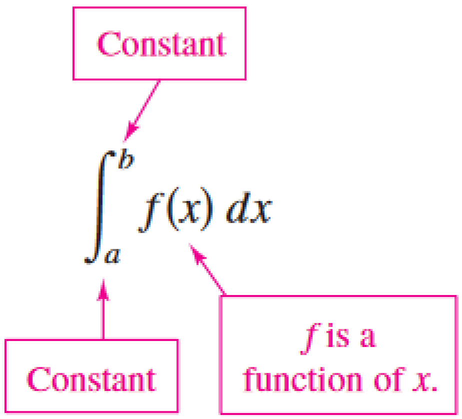
3.1 Average value of a function

If f is integrable on the closed interval [a, b], then the average value of f on the interval is

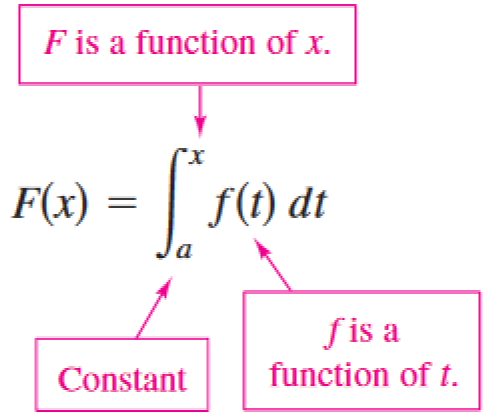
$$\frac{1}{b-a}\int_a^b f(x)dx$$

4 The second fundamental theorem of calculus

The Definite Integral as a Number



The Definite Integral as a Function of x



EXPLORATION

Use a graphing utility to graph the function

$$F(x) = \int_0^x \cos t\,dt$$

for $0 \leq x \leq \pi$. Do you recognize this graph? Explain.

EXAMPLE 6 The Definite Integral as a Function

Evaluate the function

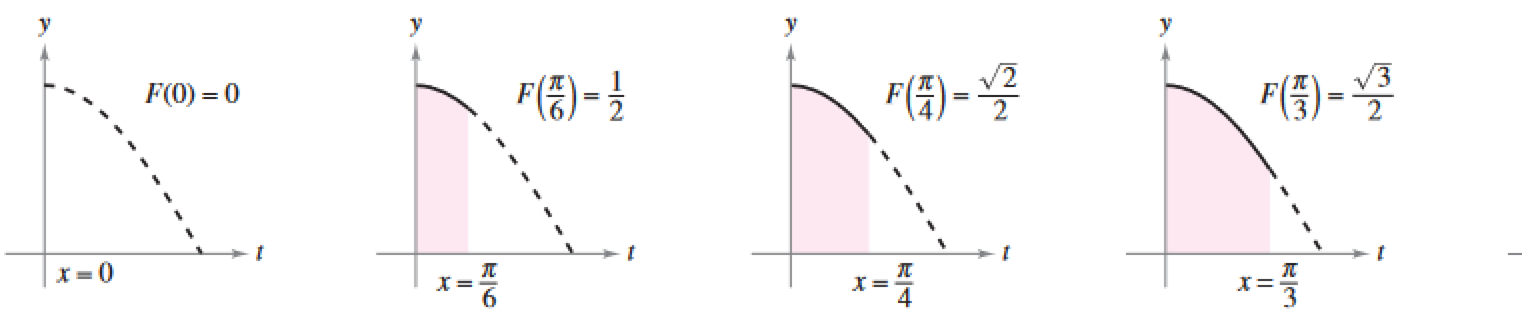
$$F(x) = \int_0^x \cos t\,dt$$

at $x = 0, \pi/6, \pi/4, \pi/3$, and $\pi/2$.

Solution You could evaluate five different definite integrals, on
given upper limits. However, it is much simpler to fix x (as a con
to obtain

$$\int_0^x \cos t\,dt = \sin t\Big]_0^x = \sin x - \sin 0 = \sin x.$$

Now, using $F(x) = \sin x$, you can obtain the results shown in Figur



If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}(F(x) - F(a)) = \frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

5 Net change theorem

$$\int_a^b F'(x)dx = F(b) - F(a)$$

EXAMPLE 9 Using the Net Change Theorem

A chemical flows into a storage tank at a rate of $180 + 3t$ liters per minu
$0 \leq t \leq 60$. Find the amount of the chemical that flows into the tank durin
20 minutes.

