5.3 Inverse functions

Juan J. Moreno Santos

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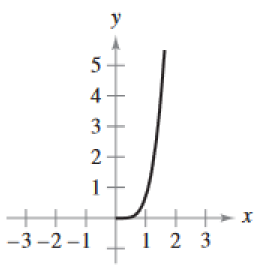
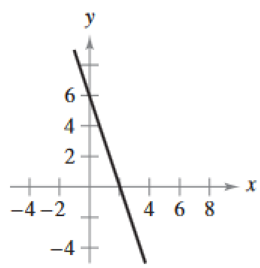
1 The graph of the function with the graph of its inverse function. [The graphs of the inverse functions are labeled (a), (b), (c), and (d).]

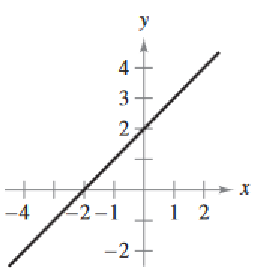
(a)



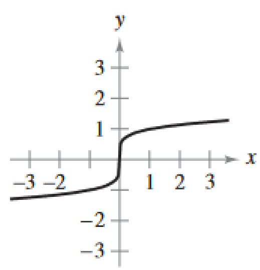
(b)



(c)



(d)



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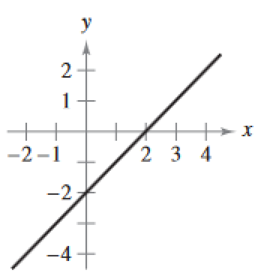

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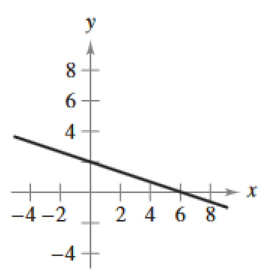

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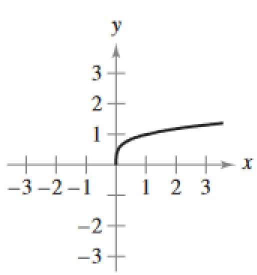

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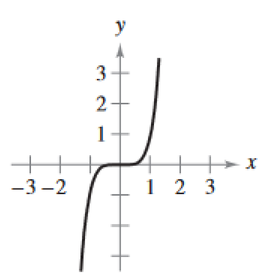


10 matches (b) and 12 matches (d)

2 Use a graphing utility to graph the function. Then use the Horizontal Line Test to determine whether the function is one-to-one on its entire domain and therefore has an inverse function.

16. $f(x) = \frac{x^2}{x^2+4}$
The function is not one-to-one and doesn't have an inverse.

3 Find the inverse function of f, (b) graph f and f^{-1} on the same set of coordinate axes, (c) describe the relationship between the graphs, and (d) state the domain and range of f and f^{-1}.

28.

$$f(x) = x^2 = y,\ x \geq 0 \quad (1)$$
$$x = \sqrt{y} \quad (2)$$
$$y = \sqrt{y} \quad (3)$$
$$f^{-1}(x) = \sqrt{x} \quad (4)$$

f and f^{-1} reflect each other on the line $y = x$. The domain and range of f are $x \geq 0$ and $y \geq 0$ respectively, and the domain and range of f^{-1} are $x \geq 0$ and $y \geq 0$ respectively.

4 Word problems

40. The formula $C = \frac{5}{9}(F - 32)$, where $F \geq 459.6$, represents Celsius temperature C as a function of Fahrenheit temperature F.

(a) Find the inverse function of C

$$\frac{9}{5}F - 32 \quad (1)$$
$$F = 32 + \frac{9}{5}C \quad (2)$$

(b) What does the inverse function represent? It represents the temperature conversion from degrees Celsius to Fahrenheit.

(c) What is the domain of the inverse function¿ Validate or explain your answer using the context of the problem.

$$F \geq -459.6,\ C = \frac{5}{9}(F - 32) \geq -273.11 \therefore \text{Domain:} C \geq -273.1 = -273\frac{1}{9} \quad (1)$$

(d) The temperature is 22°C. What is the corresponding temperature in degrees Fahrenheit?

$$F = 32 + \frac{9}{5}(22) = 71.6°F \quad (1)$$

5 Use the derivative to determine whether the function is strictly monotonic on its entire domain and therefore has an inverse function.

42.

$$f(x) = x^3 - 6x^2 + 12x \quad (1)$$
$$f'(x) = 3x^2 - 12x + 12 = 3(x - 2)^2 \quad (2)$$

f is strictly monotonic and has an inverse because it's increasing on $(-\infty, \infty)$.

46.

$$f(x) = \cos\frac{3x}{2} \tag{1}$$
$$f'(x) = -\frac{3}{2}\sin\frac{3x}{2} = 0 \text{ when } x = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \cdots \tag{2}$$

6 Determine whether the functions is one-to-one. If it is, find its inverse function.

60. $f(x) = -3$

Doesn't have an inverse since it's not one-to-one.

