5.2 The Natural Logarithmic Function: Integration

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1 Find the indefinite integral

2.

$$\int \frac{10}{x} dx \tag{1}$$
$$= 10 \int \frac{1}{x} dx \tag{2}$$
$$= 10 \ln |x| + C \tag{3}$$

6.

$$\int \frac{1}{4-3x} dx,\ u = 4 - 3x,\ du = -3dx \tag{1}$$
$$= -\frac{1}{3} \int \frac{1}{4-3x} (-3) dx \tag{2}$$
$$= -\frac{1}{3} \ln |4-3x| + C \tag{3}$$

10.

$$\int \frac{x^2-2x}{x^3-3x^2} dx,\ u = x^3 - 3x^2,\ du = (3x^2-6x)dx = 3(x^2-2x)dx \tag{1}$$
$$= \frac{1}{3} \int \frac{1}{x^3-3x^2} (3x^2-6x) dx \tag{2}$$
$$= \frac{1}{3} \ln |x^3-3x^2| + C \tag{3}$$

14.

$$\int \frac{x(x+2)}{x^3+3x^2-4} dx,\ u = x^3 + 3x^2 - 4,\ du = (3x^2+6x)dx \tag{1}$$
$$= \frac{1}{3} \int \frac{3x^2+6x}{x^3+3x^2-4} dx \tag{2}$$
$$= \frac{1}{3} \ln |x^3+3x^2-4| + C \tag{3}$$

18.

$$
\begin{align}
&\int \frac{x^3 - 6x - 20}{x+5} dx \tag{1}\\
&= \int \left(x^2 - 5x + 19 - \frac{115}{x+5} \right) dx \tag{2}\\
&= \frac{x^3}{3} - \frac{5x^2}{2} + 19x - 115 \ln|x+5| + C \tag{3}
\end{align}
$$

22.

$$
\begin{align}
&\int \frac{1}{x \ln(x^3)} dx \tag{1}\\
&= \frac{1}{3} \int \frac{1}{\ln x} \cdot \frac{1}{x} dx \tag{2}\\
&= \frac{1}{3} \ln|\ln|x|| + C \tag{3}
\end{align}
$$

26.

$$
\begin{align}
&\int \frac{x(x-2)}{(x-1)^3} dx \tag{1}\\
&= \int \frac{x^2 - 2x + 1 - 1}{(x-1)^3} dx \tag{2}\\
&= \int \frac{(x-1)^2}{(x-1)^3} dx - \int \frac{1}{(x-1)^3} dx \tag{3}\\
&= \int \frac{1}{x-1} dx - \int \frac{1}{(x-1)^3} dx \tag{4}\\
&= \ln|x-1| + \frac{1}{2(x-1)^2} + C \tag{5}
\end{align}
$$

28.

$$
\begin{align}
&\int \frac{1}{1+\sqrt{3x}} dx,\ u = 1 + \sqrt{3x},\ du = \frac{3}{2\sqrt{3x}} dx \Rightarrow dx = \frac{2}{3}(u-1) du \tag{1}\\
&= \int \frac{1}{u} \cdot \frac{2}{3}(u-1) du \tag{2}\\
&= \frac{2}{3} \int \left(1 - \frac{1}{u} \right) du \tag{3}\\
&= \frac{2}{3}(u - \ln|u|) + C \tag{4}\\
&= \frac{2}{3}(1 + \sqrt{3x} - \ln(1 + \sqrt{3x})) + C \tag{5}\\
&= \frac{2}{3}\sqrt{3x} - \frac{2}{3} \ln(1 + \sqrt{3x}) + C \tag{6}
\end{align}
$$

30.

