3.3 Increasing and Decreasing Functions and the First Derivative Test

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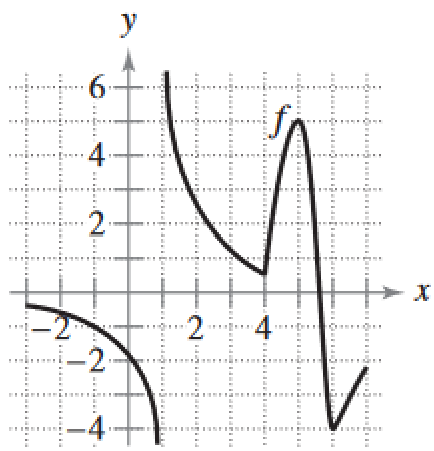
1 Use the graph of f to find (a) the largest open interval on which f is increasing, and (b) the largest open interval on which f is decreasing.

2.



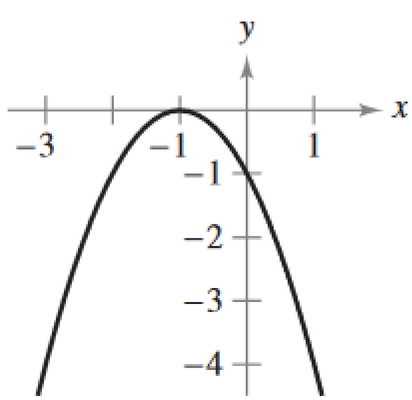
(a) The intervals in which f is increasing are (4,5) and (6, 7). They are also the largest.

(b) The intervals in which f is decreasing are (-3, 1), (1, 4), and (5, 6). (-3, 1) is the largest.

2 Use the graph to estimate the open intervals on which the function is increasing or decreasing. Then find the open intervals analytically.

4.

$$y = -(x+1)^2$$

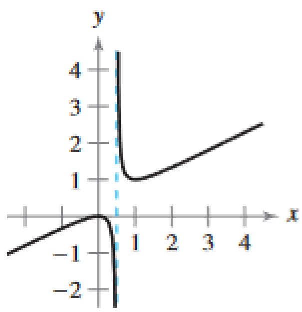


Graphically, f increases on $(-\infty, -1)$ and decreases on $(-1, \infty)$.
Analytically, $y' = -2(x+1) \therefore x = -1$ is a critical number.

Test intervals	$-\infty < x < -1$	$-1 < x < \infty$
y' sign	$y' > 0$	$y' < 0$
Conclusion	Increasing	Decreasing

8.

$$y = \frac{x^2}{2x-1}$$



Graphically, y increases on $(-\infty, 0)$ and $(1, \infty)$, and decreases on $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$.
Analytically, $y' = \frac{(2x-1)2x-x^2(2)}{(2x-1)^2} = \frac{2x^2-2x}{(2x-1)^2} = \frac{2x(x-1)}{(2x-1)^2}$ $\therefore x = 0, 1$ are critical numbers, and there is a discontinuity at $x = \frac{1}{2}$.

Test intervals	$-\infty < x < 0$	$0 < x < \frac{1}{2}$	$\frac{1}{2} < x < 1$	$1 < x < \infty$
y' sign	$y' > 0$	$y' < 0$	$y' < 0$	$y' > 0$
Conclusion	Increasing	Decreasing	Decreasing	Increasing

3 Identify the open intervals on which the function is increasing or decreasing.

10.

$$\begin{aligned} h(x) &= 27x - x^3 && (1) \\ h'(x) &= 27 - 3x^2 && (2) \\ &= 3(3-x)(3+x) && (3) \\ &= 0 && (4) \end{aligned}$$

$x = \pm 3$ are critical numbers.

Test intervals	$-\infty < x < -3$	$-3 < x < 3$	$3 < x < \infty$
h' sign	$h' < 0$	$h' > 0$	$h' < 0$
Conclusion	Decreasing	Increasing	Decreasing

The function is increasing on (-3, 3) and decreasing on $(-\infty, -3)$ and $(3, \infty)$.

