3.4 Concavity and the Second Derivative Test

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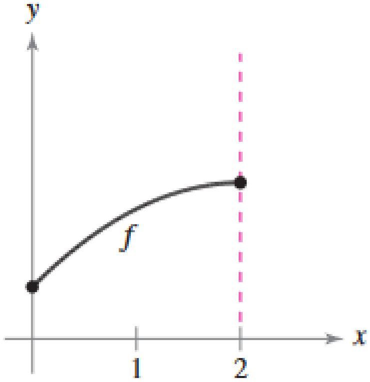
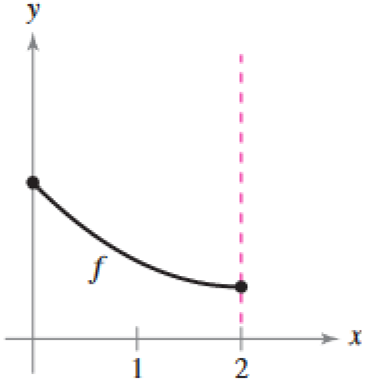
1 The graph of f is shown. State the signs of f' and f'' on the interval (0, 2)

2.



$f' > 0$ and $f'' < 0$

4.



$f' < 0$ and $f'' > 0$

2 Determine the open intervals on which the graph is concave upward or concave downward.

6.

$$y = -x^3 + 3x^2 - 2 \tag{1}$$
$$y' = -3x^2 + 6x \tag{2}$$
$$y'' = -6x + 6 \tag{3}$$

The graph is concave upward on $(-\infty, 1)$ and downward on $(1, \infty)$.

14.

$$y = \frac{1}{270}(-3x^5 + 40x^3 + 135x) \tag{1}$$
$$y' = \frac{1}{270}(-15x^4 + 120x^2 + 135) \tag{2}$$
$$y'' = -\frac{2}{9}x(x-2)(x+2) \tag{3}$$

The graph is concave upward on $(-\infty, -2)$ and $(0, 2)$, and downward on (-2, 0) and $(2.\infty)$.

18.

$$y = x + \frac{2}{\sin x}, \quad (-\pi, \pi) \tag{1}$$
$$y' = 1 - 2\csc x \cot x \tag{2}$$
$$y'' = -2\csc x(-\csc^2 x) - 2\cot x(-\csc x \cot x) \tag{3}$$
$$= 2(\csc^3 x + \csc x \cot^2 x) \tag{4}$$

The graph is concave upward on $(0, \pi)$ and downward on $(-\pi, 0)$.

3 Find the points of inflection and discuss the concavity of the graph of the function.

20.

$$f(x) = -x^4 + 24x^2 \tag{1}$$
$$f'(x) = -4x^3 + 48x \tag{2}$$
$$f''(x) = -12x^2 + 48 \tag{3}$$
$$= 12(4 - x^2) \tag{4}$$
$$= 12(2 + x)(2 - x) \tag{5}$$
$$f'' = 0 \text{ for } x = -2.2 \tag{6}$$

The points of inflection are (-2, 80) and (2, 80).
The graph is concave upward on (-2, 2) and downward on $(-\infty, -2)$ and $(2, \infty)$.

24.

$$f(x) = 2x^4 - 8x + 3 \tag{1}$$
$$f'(x) = 8x^3 - 8 \tag{2}$$
$$f''(x) = 24x^2 \tag{3}$$
$$= 0 \text{ when } x = 0 \tag{4}$$

The graph has no inflection points since it is concave upward on $(-\infty, \infty)$.

28.

$$f(x) = x\sqrt{9-x} \tag{1}$$
$$f'(x) = \frac{3(6-x)}{2\sqrt{9-x}} \tag{2}$$
$$f''(x) = \frac{3(x-12)}{4(9-x)^{3/2}} \tag{3}$$

The graph has no inflection points and is concave downward on $(-\infty, 9)$.

32.

$$f(x) = 2\csc\frac{3x}{2}, \quad (0, 2\pi) \tag{1}$$
$$f'(x) = -3\csc\frac{3x}{2}\cot\frac{3x}{2} \tag{2}$$
$$f''(x) = \frac{9}{2}\left(\csc^3\frac{3x}{2} + \csc\frac{3x}{2}\cot^2\frac{3x}{2}\right) \neq 0 \text{ for all x in } f\text{'s domain.} \tag{3}$$

The graph is concave upward on $\left(0, \frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{3}, 2\pi\right)$, and downward on $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$.