$$\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}dx,\ u = x^{\frac{1}{3}} - 1,\ du = \frac{1}{3x^{\frac{2}{3}}}dx \Rightarrow dx = 3(u+1)^2 du \tag{1}$$
$$= \int \frac{u+1}{u} 3(u+1)^2 du \tag{2}$$
$$= 3\int \frac{u+1}{u}(u^2+2u+1)du \tag{3}$$
$$= 3\int \left(u^2+3u+3=\frac{1}{u}\right) du \tag{4}$$
$$= 3\left(\frac{u^3}{3}+\frac{3u^2}{2}+3u+\ln|u|\right)+C \tag{5}$$
$$= 3\left(\frac{(x^{\frac{1}{3}}-1)^3}{3}+\frac{3(x^{\frac{1}{3}}-1^2)}{2}+3(x^{\frac{1}{3}}-1)+\ln|x^{\frac{1}{3}}-1|\right)+C \tag{6}$$
$$= 3\ln|x^{\frac{1}{3}}-1|+\frac{3x^{\frac{2}{3}}}{2}+3x^{\frac{1}{3}}+x+C \tag{7}$$

32.

$$\int \tan 5\theta d\theta \tag{1}$$
$$= \frac{1}{5}\int \frac{5\sin 5\theta}{\cos 5\theta}d\theta \tag{2}$$
$$= -\frac{1}{5}\ln|\cos 5\theta|+C \tag{3}$$

36.

$$\int \left(2-\tan\frac{\theta}{4}\right)d\theta \tag{1}$$
$$= \int 2d\theta - 4\int \tan\frac{\theta}{4}\left(\frac{1}{4}\right)d\theta \tag{2}$$
$$= 2\theta + 4\ln\left|\cos\frac{\theta}{4}\right|+C \tag{3}$$

40.

$$\int (\sec 2x = \tan 2x)dx \tag{1}$$
$$= \frac{1}{2}\int (\sec 2x+\tan 2x)(2)dx \tag{2}$$
$$= \frac{1}{2}(\ln|\sec 2x+\tan 2x| - \ln|\cos 2x)+C \tag{3}$$

2 Solve the differential equation.

42.

$$\frac{dy}{dx} = \frac{x-2}{x},\ (-1,\ 0) \tag{1}$$
$$y = \int \frac{x-2}{x} dx \tag{2}$$
$$= \int \left(1 - \frac{2}{x}\right) dx \tag{3}$$
$$= x - 2\ln|x| + C \tag{4}$$
$$0 = -1 - 2\ln|-1| + C = -1 + C \Rightarrow C = 1 \tag{5}$$
$$y = x - 2\ln|x| + 1 \tag{6}$$

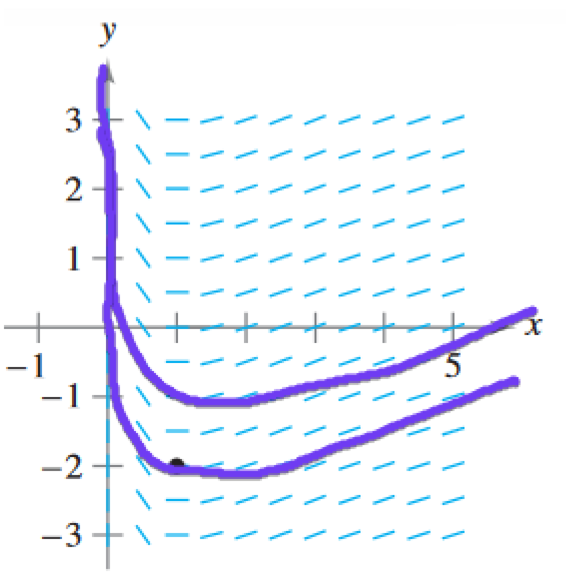
46.

$$\frac{dr}{dt} = \frac{\sec^2 t}{\tan t + 1} \tag{1}$$

3 A differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a).

49.

$$\frac{dy}{dx} = \frac{1}{x+2},\ (0,\ 1) \tag{1}$$



(a)

(b)

$$y = \int \frac{\ln x}{x} dx = \frac{(\ln x)^2}{2} + C \tag{1}$$

$$y(1) = -2 \Rightarrow -2 = \frac{(\ln 1)^2}{2} + C \therefore C = -2 \tag{2}$$

$$y = \frac{(\ln x)^2}{2} - 2 \tag{3}$$

4 Evaluate the definite integral.

58.

$$\int_0^1 \frac{x-1}{x+1} dx \tag{1}$$

$$= \int_0^1 1 dx + \int_0^1 \frac{-2}{x+1} dx \tag{2}$$

$$= (x - 2\ln|x+1|)_0^1 = 1 - 2\ln 2 \tag{3}$$

$$\approx -0.386 \tag{4}$$

5 Find $F'(x)$.

68.

$$F(x) = \int_0^x \tan t dt \tag{1}$$

$$F'(x) = \tan x \tag{2}$$

70.

$$F(x) = \int_1^{x^2} \frac{1}{t} dt \tag{1}$$

$$F'(x) = \frac{2x}{x^2} = \frac{2}{x} \tag{2}$$

72.

