4.2 Area

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1 Find the sum. Use the summation capabilities of a graphing utility to verify your result.

2.

$$\sum_{k=5}^{8} k(k-4) \tag{1}$$
$$= 5(1) + 6(2) + 7(3) + 8(4) = 70 \tag{2}$$

6.

$$\sum_{i=1}^{4}((i-1)^2 + (i+1)^3) \tag{1}$$
$$= (0+8) + (1+27) + (4+64) + (9+125) \tag{2}$$
$$= 238 \tag{3}$$
$$\tag{4}$$

2 Use sigma notation to write the sum.

8.

$$\frac{9}{1+1} + \frac{9}{1+2} + \frac{9}{1+3} \cdots + \frac{9}{1+14} \tag{1}$$
$$= \sum_{i=1}^{14} \frac{9}{1+i} \tag{2}$$

12.

$$\left(1 - \left(\frac{2}{n} - 1\right)^2\right)\left(\frac{2}{n}\right) + \cdots + \left(1 - \left(\frac{2n}{n} - 1\right)^2\right)\left(\frac{2}{n}\right) \tag{1}$$
$$= \frac{2}{n}\sum_{i=1}^{n}\left(1 - \left(\frac{2i}{n} - 1\right)^2\right) \tag{2}$$

3 Use the properties of summation and Theorem 4.2 (summation formulas) to evaluate the sum. Use the summation capabili- ties of a graphing utility to verify your result.

16.

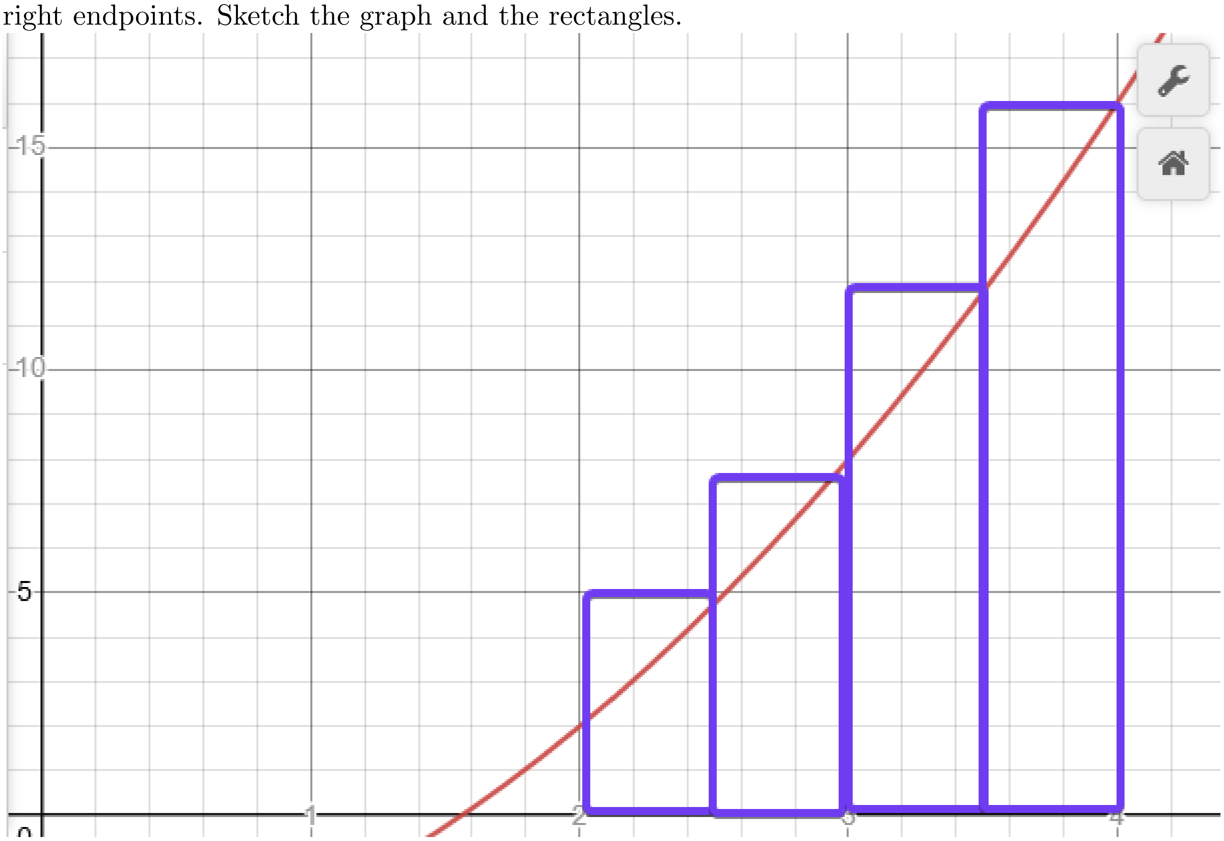
$$\sum_{i=1}^{30} -18 \tag{1}$$
$$= (-18)(30) \tag{2}$$
$$= -540 \tag{3}$$

