3.5 Limits at Infinity

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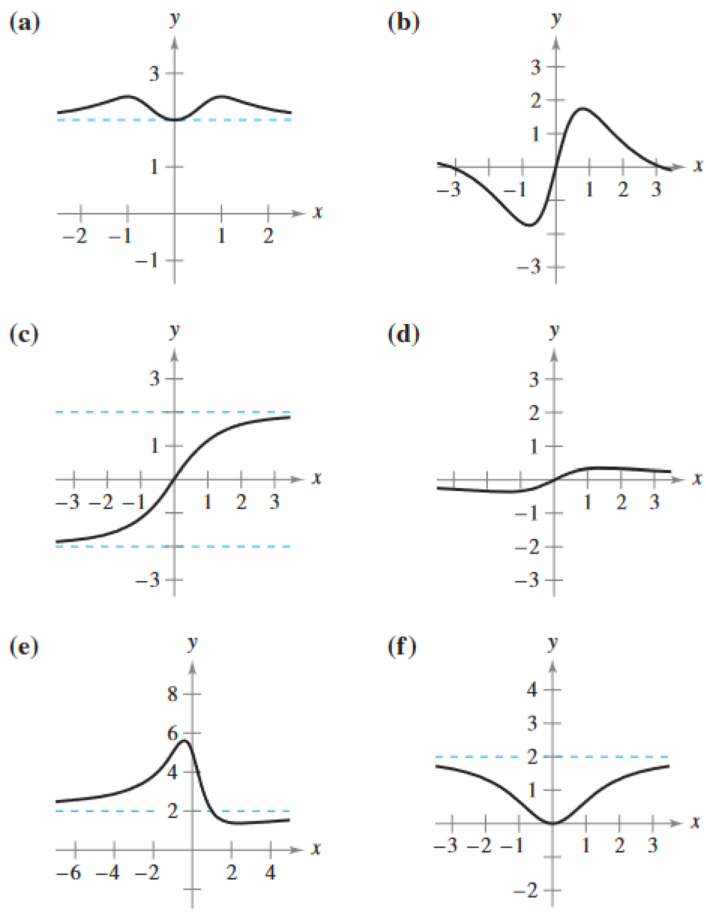
1 Match the function with one of the graphs [(a), (b), (c), (d), (e), or (f)] using horizontal asymptotes as an aid.



2.

$$f(x) = \frac{2x}{\sqrt{x^2+2}}$$

The function has a horizontal asymptote at $y = \pm 2$. Therefore, it matches (c).

4.

$$f(x) = 2 + \frac{x^2}{x^4+1}$$

The function has a vertical asymptote at $y = 2$. Therefore, it matches (a).

6.

$$f(x) = \frac{2x^2 - 3x + 5}{x^2 + 1}$$

The function has a horizontal asymptote at $y = 2$. Therefore, it matches (e).

2 Use a graphing utility to complete the table and estimate the limit as x approaches infinity. Then use a graphing utility to graph the funciton and estimate the limit graphically.

8.

$$f(x) = \frac{2x^2}{x + 1}$$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	18.18	198.02	1998.02	19998	199998	1999998

$\lim_{x\to\infty} f(x) = \infty \therefore$ it doesn't exist.

12.

$$f(x) = 4 + \frac{3}{x^2 + 2}$$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	5	4.03	4.0003	4	4	4	4

$\lim_{x\to\infty} f(x) = 4$

3 Find $\lim_{x\to\infty} h(x)$, if possible.

14.

$$f(x) = -4x^2 + 2x - 5$$

(a) $h(x) = \frac{f(x)}{x}$

$$= \frac{-4x^2 + 2x - 5}{x} \tag{1}$$

$$= -4x + 2 - \frac{5}{x} \tag{2}$$

$$\lim_{x\to\infty} h(x) = -\infty \therefore \text{ the limit doesn't exist.} \tag{3}$$

(b) $h(x) = \frac{f(x)}{x^2}$

$$= \frac{-4x^2 + 2x - 5}{x^2} \tag{1}$$

$$= -4 + \frac{2}{x} - \frac{5}{x^2} \tag{2}$$

$$\lim_{x\to\infty} h(x) = -4 \tag{3}$$

(c) $h(x) = \frac{f(x)}{x^3}$

$$= \frac{-4x^2 + 2x - 5}{x^3} \tag{1}$$
$$= -\frac{4}{x} + \frac{2}{x^2} - \frac{5}{x^3} \tag{2}$$
$$\lim_{x\to\infty} h(x) = 0 \tag{3}$$

4 Find each limit, if possible.

18.