6 Determine which value best approximates the area of the region between the x-axis and the graph of the function over the given interval. (Make your selection on the basis of a sketch of the region and not by performing any calculations.)

72. $f(x) = \frac{2x}{x^2+1}$, $[0, 4]$

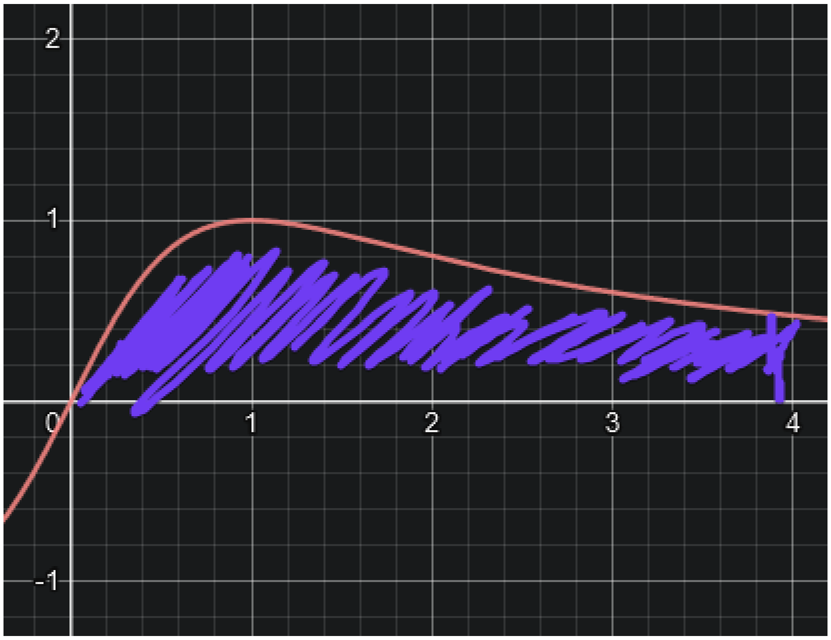
(a) 3

(b) 7

(c) 2

(d) 5

(e) 1

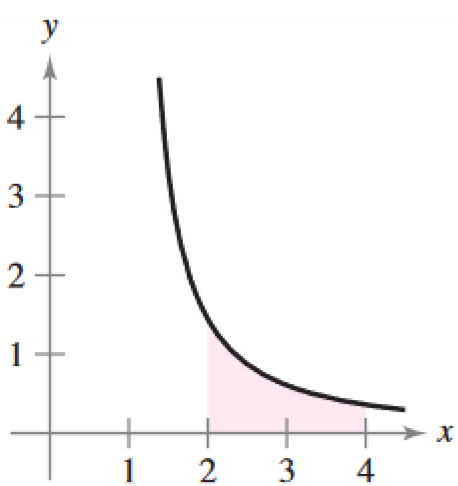


$A \approx 3 \therefore$ it matches (a).

7 Find the area of the given region.

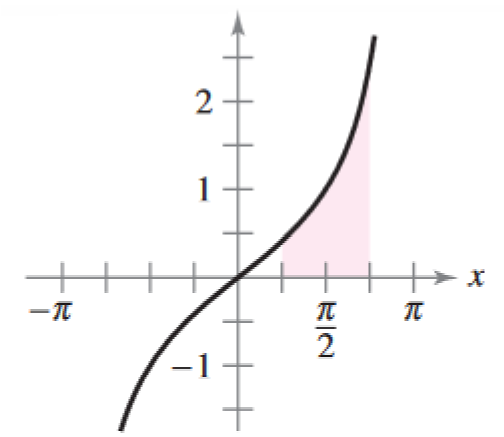
74.

