3.7 Optimization Problems

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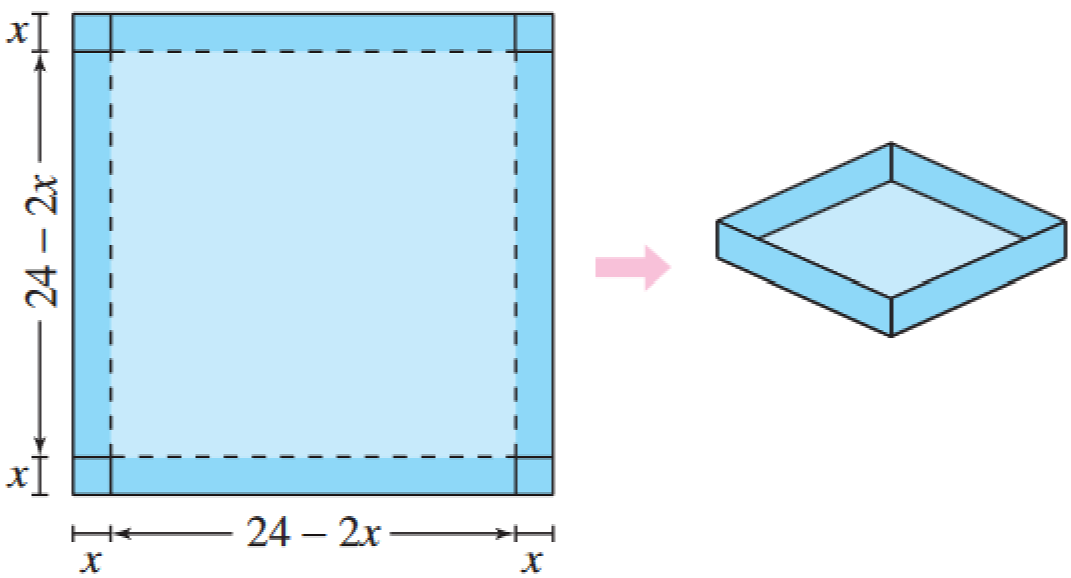
1 Numerical, Graphical and Analytic Analysis

2. An open box of maximum volume is to be made from a square piece of material, 24 inches on a side, by cutting equal squares from the cornes and turning up the sides (see figure).



(a) Analytically complete six rows of a table such as the one below. (The first two rows are shown.) Use the table to guess the maximum volume.

Height x	Length and Width	Volume V
1	$24 - 2(1)$	$1(24 - 2(1))^2 = 484$
2	$24 - 2(2)$	$2(24 - 2(2))^2 = 800$
3	$24 - 2(3)$	$3(24 - 2(3))^2 = 972$
4	$24 - 2(4)$	$4(24 - 2(4))^2 = 1024$
5	$24 - 2(5)$	$5(24 - 2(5))^2 = 980$
6	$24 - 2(6)$	$6(24 - 2(6))^2 = 864$

The maximum volume is given at $x = 4$

(b) Write the volume V as a function of x.

$$V = x(24 - 2x)^2,\ 0 < x < 12$$

(c) Use calculus to find the critical number of the function in part (b) and find the maximum value.

$$\frac{dV}{dx} = 2x(24-2x)(-2) + (24-2x)^2 \tag{1}$$
$$= (24-2x)(24-6x) \tag{2}$$
$$= 12(12-x)(4-x) = 0 \text{ when } x = 12,\ 4 \tag{3}$$

12 is not in the domain.

$$\frac{d^2V}{dx^2} = 12(2x-16) \therefore \frac{d^2V}{dx^2} < 0 \text{ when } x = 4 \tag{4}$$

When $x = 4$. $V = 1024$ is the maximum volume.

(d) Use a graphing utility to graph the function in part(b) and verify the maximum volume from the graph. The maximum volume is 1024.

