4.5 Integration by Substitution

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1 Complete the table by identifying and for the integral.

2.

$$\int x^2\sqrt{x^3+1}dx \tag{1}$$
$$u = x^3 + 1 \tag{2}$$
$$du = 3x^2dx \tag{3}$$

4.

$$\int \sec 2x \tan 2x dx \tag{1}$$
$$u = 2x \tag{2}$$
$$du = 2dx \tag{3}$$

2 Determine whether it is necessary to use substitution to evaluate the integral. (Do not evaluate the integral.)

8.

$$\int x\sqrt{x+4}dx$$

Substitution is necessary. $u = x + 4$

10.

$$\int x \cos x^2 dx$$

Substitution is necessary. $u = x^2$

3 Find the indefinite integral and check the result by differentiation.

12.

$$\int (x^2-9)^3(2x)dx = \frac{(x^2-9)^4}{4} + C \tag{1}$$
$$\frac{d}{dx}\left(\frac{(x^2-9)^4}{4} + C\right) = \frac{4(x^2-9)^3}{4}(2x) \tag{2}$$
$$= (x^2-9)^3(2x) \tag{3}$$

14.

$$\begin{aligned}
\int \sqrt[3]{3-4x^2}(-8x)dx &= \int (3-4x^2)^{1/3}(-8x)dx && (1)\\
&= \frac{(3-4x^2)^{4/3}}{\frac{4}{3}} + C && (2)\\
&= \frac{3}{4}(3-4x^2)^{4/3} + C && (3)\\
\frac{d}{dx}\left(\frac{3}{4}(3-4x^2)^{4/3} + C\right) &= \frac{3}{4}\left(\frac{4}{3}\right)(3-4x^2)^{1/3}(-8x) && (4)\\
&= (3-4x^2)^{1/3}(-8x) && (5)
\end{aligned}$$

16.

$$\begin{aligned}
\int x^2(x^3+5)^4 dx &= \frac{1}{3}\int (x^3+5)^4(3x^2)dx && (1)\\
&= \frac{1}{3}\frac{(x^3+5)^5}{5} + C && (2)\\
&= \frac{(x^3+5)^5}{15} + C && (3)\\
\frac{d}{dx}\left(\frac{(x^3+5)^5}{15} + C\right) &= \frac{5(x^3+5)^4(3x^2)}{15} && (4)\\
&= (x^3+5)^4 x^2 && (5)
\end{aligned}$$

20.

$$\begin{aligned}
\int t^3\sqrt{t^4+5}dt &= \frac{1}{4}\int (t^4+5)^{1/2}(4t^3)dt && (1)\\
&= \frac{1}{4}\frac{(t^4+5)^{3/2}}{\frac{3}{2}} + C && (2)\\
&= \frac{1}{6}(t^4+5)^{3/2} + C && (3)\\
\frac{d}{dt}\left(\frac{1}{6}(t^4+5)^{3/2} + C\right) &= \frac{1}{6}\cdot\frac{3}{2}(t^4+5)^{1/2}(4t^3) && (4)\\
&= (t^4+5)^{1/2}(t^3) && (5)
\end{aligned}$$

30.

$$\int \left(x^2 + \frac{1}{(3x)^2}\right) dx = \int \left(x^2 + \frac{1}{9}x^{-2}\right) dx \tag{1}$$

$$= \frac{x^3}{3} + \frac{1}{9}\left(\frac{x^{-1}}{-1}\right) + C \tag{2}$$

$$= \frac{x^3}{3} - \frac{1}{9x} + C \tag{3}$$

$$= \frac{3x^4 - 1}{9x} + C \tag{4}$$

$$\frac{d}{dx}\left(\frac{1}{3}x^3 - \frac{1}{9}x^{-1} + C\right) = x^2 + \frac{1}{9}x^{-2} \tag{5}$$

$$= x^2 + \frac{1}{(3x)^2} \tag{6}$$

34.