14.

$$h(x) = \cos\frac{x}{2}, \quad 0 < x < 2\pi \tag{5}$$
$$h'(x) = -\frac{1}{2}\sin\frac{x}{2} \tag{6}$$

The function has no critical numbers.

Test intervals	$0 < x < 2\pi$
h' sign	$h' < 0$
Conclusion	Decreasing

The function is decreasing on $0 < x < 2\pi$.

4 (a) Find the critical numbers fof f (if any), (b) find the open interval(s) on which the function is increasing or decreasing, (c) apply the First Derivative Test to identify all relative extrema.

18.

(a)

$$f(x) = x^2 + 6x + 10 \tag{1}$$
$$f'(x) = 2x + 6 \tag{2}$$

$x = -3$ is a critical number.

Test intervals	$-\infty < x < -3$	$-3 < x < \infty$
f' sign	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing

(b) The function is decreasing on $(-\infty, -3)$ and increasing on $(3, \infty)$

(c) (-3, 1) is the relative minimum.

22.

(a)

$$f(x) = x^3 - 6x^2 + 15 \tag{1}$$
$$f'(x) = 3x^2 - 12x \tag{2}$$
$$= 3x(x-4) \tag{3}$$

$x = 0, 4$ are critical numbers.

Test intervals	$-\infty < x < 0$	$0 < x < 4$	$4 < x < \infty$
f' sign	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion	Increasing	Decreasing	Increasing

(b) The function is decreasing on $(0, 4)$ and increasing on $(-\infty, 0)$ and $(4, \infty)$.

(c) (4, -17) is the relative minimum and (0, 15) the relative maximum.

26.

(a)

$$\begin{aligned} f(x) &= x^4 - 32x + 4 && (1) \\ f'(x) &= 4x^3 - 32 && (2) \\ &= 4(x^3 - 8) && (3) \end{aligned}$$

$x = 2$ is a critical number.

Test intervals	$-\infty < x < 2$	$2 < x < \infty$
f' sign	$f' < 0$	$f' > 0$
Conclusion	Decreasing	Increasing

(b) The function is decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$

(c) (2, -44) is the relative minimum.

30.

(a)

$$\begin{aligned} f(x) &= (x - 3)^{1/3} && (1) \\ f'(x) &= \frac{1}{3}(x - 3)^{-2/3} && (2) \\ &= \frac{1}{3(x - 3)^2/3} && (3) \end{aligned}$$

$x = 3$ is a critical number.

Test intervals	$-\infty < x < 3$	$3 < x < \infty$
f' sign	$f' > 0$	$f' > 0$
Conclusion	Increasing	Increasing

(b) The function is increasing on $(-\infty, \infty)$

(c) The function has no relative extrema.

34.

(a)

$$f(x) = \frac{x}{x+3} \tag{1}$$
$$f'(x) = \frac{(x+3)-x}{(x+3)^2} \tag{2}$$
$$= \frac{3}{(x+3)^2} \tag{3}$$

The function has no critical numbers but a discontinuity at $x = -3$.

Test intervals	$-\infty < x < -3$	$-3 < x < \infty$
f' sign	$f' > 0$	$f' > 0$
Conclusion	Increasing	Increasing

(b) The function is increasing on $(-\infty, -3)$ and $(-3, \infty)$.

(c) The function has no relative extrema.

38.

(a)

$$f(x) = \frac{x^2 - 3x - 4}{x - 3} \tag{1}$$
$$f'(x) = \frac{(2x-3)(x-2) - (x^2 - 3x - 4)}{(x-2)^2} \tag{2}$$
$$= \frac{x^2 - 4x + 10}{(x-2)^2} \tag{3}$$

The function has a discontinuity at $x = 2$.

Test intervals	$-\infty < x < 2$	$2 < x < \infty$
f' sign	$f' > 0$	$f' > 0$
Conclusion	Increasing	Increasing

(b) The function is increasing on $(-\infty, 2)$ and $(2, \infty)$.

(c) The function has no relative extrema.

42.