(a)

$$\lim_{x\to\infty} \frac{5x^{3/2}}{4x^2+1} = 0$$

(b)

$$\lim_{x\to\infty} \frac{5x^{3/2}}{4x^{3/2}+1} = \frac{5}{4}$$

(c)

$$\lim_{x\to\infty} \frac{5x^{3/2}}{4\sqrt{x}+1} = \infty \therefore \text{the limit doesn't exist.}$$

(d)

5 Find the limit

22.

$$\lim_{x\to\infty} \frac{x^2+3}{2x^2-1}$$

$$= \lim_{x\to\infty} \frac{1+\frac{3}{x^2}}{2-\frac{1}{x}} \tag{1}$$
$$= \frac{1+0}{2-0} \tag{2}$$
$$= \frac{1}{2} \tag{3}$$

24.

$$\lim_{x\to\infty} \frac{5x^3+1}{10x^3-3x^2+7}$$

$$= \lim_{x\to\infty} \frac{5+\frac{1}{x^3}}{10-\frac{3}{x}+\frac{7}{x^3}} \tag{1}$$
$$= \frac{5+0}{10-0} \tag{2}$$
$$= \frac{1}{2} \tag{3}$$

28.

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+1}}$$

$$= \lim_{x\to-\infty} \frac{1}{\left(\frac{\sqrt{x^2+1}}{-\sqrt{x^2}}\right)} \tag{1}$$

$$= \lim_{x\to-\infty} \frac{-1}{\sqrt{1+\frac{1}{x^2}}} = -1 \tag{2}$$

32.

$$\lim_{x\to\infty} \frac{\sqrt{x^4-1}}{x^3-1}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x^4-1}}{x^3-1} \left(\frac{\frac{1}{-\sqrt{x^6}}}{\frac{1}{x^3}}\right) \tag{1}$$

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{1}{x^2}-\frac{1}{x^6}}}{-1+\frac{1}{x^3}} \tag{2}$$

$$= 0 \tag{3}$$

36.

$$\lim_{x\to\infty} \cos\frac{1}{x}$$

$$= \cos 0 \tag{1}$$

$$= 1 \tag{2}$$

6 Use a graphing utility to graph the function and identify any horizontal asymptotes.

40.

$$f(x) = \frac{|3x+2|}{x-2}$$

$y = 3$ is a horizontal asymptote from the right and $y = -3$ from the left.

42.

$$f(x) = \frac{\sqrt{9x^2-2}}{2x+1}$$

$y = \frac{3}{2}$ is a horizontal asymptote from the right and $y = -\frac{3}{2}$ from the left.

7 Find the limit.

44.

$$\lim_{x\to\infty} x\tan\frac{1}{x}$$

Let $x = \frac{1}{t}$

$$= \lim_{x\to 0^+} \frac{\tan t}{t} \tag{1}$$

$$= \lim_{x\to 0^+} \left(\frac{\sin t}{t} \cdot \frac{1}{\cos t}\right) \tag{2}$$

$$= (1)(1) \qquad = 1 \tag{3}$$

48.

$$\lim_{x\to\infty}(4x - \sqrt{16x^2 - x})$$

$$
\begin{aligned}
&= \lim_{x\to\infty}(4x - \sqrt{16x^2 - x})\frac{4x - \sqrt{16x^2 - x}}{4x - \sqrt{16x^2 - x}} && (1)\\
&= \lim_{x\to\infty}\frac{16x^2 - (16x^2 - x)}{4x + \sqrt{(16x^2 - x)}} && (2)\\
&= \lim_{x\to\infty}\frac{x}{4x + \sqrt{16x^2 - x}} && (3)\\
&= \lim_{x\to\infty}\frac{1}{4 + \sqrt{16x^2 - x}} && (4)\\
&= \frac{1}{4+4} \qquad = \frac{1}{8} && (5)
\end{aligned}
$$

8 Use a graphing utility to complete the table and estimate the limit as x approaches infinity. Then user a graphing utility to graph the function and estimate the limit. Finally, find the limit analitically and compare your results with the estimates.

