4.6 Numerical Integration

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1 Use the Trapezoidal Rule and Simpson's Rule to approximate the value of the definite integral for the given value of Round your answer to four decimal places and compare the results with the exact value of the definite integral.

Equations (1), (2), and (3) will be the exact value, the Trapezoidal Rule, and Simpson's Rule respectively.

2.

$$\int_1^2 \left(\frac{x^2}{4}+1\right)dx = \left(\frac{x^3}{12}+x\right)_1^2 = \frac{19}{12} \approx 1.5833 \tag{1}$$

$$\int_1^2 \left(\frac{x^2}{4}+1\right)dx \approx \frac{1}{8}\left(\left(\frac{1^2}{4}+1\right)+2\left(\frac{\left(\frac{5}{4}\right)^2}{4}+1\right)+2\left(\frac{\left(\frac{3}{2}\right)^2}{4}+1\right)+2\left(\frac{\left(\frac{7}{4}\right)^2}{4}+1\right)+\left(\frac{2^2}{4}+1\right)\right) = \frac{203}{128} \approx 1.5859 \tag{2}$$

$$\int_0^1 \left(\frac{x^2}{4}+1\right)dx \approx \frac{1}{12}\left(\left(\frac{1^2}{4}+1\right)+4\left(\frac{\left(\frac{5}{4}\right)^2}{4}+1\right)+2\left(\frac{\left(\frac{3}{2}\right)^2}{4}+1\right)+4\left(\frac{\left(\frac{7}{4}\right)^2}{4}+1\right)+\left(\frac{2^2}{4}+1\right)\right) = \frac{19}{12} \approx 1.5833 \tag{3}$$

6.

$$\int_0^8 \sqrt[3]{x}dx = \left(\frac{3}{4}x^{4/3}\right)_0^8 = 12 \tag{1}$$

$$\int_0^8 \sqrt[3]{x}dx \approx \frac{1}{2}(0+2+2\sqrt[3]{2}+2\sqrt[3]{3}+2\sqrt[3]{4}+2\sqrt[3]{5}+2\sqrt[3]{6}+2\sqrt[3]{7}+2) \approx 11.7296 \tag{2}$$

$$\int_0^8 \sqrt[3]{x}dx \approx \frac{1}{3}(0+4+2\sqrt[3]{2}+4\sqrt[3]{3}+2\sqrt[3]{4}+4\sqrt[3]{5}+2\sqrt[3]{6}+4\sqrt[3]{7}+2) \approx 11.8632 \tag{3}$$

10.

$$\int_0^2 x\sqrt{x^2+1}dx = \frac{1}{3}((x^2+1)^{3/2})_0^2 = \frac{1}{3}(5^{3/2}-1) \approx 3.393 \tag{1}$$

$$\int_0^2 x\sqrt{x^2+1}dx \approx \frac{1}{4}\left(0+2\left(\frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2+1}+2(1)\sqrt{(1)^2+1}+2\left(\frac{3}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2+1}+2\sqrt{(2)^2+1}\right) \approx 3.457 \tag{2}$$

$$\int_0^2 x\sqrt{x^2+1}dx \approx \frac{1}{6}\left(0+4\left(\frac{1}{2}\right)\sqrt{\left(\frac{1}{2}\right)^2+1}+2(1)\sqrt{(1)^2+1}+4\left(\frac{3}{2}\right)\sqrt{\left(\frac{3}{2}\right)^2+1}+2\sqrt{(2)^2+1}\right) \approx 3.392 \tag{3}$$

2 Approximate the definite integral using the Trapezoidal Rule and Simpson's Rule with $n = 4$. Compare these results with the approximation of the integral using a graphing utility.

Equations (1), (2), and (3) will be the exact value, the Trapezoidal Rule, and Simpson's Rule respectively.
12.

$$\int_0^2 \frac{1}{\sqrt{1+x^3}}dx = 1.402 \tag{1}$$

$$\int_0^2 \frac{1}{\sqrt{1+x^3}}dx \approx \frac{1}{4}\left(1 + 2\left(\frac{1}{\sqrt{1+\left(\frac{1}{2}\right)^3}}\right) + 2\left(\frac{1}{\sqrt{1+(1)^3}}\right) + 2\left(\frac{1}{\sqrt{1+\left(\frac{3}{2}\right)^3}}\right) + \frac{1}{3}\right) \approx 1.397 \tag{2}$$

$$\int_0^2 \frac{1}{\sqrt{1+x^3}}dx \approx \frac{1}{6}\left(1 + 4\left(\frac{1}{\sqrt{1+\left(\frac{1}{2}\right)^3}}\right) + 2\left(\frac{1}{\sqrt{1+(1)^3}}\right) + 4\left(\frac{1}{\sqrt{1+\left(\frac{3}{2}\right)^3}}\right) + \frac{1}{3}\right) \approx 1.405 \tag{3}$$

16.