(a)

$$f(x) = \begin{cases} -x^3+1, & x \le 0 \\ -x^2+2x, & x > 0 \end{cases} \tag{1}$$

$$f'(x) = \begin{cases} -3x^2, & x < 0 \\ -3x+2, & x > 0 \end{cases} \tag{2}$$

$x = 0, 1$ are critical numbers.

Test intervals	$-\infty < x < 0$	$0 < x < 1$	$1 < x < \infty$
f' sign	$f' < 0$	$f' > 0$	$f' < 0$
Conclusion	Decreasing	Increasing	Decreasing

(b) The function is increasing on $0, 1)$ and decreasing on $(-\infty, 0)$ and $(1, \infty)$

(c) (1, 1) is a relative maximum.

5 Consider the function on the interval $(0, 2\pi)$. For each function, (a) find the open interval(s) in which the function is increasding or decreasing, and (b) apply the First Derivative Test to identify all relative extrema.

44.

(a)

$$f(x) = \sin x \cos x + 5 \tag{1}$$

$$= \frac{1}{2}\sin 2x + 5, \quad 0 < x < 2\pi \tag{2}$$

$$f'(x) = \cos 2x \tag{3}$$

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ are critical numbers.

Test intervals	$0 < x < \frac{\pi}{4}$	$\frac{\pi}{4} < x < \frac{3\pi}{4}$	$\frac{3\pi}{4} < x < \frac{5\pi}{4}$	$\frac{5\pi}{4} < x < \frac{7\pi}{4}$	$\frac{7\pi}{4} < x < 2\pi$
f' sign	$f' > 0$	$f' < 0$	$f' > 0$	$f' < 0$	$f' > 0$
Conclusion	Increasing	Decreasing	Increasing	Decreasing	Increasing

The function is increasing on $\left(0, \frac{\pi}{4}\right)$, $\left(\frac{3\pi}{4}, \frac{5\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$, and decreasing on $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$.

(b) $\left(\frac{\pi}{4}, \frac{11}{2}\right)$ and $\left(\frac{5\pi}{4}, \frac{11}{2}\right)$ are the relative maxima, and $\left(\frac{3\pi}{4}, \frac{9}{2}\right)$ and $\left(\frac{7\pi}{4}, \frac{9}{2}\right)$ the relative minima.

48.

(a)

$$f(x) = \sqrt{3}\sin x + \cos x \tag{1}$$

$$f'(x) = \sqrt{3}\cos x - \sin x = 0 \tag{2}$$

$$\tan x = \sqrt{3} \tag{3}$$

$x = \frac{\pi}{3}, \frac{4\pi}{3}$ are critical numbers.

Test intervals	$0 < x < \frac{\pi}{3}$	$\frac{\pi}{3} < \frac{4\pi}{3}$	$\frac{4\pi}{3} < x < 2\pi$
f' sign	$f' > 0$	$f' < 0$	$f' < 0$
Conclusion	Increasing	Decreasing	Decreasing

The function is increasing on $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{4\pi}{3}, 2\pi\right)$, and decreasing on $\left(\frac{\pi}{3}, \frac{4\pi}{3}\right)$

(b) $\left(\frac{\pi}{3}, 2\right)$ is the relative maximum, and $\left(\frac{4\pi}{3}, -2\right)$ the relative minimum.

