Notes - 3.6 A Summary of Curve Sketching

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So far, we have reviewed the following characteristic of a function in each of the following sections:

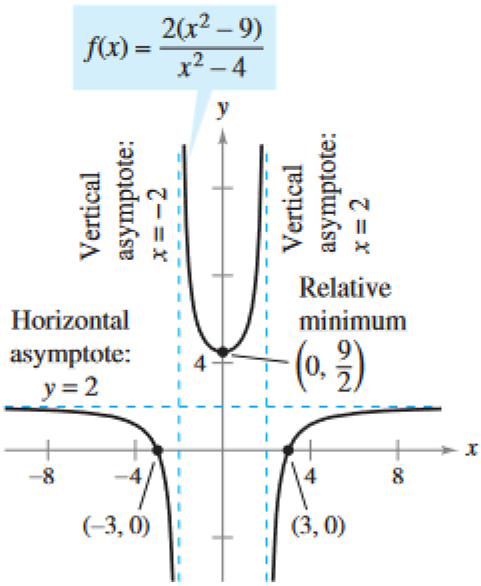
- x-intercepts and y- intercepts (Section P.1)
- Symmetry (Section P.1)
- Domain and range (Section P.3)
- Continuity (Section 1.4)
- Vertical asymptotes (Section 1.5)
- Differentiability (Section 2.1)
- Relative extrema (Section 3.1)
- Concavity (Section 3.4)
- Points of inflection (Section 3.4)
- Horizontal asymptotes (Section 3.5)
- Infinite limits at infinity (Section 3.5)

1 Guidelines for analyzing the graph of a function.

1. Determine the domain and range of the function.
2. Determine the intercepts, asymptotes, and symmetry of the graph.
3. Locate the x-values for which $f'(x)$ and $f''(x)$ either are zero or do not exist. Use those results to determine relative extrema and inflection points.

2 Examples

2.1 Sketching the graph of a rational function



Using calculus, you can be certain that you have determined all characteristics of the graph of f.
Figure 3.45

EXAMPLE 1 Sketching the Graph of a Rational Functio

Analyze and sketch the graph of $f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$.

Solution

First derivative: $f'(x) = \dfrac{20x}{(x^2 - 4)^2}$

Second derivative: $f''(x) = \dfrac{-20(3x^2 + 4)}{(x^2 - 4)^3}$

x-intercepts: $(-3, 0), (3, 0)$

y-intercept: $\left(0, \frac{9}{2}\right)$

Vertical asymptotes: $x = -2, x = 2$

Horizontal asymptote: $y = 2$

Critical number: $x = 0$

Possible points of inflection: None

Domain: All real numbers except $x = \pm$

Symmetry: With respect to y-axis

Test intervals: $(-\infty, -2), (-2, 0), (0, 2), (2$

The table shows how the test intervals are used to determine sever the graph. The graph of f is shown in Figure 3.45.

	$f(x)$	$f'(x)$	$f''(x)$	Characteri
$-\infty < x < -2$		$-$	$-$	Decreasing, co
$x = -2$	Undef.	Undef.	Undef.	Vertical
$-2 < x < 0$		$-$	$+$	Decreasing, c
$x = 0$	$\frac{9}{2}$	0	$+$	Relative
$0 < x < 2$		$+$	$+$	Increasing, c
$x = 2$	Undef.	Undef.	Undef.	Vertical
$2 < x < \infty$		$+$	$-$	Increasing, co

EXAMPLE 2 Sketching the Graph of a Rational Fu

Analyze and sketch the graph of $f(x) = \dfrac{x^2 - 2x + 4}{x - 2}$.

Solution

First derivative: $f'(x) = \dfrac{x(x-4)}{(x-2)^2}$

Second derivative: $f''(x) = \dfrac{8}{(x-2)^3}$

x-intercepts: None

y-intercept: $(0, -2)$

Vertical asymptote: $x = 2$

Horizontal asymptotes: None

End behavior: $\lim\limits_{x \to -\infty} f(x) = -\infty$, lin

Critical numbers: $x = 0, x = 4$

Possible points of inflection: None

Domain: All real numbers except

Test intervals: $(-\infty, 0), (0, 2), (2, 4),$ (

The analysis of the graph of f is shown in the table, an Figure 3.47.