$$\int \frac{t - 9t^2}{\sqrt{t}} dt = \int (t^{1/2} - 9t^{3/2}) dt \tag{1}$$

$$= \frac{2}{3}t^{3/2} - \frac{18}{5}t^{5/2} + C \tag{2}$$

$$= \frac{2}{15}{}^{3/2}(5 - 27t) + C \tag{3}$$

$$\frac{d}{dt}\left(\frac{2}{3}t^{3/2} - \frac{18}{5}t^{5/2} + C\right) = t^{1/2} - 9t^{3/2} \tag{4}$$

$$= \frac{t - 9t^2}{\sqrt{t}} \tag{5}$$

38.

$$\int 4\pi y(6 + y^{3/2}) dy = \int (24\pi + 4\pi y^{5/2}) dy \tag{1}$$

$$= 12\pi y^2 + \frac{8\pi}{7}y^{7/2} + C \tag{2}$$

$$\frac{d}{dy}\left(12\pi y^2 + \frac{8\pi}{7}y^{7/2} + C\right) = 24\pi y + 4\pi y^{5/2} \tag{3}$$

$$= 4\pi y(6 + y^{3/2}) \tag{4}$$

4 Solve the differential equation.

40.

$$\frac{dy}{dx} = \frac{10x^2}{\sqrt{1+x^3}} \tag{1}$$

$$y = \int \frac{10x^2}{\sqrt{1+x^3}}dx \tag{2}$$

$$= \frac{10}{3}\int (1+x^3)^{-1/2}(3x^2)dx \tag{3}$$

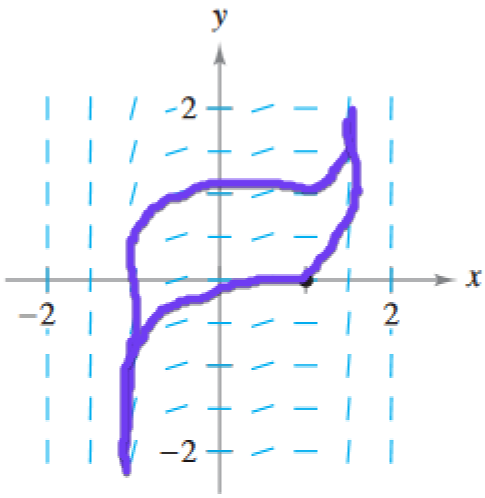
$$= \frac{10}{3}\left(\frac{(1+x^3)^{1/2}}{1/2}\right) + C \tag{4}$$

$$= \frac{20}{3}\sqrt{1+x^3} + C \tag{5}$$

5 A differential equation, a point, and a slope field are given. A $slopefield$ consists of line segments with slopes given by the differential equation. These line segments give a visual perspective of the directions of the solutions of the differential equation. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketches in part (a).

44.

$$\frac{dy}{dx} = x^2(x^3-1)^2,\ (1,0)$$



$$\frac{dy}{dx} = x^2(x^3-1)^2,\ (1,0) \quad (1)$$
$$y = \int x^2(x^3-1)^2 dx \quad (2)$$
$$(u = x^3-1) = \frac{1}{3}\int (x^3-1)^2(3x^2 dx) \quad (3)$$
$$= \frac{1}{3}\frac{(x^3-1)^3}{3} + C = \frac{1}{9}(x^3-1)^3 + C \quad (4)$$
$$0 = C \quad (5)$$
$$y = \frac{1}{9}(x^3-1)^3 \quad (6)$$

6 Find the indefinite integral.

48.

$$\int 4x^3 \sin x^4 dx \quad (1)$$
$$= \int \sin x^4(4x^3)dx \quad (2)$$
$$= -\cos x^4 + C \quad (3)$$

52.

