Notes - 4.1 Antiderivatives and Indefinite Integration

Juan J. Moreno Santos

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1 Warm-up

1. Let $y = x^2 \cos x$. $\frac{dy}{dx} =$
Apply the product rule:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

Therefore,

$$\begin{aligned} y' &= 2x\cos x + x^2(-\sin x) && (1) \\ &= 2x\cos x - x^2 \sin x && (2) \end{aligned}$$

2. If $y = \frac{\sin x}{\cos x}$, then $\frac{dy}{dx} =$ Remember the trigonometric identity:

$$\frac{\sin x}{\cos x} = \tan x$$

Therefore,

$$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{d}{dx}(\tan x) = \sec^2 x$$

2 Reversing derivatives

$$\begin{aligned} 2x &= \frac{d}{dx}(x^2) = \int 2x dx = x^2 + C && (1) \\ 6x &= \frac{d}{dx}(3x^2) = \int x^{-2/3} dx = 3x^{1/3} + C && (2) \\ x^{-2/3} &= \frac{d}{dx}(3x^{1/3}) = \int x dx = \frac{1}{2}x^2 + C && (3) \\ \frac{5}{3}x^{2/3} &= \frac{d}{dx}(x^{5/3}) = \int \frac{5}{3}x^{2/3} dx = x^{5/3} + C && (4) \\ -\sin x &= \frac{d}{dx}(\cos x) && (5) \\ \sin x &= \frac{d}{dx}(-\cos x) \therefore \int \sin x dx = -\cos x + C && (6) \end{aligned}$$

3 Applying an integral:

$$\begin{aligned} &\int (3x^2 + \frac{1}{2}x^2 + 4) dx && (1) \\ &= x^3 + \frac{1}{6}x^3 + 4x + C && (2) \end{aligned}$$

4 The fundamental theorem of Calculus:

$$y = \int f(x)dx = F(x) + C$$

Where

1. f is the integran.
2. dx is the variable of integration.
3. F is an antiderivative of $f(x)$.
4. C is the constant of integration.

Note the capital F.
Integral is a synonym of antiderivative.

5 Integrals inside a derivative:

If $\int f(x)dx = F(x) + C$, then:

$$\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$$

In conclusion, differentiation is the "inverse" of integration and vice versa.

BASIC INTEGRATION RULES

Differentiation Formula	*Integration Formula*	
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$	
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$	
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf(x)\,dx = k\int f(x)\,dx$	
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$	
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$	Power Rule
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x\,dx = \sin x + C$	
$\frac{d}{dx}[\cos x] = -\sin x$	$\int \sin x\,dx = -\cos x + C$	
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$	
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$	
$\frac{d}{dx}[\cot x] = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$	
$\frac{d}{dx}[\csc x] = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$	

6 Warm-up 11/13/2023

1. Let $y = \tan^2(3x)$. $\frac{dy}{dx} =$

$$\begin{aligned}
\frac{dy}{dx} &= 2(\tan(3x))\frac{d}{dx}(\tan(3x))\frac{d}{dx}(3x) && (1)\\
&= 2(\tan(3x))\sec^2(3x)\cdot 3 && (2)\\
&= 6(\tan(3x))\sec^2(3x) && (3)
\end{aligned}$$