62.

$$f(x) = ax + b \tag{1}$$

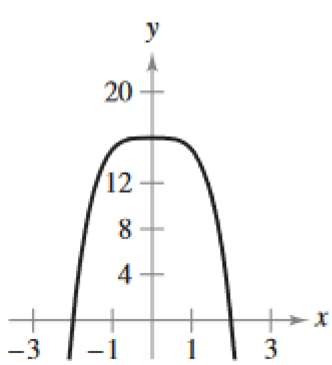
f is one-to-one and has an inverse.

$$ax + b = y \tag{1}$$
$$x = \frac{y-b}{a} \tag{2}$$
$$y = \frac{x-b}{a} \tag{3}$$

7 Delete part of the domain so that the function that remains is one-to-one. Find the inverse function of the remaining function and give the domain of the inverse function.

64.

$$f(x) = 16 - x^4 \text{ will be one-to-one for } x \geq 0 \tag{1}$$
$$16 - x^4 = y \tag{2}$$
$$16 - y = x^4 \tag{3}$$
$$\sqrt[4]{16-y} = x^4 \tag{4}$$
$$\sqrt[4]{16-x} = y \tag{5}$$
$$f^{-1}(x) = \sqrt[4]{16-x},\ x \leq 16 \tag{6}$$



8 Decide whether the function has an inverse function. If so, what is the inverse function?

68. $h(t)$ is the height of the tide t hours after midnight, where $0 \leq t \leq 24$.

The function doesn't have an inverse because there could be two times $t_1 \neq t_2$ for which $h(t_1) = h(t_2)$.

70. $A(r)$ is the area of a circle of radius r.

Yes. The function is one-to-one since it's increasing. Its inverse yields the radius r that corresponds to the area A.

9 Verify that f has an inverse. Then use the function f and the given real number a to find $(f^{-1})'(a)$.

72.

$$f(x) = 5 - 2x^3,\ a = 7 \tag{1}$$
$$f'(x) = -6x^2 \tag{2}$$

f is decreasing on $(-\infty,\ \infty)$. Therefore, f has an inverse and is monotonic.

$$f(-1) = 7 \Rightarrow f^{-1}(7) = -1 \tag{3}$$
$$(f^{-1})'(7) = \frac{1}{f'(f^{-1}(7))} = \frac{1}{f'(-1)} = \frac{1}{-6(-1)^2} = \frac{-1}{6} \tag{4}$$

76.

$$f(x) = \cos 2x,\ a = 1,\ 0 \leq x \leq \frac{\pi}{2} \tag{1}$$
$$f'(x) = -2\sin 2x < 0 \text{ on } (0,\ \frac{\pi}{2}) \tag{2}$$

f has an inverse since it's monotonic on $[0,\ \frac{\pi}{2}]$.

$$f(0) = 1 \Rightarrow f^{-1}(1) = 0 \tag{3}$$
$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{-2\sin 0} = \frac{1}{0} = \text{undefined} \tag{4}$$

80.

$$f(x) = \sqrt{x-4},\ a = 2,\ x \geq 4 \tag{1}$$
$$f'(x) = \frac{1}{2\sqrt{x-4}} > 0 \text{ on } (4,\ \infty) \tag{2}$$

f has an inverse since it's monotonic on $[4,\ \infty]$

10 Find the domains of f and f^{-1}, find the ranges of f and f^{-1}, graph f and f^{-1}, and show that the slopes of the graphs of f and f^{-1} are reciprocals at the given point.

82.

$$f(x) = 3 - 4x,\ (1,\ -1);\ f^{-1} = \frac{3-x}{4},\ (-1,\ 1) \tag{1}$$
$$\text{Domain } f = \text{Domain } f^{-1} = (-\infty,\ \infty) \tag{2}$$
$$\text{Range } f = \text{Range } f^{-1} = (-\infty,\ \infty) \tag{3}$$
$$f'(x) = -4 \tag{4}$$
$$f'(-1) = -4 \tag{5}$$
$$(f^{-1})'(x) = -\frac{1}{4} \tag{6}$$
$$(f^{-1})'(-1) = -\frac{1}{4} \tag{7}$$

11 Find $\frac{dy}{dx}$ for the equation at the given point.

86.

$$x = 2\ln(y^2 - 3),\ (0,\ 2) \tag{1}$$
$$1 = 2\frac{1}{y^2-3}2y\frac{dy}{dx} \tag{2}$$
$$\frac{dy}{dx} = \frac{y^2-3}{4y} \tag{3}$$
$$= \frac{4-3}{8} = \frac{1}{8} \tag{4}$$

12 Use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the given value.

88.

$$(g^{-1} \circ f^{-1})(-3) \tag{1}$$
$$= g^{-1}(f^{-1}(-3)) \tag{2}$$
$$= g^{-1}(0) = 0 \tag{3}$$

13 Use the functions $f(x)x + 4$ and $g(x) = 2x - 5$ to find the given function.

94.

$$(g \circ f)(x) = g(f(x)) \tag{1}$$
$$= g(x+4) \tag{2}$$
$$= 2(x+4) - 5 \tag{3}$$
$$= 2x + 3 \tag{4}$$
$$(g \circ f)^{-1}(x) = \frac{x-3}{2} \tag{5}$$

14 Determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

101. If f is an even function, then f^{-1} exists.
False. This isn't true with $f(x) = x^2$.

102. If the inverse function of f exists, then the y-intercept of f is an x-intercept of f^{-1}.
True.

103. If $f(x) = x^n$, where n is odd, then f^{-1} exists.
True.

104. There exists no function f such that $f = f^{-1}$.
False. One of these function is $f(x) = x$.