$$\int x \sin x^2 dx \quad (1)$$
$$= \frac{1}{2}\int (\sin x^2)(2x)dx \quad (2)$$
$$= -\frac{1}{2}\cos x^2 + C \quad (3)$$

56.

$$\int \sqrt{tanx} \sec^2 x dx \quad (1)$$
$$= \frac{(\tan x)^{3/2}}{\frac{3}{2}} + C \quad (2)$$
$$= \frac{2}{3}(\tan x)^{3/2} + C \quad (3)$$

60.

$$\int \csc^2\left(\frac{x}{2}\right) dx \tag{1}$$

$$= 2\int \csc^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right) dx \tag{2}$$

$$= -2\cot\left(\frac{x}{2}\right) + C \tag{3}$$

7 Find an equation for the function that has the given derivative and whose graph passes through the given point.

62.

$$f'(x) = \pi \sec \pi x \tan \pi x,\ \left(\frac{1}{3}, 1\right) \tag{1}$$

$$f(x) = \int \pi \sec \pi x \tan \pi x dx \tag{2}$$

$$= \sec \pi x + C \tag{3}$$

$$f(x) = \sec \pi x - 1 \because f\left(\frac{1}{3}\right) = 1 \tag{4}$$

66.

$$f'(x) = -2x\sqrt{8 - x^2},\ (2, 7) \tag{1}$$

$$f(x) = \frac{2(8 - x^2)^{3/2}}{3} + C \tag{2}$$

$$f(2) = \frac{2(4)^{3/2}}{3} + C \tag{3}$$

$$= \frac{16}{3} + C \tag{4}$$

$$= 7 \Rightarrow C = \frac{5}{3} \tag{5}$$

8 Evaluate the definite integral. Use a graphing utility to verify your result.

76.

$$\int_{-2}^{4} x^2(x^3 + 8)^2 dx,\ u = x^3 + 8,\ du = 3x^2 dx \tag{1}$$

$$= \frac{1}{3}\int_{-2}^{4} (x^3 + 8)^2(3x^2) dx \tag{2}$$

$$= \left(\frac{1}{3}\frac{(x^3 + 8)^3}{3}\right)_{-2}^{4} \tag{3}$$

$$= \frac{1}{9}\left((64 + 8)^3 - (-8 + 8)^3\right) = 41.472 \tag{4}$$

82.

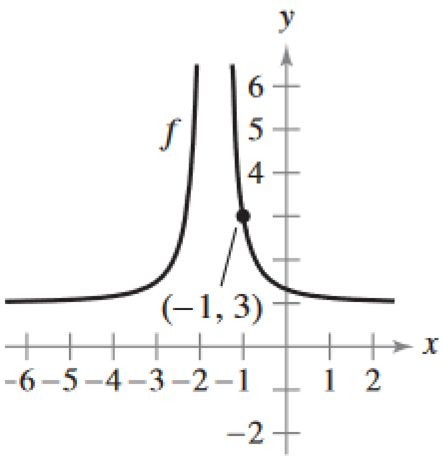
$$\int_0^2 x\sqrt[3]{4+x^2}dx,\ u=4+x^2,\ du=2xdx \tag{1}$$
$$=\frac{1}{2}\int_0^2 (4+x^2)^{1/3}(2x)dx \tag{2}$$
$$=\left(\frac{3}{8}(4+x^2)^{4/3}\right)_0^2 \tag{3}$$
$$=\frac{3}{8}(8^{4/3}-4^{4/3}) \tag{4}$$
$$=6-\frac{3}{2}\sqrt[3]{4}\approx 3.619 \tag{5}$$

86.

$$\int_{\pi/3}^{\pi/2}(x+\cos x)dx \tag{1}$$
$$=\left(\frac{x^2}{2}+\sin x\right)_{\pi/3}^{\pi/2} \tag{2}$$
$$=\left(\frac{\pi^2}{8}+1\right)-\left(\frac{\pi^2}{18}+\frac{\sqrt{3}}{2}\right) \tag{3}$$
$$=\frac{5\pi^2}{72}+\frac{2-\sqrt{3}}{2} \tag{4}$$