52.

$$f(x) = \frac{x+1}{x\sqrt{x}}$$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	2	0.348	0.101	0.032	0.010	0.003	0.001

$$\lim_{x\to\infty}\frac{x+1}{x\sqrt{x}} = 0$$

9 Describe in your own words what the statement means.

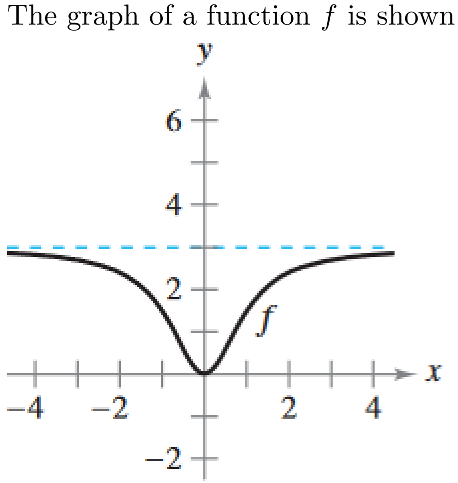
54.

$$\lim_{x\to-\infty} f(x) = 2$$

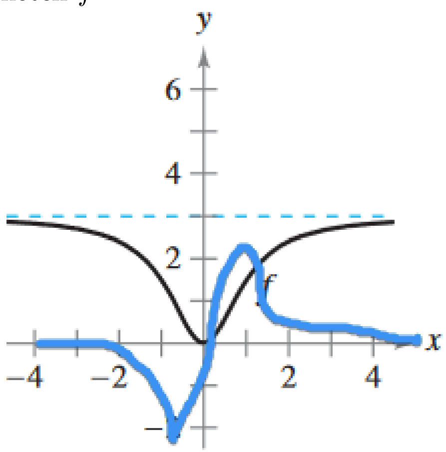
It means that the function approaches 2 as x becomes an infinite and negative value.

10 Capstone

58. The graph of a function f is shown below.



(a) Sketch f'



(b) Use the graphs to estimate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$.

$$\lim_{x\to\infty} f(x) = 3$$

and

$$\lim_{x\to\infty} f'(x) = 0$$

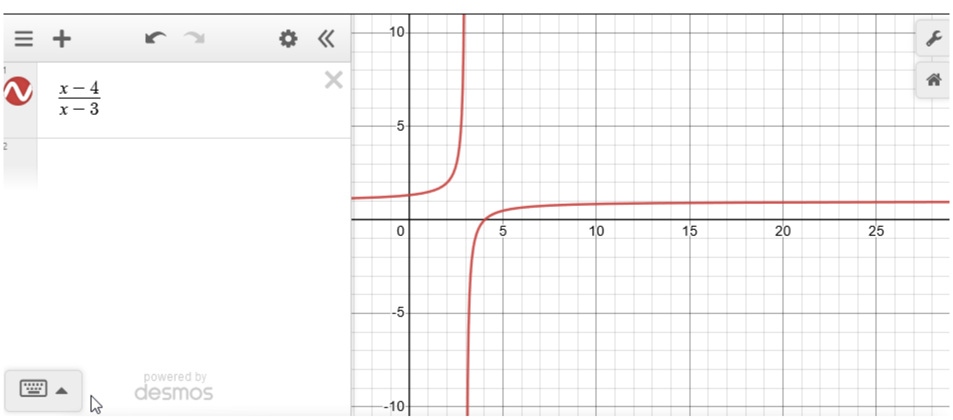
(c) Because the first limit approaches a constant, we can deduce that the graph is approaching that of a line with slope of zero: a horizontal line.

11 Sketch the graph of the equation using extrema, intercepts, symmetry and asymptotes. The use a graphing utility to verify your result.