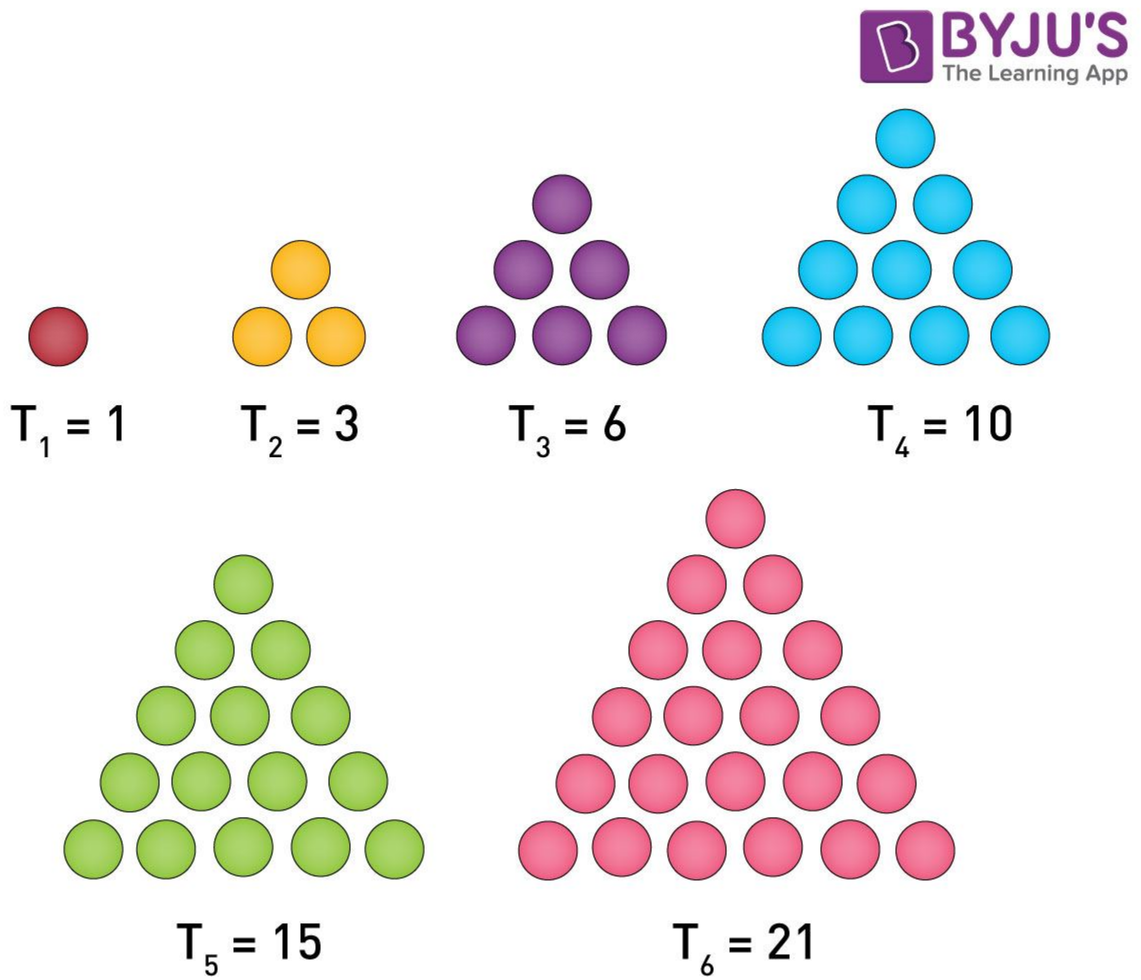
2.

$$\lim_{h\to 0}\frac{\sin(\pi+h)-\sin(\pi)}{h}$$

$$\begin{aligned}
&= \frac{d}{dx}(\sin(\pi)) && (1)\\
&= \cos(\pi) && (2)\\
&= -1 && (3)
\end{aligned}$$

7 Triangular numbers

https://en.wikipedia.org/wiki/Triangular_number



Triangular numbers are given by the following formulas:

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n$$

$$\frac{n(n+1)}{2}$$

$$\frac{n^2+n}{2}$$

$$\left(\frac{n+1}{2}\right)$$

The first equation can be illustrated using a visual proof.[1] For every triangular number T_n, imagine a "half-rectangle" arrangement of objects corresponding to the triangular number, as in the figure below. Copying this arrangement and rotating it to create a rectangular figure doubles the number of objects, producing a rectangle with dimensions $n \times (n+1)$, which is also the number of objects in the rectangle. Clearly, the triangular number itself is always exactly half of the number of objects in such a figure, or: $T_n = \frac{n(n+1)}{2}$. The example T_4 follows:

$$2^{1/3} = \sqrt[3]{2} \approx 1.26$$

$$2^{-2/7} = \frac{1}{\sqrt[7]{(2)^2}} = \approx 0.82$$