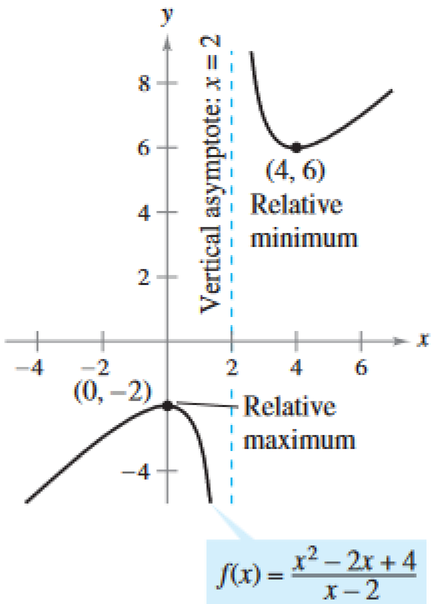
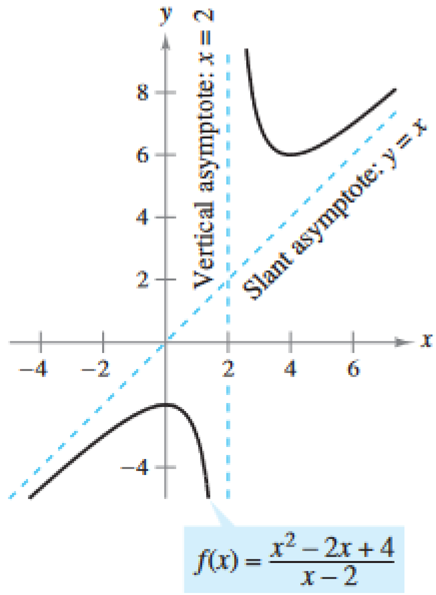


Figure 3.47

	$f(x)$	$f'(x)$	$f''(x)$	Chara
$-\infty < x < 0$		$+$	$-$	Increasing
$x = 0$	-2	0	$-$	Rel
$0 < x < 2$		$-$	$-$	Decreasin
$x = 2$	Undef.	Undef.	Undef.	Ver
$2 < x < 4$		$-$	$+$	Decreasi
$x = 4$	6	0	$+$	Rel
$4 < x < \infty$		$+$	$+$	Increasi



A slant asymptote
Figure 3.48

Although the graph of the function in Example 2 has no h
does have a slant asymptote. The graph of a rational functio
factors and whose denominator is of degree 1 or greater) has a
degree of the numerator exceeds the degree of the denominato
the slant asymptote, use long division to rewrite the rational fu
first-degree polynomial and another rational function.

$$f(x) = \frac{x^2 - 2x + 4}{x - 2} \qquad \text{Write original equation.}$$

$$= x + \frac{4}{x - 2} \qquad \text{Rewrite using long division.}$$

In Figure 3.48, note that the graph of f approaches the slant
approaches $-\infty$ or ∞.

2.2 Sketching the graph of a radical function

EXAMPLE 3 Sketching the Graph of a Radical Function

Analyze and sketch the graph of $f(x) = \dfrac{x}{\sqrt{x^2 + 2}}$.

Solution

$$f'(x) = \frac{2}{(x^2 + 2)^{3/2}} \qquad f''(x) = -\frac{6x}{(x^2 + 2)^{5/2}}$$

The graph has only one intercept, (0, 0). It has no vertical asympto
horizontal asymptotes: $y = 1$ (to the right) and $y = -1$ (to the left)
no critical numbers and one possible point of inflection (at $x = 0$). T
function is all real numbers, and the graph is symmetric with respect
analysis of the graph of f is shown in the table, and the graph is show

	$f(x)$	$f'(x)$	$f''(x)$	Characteris
$-\infty < x < 0$		+	+	Increasing, co
$x = 0$	0	$\dfrac{1}{\sqrt{2}}$	0	Point of i
$0 < x < \infty$		+	−	Increasing, con

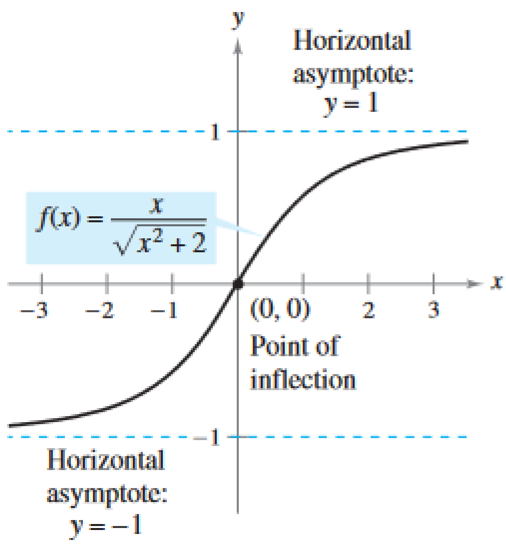


Figure 3.49

EXAMPLE 4 Sketching the Graph of a Radical Functi

Analyze and sketch the graph of $f(x) = 2x^{5/3} - 5x^{4/3}$.