60.

$$y = \frac{x-4}{x-3}$$

1. Intercepts: $(0, \frac{4}{3})$, $(4, 0)$
2. No symmetry
3. Horizontal asymptotes: $y = 1$
4. Vertical asymptotes: $x = 3$

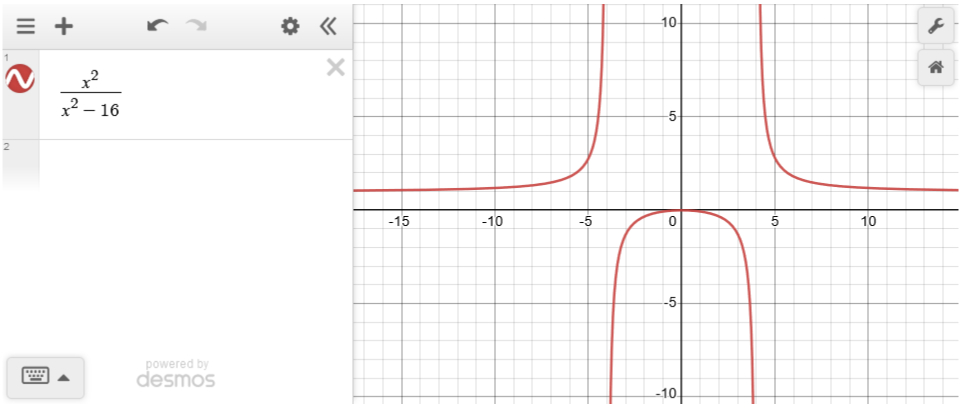


64.

$$y = \frac{x^2}{x^2 - 16}$$

1. Intercepts: $(0, 0)$

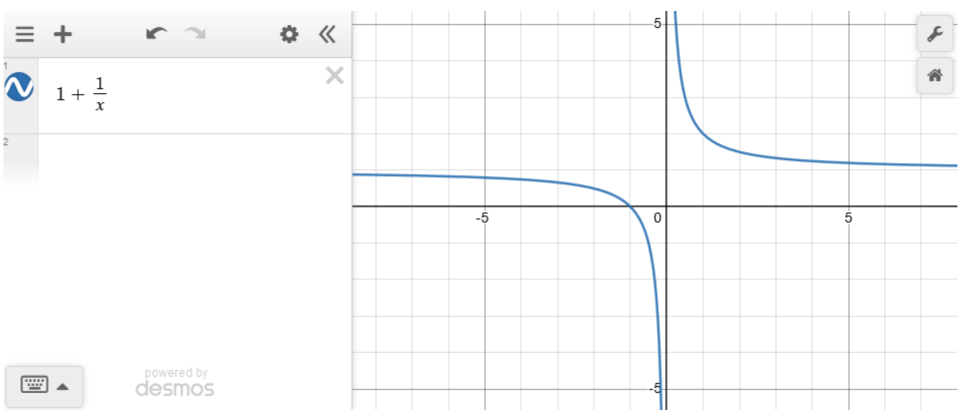
2. Symmetry in the y-axis
3. Horizontal asymptotes: $y = 1$
4. Vertical asymptotes: $x = \pm 4$
5. $y' = \frac{-32x}{(x^2-16)^2}$
6. Relative maxima: (0, 0)



72.

$$y = \frac{1}{x}$$

1. Intercepts: $(-1, 0)$
2. No symmetry
3. Horizontal asymptotes: $y = 1$
4. Vertical asymptotes: $x = 0$



12 (a) Use a graphing utility to graph f and g in the same viewing window, (b) verify algebraically that f and g represent the same function, and (c) zoom out sufficiently far so that the graph appears as a line. What equation does this line appear to have? (Note that the points at which function is not continuous are not readily seen when you zoom out.)

86.

$$f(x) = -\frac{x^3 - 2x^2 + 2}{2x^2}$$
$$g(x) = -\frac{1}{2}x + 1 - \frac{1}{x^2}$$

(b)

$$f(x) = -\frac{x^3 - 2x^2 + 2}{2x^2} \tag{1}$$
$$= -\left(\frac{x^3}{2x^2} - \frac{2x^2}{2x^2} + \frac{2}{2x^2}\right) \tag{2}$$
$$= -\frac{1}{2}x + 1 - \frac{1}{x^2} \tag{3}$$
$$= g(x) \tag{4}$$