20.

$$\sum_{i=1}^{10} (i^2 - 1) \tag{1}$$
$$= \sum_{i=1}^{10} i^2 - \sum_{i=1}^{10} 1 \tag{2}$$
$$= \left(\frac{10(11)(21)}{6}\right) - 10 \tag{3}$$
$$= 375 \tag{4}$$

26. Consider the function $g(x) = x^2 + x - 4$.

(a) Estimate the are between the graph of g and the x-axis between $x = 2$ and $x = 4$ using four rectangles and right endpoints. Sketch the graph and the rectangles.



The Δx width of each rectangle is $\frac{1}{2}$. The right endpoints yield the heights.

$$\begin{aligned}
\text{Area} &\approx \frac{1}{2}\left(\left(\left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right) - 4\right) + (3^2 + 3 - 4) + \left(\left(\frac{7}{2}\right)^2 + \frac{7}{2} - 4\right) + (4^2 + 4 - 4)\right) && (1)\\
&= \frac{81}{4} && (2)\\
&= 20.25 && (3)
\end{aligned}$$

4 Use left and right endpoints and the given number of rectangles to find two approximations of the area of the region between the graph of the function and the -axis over the given interval.

28.$f(x) = 9 - x,\ [2,4],\ 6$ rectangles

$$\begin{aligned}
\Delta x = \frac{4-2}{6} &= \frac{1}{3} && (1)\\
\text{Left endpoints: Area} &\approx \frac{1}{3}\left(7 + \frac{20}{3} + \frac{19}{3} + 6 + \frac{17}{3} + \frac{16}{3}\right) = \frac{37}{3} && (2)\\
\text{Left endpoints: Area} &\approx \frac{1}{3}\left(\frac{20}{3} + \frac{19}{3} + 6 + \frac{17}{3} + \frac{16}{3} + \frac{15}{3}\right) = \frac{35}{3} && (3)\\
\frac{35}{3} &< \text{Area} < \frac{37}{3} && (4)
\end{aligned}$$

30.$g(x) = x^2 + 1,\ [1,3],\ 8$ rectangles

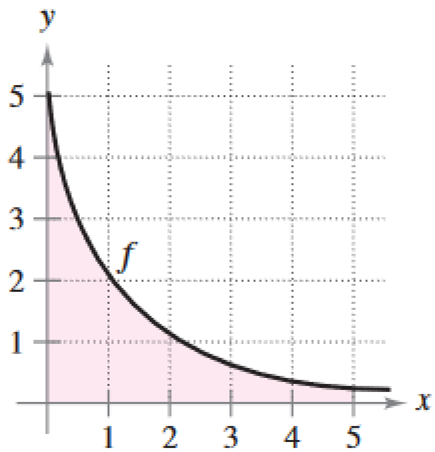
$$\begin{aligned}
\Delta x = \frac{3-1}{8} &= \frac{1}{4} && (1)\\
\text{Left endpoints: Area} &\approx \frac{1}{4}\left(2 + \frac{41}{16} + \frac{13}{4} + \frac{65}{16} + 5 + \frac{97}{16} + \frac{29}{4} + \frac{137}{16}\right) = \frac{155}{16} = 9.6875 && (2)\\
\text{Left endpoints: Area} &\approx \frac{1}{4}\left(\frac{41}{16} + \frac{13}{4} + \frac{65}{16} + 5 + \frac{97}{16} + \frac{29}{4} + \frac{137}{16} + 10\right) = 11.6875 && (3)\\
9.6875 &< \text{Area} < 11.6875 && (4)
\end{aligned}$$

32.$g(x) = \sin x,\ [0,\pi],\ 4$ rectangles

$$\begin{aligned}
\Delta x = \frac{\pi - 0}{6} &= \frac{\pi}{6} && (1)\\
\text{Left endpoints: Area} &\approx \frac{\pi}{6}\left(\sin 0 + \sin\frac{\pi}{6} + \sin\frac{\pi}{3} + \sin\frac{\pi}{2} + \sin\frac{2\pi}{3} + \sin\frac{5\pi}{6}\right) \approx 1.9541 && (2)\\
\text{Left endpoints: Area} &\approx \frac{\pi}{6}\left(\sin\frac{\pi}{6} + \sin\frac{\pi}{3} + \sin\frac{\pi}{2} + \sin\frac{2\pi}{3} + \sin\frac{5\pi}{6} + \sin\pi\right) \approx 1.9541 && (3)\\
&\text{The answers are the same by symmetry. The exact area of 2 is larger.} && (4)
\end{aligned}$$

5 Bound the area of the shaded region by approximating the upper and lower sums. Use rectangles of width 1.

36.



