Notes - 4.5 Integration by Substitution

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The Chain Rule states that

$$\frac{d}{dx}(F(g(x))) = F'(g(x))g'(x)$$

Defining an antiderivative follows that

$$\int F'(g(x))g'(x)dx = F(g(x)) + C$$

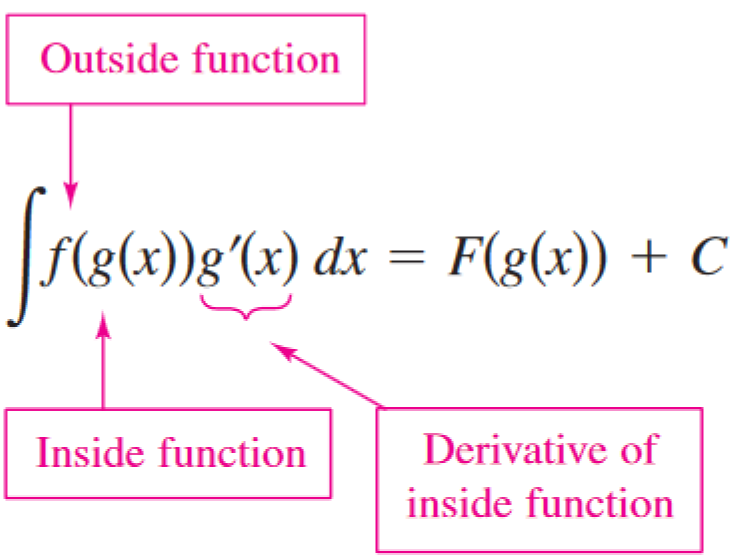
1 Antidifferentiation of a composite function

Let be g a function whose range is an interval I and let f be a function that is continuous on I If g is differentiable on its domain and F is an antiderivative of f on I then

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

By letting $u = g(x)$ gives $du = g'(x)$ and

$$\int f(u)du = F(u) + C$$



1.1 Recognizing the $f(g(x))g'(x)$ pattern

Find $\int (x^2 + 1)^2(2x)\, dx$.

Solution Letting $g(x) = x^2 + 1$, you obtain

$$g'(x) = 2x$$

and

$$f(g(x)) = f(x^2 + 1) = (x^2 + 1)^2.$$

From this, you can recognize that the integrand follows the $f(g(x))g'(x)$ pattern. Us the Power Rule for Integration and Theorem 4.13, you can write

$$\int \overbrace{(x^2 + 1)^2}^{f(g(x))}\overbrace{(2x)}^{g'(x)}\, dx = \frac{1}{3}(x^2 + 1)^3 + C.$$

Try using the Chain Rule to check that the derivative of $\frac{1}{3}(x^2 + 1)^3 + C$ is integrand of the original integral.

Class solution ("Is the derivative of the inside on the outside?)

$$\int (x^2+1)^2(2x)dx \tag{1}$$
$$u = x^2+1 \tag{2}$$
$$du = 2xdx \tag{3}$$
$$\int u^2 du \tag{4}$$
$$= \frac{1}{3}(u)^3 + C \tag{5}$$
$$= \frac{1}{3}(x^2+1)^3 + C \tag{6}$$

You have done u-substitution correctly if you don't have any more of the original variables.

2. Find $\int 5\cos 5dx$

$$u = 5x \tag{1}$$
$$du = 5dx \tag{2}$$
$$\int 5\cos 5dx = \int \cos u du = \sin 5x + C \tag{3}$$

Restating the Constant Multiple Rule

$$\int kf(x)dx = k\int f(x)dx$$

EXAMPLE 3 Multiplying and Dividing by a Constant

Find $\int x(x^2+1)^2\,dx$.