Solution

$$f'(x) = \frac{10}{3}x^{1/3}(x^{1/3} - 2) \qquad f''(x) = \frac{20(x^{1/3} - 1)}{9x^{2/3}}$$

The function has two intercepts: $(0, 0)$ and $\left(\frac{125}{8}, 0\right)$. There are n
cal asymptotes. The function has two critical numbers ($x = 0$
possible points of inflection ($x = 0$ and $x = 1$). The domain is a
analysis of the graph of f is shown in the table, and the graph is s

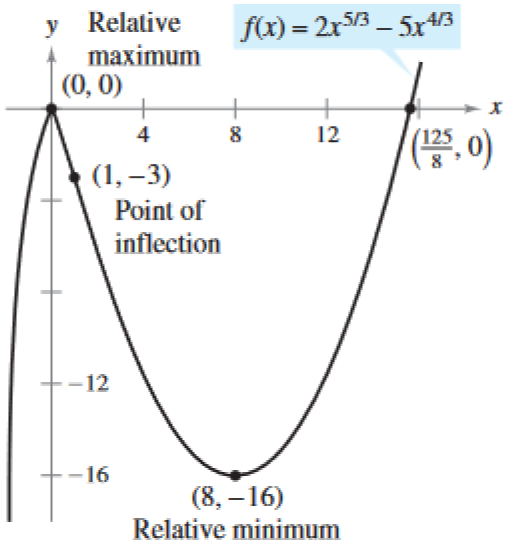


Figure 3.50

	$f(x)$	$f'(x)$	$f''(x)$	Characte
$-\infty < x < 0$		$+$	$-$	Increasing, c
$x = 0$	0	0	Undef.	Relativ
$0 < x < 1$		$-$	$-$	Decreasing, c
$x = 1$	-3	$-$	0	Point c
$1 < x < 8$		$-$	$+$	Decreasing
$x = 8$	-16	0	$+$	Relativ
$8 < x < \infty$		$+$	$+$	Increasing,

2.3 Sketching the graph of a polynomial function

EXAMPLE 5 Sketching the Graph of a Polynomial Functi

Analyze and sketch the graph of $f(x) = x^4 - 12x^3 + 48x^2 - 64x$.

Solution Begin by factoring to obtain

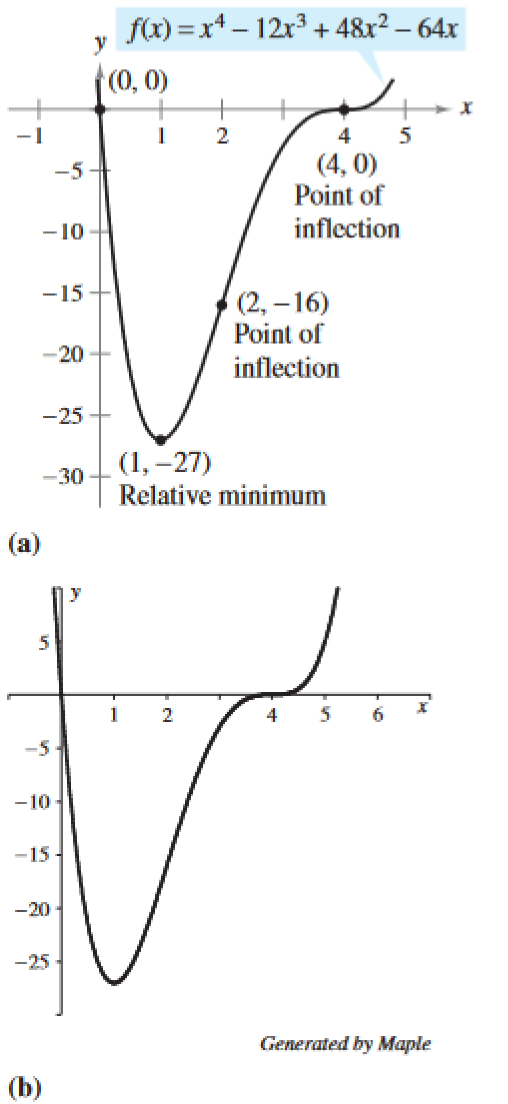
$$\begin{aligned} f(x) &= x^4 - 12x^3 + 48x^2 - 64x \\ &= x(x-4)^3. \end{aligned}$$

Then, using the factored form of $f(x)$, you can perform the followi

First derivative:	$f'(x) = 4(x-1)(x-4)^2$
Second derivative:	$f''(x) = 12(x-4)(x-2)$
x-intercepts:	$(0, 0), (4, 0)$
y-intercept:	$(0, 0)$
Vertical asymptotes:	None
Horizontal asymptotes:	None
End behavior:	$\lim_{x\to-\infty} f(x) = \infty, \lim_{x\to\infty} f(x) =$
Critical numbers:	$x = 1, x = 4$
Possible points of inflection:	$x = 2, x = 4$
Domain:	All real numbers
Test intervals:	$(-\infty, 1), (1, 2), (2, 4), (4, \infty)$

The analysis of the graph of f is shown in the table, and the graph i
3.51(a). Using a computer algebra system such as *Maple* [see Figure
you verify your analysis.

	$f(x)$	$f'(x)$	$f''(x)$	Characteris
$-\infty < x < 1$		$-$	$+$	Decreasing, c
$x = 1$	-27	0	$+$	Relative
$1 < x < 2$		$+$	$+$	Increasing, c
$x = 2$	-16	$+$	0	Point of
$2 < x < 4$		$+$	$-$	Increasing, con
$x = 4$	0	0	0	Point of
$4 < x < \infty$		$+$	$+$	Increasing, c



A polynomial function of even degree must have at least one relative extremum.

Figure 3.51

2.4