Solution Let $c(t)$ be the amount of the chemical in the tank at time t. T
represents the rate at which the chemical flows into the tank at time t. Durin
20 minutes, the amount that flows into the tank is

$$\begin{aligned}\int_0^{20} c'(t)\,dt &= \int_0^{20} (180 + 3t)\,dt \\ &= \left[180t + \frac{3}{2}t^2\right]_0^{20} \\ &= 3600 + 600 = 4200.\end{aligned}$$

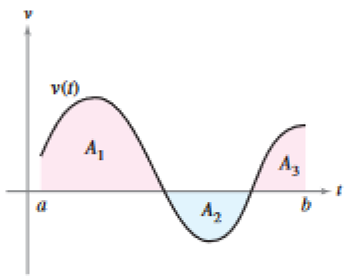
So, the amount that flows into the tank during the first 20 minutes is 4200 l

When calculating the *total* distance traveled by the particle, you must consider the intervals where $v(t) \le 0$ and the intervals where $v(t) \ge 0$. When $v(t) \le 0$, the particle moves to the left, and when $v(t) \ge 0$, the particle moves to the right. To calculate the total distance traveled, integrate the absolute value of velocity $|v(t)|$. So, the displacement of a particle and the total distance traveled by a particle over $[a, b]$ can be written as

$$\textbf{Displacement on } [a, b] = \int_a^b v(t)\,dt = A_1 - A_2 + A_3$$

$$\textbf{Total distance traveled on } [a, b] = \int_a^b |v(t)|\,dt = A_1 + A_2 + A_3$$

(see Figure 4.36).



A_1, A_2, and A_3 are the areas of the shaded regions.
Figure 4.36

EXAMPLE 10 Solving a Particle Motion Problem

A particle is moving along a line so that its velocity is $v(t) = t^3 - 10t^2 + 29t - 20$ feet per second at time t.

a. What is the displacement of the particle on the time interval $1 \le t \le 5$?

b. What is the total distance traveled by the particle on the time interval $1 \le t \le 5$?

Solution

a. By definition, you know that the displacement is

$$\begin{aligned}\int_1^5 v(t)\,dt &= \int_1^5 (t^3 - 10t^2 + 29t - 20)\,dt \\ &= \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_1^5 \\ &= \frac{25}{12} - \left(-\frac{103}{12}\right) \\ &= \frac{128}{12} \\ &= \frac{32}{3}.\end{aligned}$$

So, the particle moves $\frac{32}{3}$ feet to the right.

b. To find the total distance traveled, calculate $\int_1^5 |v(t)|\,dt$. Using Figure 4.37 and the fact that $v(t)$ can be factored as $(t - 1)(t - 4)(t - 5)$, you can determine that $v(t) \ge 0$ on $[1, 4]$ and $v(t) \le 0$ on $[4, 5]$. So, the total distance traveled is

$$\begin{aligned}\int_1^5 |v(t)|\,dt &= \int_1^4 v(t)\,dt - \int_4^5 v(t)\,dt \\ &= \int_1^4 (t^3 - 10t^2 + 29t - 20)\,dt - \int_4^5 (t^3 - 10t^2 + 29t - 20)\,dt \\ &= \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_1^4 - \left[\frac{t^4}{4} - \frac{10}{3}t^3 + \frac{29}{2}t^2 - 20t\right]_4^5 \\ &= \frac{45}{4} - \left(-\frac{7}{12}\right) \\ &= \frac{71}{6} \text{ feet.}\end{aligned}$$

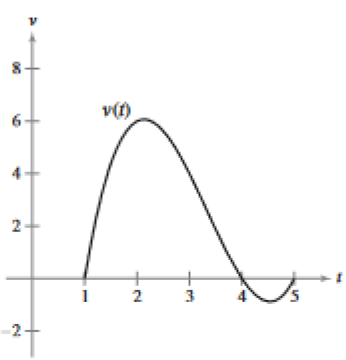


Figure 4.37

6 11/30/2023 - Warm-up

1. If $\frac{dy}{dx} = \cos(2x)$, then $y =$

$$
\begin{align}
dy &= \cos(2x)dx \tag{1}\\
&= \int cos(2x)dx \tag{2}\\
\tag{3}
\end{align}
$$

Using u-substitution:

$$
\begin{align}
u &= 2x \tag{4}\\
du &= 2dx \tag{5}\\
\frac{du}{2} &= dx \tag{6}
\end{align}
$$

Therefore, following euqation 3

$$
\begin{align}
&= \int \frac{1}{2}\cos(u)du \tag{7}\\
y &= \frac{1}{2}\int \cos u du \tag{8}\\
&= \frac{1}{2}\int \cos u du + C \tag{9}\\
\tag{10}
\end{align}
$$

Substituting back the u:

$$
\begin{align}
y &= \frac{1}{2}\sin u + C \tag{11}\\
&= \frac{1}{2}\sin(2x) + C \tag{12}
\end{align}
$$

2. The absolute maximum value of $f(x) = x^3 - 3x^2 + 12$ on the closed interval occurs when $x =$

$$
\begin{align}
f'(x) &= 3x^2 - 6x = 0,\ (-2, 4) \tag{1}\\
&= 3x(x-2) = 0 \tag{2}\\
3x &= 0 \text{ and } x - 2 = 0 \tag{3}\\
x &= 0,\ 2 \text{ are critical numbers} \tag{4}
\end{align}
$$

First Derivative Test:

$$
f'(0) < 12 f'(4) > 12 \tag{5}
$$

Therefore, $x = 4$ is the absolute maximum.