36.

$$f(x) = x + 2\cos x, \quad [0, 2\pi] \tag{1}$$
$$f'(x) = 1 - 2\sin x \tag{2}$$
$$f''(x) = -2\cos x \tag{3}$$
$$f'''(x) = 0 \text{ when } x = \frac{\pi}{2}, \frac{3\pi}{2}. \tag{4}$$

Test intervals	$0 < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
y' sign	$f'' < 0$	$f'' > 0$	$f'' < 0$
Concavity	Downward	Upward	Downward

The points of inflection are $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, \frac{3\pi}{2}\right)$.

4 Find all relative extrema. Use the Second Derivative Test when applicable.

38.

$$f(x) = -(x-5)^2 \tag{1}$$
$$f'(x) = -2(x-5) \tag{2}$$
$$f''(x) = -2 \tag{3}$$

$x = 5$ is a critical number and $f''(5) < 0 \therefore$ (5, 0) is a relative maximum.

42.

$$f(x) = x^3 - 5x^2 + 7x \tag{1}$$
$$f'(x) = 3x^2 - 10x + 7 \tag{2}$$
$$= (3x-7)(x-1) \tag{3}$$
$$f''(x) = 6x - 10 \tag{4}$$

$x = \frac{7}{3}, 1$ are critical numbers.

$$f''\left(\frac{7}{3}\right) = 4 > 0 \therefore \left(\frac{7}{3}, \frac{49}{27}\right) \text{ is a relative minimum.} \tag{1}$$
$$f''(1) = 4 > 0 \therefore (1, 3) \text{ is a relative maximum.} \tag{2}$$

46.

$$g(x) = -\frac{1}{8}(x+2)^2(x-4)^2 \tag{1}$$
$$g'(x) = \frac{-(x-4)(x-1)(x+2)}{2} \tag{2}$$
$$g''(x) = 3 + 3x - \frac{3}{2}x^2 \tag{3}$$

$x = -2, 1, 4$ are critical numbers.

$$g''(-2) = -9 < 0 \tag{1}$$
$$g''(1) = \frac{9}{2} > 0 \tag{2}$$
$$g''(4) = -9 < 0 \tag{3}$$

(-2, 0) and (4, 0) are relative maxima, and (1, -10.125) the relative minimum.

50.

$$f(x) = \frac{x}{x-1} \tag{1}$$
$$f'(x) = \frac{-1}{(x-1)^2} \tag{2}$$

$x = 1$ is not in the domain, and the function doesn't have any critical numbers. Therefore, there are no relative extrema.

52

$$f(x) = 2\sin x + \cos 2x, \quad [0, 2\pi] \tag{1}$$
$$f'(x) = 2\cos x - 2\sin 2x = 2\cos x - 4\sin x \cos x \tag{2}$$
$$= 2\cos x(1 - 2\sin x) \tag{3}$$
$$= 0 \text{ when } x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}. \tag{4}$$
$$f'''(x) = -2\sin x - 4\cos 2x \tag{5}$$
$$f'''(\frac{\pi}{6}) < 0, \quad f'''(\frac{\pi}{2}), \quad f'''(\frac{5\pi}{6}) < 0, \quad f'''(\frac{3\pi}{2}) > 0 \tag{6}$$

$\left(\frac{\pi}{6}, \frac{3}{2}\right), \left(\frac{5\pi}{6}, \frac{3}{2}\right)$ are relative maxima and $\left(\frac{\pi}{2}, 1\right), \left(\frac{3\pi}{2}, -3\right)$ relative minima.