2 Find two positive numbers that satisfy the given requirements.

4. The product is 185 and the sum is a minimum.

$$S = x + y = x + \frac{185}{x} \tag{1}$$
$$\frac{dS}{dx} = 1 - \frac{185}{x^2} = 0 \text{ when } x = \sqrt{185} \tag{2}$$
$$\frac{d^2S}{dx^2} = \frac{370}{x^3} > 0 \text{ when } x = 185 \tag{3}$$

The two numbers are $\sqrt{185}$.

8. The sum of the first number squared and the second number is 54 and the product is a maximum.

$$x^2 + y = 54 \tag{1}$$
$$S = xy = x(54 - x^2) = 54x - x^3 \tag{2}$$
$$\frac{dS}{dx} = 54x - 3x^2 = 0 \text{ when } x = 3\sqrt{2} \tag{3}$$
$$\frac{d^2P}{dx^2} = -6x < 0 \text{ when } x = 3\sqrt{2} \tag{4}$$

The two numbers are $x = 3\sqrt{2}$ and $y = 36$.

3 Find the length and width of a rectangle that has the given perimeter and a maximum area.

10. Perimeter: P units

$$2x + 2y = P \tag{1}$$
$$y = \frac{P-2x}{2} = \frac{P}{2} - x \tag{2}$$
$$A + xy = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2 \tag{3}$$
$$\frac{dA}{dx} = \frac{P}{2} - 2x = 0 \text{ when } x = \frac{P}{4} \tag{4}$$
$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = \frac{P}{4} \tag{5}$$

A is maximum when $x = y = \frac{P}{4}$ units. Therefore, this rectangle is a square.

4 Find the length and width of a rectangle that has the given perimeter and a minimum perimeter.

12. Area: A square centimeters

$$xy = A \tag{1}$$
$$y = \frac{A}{x} \tag{2}$$
$$P = 2x + 2y = 2x + 2\left(\frac{A}{x}\right) = 2x + \frac{2A}{x} \tag{3}$$
$$\frac{dP}{dx} = 2 - \frac{2A}{x^2} = 0 \text{ when } x = \sqrt{A} \tag{4}$$
$$\frac{d^2P}{dx} = \frac{4A}{x^3} > 0 \text{ when } x = \sqrt{A} \tag{5}$$

P is minimum when $x = y = \sqrt{A}$cm. Therefore, this rectangle is a square.

5 Find the point on the graph of the function that is closest to the given point.

14.

$$f(x) = (x-1)^2,\ (-5,3) \tag{1}$$
$$d = \sqrt{(x = 5^2) + ((x-1)^2 - 3)^2} \tag{2}$$
$$= \sqrt{(x^2 + 10x + 25) + (x^4 - 4x^3 + 8x + 4)} \tag{3}$$
$$= \sqrt{x^4 - 4x^3 + x^2 + 18x + 29} \tag{4}$$

d is the smallest when the expression inside this radical is the smallest. Now, we need to find the critical numbers of

$$g(x) = x^4 - 4x^3 + x^2 + 18x + 29 \tag{1}$$
$$g'(x) = 4x^3 - 12x^2 + 2x + 18 \tag{2}$$
$$= 2(x+1)(2x^2 - 8x + 9) = 0 \tag{3}$$
$$x = -1 \tag{4}$$

Since $x = -1$ yields a minimum, (-1, 4) is the closest point to (-5, 3).

6 Area

18. A rectangular page is to contain 36 square inches of print. The margins of each side are $1\frac{1}{2}$ inches. Find the dimensions of the page such that the least amount of paper is used.

$$xy = 36 \Rightarrow y = \frac{36}{x} \tag{1}$$
$$A = (x+3)(y+3) = (x+3)\left(\frac{36}{x} + 3\right) \tag{2}$$
$$= 36 + \frac{108}{x} + 3x + 9 \tag{3}$$
$$\frac{dA}{dx} = -\frac{108}{x^2} + 3 = 0 \Rightarrow 3x^2 = 108 \Rightarrow x = 6,\ y = 6 \tag{4}$$

Therefore, the dimensions of the page are 9x9 inches.