Solution This is similar to the integral given in Example 1, except that the integr
is missing a factor of 2. Recognizing that $2x$ is the derivative of x^2+1, you car
$g(x)=x^2+1$ and supply the $2x$ as follows.

$$\int x(x^2+1)^2\,dx = \int (x^2+1)^2\left(\frac{1}{2}\right)(2x)\,dx \qquad \text{Multiply and divide by 2.}$$

$$= \frac{1}{2}\int \overbrace{(x^2+1)^2}^{f(g(x))}\,\overbrace{(2x)}^{g'(x)}\,dx \qquad \text{Constant Multiple Rule}$$

$$= \frac{1}{2}\left[\frac{(x^2+1)^3}{3}\right]+C \qquad \text{Integrate.}$$

$$= \frac{1}{6}(x^2+1)^3+C \qquad \text{Simplify.}$$

In practice, most people would not write as many steps as are shown in Exampl
For instance, you could evaluate the integral by simply writing

$$\int x(x^2+1)^2\,dx = \frac{1}{2}\int (x^2+1)^2\,2x\,dx$$

$$= \frac{1}{2}\left[\frac{(x^2+1)^3}{3}\right]+C$$

$$= \frac{1}{6}(x^2+1)^3+C.$$

NOTE Be sure you see that the *Constant* Multiple Rule applies only to *constants*. You ca

2 Change of variables

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u)+C$$

2.1 Example

$$u = 2x - 1 \Rightarrow x = \frac{1}{2}u + \frac{1}{2} \tag{1}$$

$$du = 2dx \Rightarrow \frac{1}{2}du = dx \tag{2}$$

$$\int x\sqrt{2x-1}dx \tag{3}$$

$$= \int \left(\frac{1}{2}u + \frac{1}{2}\right)\sqrt{u}\frac{1}{2}du \tag{4}$$

$$= \frac{1}{2}(u+1)u^{1/2}du \tag{5}$$

$$= \frac{1}{4}\int u^{3/2} + u^{1/2}du \tag{6}$$

$$= \frac{1}{4}\left(\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C \tag{7}$$

$$= \frac{1}{10}(2x-1)^{5/2} + \frac{1}{6}(2x-1)^{3/2} + C \tag{8}$$

2.2 Example 2

$$\int \sin^2 3x \cos 3x dx \tag{9}$$

$$u = \sin 3x \tag{10}$$

$$du = \cos(3x) \cdot 3dx \Rightarrow \frac{du}{3} = \cos(3x)dx \tag{11}$$

$$\int \sin^2 3x \cos 3x dx \tag{12}$$

$$= \int u^2 \cdot \frac{du}{3} \tag{13}$$

$$= \frac{1}{3}\int u^2 du \tag{14}$$

$$= \frac{1}{3}\left(\frac{u^3}{3}\right) + C \tag{15}$$

$$= \frac{1}{9}u^3 + C \tag{16}$$

$$= \frac{1}{9}\sin^3 3x + C \tag{17}$$

3 12/01/2023 Warm-up

$$\lim_{x\to\infty} \frac{x^2-4}{2+x^4-2} \tag{1}$$

$$= \lim_{x\to\infty} \frac{x^2-4}{2+x-4x^2}\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right) \tag{2}$$

$$= \lim_{x\to\infty} \frac{1-\frac{4}{x^2}}{-4+\frac{1}{x}+\frac{1}{x^2}} \tag{3}$$

$$= 0 \tag{4}$$

4 4.5 Problem 2

$$u = x^3 + 1,\ du = 3x^2 dx \Rightarrow \frac{du}{3} = x^2 dx \tag{5}$$

$$\int x^2 \sqrt{x^3 + 1} dx \tag{6}$$

$$= \int \frac{\sqrt{u}}{3} dx \tag{7}$$

$$= \frac{1}{3} \int u^{1/2} du \tag{8}$$

$$= \frac{1}{3} \frac{u^{3/2}}{\frac{3}{2}} + C \tag{9}$$

$$= \frac{2}{9} (x^3 + 1)^{3/2} + C \tag{10}$$