5 Writing about concepts

60. S represents weekly sales of a product. What can be said of S' and S'' for each of the following statements?

(a) The rate of change of sales is increasing.
$S'' > 0$

(b) Sales are increasing at a slower rate.
$S' > 0, \quad S'' < 0$

(c) The reate of change of sales is constant.
$S' = C, \quad S'' = 0$

(d) Sales are steady.
$S' = 0, \quad S'' = 0$

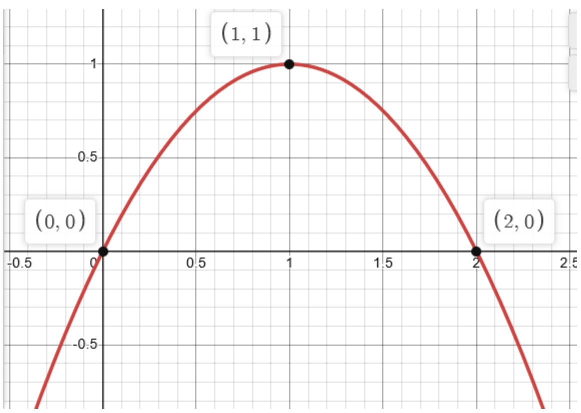
(e) Sales are declining, but at a lower rate.
$S' < 0, \quad S'' > 0$

(f) Sales have bottomed out and have started to rise.
$S' > 0$

6 Sketch a graph of the function f having the given characteristics.

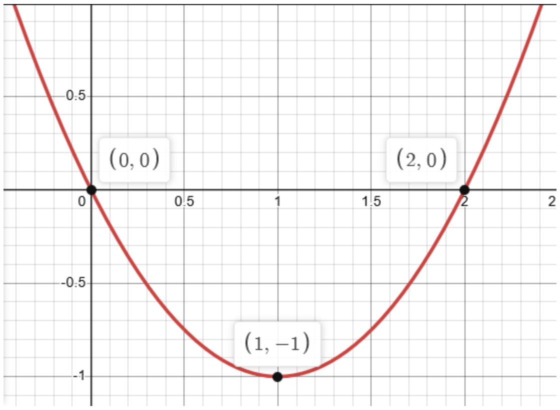
66.

$$f(0) = f(2) = 0 \tag{1}$$
$$f'(x) > 0, \quad x < 1 \tag{2}$$
$$f'(1) = 0 \tag{3}$$
$$f'(x) < 0, \quad x > 1 \tag{4}$$
$$f''(x) = 0 \tag{5}$$



68.

$$f(0) = f(2) = 0 \tag{1}$$
$$f'(x) < 0, \quad x < 1 \tag{2}$$
$$f'(1) = 0 \tag{3}$$
$$f'(x) > 0, \quad x > 1 \tag{4}$$
$$f''(x) > 0 \tag{5}$$



7 Capstone

70. Water is running into the vase shown in the figure at a constant rate.

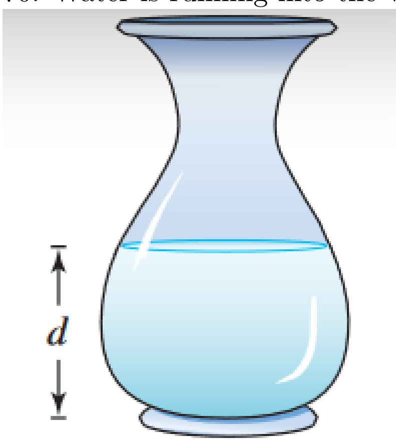
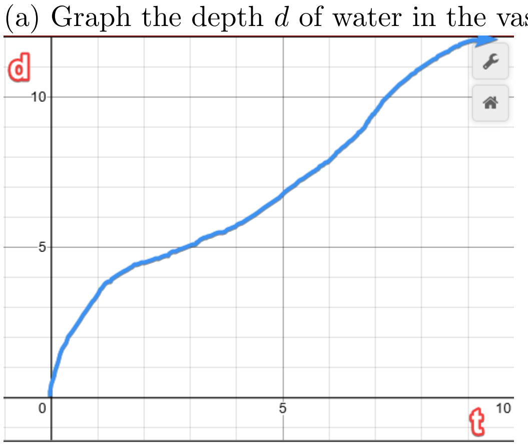


Figure for 70

(a) Graph the depth d of water in the vase as a function of time.



(b) Does the function have any extrema? Explain.

There are no relative extrema since the depth is always increasing.

(c) Interpret the inflection points of the graph of d.

The rate of changes d decreases until the liquid reaches the vase's widest point, then the rate increases in the vase's narrowest point, and the rate decreases again when the top of vase is reached.

8 Find $a, b, c,$ and d such that the cubic $f(x) = ax^3 + bx^2 + cx + d$ satisfies the given conditions.

74.