7 Traffic control

20. On a given day, the flow rate F (cars per hour) on a congested roadway is

$$F = \frac{v}{22 + 0.02v^2}$$

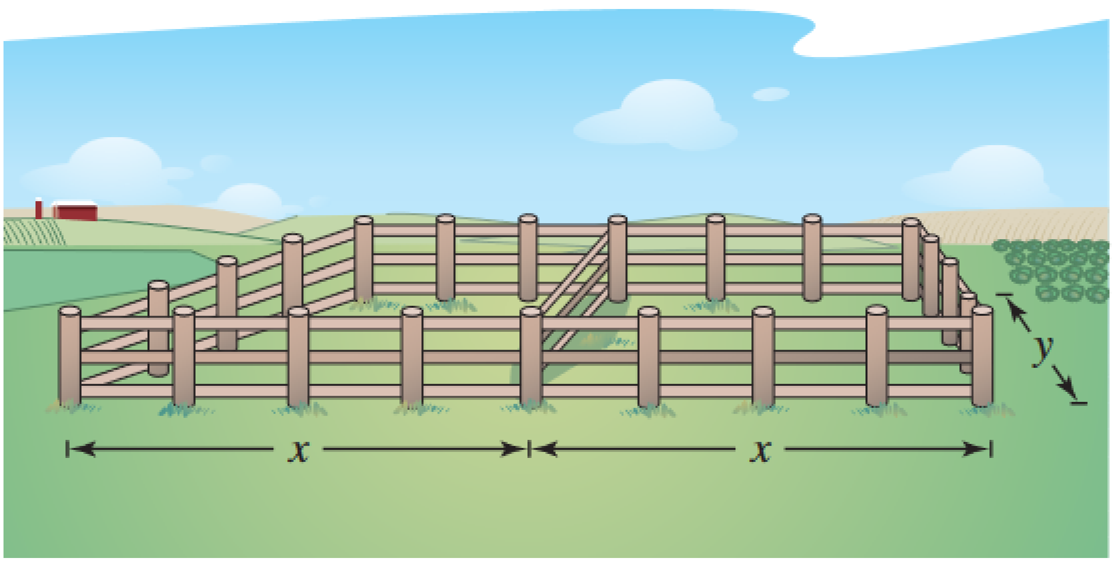
where v is the speed of the traffic in miles per hour. What speed will maximize the flow rate on the road?

$$\begin{aligned} \frac{dF}{dV} &= \frac{22 - 0.02v^2}{(22 + 0.02v^2)^2} \qquad (1) \\ &= 0 \text{ when } v = \sqrt{1100} \approx 33.17 \qquad (2) \end{aligned}$$

The flow rate on the road is maximized when the speed of traffic is 33mi/h.

8 Maximum Area

22. A rancher has 400 feet of fencing in which to enclose two adjacent rectangular corrals (see figure). What dimensions should be used so that the enclosed are will be a maximum?



$$\begin{aligned} 400ft &= 4x + 3y \qquad (1) \\ x &= \frac{400 - 3y}{4} \qquad (2) \\ Area = A &= 2x \cdot y \qquad (3) \\ &= 2\left(\frac{400 - 3y}{4}\right)y \qquad (4) \\ &= 200y - \frac{3}{2}y^2 \qquad (5) \\ A' &= 200 - 3y = 0 y = \frac{200}{3} \qquad (6) \end{aligned}$$

Solving for x:

$$\begin{aligned} 400 &= 3\left(\frac{200}{3}\right) + 4x \qquad (7) \\ 200 &= 4x \qquad (8) \\ 50 &= x \qquad (9) \end{aligned}$$

Since the fence is $2x$, the dimensions of the enclosed area will be 100ft by $\frac{200}{3}$ft.

