3.2 Rolle's Theorem and the Mean Value Theorem

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1 Explain why Rolle's Theorem does not apply to the function even though there exist a and b such that $f(a) = f(b)$.

2.

$$f(x) = \cot \frac{x}{2}, \;\; [\pi, 3\pi]$$

Rolle's Theorem does not apply because f is not continuous at $x = 2\pi$.

4.

$$f(x) = \sqrt{(2 - x^{2/3})} \tag{1}$$
$$f(-1) = 1 = f(1) \tag{2}$$
$$f'(x) = \frac{-\sqrt{(2 - x^{2/3})^3}}{x^{1/3}} \tag{3}$$

Rolle's Theorem does not apply because f is not differentiable at $x = 0$

2 Find the two x-intercepts of the function f and show that $f'(x) = 0$ at some point between the two x-intercepts.

6.

$$f(x) = x(x - 3)$$

(0,0) and (3, 0) are x-intercepts

$$f'(x) = 2x - 3 = 0 \quad \text{at} \quad x = \frac{3}{2}$$

8.

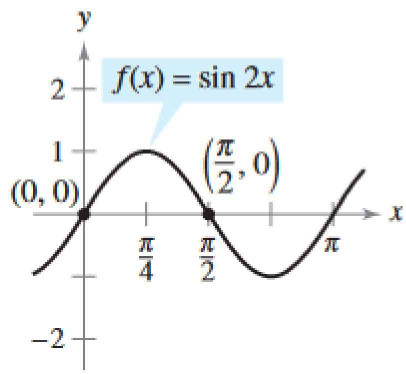
$$f(x) = -3x\sqrt{x + 1}$$

(-1, 0) and (0, 0) are x-intercepts

$$f'(x) \quad = -3x\tfrac{1}{2}(x+1)^{-1/2} - 3(x+1)^{1/2} \tag{1}$$
$$= -3x(x+1)^{-1/2}\left(\tfrac{x}{2} + (x+1)\right) \tag{2}$$
$$= -3(x+1)^{-1/2}\left(\tfrac{3}{2}x + 1\right) \quad \text{at} \quad x = \tfrac{2}{3} \tag{3}$$

3 The graph of f is shown. Apply Rolle's Theorem and find all values of c such that $f'(c) = 0$ at some point between the labeled intercepts.

10.



$$f(x) = \sin 2x \tag{1}$$
$$f(0) = f\left(\tfrac{\pi}{2}\right) = 0 \tag{2}$$
$$f'(x) = 2\cos 2x = 0 \text{ at } x = \tfrac{\pi}{4} \tag{3}$$

4 Determine whether Rolle's Theorem can be applied. If Rolle's Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$. If Rolle's Theorem cannot be applied, explain why not.

12.

$$f(x) = x^2 - 5x + 4, \quad [1, 4] \tag{1}$$
$$f(1) = 0 = f(4) \tag{2}$$
$$\tag{3}$$

Rolle's Theorem applies since f is continuous on $[1, 4]$ and differentiable on (1, 4).

$$f'(x) = 2x - 5 \tag{4}$$
$$2x - 5 = 0 \therefore x = \tfrac{5}{2} = c \tag{5}$$

14.

$$f(x) = (x-3)(x+1)^2, \quad [1, 3] \tag{1}$$
$$f(-1) = f(3) = 0 \tag{2}$$

Rolle's Theorem applies since f is continuous on [1, 3] and differentiable on (-1, 3).

$$f'(x) = (2)(x-3)(x+1) + (x+1)^2 \tag{3}$$
$$= (x+1)(2x - 6 + x + 1) \tag{4}$$
$$= (x+1)(3x-5) \therefore c = \tfrac{5}{3} \tag{5}$$

18.

$$f(x) = \frac{x^2 - 1}{x}, \quad [-1, 3] f(-1) = 0 = f(1) \tag{1}$$

Rolle's Theorem doesn't apply since f is not continuous on [-1, 1] because $f(0)$ is undefined.