$$y = \frac{2}{x \ln x} \tag{1}$$
$$A = \int_2^4 \frac{2}{x \ln x} \tag{2}$$
$$= 2\int_2^4 \frac{1}{\ln x} \cdot \frac{1}{x} dx \tag{3}$$
$$= 2\ln|\ln x|)_2^4 \tag{4}$$
$$= 2(\ln(\ln 4) - \ln(\ln 2)) \tag{5}$$
$$= 2\ln\left(\frac{2\ln 2}{\ln 2}\right) \tag{6}$$
$$= 2\ln 2 \tag{7}$$



76.

$$
\begin{align}
y &= \frac{\sin x}{1+\cos x} \tag{1}\\
A &= \frac{\frac{\pi}{4}}{\frac{3\pi}{4}} \frac{\sin x}{1+\cos x} dx \tag{2}\\
&= -\ln|1+\cos x|)^{\frac{3\pi}{4}}_{\frac{\pi}{4}} \tag{3}\\
&= -\ln\left(1-\frac{\sqrt{2}}{2}\right)+\ln\left(1+\frac{\sqrt{2}}{2}\right) \tag{4}\\
&= \ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) \tag{5}\\
&= \ln(3+2\sqrt{2}) \tag{6}
\end{align}
$$



8 Find the area of the region bounded by the graphs of the equations.

78.

$$
\begin{align}
y &= \frac{x+6}{x},\ x=1,\ x=5,\ y=0 \tag{1}\\
A &= \int_1^5 \frac{x+6}{x} dx \tag{2}\\
&= \int_1^5 \left(1+\frac{6}{x}\right) dx \tag{3}\\
&= (x+6\ln x)_1^5 \tag{4}\\
&= 5+6\ln 5-1 \tag{5}\\
&= 4+6\ln 5 \tag{6}\\
&\approx 13.657 \tag{7}
\end{align}
$$

80.

$$y = 2x - \tan 0.3x,\ x = 1,\ x = 4,\ y = 0 \tag{1}$$

$$\int_1^4 (2x - \tan(0.3x))dx \tag{2}$$

$$= \left(x^2 + \frac{10}{3}\ln|\cos(0.3x)|\right)_1^4 \tag{3}$$

$$= \left(16 + \frac{10}{3}\ln\cos(1.2)\right) - \left(1 + \frac{10}{3}\ln\cos(0.3)\right) \tag{4}$$

$$\approx 11.769 \tag{5}$$

9 Use the Trapezoidal Rule and Simpson's Rule to approximate the value of the definite integral. Let $n = 4$ and round your answer to four decimal places.

82.

$$f(x) = \frac{8x}{x^2+4},\ b - 4 = 4 - 0 - 4,\ n = 4 \tag{1}$$

$$\text{Trapezoidal Rule:} \frac{4}{2(4)}(f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)) = \frac{1}{2}(0 + 3.2 + 4 + 3.6923 + 1.6) \approx 6.2462 \tag{2}$$

$$\text{Simpson's Rule:} \frac{4}{3(4)}(f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)) \approx 6.4615 \tag{3}$$

10 Capstone

90. Find a value of x such that $\int_1^x \frac{1}{t}dt$ is equal to (a) $\ln 5$ and (b) 1.

$$\int_1^x \frac{1}{t}dt = (\ln|t|)_1^x = \ln x \tag{1}$$

(a)

$$\ln x = \ln 5 \Rightarrow x = 5 \tag{2}$$

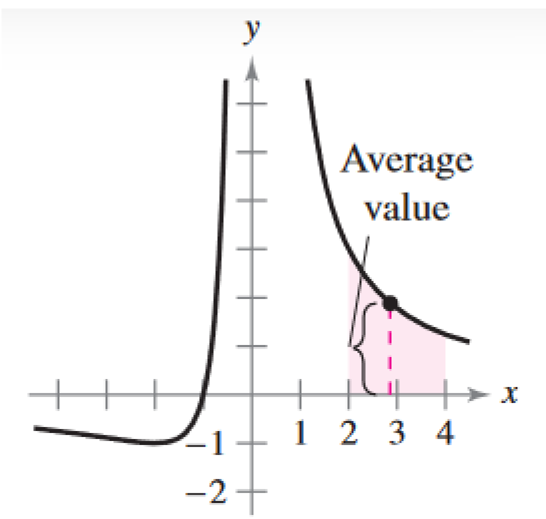
(b)

$$\ln x = 1 \Rightarrow x = e \tag{3}$$

11 Find the average value of the function over the given interval.

98.