9 Maximum Volume

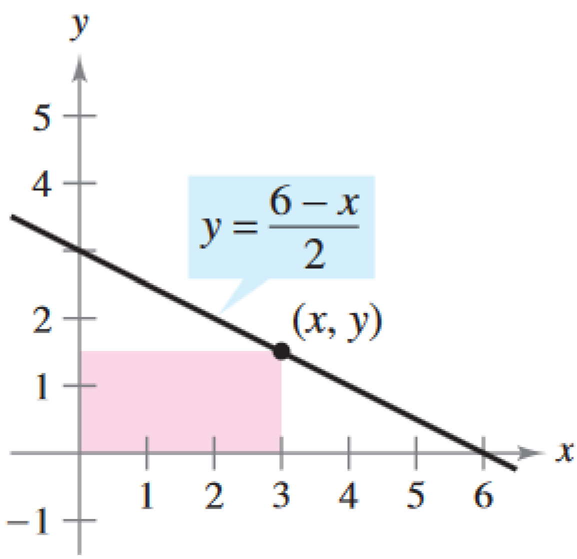
24. Determine the dimensions of a rectangular solid (with a square base) with maximum volume if its surgace area is 377.5 square centimeters.

$$S = 2x^2 + 4xy = 337.5 \tag{1}$$
$$y = \frac{337.5 - 3x^2}{4x} \tag{2}$$
$$V = x^2 y = x^2\left(\frac{337.5 - 2x^2}{4x}\right) = 84.275 - \frac{1}{2}x^3 \tag{3}$$
$$\frac{dV}{dx} = 84.375 - \frac{3}{2}x^2 = 0 \Rightarrow x^2 = 56.25 \Rightarrow 56.25 \Rightarrow x = 7.5 \text{ and } y = 7.5 \tag{4}$$
$$\frac{d^V}{dx^2} = -3x < 0 \text{ when } x = 7.5 \tag{5}$$

The maximum volume occurs when $x = y = 7.5$cm.

10 Maximum Area

26. A rectangle is bounded by the $x-$ and $y-$axes and the graph of $y = \frac{6-x}{2}$ (see figure). What length and width should the rectangle have so that its area is a maximum?

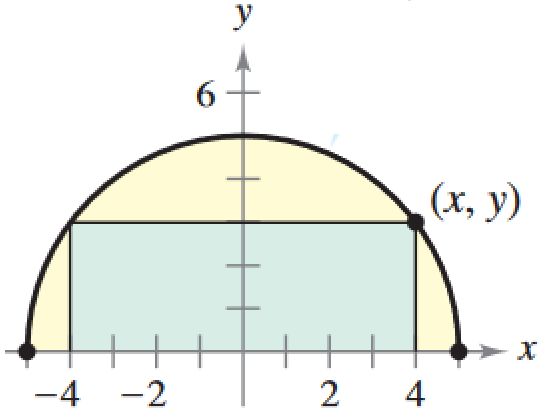


$$A = xy \text{ and } y = \frac{6-x}{2} \tag{1}$$
$$A = x\left(\frac{6-x}{2}\right) = \frac{1}{2}(6x - x^2) \tag{2}$$
$$\frac{dA}{dx} = \frac{1}{2}(6 - 2x) = 0 \text{ when } x = 3 \tag{3}$$
$$\frac{d^2A}{dx^2} = -1 < 0 \text{ when } x = 3 \tag{4}$$

The rectangle should have a length and width of $x = 3$ and $y = \frac{3}{2}$.

11 Area

30. Find the dimensions of the largest rectangle that can be inscribed in a semicircle of radius r.



$$A = 2xy = 2x\sqrt{r^2 - x^2} \tag{5}$$

$$\frac{dA}{dx} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}} = 0 \text{ when } x = \frac{\sqrt{2}r}{2} \tag{6}$$

The largest rectangle has dimensions $\sqrt{2}r$ by $\frac{\sqrt{2}r}{2}$.

12 Numerical, Graphical, and Analytic Analysis

A right circular cylinder is to be designed to hold 22 cubic inches of a soft drink (approximately 12 fluid ounces).

(a) Analytically complete six rows of a table such as the one below (The first two rows are shown).

