4.6 Numerical Integration

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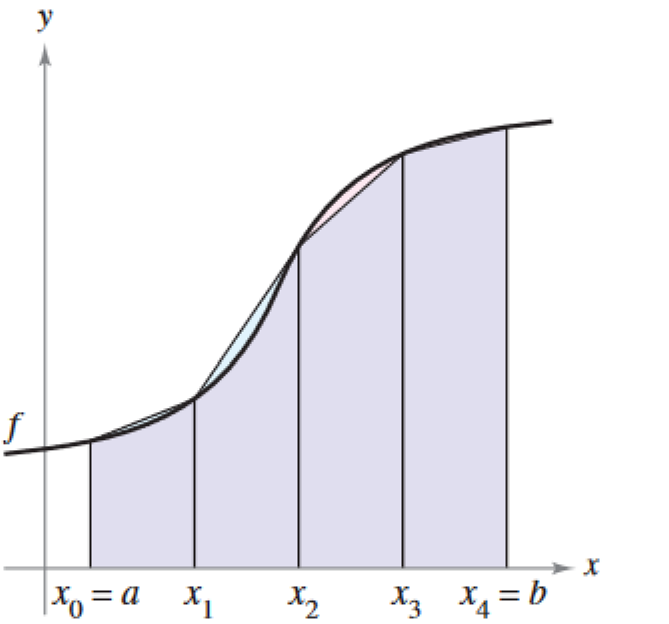
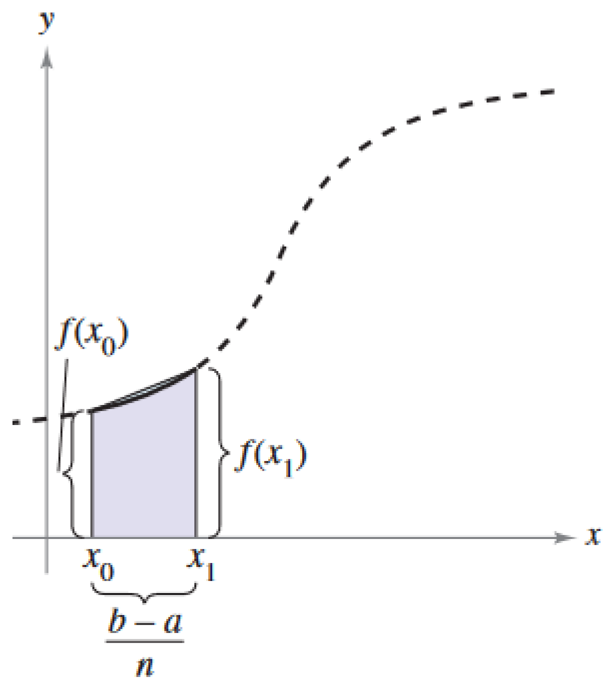
December 2023

1 Why trapezoids are better than rectangles when approximating area



The area of the region can be approximated using four trapezoids.



Let b_n be the height of the trapezoid at each x_n point. The area of the i-th trapezoid is

$$\left(\frac{f(x_{i-1}+f(x_i))}{2}\right)\left(\frac{b-a}{n}\right)$$

The area of a trapezoid is

$$\left(\frac{b_1+b_2}{2}\right)h$$

This implies that the sum of the areas of the n trapezoids is

$$\begin{aligned}
\text{Area} &= \left(\frac{b_1+b_2}{2}\right)h+\left(\frac{b_2+b_3}{2}\right)h+\left(\frac{b_3+b_4}{2}\right)h+\left(\frac{b_4+b_5}{2}\right)h && (1)\\
&= \frac{h}{2}(b_1+b_2+b_2+b_3+b_3+b_4+b_5) && (2)\\
&= \frac{b-a}{2n}(b_1+2b_2+2b_3+2b_4+b5);\ \Delta x=\frac{b-a}{n}=\frac{h}{2}\Rightarrow h=\frac{b-a}{2n}; && (3)\\
&= \frac{b-a}{2n}(f(x_0)=2f(x_1)+2f(x_2)+2f(x_3)+\cdots+2f(x_{n-1})+f(x_n)) && (4)
\end{aligned}$$