$$f(x) = \frac{4(x+1)}{x^2},\ [2,\ 4] \tag{1}$$
$$\text{Average} = \frac{1}{4-2}\int_2^4 \frac{4(x+1)}{x^2}dx \tag{2}$$
$$= 2\int_2^4 \left(\frac{1}{x} + \frac{1}{x^2}\right) dx \tag{3}$$
$$= 2\left(\ln x - \frac{1}{x}\right)_2^4 \tag{4}$$
$$= 2\left(\ln 4 - \frac{1}{4} - \ln 2 + \frac{1}{2}\right) \tag{5}$$
$$= 2\left(\ln 2 + \frac{1}{4}\right) \tag{6}$$
$$= \ln 4 + \frac{1}{2} \tag{7}$$
$$\approx 1.8863 \tag{8}$$



100.

$$f(x) = \sec\frac{\pi x}{6},\ [0,\ 2] \tag{1}$$
$$\text{Average} = \frac{1}{2-0}\int_0^2 \sec\frac{\pi x}{6}dx \tag{2}$$
$$= \left(\frac{1}{2}\left(\frac{6}{\pi}\ln\left|\sec\frac{\pi x}{6} + \tan\frac{\pi x}{6}\right|\right)\right)_0^2 \tag{3}$$
$$= \frac{3}{\pi}(\ln(2+\sqrt{3}) - \ln(1+0)) \tag{4}$$
$$= \frac{3}{\pi}\ln(2+\sqrt{3}) \tag{5}$$

12 Word problems

102. Find the time required for an object to cool from $300°F$ to $250°F$ by evaluating $t = \frac{10}{\ln 2}\int_{250}^{300} \frac{1}{T-100} dT$ where t is time in minutes.

$$\begin{aligned}
t &= \frac{10}{\ln 2}\int_{250}^{300} \frac{1}{T-100} dT && (1)\\
&= \frac{10}{\ln 2}(\ln(T-100))_{250}^{300} && (2)\\
&= \frac{10}{\ln 2}(\ln 200 - \ln 150) && (3)\\
&= \frac{10}{\ln 2}\left(\ln\left(\frac{4}{3}\right)\right) \approx 4.1504\text{min} && (4)
\end{aligned}$$

104. The rate of change in sales S is inversely proportional to time $t(t > 1)$ measured in weeks. Find S as a function of t if sales after 2 and 4 weeks are 200 units and 300 units, respectively.

$$\begin{aligned}
\frac{dS}{dt} &= \frac{k}{t} && (1)\\
S(t) &= \int \frac{k}{t} dt && (2)\\
&= k \ln|t| + C && (3)\\
&= k \ln t + C \because t > 1 && (4)\\
S(2) &= k \ln 2 + C = 200 && (5)\\
S(4) &= k \ln 4 + C = 300 \therefore k = \frac{100}{\ln 2},\ C = 100 && (6)\\
S(t) &= \frac{100 \ln t}{\ln 2} + 100 && (7)\\
&= 100\left(\frac{\ln t}{\ln 2} + 1\right) && (8)
\end{aligned}$$

13 Determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

105. $(\ln x)^{1/2} = \frac{1}{2}(\ln x)$
False because $\frac{1}{2}(\ln x) = \ln(x^{1/2}) \neq (\ln x)^{1/2}$.

106. $\int \ln x dx = \left(\frac{1}{x}\right) + C$
False because $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

107. $\int \frac{1}{x} dx = \ln|cx|,\ c \neq 0$
True.

108. $\int_{-1}^{2} \frac{1}{x} dx = (\ln|x|)_{-1}^{2} = \ln 2 - \ln 1 = \ln 2$
False because $\frac{1}{x}$ is discontinuous at $x = 0$