Radius r	Height	Surface Area
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left(0.2 + \frac{22}{\pi(0.2)^2}\right) \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left(0.4 + \frac{22}{\pi(0.4)^2}\right) \approx 111$
0.6	$\frac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left(0.6 + \frac{22}{\pi(0.6)^2}\right) \approx 75.6$
0.8	$\frac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left(0.8 + \frac{22}{\pi(0.8)^2}\right) \approx 59$

(b) Use a graphing utility to generate additional rows of the table. Use the table to estimate the minimum surface area.

The minimum is 43.6 for $r = 1.6$

Radius r	Height	Surface Area
0.2	$\frac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left(0.2+\frac{22}{\pi(0.2)^2}\right) \approx 220.3$
0.4	$\frac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left(0.4+\frac{22}{\pi(0.4)^2}\right) \approx 111$
0.6	$\frac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left(0.6+\frac{22}{\pi(0.6)^2}\right) \approx 75.6$
0.8	$\frac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left(0.8+\frac{22}{\pi(0.8)^2}\right) \approx 59$
1.0	$\frac{22}{\pi(1.0)^2}$	$2\pi(1.0)\left(1.0+\frac{22}{\pi(1.0)^2}\right) \approx 50.3$
1.2	$\frac{22}{\pi(1.2)^2}$	$2\pi(1.2)\left(1.2+\frac{22}{\pi(1.2)^2}\right) \approx 45.7$
1.4	$\frac{22}{\pi(1.4)^2}$	$2\pi(1.4)\left(1.4+\frac{22}{\pi(1.4)^2}\right) \approx 43.7$
1.6	$\frac{22}{\pi(1.6)^2}$	$2\pi(1.6)\left(1.6+\frac{22}{\pi(1.6)^2}\right) \approx 43.6$
1.8	$\frac{22}{\pi(1.8)^2}$	$2\pi(1.8)\left(1.8+\frac{22}{\pi(1.8)^2}\right) \approx 44.8$
2.0	$\frac{22}{\pi(2.0)^2}$	$2\pi(2.0)\left(2.0+\frac{22}{\pi(2.0)^2}\right) \approx 47.1$

(c) Write the surface area S as a function of r.

$$\begin{aligned} S &= 2\pi r^2 + 2\pi r h &(1)\\ &= 2\pi r(r+h) &(2)\\ &= 2\pi r\left(r + \frac{22}{\pi r^2}\right) &(3)\\ &= 2\pi r^2 + \frac{44}{r} &(4)\end{aligned}$$

(d) Use a graphing utility to graph the funciton in part (c) and estimate the minimum surface area from the graph.
The minimum is about 4.5 for $r \approx 1.5$.

(e) Use calculus to find the critical number of the function in part (c) and find dimensions that will yield the minimum surface area.

$$\begin{aligned} \frac{dS}{dr} &= 4\pi r - \frac{44}{r^2} = 0 \text{ when } r = \sqrt[3]{\frac{11}{\pi}} \approx 1.52\text{in} &(1)\\ h &= \frac{22}{\pi r^2} \approx 3.04\text{in} &(2)\end{aligned}$$

13 Capstone

38. The perimeter of a rectangle is 20 feet. Of all possible dimensions, the maximum area is 25 square feet when its length and width are both 5 feet. Are there dimensions that yield a minimum area? Explain.
the isn't a minimum area. If we let the sides be x and y then

$$\begin{aligned} 2x + 2y &= 20 &(1)\\ y &= 10 - x &(2)\end{aligned}$$

The area would be

$$\begin{aligned} A(x) &= x(10-x) &(3)\\ &= 10 - x^2 &(4)\end{aligned}$$

This can be made indiscriminately small if x approaches 0.

14 Minimum cost

40. An industrial tank of the shape describe of the shape described in Exercise 30 [”two hemispheres adjoined to the ends of a right circular cilinder”] must have a volume of 4000 cubic feet. The hemispherical ends cost twice as mch per square foot of surface area as the sides Find the dimensions that will minimize cost.