2 The trapezoidal rule

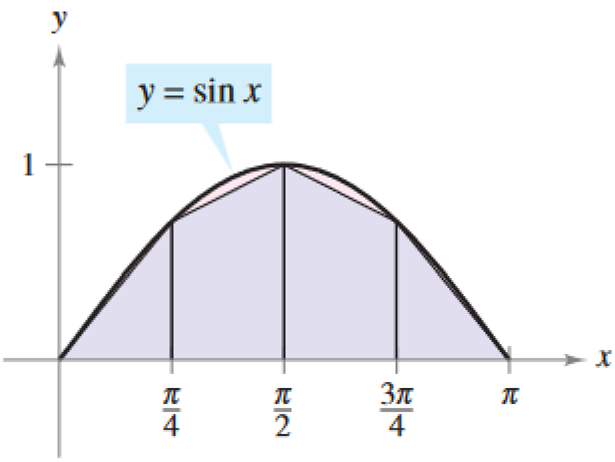
Let f be continuous on [a, b]. The Trapezoidal Rule for approximating $\int_a^b f(x)dx$ is given by

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}(f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1}+f(x_n)))$$

Moreover, as $n \to \infty$, the right-hand side approaches the integral $\int_a^b f(x)dx$.

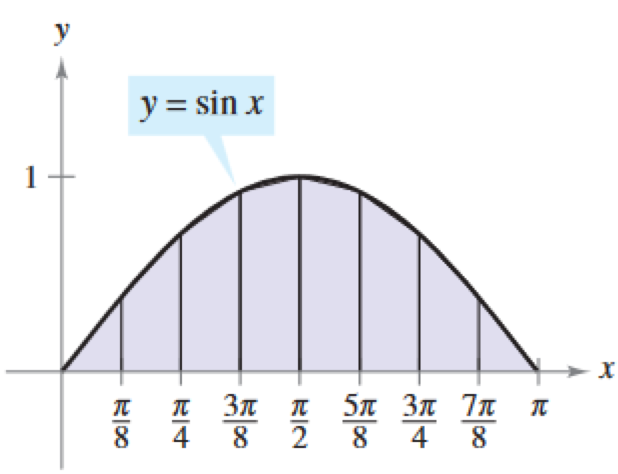
2.1 Example 1 - Approximation with the Trapezoidal Rule

Approximate $\int_0^\pi \sin x dx$ for $n = 4$ and $n = 8$



Four subintervals

$$\int_0^\pi \sin x dx,\ n = 4,\ \Delta x = \frac{\pi}{4} \tag{5}$$
$$\approx \frac{\pi = 0}{2(4)}\left(\sin 0 + 2\sin\frac{\pi}{4} + 2\sin\frac{\pi}{2} + 2\sin\frac{3\pi}{4} + \sin\pi\right) \tag{6}$$
$$= \frac{\pi}{8}(0+\sqrt{2}) + 2 + \sqrt{2} + 0 \tag{7}$$
$$= \frac{\pi}{8}(2+2\sqrt{2})u^2 \tag{8}$$



Eight subintervals

$$
\begin{align}
&\int_0^\pi \sin x dx,\ n = 8,\ \Delta x = \frac{\pi}{8} \tag{1}\\
&\approx \frac{\pi}{16}(\sin 0 + 2\sin\frac{\pi}{8} + 2\sin\frac{\pi}{4} + 2\sin\frac{3\pi}{8} + 2\sin\frac{\pi}{2} + 2\sin\frac{5\pi}{8} + 2\sin\frac{3\pi}{4} + 2\sin\frac{7\pi}{8} + \sin\pi) \tag{2}\\
&= \frac{\pi}{16}\left(0 + \sum_{i=1}^{7} 2\sin\left(\frac{i\pi}{8}\right) + 0\right) \tag{3}\\
&\approx 1.974 \tag{4}
\end{align}
$$

3 Simpson's Rule and why curves are even better than trapezoids when approximating area.

Let f be continuous on [a, b] and let n be an even integer. Simpson's Rule for approximating $\int_a^b f(x)dx$ is

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 4f(x_{n-1}) + f(x_n))$$

Moreover, as $n \to \infty$, the right-hand side approaches the integral $\int_a^b f(x)dx$.