6 Use symmetry, extrema, and zeros to sketch the graph of f. How do the functions f and g differ?

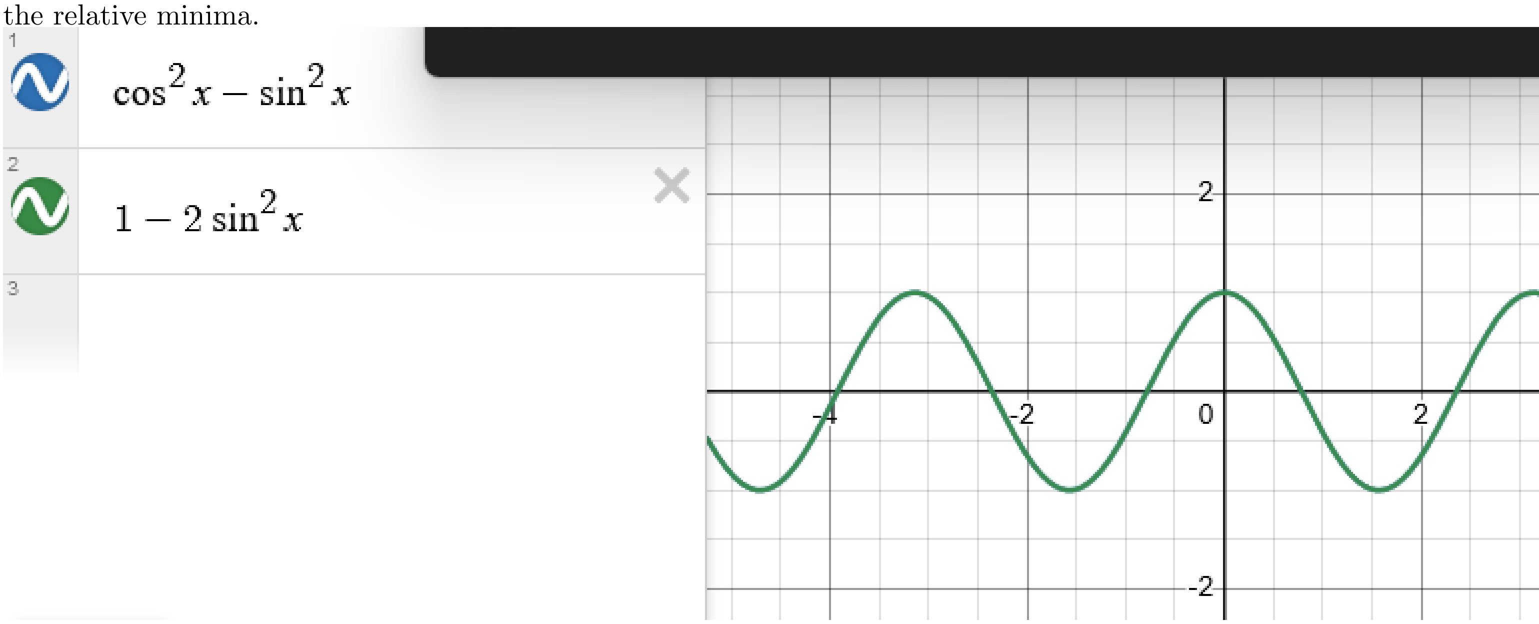
58.

$$f(t) = \cos^2 t - \sin^2 t, \quad g(t) = 1 - 2\sin^2 t$$

$$f(t) = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t = g(t)$$

$$f'(t) = -4 \sin t \cos t = -2 \sin 2t$$

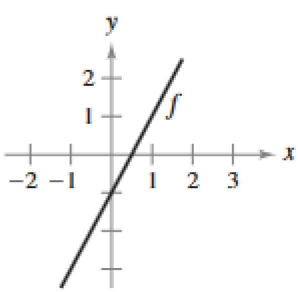
f is symmetric to the y-axis and has zeros at $\pm\frac{\pi}{4}$ (0, 1) is the relative maximum, and $\left(-\frac{\pi}{2}, -1\right)$ and $\left(\frac{\pi}{2}, -1\right)$ are the relative minima.

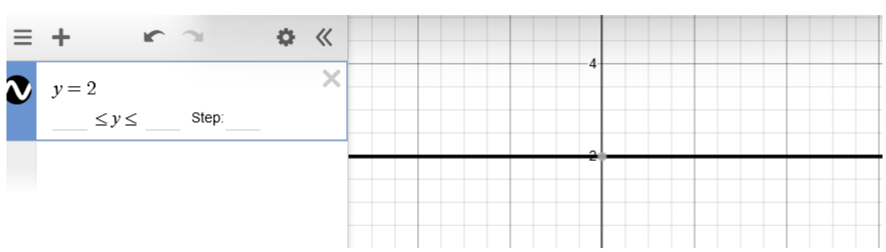


$f(x)$ and $g(x)$ are the same graph.