22.

$$f(x) = \cos 2x, \quad [-\pi, \pi] f(-pi) = 1 = f(\pi) \tag{1}$$

Rolle's Theorem applies since f is continuous on $[-\pi, \pi]$ and differentiable on $(-\pi, \pi)$.

$$\begin{aligned} f'(x) &= -2\sin 2x && (2)\\ -2\sin 2x &= 0 && (3)\\ \sin 2x &= 0 \therefore x = -\pi, -\tfrac{\pi}{2}, 0, \tfrac{\pi}{2}, \pi && (4)\\ c &= -\tfrac{\pi}{2}, 0, \tfrac{\pi}{2} && (5)\\ & && (6) \end{aligned}$$

5 Determine whether Rolle's Theorem can be applied to f on the interval and, if so, find all values of c in the open interval (a, b) such that $f'(c) = 0$

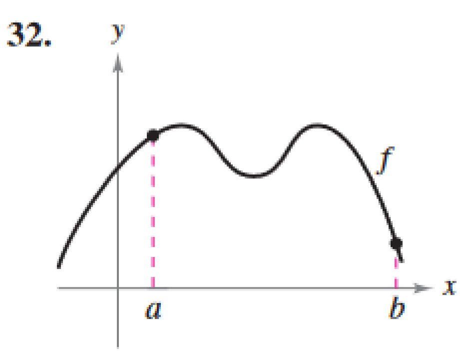
26.

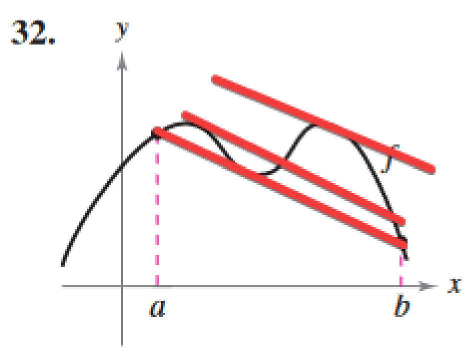
$$f(x) = x - x^{1/3}, \quad [0, 1] f(0) = 0 = f(1) \tag{1}$$

Rolle's theorem applies since f is continuous on [0, 1] and differentiable on (0, 1)

$$\begin{aligned} f'(x) &= 1 - \tfrac{1}{3\sqrt[3]{x^2}} = 0 && (2)\\ 1 &= \tfrac{1}{3\sqrt[3]{x^2}} && (3)\\ \sqrt[3]{x^2} &= \tfrac{1}{3} && (4)\\ x^2 &= \tfrac{1}{27} && (5)\\ x &= \sqrt{\tfrac{1}{27}} = \tfrac{\sqrt{3}}{9} = c && (6) \end{aligned}$$

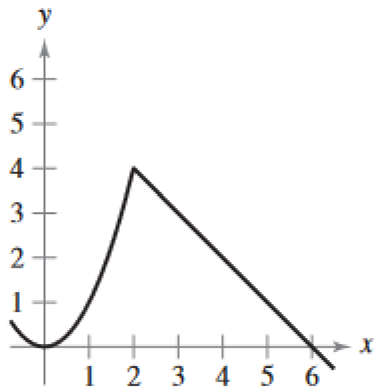
6 Copy the gaph and sketch the secant line to the graph through the points $(a, f(a))$ and $(b, f(b))$. Then sketch any tangent lines to the graph for each value of c guaranteed by the Mean Value Theorem.





7 Explain why the Mean Value Theorem does not apply to the function f on the interval $[0, 6]$

34.



The Mean Value Theorem doesn't apply since f is not differentiable at $x = 2$.

8 Determine whether the Mean Value Theorem can be applied to f on the closed interval $[a, b]$. If the Mean Value Theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = \frac{f(b)-f(a)}{b-a}$. If the Mean Value Theorem cannot be applied, explain why not.

42.

$$f(x) = x^4 = 8x, \quad [0, 2]$$

The function is differentiable on (0, 2) and continuous on [0, 2].

$$\begin{align}
& \\
\frac{f(2)-f(0)}{2-0} &= \tfrac{0}{2} = 0 \\
f'(x) &= 4x^3 - 8 = 4(x^3 - 2) = 0 \\
x^3 &= 2 \\
x &= \sqrt[3]{2} = c
\end{align}$$

44.

$$f(x) = \frac{x+1}{x}, \quad [-1, 2]$$

The Mean Value Theorem doesn't apply because $f(x)$ is not continuous at $x = 0$.