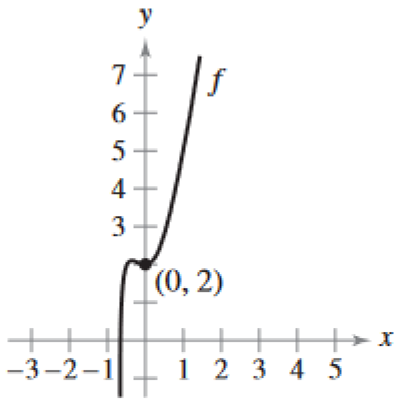
9 The graph of a function f is shown. Use the differential equation and the given point to find an equation of the function.

88.



$$
\begin{aligned}
\frac{dy}{dx} &= \frac{-48}{(3x+5)^3},\ (-1,3) && (1)\\
y &= -48\int (3x+5)^{-3}dx && (2)\\
&= (-48)\frac{1}{3}\int (3x+5)^{-3}dx && (3)\\
&= \frac{-16(3x+5)^{-2}}{-2} + C && (4)\\
3 &= \frac{8}{(3(-1)+5)^2} + C && (5)\\
&= \frac{8}{4} + C \Rightarrow 1 && (6)\\
y &= \frac{8}{(3x+5)^2} + 1 && (7)
\end{aligned}
$$

90.



$$\frac{dy}{dx} = 4x + \frac{9x^2}{(3x^3+1)^{3/2}},\ (0,2) \tag{1}$$

$$y = \int (4x + (3x^3+1)^{-3/2} 9x^2) dx \tag{2}$$

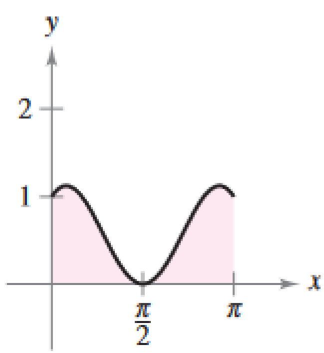
$$= 2x^2 - \frac{2}{\sqrt{3x^3+1}} + C \tag{3}$$

$$2 = 0 = \frac{2}{1} + C \Rightarrow C = 4 \tag{4}$$

$$y = 2x^2 - \frac{2}{\sqrt{3x^3+1}} + 4 \tag{5}$$

10 Find the area of the region. Use a graphing utility to verify your result.

94



$$\text{Area} = \int_0^{\pi} (\sin x + \cos 2x) dx \tag{6}$$

$$= \left(-\cos x + \frac{1}{2}\sin 2x\right)_0^{\pi} \tag{7}$$

$$= 2 \tag{8}$$

11 Use a graphing utility to evaluate the integral. Graph the region whose area is given by the definite integral.

98.

$$\int_0^2 x^3\sqrt{2x+3} dx \tag{1}$$

$$\approx 9.945 \tag{2}$$

100.

$$\int_1^5 x^2\sqrt{x-1}dx \tag{1}$$
$$\approx 67.505 \tag{2}$$

102.

$$\int_0^{\pi/6} \cos 3x dx \tag{1}$$
$$= \frac{1}{3} \tag{2}$$

12 Evaluate the integral using the properties of even and odd functions as an aid.

104.

$$f(x) = x(x^2+1)^3 \therefore \text{ odd} \tag{1}$$
$$\int_{-2}^{2} x(x^2+1)^3 dx = 0 \tag{2}$$

106.

$$f(x) = \sin x \cos x \therefore \text{odd} \tag{1}$$
$$\int_{-\pi/2}^{\pi/2} \sin x \cos x dx = 0 \tag{2}$$

108.