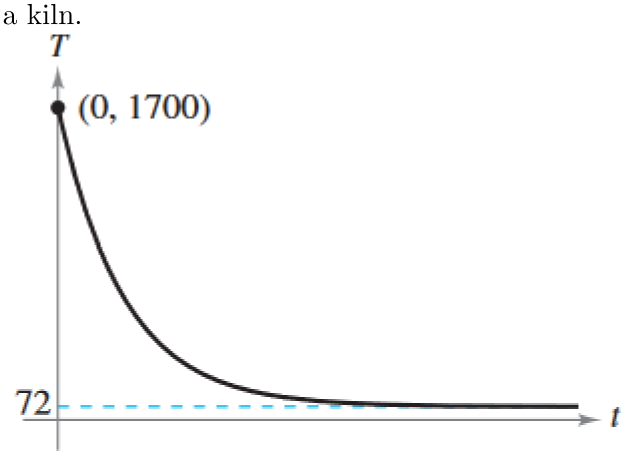
(c) The graph appears to be the line $y = -\frac{1}{2} + 1$

13 Word problems

88. A business has a cost of $C = 0.5x + 500$ for producing x units. The average cost per unit is $\overline{C} = \frac{C}{x}$. Find the limit of $\overline{C}$ as x approaches infinity.

$$\overline{C} = 0.5 + \frac{500}{x} \tag{1}$$
$$\lim_{x\to\infty}\left(0.5 + \frac{500}{x}\right) = 0.5 \tag{2}$$

90. The graph shows the temperature T, in degrees Fahrenheit, of molten glass t seconds after it is removed from a kiln.



(a) Find $\lim_{t\to 0^+} T$. What does this limit represent?
The temperature of the kiln.

(b) Find $\lim_{t\to\infty} T$. What does this limit represent?
The room's temperature.

(c) Will the temperature of the glass ever actually reach room temperature? Why?
No because $y = 72$ is a horizontal asymptote.

92. The average typing speeds S (in words per minute) of a typing student after t weeks of lessons are shown in the table.

t	5	10	15	20	25	30
S	28	56	79	90	93	94

A model for the data is $S = \frac{100t^2}{65+t^2}$, $t > 0$

(a) Use a graphing utility to plot the data and graph the model.

(b) Does there appear to be a limiting typing speed? Explain.
Yes because $\lim_{t\to\infty} S = \frac{100}{1} = 100$.

96. A line with slope m passes through the point (0, -2).

(a) Write the distance d between the line and the point (4, 2) as a function of m.
Rewriting the line's equation:

$$y + 2 = m(x - 0) \tag{1}$$
$$mx - y - 2 = 0 \tag{2}$$

Finding the function:

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} \tag{3}$$
$$= \frac{|m(4) - 1(2) - 2|}{\sqrt{m^2+1}} \tag{4}$$
$$= \frac{|4m - 4|}{\sqrt{m^2+1}} \tag{5}$$

(b) Use a graphing utlity to graph the equation in part (a).

(c) Find $\lim_{m\to\infty} d(m)$ and $\lim_{m\to-\infty} d(m)$. Interpret the results geometrically.

$$\lim_{m\to\infty} d(m) = 4; \quad \lim_{m\to-\infty} d(m) = 4$$

The distance from the given point approaches 4 because the line approaches the origin's vertical line $x = 0$.

14 Use the definition of limits at infinity to prove the limit.

104.

$$\lim_{x\to-\infty} \frac{1}{x-2} = 0$$

Let $\epsilon > 0$. We are finding $N < 0$ such that $|f(x) - L| = \frac{-1}{x-2} < \epsilon$ when $x < N$

$$\frac{-1}{x-2} < \epsilon \Rightarrow x - 2 < \frac{-1}{\epsilon} \Rightarrow x < 2 - \frac{1}{\epsilon} \tag{1}$$
$$x < N = 2 - \frac{1}{\epsilon} \tag{2}$$
$$x - 2 < \frac{-1}{\epsilon} \tag{3}$$
$$\frac{-1}{x-2} < \epsilon \tag{4}$$
$$\Rightarrow |f(x) - L| < \epsilon \tag{5}$$

15 Determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

107. If $f'(x) > 0$ for all real numbers x, then f increases without bound.
False. $f(x) = \frac{2x}{\sqrt{x^2+2}}$ from Exercise 54 proves otherwise.

108. If $f''(x)$ for all real numbers x, then f decreases without bound.
False. $f(x) = -x^2$ increases without bound even though $f''(x) = -2 < 0,\ x \in \mathbb{R}$.