48.

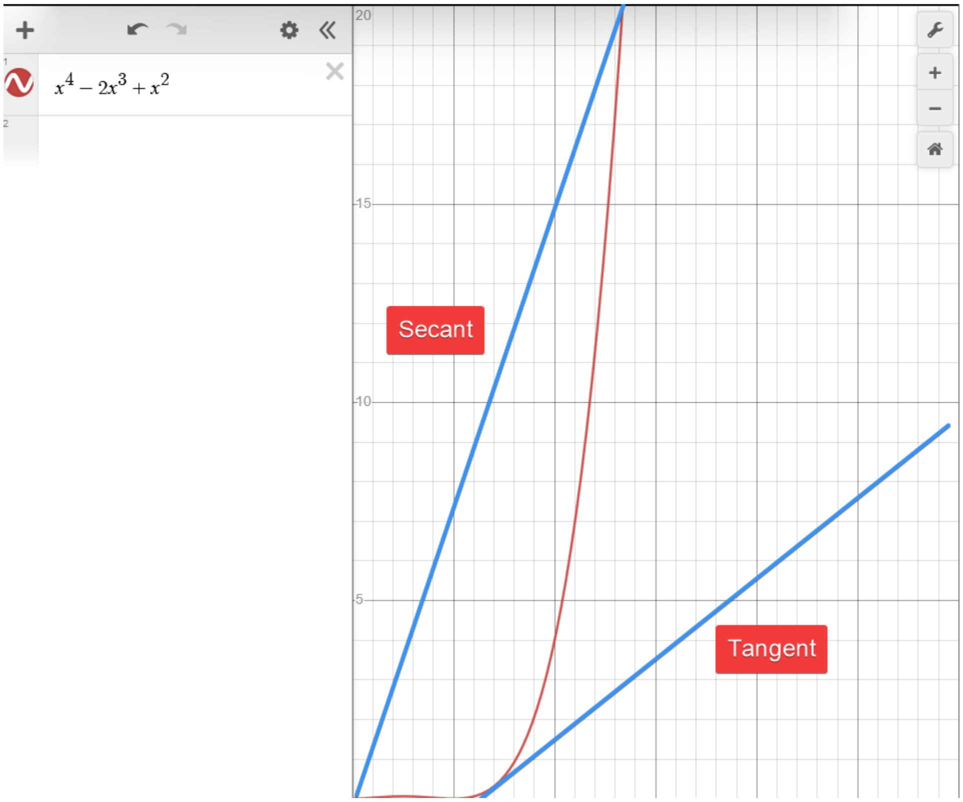
$$f(x) = \cos x + \tan x, \quad [0, \pi]$$

The Mean Value Theorem doesn't apply because $f(x)$ is not continuous at $x = \frac{pi}{2}$.

9 (a) Graph the function f on the given interval, (b) find and graph the secant line through the points on the graph of f at the endpoints of the given interval, and (c) find and graph any tangent lines to the graph of f that are parallel to the secant line.

52.

$$f(x) = x^4 - 2x^3 + x^2, \quad [0, 6]$$



10 Word problems

54. A company introduces a new product for which the number of units sold S is

$$S(t) = 200\left(5 - \frac{9}{2+t}\right)$$

where t is the time in months.
(a) Find the average rate of change of $S(t)$ during the first year.

$$\frac{S(12) - S(0)}{12 - 0} = \frac{200(5 - \frac{9}{14} - 200(5 - \frac{9}{2}))}{12} = \frac{450}{7} \tag{1}$$

(b) During what month of the first year does $S'(t)$ equal the average rate of change?

$$S'(t) = 200\left(\tfrac{9}{(2+t)^2}\right) = \tfrac{450}{7} \tag{2}$$

$$\frac{1}{(2+t)^2} = \tfrac{1}{28} \tag{3}$$

$$2 + t = 2\sqrt{7} \tag{4}$$

$$t = 2\sqrt{7} - 2 \approx 3.3\text{months} \tag{5}$$

60. When an object is removed from a furnace and placed in an environment with a constant temperature of 90°F, its core temperature is 1500°F. Five hours later the core temperature is 390°F. Explain why there must exist a time in the interval when the temperature is decreasing at a rate of 222°F per hour. Let $F(t)$ be the object's temperature.

