5.4 Exponential Functions: Differentiation and Integration

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January 2024

1 Solve for x accurate to three decimal places.

2.

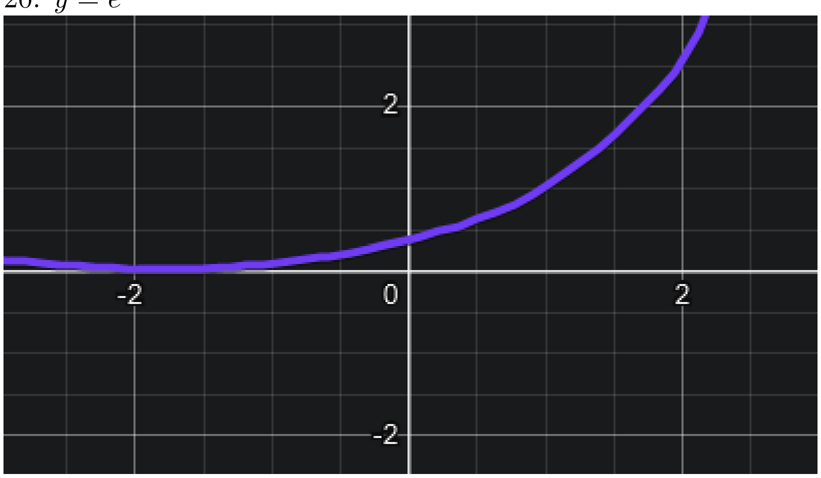
$$e^{\ln 2x} = 12 \tag{1}$$
$$2x = 12 \Rightarrow x = 6 \tag{2}$$

10.

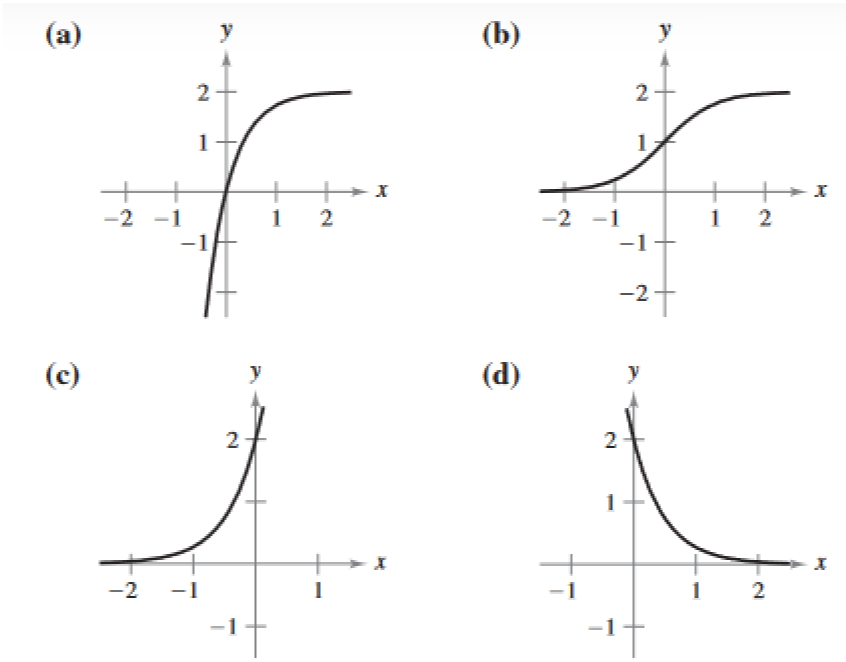
$$\frac{5000}{1+e^{2x}} = 2 \tag{1}$$
$$\frac{5000}{2} = 1 + e^{2x} \tag{2}$$
$$2499 = e^{2x} \tag{3}$$
$$\ln 2499 = 2x \tag{4}$$
$$x = \frac{1}{2}\ln 2499 \approx 3.912 \tag{5}$$

2 Sketch the graph of the function.

20. $y = e^{x-1}$



3 Match the equation with the correct graph. Assume that a and C are positive real numbers.



26.$y = Ce^{-ax}$
Horizontal asymptote at $y = 0$
Reflects the y-axis and matches (d)

28.

$$y = \frac{C}{1+e^{-ax}} \tag{1}$$
$$\lim_{x\to\infty} \frac{C}{1+e^{-ax}} = C \tag{2}$$
$$\lim_{x\to\infty} \frac{C}{1+e^{-ax}} = 0 \tag{3}$$
$$\tag{4}$$

4 Illustrate that the functions are inverses of each other by graphing both functions on the same set of coordinate axes

30.