1. Relative maximum: (2, 4)
2. Relative minimum: (4, 2)
3. Inflection point (3, 3)

$$
\begin{align}
f'(x) &= 3ax^2 + 2bx + c \tag{1}\\
f''(x) &= 6ax + 2b \tag{2}\\
f(2) &= 8a + 4b + 2c + d = 4 \tag{3}\\
f(3) &= 64a + 16b + 4c + d = 2 \tag{4}
\end{align}
$$

By equations 3 and 4, it follows that:

$$
\begin{align}
56a + 12b + 2c &= -2 \tag{5}\\
28a + 6b + c &= -1 \tag{6}\\
f'(2) = 12a + 4b + c &= 0 \tag{7}\\
f'(4) = 48a + 8b + c &= 0 \tag{8}\\
f''(3) = 18a + 2b &= 0 \tag{9}
\end{align}
$$

We have that

$$
\begin{align}
18a + 2b &= 0 \tag{10}\\
16a + 2b &= -1 \tag{11}\\
2a &= 1 \tag{12}
\end{align}
$$

and

$$
\begin{align}
28a + 6b + c &= 1 \tag{13}\\
12a + 4b + c &= 0 \tag{14}\\
16a + 2b &= -1 \tag{15}
\end{align}
$$

Therefore,

$$
\begin{align}
a &= \frac{1}{2} \tag{16}\\
b &= -\frac{9}{2} \tag{17}\\
c &= 12 \tag{18}\\
d &= -6 \tag{19}\\
f(x) &= \frac{1}{2}x^3 - \frac{9}{2}x^2 + 12x - 6 \tag{20}
\end{align}
$$

9 Word problems

74. The total cost C of ordering and storing x units is $C = 2x + \frac{300000}{x}$. What order size will produce a minimum cost?

$$
\begin{align}
C' &= 2 - \frac{300000}{x^2} \tag{1}\\
&= 0 \text{ when } x = 100\sqrt{15} = 387 \tag{2}
\end{align}
$$

An order size of 387 units will produce a minimum cost.

82. The average typing speed S (in words per minute) of a typing student after t weeks of lessons is shown in the table.

t	5	10	15	20	25	30
S	38	56	79	90	93	94

A model for the data is $S = \frac{100t^2}{65+t^2}$, $t > 0$.
(b) Use the second deritvative to determine the concavity of S. Compare the result with the graph in part (a).

$$S'(t) = \frac{13000}{(65+t^2)^2} \tag{1}$$
$$S''(t) = \frac{13000(65-3t^2)}{(65+t^2)} = 0 \tag{2}$$
$$t = 4.65 \tag{3}$$

S is concave upwards on (0, 4.65) and downwards on (4.65, 30).

(c) What is the sign of the first derivative for $t > 0$? By combining this information with the concavity of the model, what inferences can be made about the typing speed as t increases.

$$S'(t) > 0, \quad t > 0$$

The speed increases at a slower rate as t increases.

88. Show that the point of inflection of $f(x) = x(x-6)^2$ lies midway between the relative extrema of f.

$$f(x) = x(x-6)^2 \tag{1}$$
$$= x^3 - 12x^2 + 36x \tag{2}$$
$$f'(x) = 3x^2 - 24x + 36 \tag{3}$$
$$= 3(x-2)(x-6) \tag{4}$$
$$= 0 \tag{5}$$
$$f''(x) = 6x - 24 \tag{6}$$
$$= 6(x-4) = 0 \tag{7}$$

(6, 0) and (2, 32) are relative extrema. (4, 16) is the point of inflection, and it lies midway between f's relative extrema.

10 Determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

91. The graph of every cubic polynomial has precisely one point of inflection.
True.

92. The graph of $f(x) = \frac{1}{x}$ is concave downward for $x < 0$ and concave upward for $x > 0$, and thus it has a point of inflection at $x = 0$

False. $f(x)$ is discontinuous at $x = 0$.

93. If $f'(c) > 0$, then f is concave upward at $x = c$

False. Concavity is identified with f''. If $f(x) = x$ and $c = 2$, f will not be concave upward at $c = 2$ even though $f'(c) = f(2) > 0$.

94. If $f''(2)$=0, then the graph of f must have a point of inflection at $x = 2$

False. $f(x) = (x-2)^4$ doesn't have one.