$$F(0) = 1500 and F(5) = 390 \tag{1}$$

The average temperature over [0, 5] is

$$\frac{390 - 1500}{5} = -222°F/h \tag{2}$$

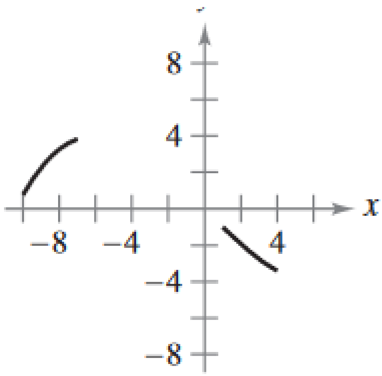
As per the Mean Value Theorem, the exists a time t_0 such that $F'(t_0) = -222°F/h, \quad 0 < t_0 < 5$.

62. At 9:13AM, a sports car is traveling 35 miles per hour. Two minutes later, the car is traveling 85 miles per hour. Prove that at some time during this two-minute interval, the car's acceleration is exactly 1500 miles per hour squared.
Let $t = 0$ be 9:13AM. As per the Mean Value Theorem, there exists a t_0 in $(0, \frac{1}{30})$ such that

$$v'(t_0) = a(t_0) = \frac{85 - 35}{\frac{1}{30}} = 1500\text{mi/h}^2$$

11 Capstone

64. The figure shows two parts of the graph of a continuous differentiable function f on [-10, 4]. The derivative of f' is also continuous.



(a) Explain why f must have at least one zero in [-10, 4]

$$f(-8) > 0, f(3) < 0$$

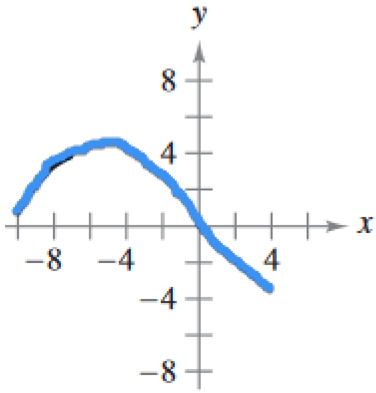
f is continuous and changes sign. By the Intermediate Value Theorem, there is at least one x in [-10, 4] that satisfies $f(x) = 0$

(b) Explain why f' must also have at least one zero in the interval [-10, 4]. What are these zeros called? There are numbers a and b such that

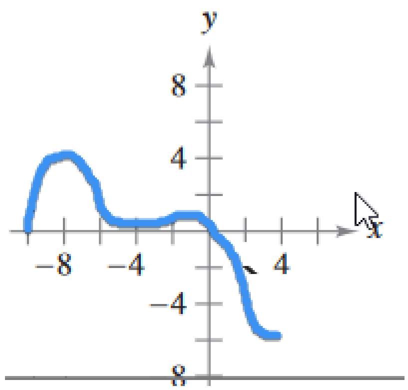
$$f(a) = 2 = f(b), \quad -10 < a < b < 4$$

By Rolle's Theorem, there is a least one c in (-10, 4) such that $f'(c) = 0$.
This zero is called a critical number.

(c) Make a possible sketch of the function with one zero of f' on the interval [-10, 4].



(d) Make a possible sketch of the function with two zeros of f' on the interval [-10, 4].



(e) Were the conditions of continuity of f and f' necessary to do parts (a) through (d)? Explain.
No. f' could have been discontinous on [-10, 4].

12 Determine whether the statement is true or flase. If it is false, explain why or give an example that shows it is false.

77. The Mean Value Theorem can be applied to $f(x) = \frac{1}{x}$ on the interval [-1, 1].
False. $f(x)$ is discontinous at $x = 0$.

78. If the graph of a function has three x-intercepts, then it must have at least two points at which its tangent line is horizontal.
False. f also has to be continuous and differentiable on each interval. For example, $f(x) = \frac{x^3-4x}{x^2-1}$.

79. If the graph of a polynomial function has three x-intercepts, then it must have at least two points at which its tangent line is horizontal.
True.

80. If $f'(x) = 0$ for all x in the domain of f, then f is a constant function.
True.