$$\int_0^{\sqrt{\frac{\pi}{2}}} \tan x^2 dx = 0.256 \tag{1}$$

$$\int_0^{\sqrt{\frac{\pi}{2}}} \tan x^2 dx \approx \frac{\sqrt{\frac{\pi}{4}}}{8}\left(\tan 0 + 2\tan\left(\frac{\sqrt{\frac{\pi}{4}}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\frac{\pi}{4}}}{2}\right)^2 + 2\tan\left(\frac{\sqrt{\frac{3\pi}{4}}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right) \approx 0.271 \tag{2}$$

$$\int_0^{\sqrt{\frac{\pi}{2}}} \tan x^2 dx \approx \frac{\sqrt{\frac{\pi}{4}}}{12}\left(\tan 0 + 4\tan\left(\frac{\sqrt{\frac{\pi}{4}}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\frac{\pi}{4}}}{2}\right)^2 + 4\tan\left(\frac{\sqrt{\frac{3\pi}{4}}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right) \approx 0.257 \tag{3}$$

20.

$$\int_0^{\pi} f(x)dx = 1.852,\ f(x) = \begin{cases} \frac{\sin x}{x},\ x > 0 \\ 1,\ x = 0 \end{cases} \tag{1}$$

$$\int_0^{\pi} \frac{\sin x}{x}dx \approx \frac{\pi}{8}\left(1 + \frac{2\sin\left(\frac{\pi}{4}\right)}{\frac{\pi}{4}} + \frac{2\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} + \frac{2\sin\left(\frac{3\pi}{4}\right)}{\frac{3\pi}{4}} + 0\right) \approx 1.836 \tag{2}$$

$$\int_0^{\pi} \frac{\sin x}{x}dx \approx \frac{\pi}{12}\left(1 + \frac{4\sin\left(\frac{\pi}{4}\right)}{\frac{\pi}{4}} + \frac{2\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} + \frac{4\sin\left(\frac{3\pi}{4}\right)}{\frac{3\pi}{4}} + 0\right) \approx 1.852 \tag{3}$$

3 Use the error formulas in Theorem 4.20 (Errors in the Trapezoidal Rule and Simpson's Rule) to estimate the errors in approximating the integral, with $n = 4$ using (a) the Trapezoidal Rule and (b) Simpson's Rule.

24.

$$\int_3^5 (5x+2)dx \tag{1}$$

$$f(x) = 5x + 2 \tag{2}$$

$$f'(x) = 5 \tag{3}$$

$$f''(x)0 \tag{4}$$

For both rules, the error is zero.

4 Use the error formulas in Theorem 4.20 to find n such that the error in the approximation of the definite integral is less than or equal to 0.00001 using (a) the Trapezoidal Rule and (b) Simpson's Rule.

30.

$$\int_0^1 \frac{1}{1+x}dx \tag{1}$$
$$f(x) = (1+x)^{-1},\ 0 \le x \le 1 \tag{2}$$
$$f'(x) = -(1+x)^{-2} \tag{3}$$
$$f''(x) = 2(1+x)^{-3} \tag{4}$$
$$f'''(x) = -6(1+x)^{-4} \tag{5}$$
$$f^{(4)}(x) = 24(1+x)^{-5} \tag{6}$$

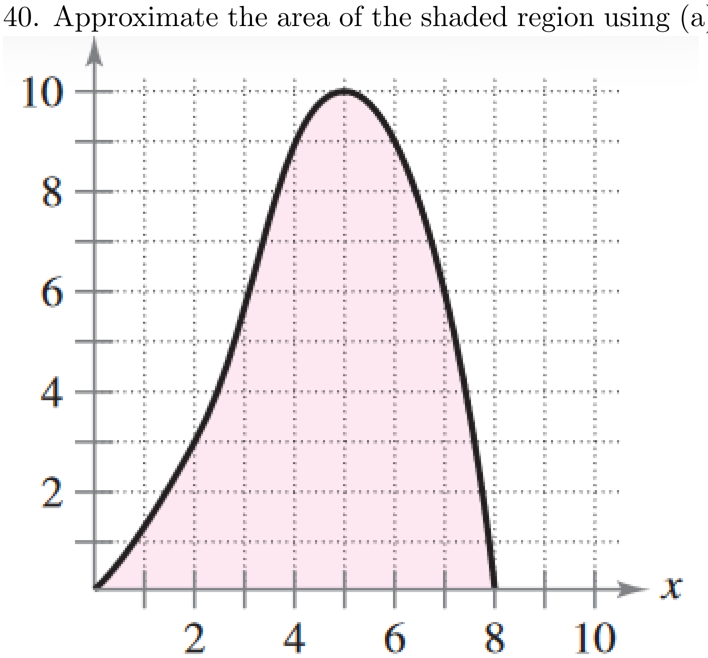
(a) Trapezoidal: Maximum of $|f''(x)| = |2(1+x)^{-3}|$ is 2.

$$\text{Error} \le \frac{1}{12n^2}(2) \le 0.00001 \tag{7}$$
$$n^2 \ge 16666.67 \tag{8}$$
$$n \ge 129.10 \tag{9}$$

(b) Simpson's: Maximum of $|f^{(4)}(x)| = |24(1+x)^{-5}|$ is 24.