Let lowercase c be the cost per surface area's ft^2.
$2c$ would be the hemispherical ends' cost per ft^2.

$$
\begin{aligned}
C &= 2c(4\pi r^2) + c(2\pi r h) \qquad (5)\\
&= c(8\pi r^2 + 2\pi r\left(\frac{4000}{\pi r^2} - \frac{4}{3}r\right)) \qquad (6)\\
&= c\left(\frac{16}{3}\pi r^2 + \frac{8000}{r}\right) \qquad (7)\\
\frac{dC}{dr} &= c\left(\frac{32}{3}\pi r - \frac{8000}{r^2}\right) = 0 \text{ when } r = \sqrt[3]{\frac{750}{\pi}} \qquad (8)\\
&\approx 6.204\text{ft and } h \approx 24.814\text{ft} \qquad (9)
\end{aligned}
$$

The dimensions that will minimize cost are $r = \sqrt[3]{\frac{750}{\pi}}$ and $h \approx 24.814$ft.

15 Minimum cost

54. An offshore oil well is 2 kilometers off the coast. The refinery is 4 kilometers down the coast. Laying pipe in the ocean is twice as expensive as on land. What path should the pipe follow in order to minimize the cost?

$$
\begin{aligned}
C(x) &= 2c\sqrt{x^2+4} + c(4-x) \qquad (1)\\
C'(x) &= \frac{2xk}{\sqrt{x^2+4}} - k = 0 \qquad (2)\\
2x &= \sqrt{x^2+4} \qquad (3)\\
4x^2 &= x^2 + 4 \qquad (4)\\
3x^2 &= 4 \qquad (5)\\
x &= \frac{2}{\sqrt{3}} \qquad (6)
\end{aligned}
$$

16 Maximum Profit

58. Assume that the amount of money deposited in a bank is proportional to the square of the interest rate the bank pays on this money. Furthermore, the bank can reinvest this money at 12%. Find the interest rate the bank should pay to maximize profit (Use the simple interest formula.)

Let m be the money desposited, p the profit, and i the interest paid.

$$
\begin{aligned}
p &= (0.12)m - im \qquad (7)\\
m &= ki^2 \qquad (8)\\
p &= (0.12)(ki^2) - i(ki^2) = k(0.12i^2 - i^3) \qquad (9)\\
\frac{dp}{di} &= k(0.24i - 3i^2) = 0 \text{ when } i = \frac{0.24}{3} = 0.08 \qquad (10)\\
\frac{d^2p}{di^2} &= k(0.24 - 6i) < 0 \text{ when } i = 0.08 \qquad (11)
\end{aligned}
$$

The profit is maximized when the interest rate is at 8%.

17 Diminishing returns

The profit P (in thousands of dollars) for a company spending an amount s (in thousands of dollars) on advertising is

$$P = -\frac{1}{10}s^3 + 6s^2 + 400$$

(a) Find the amount of money the company should spend on advertising in order to yield a maximum profit.

$$\begin{aligned}
\frac{dp}{ds} &= -\frac{3}{10}s^2 + 12s && (1)\\
&= -\frac{3}{10}s^2(s-40) && (2)\\
&= 0 \text{ when } x = 0 \text{ and } s = 40 && (3)\\
\frac{d^2p}{ds^2} &= -\frac{3}{5}s + 12 && (4)\\
\frac{d^2p}{ds^2}&(0) > 0,\ \ s = 0 \text{ gives a minimum.} && (5)\\
\frac{d^2p}{ds^2}&(40) < 0,\ \ s = 40 \text{ gives a maximum.} && (6)
\end{aligned}$$

The company should spend \$40000 (from $s = 40$) in order to yield a maximum profit of $p =$ \$3600000.

(b) The point of diminishing returns is the point at which the rate of growth of the profit funciton begins to decline. Find the point of diminishing returns.

$$\frac{d^2p}{ds^2} = -\frac{3}{5}s + 12 = 0 \text{ when } s = 20 \qquad (1)$$

The point of diminishing returns happens at $s = 20$, which is \$20000 spent on advertising.