$$f(x) = e^{x/3} \tag{1}$$
$$g(x) = \ln x^3 = 3\ln x \tag{2}$$

5 Find an equation of the tangent line to the graph of the function at the point (0, 1)

38.

$$\begin{aligned} y &= e^{2x} \quad &(1)\\ y' &= 2e^{2x} \quad &(2)\\ y'(0) &= 2 \quad &(3)\\ \text{Tangent line} &= y - 1 = 2(x - 0) \quad &(4)\\ y &= 2x + 1 \quad &(5) \end{aligned}$$

$$\begin{aligned} y &= e-2x \quad &(1)\\ y' &= -2e^{-2x} \quad &(2)\\ y'(0) &= -2 \quad &(3)\\ \text{Tangent line} &= y - 1 = -2(x - 0) \quad &(4)\\ y &= -2x + 1 \quad &(5) \end{aligned}$$

6 Find the derivative

40.

$$\begin{aligned} y &= e^{-5x} \quad &(1)\\ \frac{dy}{dx} &= -5e^{-5x} \quad &(2) \end{aligned}$$

44.

$$f(x) = 3e^{1-x^2} \tag{1}$$
$$f'(x) = 3e^{1-x^2}(-2x) \tag{2}$$
$$= -6xe^{1-x^2} \tag{3}$$

48.

$$y = x^2e^{-x} \tag{1}$$
$$y' = x^2(-e^{-x}) + 2xe^{-x} \tag{2}$$
$$= xe^{-x}(2-x) \tag{3}$$

52.

$$y = \ln\left(\frac{1+e^x}{1-e^x}\right) \tag{1}$$
$$= \ln(1+e^x) - \ln(1-e^x) \tag{2}$$
$$\frac{dy}{dx} = \frac{e^x}{1+e^x} + \frac{e^x}{1-e^x} \tag{3}$$
$$= \frac{2e^x}{1-e^{2x}} \tag{4}$$

56.

$$y = \frac{e^{2x}}{e^{2x}+1} \tag{1}$$
$$y' = \frac{(e^{2x}+1)2e^{2x} - e^{2x}(2e^{2x})}{(e^{2x}+1)^2} \tag{2}$$
$$= \frac{2e^{2x}}{(e^{2x}+1)^2} \tag{3}$$

60.

$$F(x) = \int_0^{e^{2x}} \ln(t+1)dt \tag{1}$$
$$F'(x)\ln(e^{2x}+1)2e^{2x} \tag{2}$$
$$= 2e^{2x}\ln(e^{2x}+1) \tag{3}$$

7 Find an equation of the tangent line to the graph of the function at the given point.

62.

$$y = e^{-2x+x^2},\ (2,\ 1) \tag{1}$$
$$y' = (2x-2)e^{-2x+x^2},\ y'(2) = 2 \tag{2}$$
$$\text{Tangent line} = y - 1 = 2(x-2) \tag{3}$$
$$y = 2x - 3 \tag{4}$$

66.

$$y = xe^x - e^x,\ (1,\ 0) \tag{1}$$
$$y' = xe^x + e^x - e^x = xe^x \tag{2}$$
$$y'(1) = e \tag{3}$$
$$\text{Tangent line} = y - 0 = e(x-1) \tag{4}$$
$$y = ex - e \tag{5}$$

8 Use implicit differentiation to find $\frac{dy}{dx}$.

$$e^{xy} + x^2 - y^2 = 10 \tag{1}$$
$$\left(x\frac{dy}{dx} + y\right)e^{xy} + 2x - 2y\frac{dy}{dx} = 0 \tag{2}$$
$$\frac{dy}{dx}(xe^{xy} - 2y) = -ye^{xy} - 2x \tag{3}$$
$$\frac{dy}{dx} = -\frac{ye^{xy} + 2x}{xe^{xy} - 2y} \tag{4}$$

9 Find an equation of the tangent line to the graph of the function at the given point.

72.

$$1 + \ln(xy) = e^{x-y},\ (1,\ 1) \tag{1}$$
$$\frac{1}{xy}(xy' + y) = e^{x-y}(1 - y') \tag{2}$$
$$y' + 1 = 1 - y' \tag{3}$$
$$y' = 0 \tag{4}$$

10 Find the second derivative of the function.

74.

$$g(x) = \sqrt{x} + e^x \ln x \tag{1}$$
$$g'(x) = \frac{1}{2\sqrt{x}} + \frac{e^x}{x} + e^x \ln x \tag{2}$$
$$g''(x) = -\frac{1}{4x^{3/2}} + \frac{xe^x - e^x}{x^2} + \frac{e^x}{x} + e^x \ln x \tag{3}$$
$$= -\frac{1}{4x\sqrt{x}} + \frac{e^x(2x-1)}{x^2} + e^x \ln x \tag{4}$$