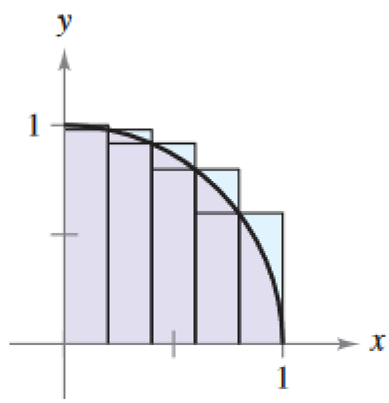
$$S = \left(5 + 2 + 1 + \frac{2}{3} + \frac{1}{2}\right)(1) = \frac{55}{6} \tag{1}$$

$$s = \left(2 + 1 + \frac{2}{3} + \frac{1}{2} + \frac{1}{3}\right)(1) = \frac{9}{2} \tag{2}$$

6 Use upper and lower sums to approximate the area of the region using the given number of subintervals (of equal width).

44.

$y = \sqrt{1 - x^2}$



$$S(5) = 1\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) \tag{1}$$

$$= \frac{1}{5}\left(1 + \frac{\sqrt{24}}{5} + \frac{\sqrt{21}}{5} + \frac{\sqrt{16}}{5} + \frac{\sqrt{9}}{5}\right) \approx 0.859 \tag{2}$$

$$s(5) = \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) + 0 \approx 0.659 \tag{3}$$

7 Use the Midpoint Rule Area $\approx \sum_{i=1}^{n} f\left(\frac{x_i+x_{i-1}}{2}\right)\Delta x$ with $n=4$ to approximate the area of the region bounded by the graph of the function and the x-axis over the given interval.

74.

$$f(x) = x^2 + 4x,\ [0,4],\ n=4 \tag{1}$$
$$\text{Let } c_i = \frac{x_i + x_{i-1}}{2} \tag{2}$$
$$\Delta x = 1,\ c_1 = \frac{1}{2},\ c_2 = \frac{3}{2},\ c_3 = \frac{5}{2},\ c_4 = \frac{7}{2} \tag{3}$$
$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i-1}^{4}(c_i^2 + 4c_i)(1) \tag{4}$$
$$= \left(\left(\frac{1}{4}+2\right) + \left(\frac{9}{4}+6\right) + \left(\frac{25}{4}+10\right) + \left(\frac{49}{4}+14\right)\right) \tag{5}$$
$$= 53 \tag{6}$$

76.

$$f(x) = \sin x,\ 0 \le x \le \frac{\pi}{2},\ n=4 \tag{1}$$
$$\text{Let } c_i = \frac{x_i + x_{i-1}}{2} \tag{2}$$
$$\Delta x = \frac{\pi}{8},\ c_1 = \frac{\pi}{16},\ c_2 = \frac{3\pi}{16},\ c_3 = \frac{5\pi}{16},\ c_4 = \frac{7\pi}{16} \tag{3}$$
$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x \tag{4}$$
$$= \sum_{i=1}^{4}(\sin c_i)\left(\frac{\pi}{8}\right) = \frac{\pi}{8}\left(\sin\frac{\pi}{16} + \sin\frac{3\pi}{16} + \sin\frac{5\pi}{16} + \sin\frac{7\pi}{16}\right) \approx 1.006 \tag{5}$$

8 Capstone

86. Consider a function $f(x)$ that is increasing on the interval $[1,4]$ The interval $[1,4]$ is divided into 12 subintervals.

(a) What are the left endpoints of the first and last subintervals?

$$\Delta x = \frac{4-1}{12} = \frac{1}{4} \tag{1}$$

The left endpoint of the first subinterval is 1, and the left endpoint of the last subinterval is $4 - \frac{1}{4} = \frac{15}{4}$.

(b) What are the right endpoints of the first two subintervals?
The right end points of the first two subintervals are $1 + \frac{1}{4} = \frac{5}{4}$ and $1 + 2\left(\frac{1}{4}\right) = \frac{3}{2}$.

(c) When using the right endpoints, will the rectangles lie above or below the graph of $f(x)$¿ Use a graph to explain your answer.
When using right endpoints, the rectangles will be above the curve.

(d) What can you conclude about the heights of the rectangles if a function is constant on the given interval?
The height of the rectangles are the same for a constant function.