$$\text{Error} \le \frac{1}{180n^4}(24) \le 0.00001 \tag{10}$$
$$n^4 \ge 133333.33 \tag{11}$$
$$n \ge 10.75 \tag{12}$$

40. Approximate the area of the shaded region using (a) the Trapezoidal Rule and (b) Simpson's Rule with $n = 8$



$$n = 8,\ b - a = 8 - 0 = 8 \tag{1}$$
$$\int_0^8 f(x)dx \approx \frac{8}{16}(0 + 2(1.5) + 2(3) + 2(5.5) + 2(9) + 2(10) + 2(9) + 2(6) + 0) = \frac{1}{2}(88) = 44 \tag{2}$$
$$\int_0^8 f(x)dx \approx \frac{8}{24}(0 + 4(1.5) + 2(3) + 4(5.5) + 2(9) + 4(10) + 2(9) + 4(6) + 0) = \frac{134}{3} \tag{3}$$

5 Capstone

46. Consider a function $f(x)$ that is concave upward on the interval [0, 2] and a function $g(x)$ that is concave downward on [0, 2].

(a) Using the Trapezoidal Rule, which integral would be overestimated? Which integral would be underestimated? Assume $n = 4$ Use graphs to explain your answer.
$\int_0^2 f(x)dx$ would be overestimated, and $\int_0^2 g(x)dx$ would be underestimated.

(b) Which rule would you use for more accurate approxi- mations of $\int_0^2 f(x)dx$ and $\int_0^2 g(x)dx$ the Trapezoidal Rule or Simpson's Rule? Explain your reasoning.
Simpson's Rule would be more accurate because it considers the graph's curvature.

6 Work

48. To determine the size of the motor required to operate a press, a company must know the amount of work done when the press moves an object linearly 5 feet. The variable force to move the object is $F(x) = 100x\sqrt{25 - x^3}$ where F is given in pounds and x gives the position of the unit in feet. Use Simpson's Rule with $n = 12$ to approximate the work W (in foot-pounds) done through one cycle if $W = \int_0^5 F(x)dx$.

$$W = \int_0^5 100x\sqrt{125 - x^3}dx,\ n = 12 \tag{1}$$
$$\int_0^5 100x\sqrt{125 - x^3}dx \approx \frac{5}{3(12)}\left(0 + 400\left(\frac{5}{12}\right)\sqrt{125 - \left(\frac{5}{12}\right)^3} + 200\left(\frac{10}{12}\right)\sqrt{125 - \left(\frac{10}{12}\right)^3}\right.$$
$$\left. + 400\left(\frac{15}{12}\sqrt{125 - \left(\frac{15}{12}\right)^3} + \cdots + 0\right) \approx 10233.58\text{ft-lb} \tag{2}$$

7 Use Simpson's Rule with $n = 6$ to approximate π using the given equation.

50.

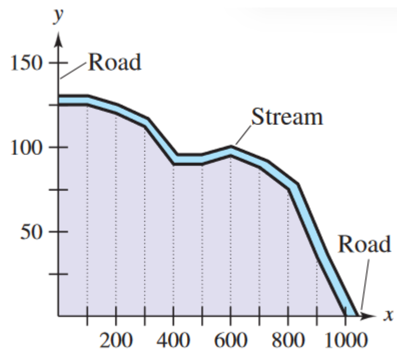
$$\int_0^{\frac{1}{2}} \frac{6}{\sqrt{1 - x^2}}dx,\ n = 6 \tag{1}$$
$$\pi \approx \frac{\left(\frac{1}{2} - 0\right)}{3(6)}(6 + 4(6.0209) + 2(6.0851) + 4(6.1968) + 2(6.3640) + 4(6.6002) + 6.9282) \approx \frac{1}{36}(113.098) \approx 3.1416 \tag{2}$$

8 Use the Trapezoidal Rule to estimate the number of square meters of land in a lot where x and y are measured in meters, as shown in the figures. The land is bounded by a stream and two straight roads that meet at right angles.

52.

x	0	100	200	300	400	500
y	125	125	120	112	90	90

x	600	700	800	900	1000
y	95	88	75	35	0



$$\text{Area} \approx \frac{1000}{2(10)}(125 + 2(125) + 2(120) + 2(112) + 2(90) + 2(90) + 2(95) + 2(88) + 2(75) + 2(35)) = 89.250\text{m}^2 \tag{1}$$

9 Warm-up 12/11/2023

The sides of the rectangle increase in such a way that $\frac{dz}{dt} = 1$ and $\frac{dx}{dt} = 3\frac{dy}{dt}$. At the instant that $x = 4$ and $y = 3$ what is the value of $\frac{dx}{dt}$?

$$x^2 + y^2 = z^2 \tag{1}$$

$$4(3\frac{dy}{dt}) + 3(\frac{dy}{dt}) = 5(1) \tag{2}$$

$$\frac{dx}{dt} = 3\frac{dy}{dt} \therefore 15\frac{dy}{dt} = \frac{5}{5 \cdot 3} = \frac{1}{3}\frac{dy}{dt} \tag{3}$$