11 Find the extrema and the points of inflection (if any exist) of the function

80.

$$f(x) = \frac{e^x - e^{-x}}{2} \tag{1}$$
$$f'(x) = \frac{e^x + e^{-x}}{2} > 0 \tag{2}$$
$$f''(x) = \frac{e^x - e^{-x}}{2} = 0 \text{ when } x = 0 \tag{3}$$
$$\text{Inflection point at } (0,\ 0) \tag{4}$$

84.

$$
\begin{aligned}
f(x) &= xe^{-x} && (1)\\
f'(x) &= -xe^{-x} + e^{-x} && (2)\\
&= e^{-x}(1-x) = 0 \text{ when } x = 1 && (3)\\
f''(x) &= -e^{-x} + (-e^{-x})(1-x) && (4)\\
&= e^{-x}(x-2) = 0 \text{ when } x = 2 && (5)\\
&\text{Relative maximum at } (1, e^{-1}) && (6)\\
&\text{Inflection point at } (2, 2e^{-2}) && (7)
\end{aligned}
$$

12 Word problems

90. Find the point on the graph of $y = e^{-x}$ where the normal line to the curve passes through the origin.
Let $(x_0,\, y_0)$ be the coordinate on the graph.

$$
\begin{aligned}
y &= e^{-x} && (1)\\
y' &= -e^{-x} && (2)\\
-\frac{1}{y'} &= e^{x} && (3)\\
y - e^{-x_0} &= e^{x_0}(x - x_0) && (4)\\
& && (5)
\end{aligned}
$$

Because the curve passes through the origin,

$$
\begin{aligned}
-e^{-x_0} &= -x_0 e^{x_0} && (6)\\
1 &= x_0 e^{2x_0} && (7)\\
x_0 e^{2x_0} - 1 &= 0 \therefore x_0 \approx 0.4263 && (8)
\end{aligned}
$$

92. The displacement from equilibrium of a mass oscillating on the end of a spring suspended from a ceiling is $y = 1.56e^{-0.22t}\cos 4.9t$, where y is the displacement in feet and t is the time in seconds. Use a graphing utility to graph the displacement function on the interval [0, 10] Find a value of t past which the displacement is less than 3 inches from equilibrium.
$1.56e^{-0.22t}\cos 4.9t \leq 0.25$. With a calculator, it is determined that $t \geq 7.79$ seconds.

94. The table lists the approximate values V of a mid-sized sedan for the years 2003 through 2009. The variable t represents the time in years, with $t = 3$ corresponding to 2003.

t	3	4	5	6
V	\$23,046	\$20,596	\$18,851	\$17,001

t	7	8	9
V	\$15,226	\$14,101	\$12,841

(a) Use the regression capabilities of a graphing utility to fit linear and quadratic models to the data. Plot the data and graph the models.

$$\text{Linear model: } V = -1686.8t + 27501 \tag{9}$$
$$\text{Quadratic model: } V = 109.52t^2 - 3001.1t + 31006 \tag{10}$$

(b) What does the slope represent in the linear model in part (a)?
The average value loss each year.

(c) Use the regression capabilities of a graphing utility to fit an exponential model to the data.
$V = 30582.68(0.90724)^t = 30582.68e^{-0.09735t}$

(d) Determine the horizontal asymptote of the exponential model found in part (c). Interpret its meaning in the context of the problem.
As $t \to 0$, $V \to 0$ in the model, indicating that the value tends to zero.

(e) Find the rate of decrease in the value of the sedan when $t = 4$ and $t = 8$ using the exponential model.

$$\text{When } t = 4,\ V' \approx -2017 \text{ dollars/year} \tag{1}$$
$$\text{When } t = 8,\ V' \approx -1366 \text{ dollars/year} \tag{2}$$

13 Find the indefinite integral.

100.

$$\int e^{-x^4}(-4x^3)dx,\ u = -x^4,\ du = -4x^3dx \tag{1}$$
$$= e^{-x^4} + C \tag{2}$$

104.

$$\int e^x(e^x+1)^2dx,\ u = e^x + 1,\ du = e^x dx \tag{1}$$
$$= \int (e^x+1)^2(e^x)dx \tag{2}$$
$$= \frac{(e^x+1)^3}{3} + C \tag{3}$$

108.