13 Use the symmetry of the graphs of the sine and cosine functions as an aid in evaluating each definite integral.

(a) $\int_{-\pi/4}^{\pi/4} \sin x dx = 0$ because $\sin x$ is symmetric to the origin.

(b) $\int_{-\pi/4}^{\pi/4} \cos x dx = 2\int_0^{\pi/4} \cos x dx = (2\sin x)_0^{\pi/4} = \sqrt{2}$ because $\cos x$ is symmetric to the y-axis

(c) $\int_{-\pi/2}^{\pi/2} \cos x dx = 2\int_0^{\pi/2} \cos x dx = (2\sin x)_0^{\pi/2} = 2$

(d) $\int_{-\pi/2}^{\pi/2} \sin x \cos x dx = 0$ because $\sin(-x)\cos(-x) = -\sin x \cos x$, making it symmetric to the origin.

116 can't be completed due to a formatting error in the textbook (the initial and decreasing values aren't specified).

116. *Depreciation* The rate of depreciation dV/dt of a machine is inversely proportional to the square of $t+1$, where V is the value of the machine t years after it was purchased. The initial value of the machine was \$00,000, and its value decreased \$00,000 in the first year. Estimate its value after 4 years.

14 Sales

118. The sales S (in thousands of units) of a seasonal product are given by the model

$$S = 74.50 + 43.75 \sin \frac{\pi t}{6}$$

where t is the time in months, with $t = 1$ corresponding to January. Find the average sales for each time period.

(a) The first quarter $(0 \le t \le 3)$

$$\frac{1}{b-a}\int_a^b \left(74.5 + 43.75 \sin \frac{\pi t}{6}\right) dt = \frac{1}{b-a}\left(74.5t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right)_a^b \tag{1}$$

$$\frac{1}{3}\left(74.5t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right)_0^3 = \frac{1}{3}\left(223.5 + \frac{262.5}{\pi}\right) \approx 102.352 \text{ thousand units} \tag{2}$$

(b) The second quarter $(3 \le t \le 6)$

$$\frac{1}{3}\left(74.5t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right)_3^6 = \frac{1}{3}\left(447 + \frac{262.5}{\pi} - 223.5\right) \approx 102.352 \text{ thousand units} \tag{1}$$

(c) The entire year $(0 \le t \le 12)$

$$\frac{1}{12}\left(74.5t - \frac{262.5}{\pi}\cos\frac{\pi t}{6}\right)_0^{12} = \frac{1}{12}\left(894 - \frac{262.5}{\pi} + \frac{262.5}{\pi}\right) \approx 74.5 \text{ thousand units} \tag{1}$$

15 Determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

129. $\int (2x+1)^2 dx = \frac{1}{3}(2x+1)^3 + C$
False because $\int (2x+1)^2 dx = \frac{1}{2}\int (2x+1)^2 2dx = \frac{1}{6}(2x+1)^3 + C$.

130. $\int x(x^2+1)dx = \frac{1}{2}x^2\left(\frac{1}{3}x^3 + x\right) + C$
False because $\int x(x^2+1)dx = \frac{1}{2}\int (x^2+1)(2x)dx = \frac{1}{4}(x^2+1)^2 + C$

131. $\int_{-10}^{10}(ax^3 + bx^2 + cx + d)dx = 2\int_0^{10}(bx^2 + d)dx$
True.

132. $\int_a^b \sin x dx = \int_a^{b+2\pi} \sin x dx$
True.

133.4 $\int \sin x \cos x dx = -\cos 2x + C$

True.

134.$\int \sin^2 2x \cos 2x dx = \frac{1}{3} \sin^3 2x + C$

False.

$$\int \sin^2 2x \cos 2x dx = \frac{1}{2} \int (\sin 2x)^2 (2 \cos 2x) dx \tag{1}$$

$$= \frac{1}{2} \frac{(\sin 2x)^3}{3} + C \tag{2}$$

$$= \frac{1}{6} \sin^3 2x + C \tag{3}$$