7 The graph of f is shown. Sketch a graph of the derivative of f.

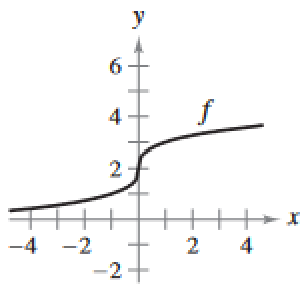
60.

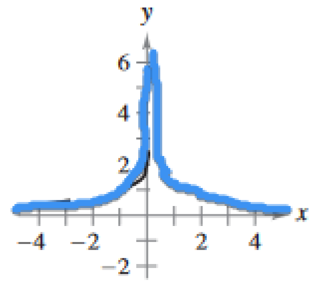




$$f'(x) = 2$$

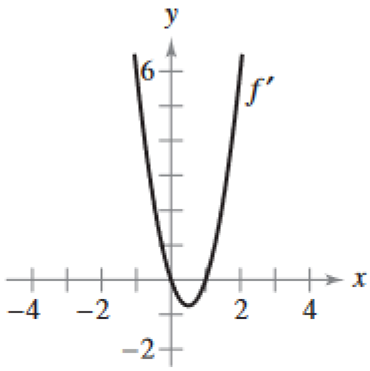
64.





8 Use the graph of f' to (a) identify the interval(s) on which f is increasing or decreasing, and (b) estimate the value(s) of x at which f has a relative maximum or minimum.

66.

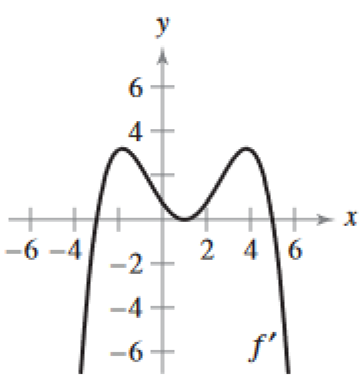


(a) f is increasing on $(-\infty, 0)$ and $(1, \infty)$, and decreasing on (0, 1).

(b) f has a relative maximum and minimum and $x = 0$ and $x = 1$ respectively.

9 Use the graph of f' to (a) identify the critical numbers of f, and (b) determine whether f has a relative maximum, a relative minimum, or neither at each critical number.

70.



(a) $f' = 0$ at $x = -3, 1, 5$

(b)
  i. $x = -3$ is a relative minimum.
  ii. $x = 5$ is a relative maximum.
  iii. $x = 1$ is neither.

10 The function f is differentiable on the indicated interval. The table shows $f'(x)$ for selected values of x. (a) Sketch the graph of f, (b) approximate the critical numbers, and (c) identify the relative extrema.

80. f is differentiable on $[0, \pi]$

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$f'(x)$	3.14	−0.23	−2.45	−3.11	0.69

x	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$f'(x)$	3.00	1.37	−1.14	−2.84

(b) The critical numbers are in the intervals $\left(0, \frac{\pi}{6}\right)$, $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$, and $\left(\frac{3\pi}{4}, \frac{5\pi}{6}\right)$.
f is increasing on $\left(0, \frac{\pi}{7}\right)$ and $\left(\frac{3\pi}{7}, \frac{6\pi}{7}\right)$, and decreasing on $\left(\frac{\pi}{7}, \frac{3\pi}{7}\right)$ and $\left(\frac{6\pi}{7}, \pi\right)$.

(c) Relative minima occur when $x = \frac{3\pi}{7}, \pi$ and relative maxima occur when $x = \frac{\pi}{7}, \frac{6\pi}{7}$.

11 Word problems

82. The concentration C of a chemical in the bloodstream t hours after the injection into muscle tissue is

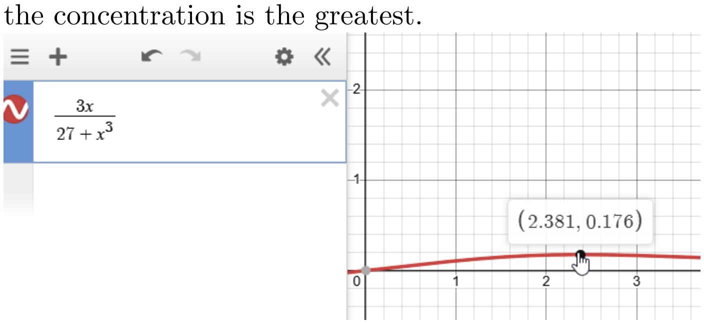
$$C(t) = \frac{3t}{27 + t^3}, \quad t \geq 0$$

(a) Complete the table and use it to approximate the time when the concentration is greatest.

t	0	0.5	1	1.5	2	2.5	3
$C(t)$	0	0.055	0.107	0.148	0.171	0.176	0.167

The concentration is greatest at $t = 2.5$ hours.

(b) Use a graphing utility to graph the concentration function and use the graph to approximate the time when the concentration is the greatest.



The concentration is greatest at $t \approx 2.381$ hours.

(c) Use calculus to determine analytically the time when the concentration is the greatest.

$$C' = \frac{(3)(27+t^3)-(3t)(3t^2)}{(27+t^3)^2} \tag{1}$$

$$= \frac{3(27-2t^3)}{(27+t^3)^2} \tag{2}$$

$$= 0 \text{ when } t = \frac{3}{\sqrt[3]{2}} \approx 2.381 \text{ hours} \tag{3}$$

This is a maximum according to the First Derivative Test.

88. The end-of-year assets of the Medicare Hospital Insurance Trust Fund (in billions of dollars) for the years 1995 through 2006 are shown.

1995: 130.3; 1996: 124.9; 1997: 115.6; 1998: 120.4;

1999: 141.4; 2000: 177.5; 2001: 208.7; 2002: 234.8;

2003: 256.0; 2004: 269.3; 2005: 285.8; 2006: 305.4

(Source: U.S. Centers for Medicare and Medicaid Services)

(a) Use the regression capabilities of a graphing utility to find a model of the form $M = at^4 + bt^3 + ct^2 + dt + e$ for the data (Let $t = 5$ represent 1995.).

$$M = 0.03723t^4 - 1.993t^3 + 37.986t^2 - 282.74t + 825.7$$

(c) Find the minimum value of the model and compare the result with the actual data.
The model's minimum value is (6.5, 111.9), and the data's is (7, 115.6).

12 The function $s(t)$ describes the motion of a particle along a line. For each function, (a) find the velocity function of a particle at any time $t \geq 0$. (b) identify the time interval(s) in which the particle is moving in a positive direction, (c) identify the time interval(s) in which the particle is moving in a negative direction. and (d) identify the time(s) at which the particle changes direction.

90.

$$s(t) = t^2 - 7t + 10, \quad t \geq 0$$

(a)

$$v(t) = s'(t) = 2t - 7$$

(b) $v(t) = 0$ when $t = \frac{7}{2} \therefore$ the particle is moving in the positive direciton for $t > \frac{7}{2} \because v'(t) > 0$ on $\left(\frac{7}{2}, \infty\right)$.

(c) The particle is moving in the negative direction on $\left[0, \frac{7}{2}\right)$.

(d) The particle is changing direction at $t = \frac{7}{2}$.

13 Determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

99. The sum of two increasing functions is increasing.
    True.

100. The product of two increasing functions is incrasing.
    False. Let $f(x) = g(x)h(x)$ where $g(x) = h(x) = x$. $f(x) = x^2$ is decreasing on $(-\infty, 0)$.

101. Every nth-degree polynomial has $(n-1)$ critical numbers.
    False. Let $f(x) = x^3$. $f(x) = 3x^2$ and the function only has one critical number.

102. An nth-degree-polynomial has at most $(n-1)$ critical numbers.
    True.

103. There is a relative maximum or minimum at each critical number.
    False. $f(x) = x^3$ doesn't have relative extrema at $x = 0$, which is a critical number.