$$\frac{e^{2x}}{1+e^{2x}}dx,\ u = 1 + e^{2x},\ du = 2e^{2x}dx \tag{1}$$
$$= \frac{1}{2}\int \frac{2e^{2x}}{1+e^{2x}}dx \tag{2}$$
$$= \frac{1}{2}\ln(1+e^{2x}) + C \tag{3}$$

112.

$$\int \frac{2e^x - 2e^{-x}}{(e^x + e^{-x})^2} dx \tag{1}$$
$$= 2\int (e^x + e^{-x})^{-2}(e^x - e^{-2})dx \tag{2}$$
$$= \frac{-2}{e^x + e^{-x}} + C \tag{3}$$

116.

$$\int \ln(e^{2x-1})dx \tag{1}$$
$$= \int (2x - 1)dx \tag{2}$$
$$= x^2 - x + C \tag{3}$$

14 Evaluate the definite integral.

118.

$$\int_3^4 e^{3-x}dx \tag{1}$$
$$= [-e^{3-x}]_3^4 \tag{2}$$
$$= -e^{-1} + 1 \tag{3}$$
$$= 1 - \frac{1}{e} \tag{4}$$

122.

$$\int_0^{\sqrt{2}} xe^{-x^2/2}dx,\ u = \frac{-x^2}{2},\ du = -xdx \tag{1}$$
$$= -\int_0^{\sqrt{2}} e^{-x^2/2}(-x)dx \tag{2}$$
$$= [-e^{-x^2/2}]_0^{\sqrt{2}} \tag{3}$$
$$= 1 - e^{-1} \tag{4}$$

126.

$$\int_{\pi/3}^{\pi/2} e^{\sec 2x} \sec 2x \tan 2x dx,\ u = \sec 2x,\ du = 2\sec 2x \tan 2x dx \tag{1}$$
$$= \frac{1}{2}\int_{\pi/3}^{\pi/2} e^{\sec 2x}(2\sec 2x \tan 2x)dx \tag{2}$$
$$= \frac{1}{2}[e^{\sec 2x}]_{\pi/3}^{\pi/2} \tag{3}$$
$$= \frac{1}{2}(e^{-1} - e^{-2}) \tag{4}$$
$$= \frac{1}{2}\left(\frac{1}{e} - \frac{1}{e^2}\right) \tag{5}$$
$$= \frac{e-1}{2e^2} \tag{6}$$

15 Solve the differential equation.

128.

$$\frac{dy}{dx} = (e^x - e^{-x})^2 \tag{1}$$
$$y = \int (e^x - e^{-x})^2 dx \tag{2}$$
$$= \int (e^{2x} - 2 + e^{-2x}) dx \tag{3}$$
$$= \frac{1}{2}e^{2x} - 2x - \frac{1}{2}e^{-2x} + C \tag{4}$$

16 Find the particular solution that satisfies the initial conditions.

130.

$$f''(x) = \sin x + e^{2x} \tag{1}$$
$$f(0) = \frac{1}{4} \tag{2}$$
$$f'(0) = \frac{1}{2} \tag{3}$$
$$f'(x) = \int (\sin x + e^{2x}) dx = -\cos x + \frac{1}{2}e^{2x} + C_1 \tag{4}$$
$$f'(0) = -1 + \frac{1}{2} + C_1 = \frac{1}{2} \therefore C_1 = 1 \tag{5}$$
$$f'(x) = -\cos x + \frac{1}{2}e^{2x} + 1 \tag{6}$$
$$f(x) = \int (-\cos x + \frac{1}{2}e^{2x} + 1) dx \tag{7}$$
$$= -\sin x + \frac{1}{4}e^{2x} + x + C_2 \tag{8}$$
$$f(0) = \frac{1}{4} + C_2 = \frac{1}{4} \therefore C_2 = 0 \tag{9}$$
$$f(x) = x - \sin x + \frac{1}{4}e^{2x} \tag{10}$$

17 Find the area of the region bounded by the graphs of the equations.

134.

$$y = e^{-x},\ y = 0,\ x = a,\ x = b \tag{1}$$
$$\int_a^b e^{-x} dx = [-e^{-x}]_a^b = e^a - e^b \tag{2}$$

136.

$$y = e^{-2x} + 2x,\ y = 0,\ x = 0,\ x = 2 \tag{1}$$
$$\int_0^2 (e^{-2x} + 2) dx = \left[-\frac{1}{2}e^{-2x} + 2x\right]_0^2 \tag{2}$$
$$= -\frac{1}{2}e^{-4} + 4 + \frac{1}{2} \approx 4.491 \tag{3}$$

18 Approximate the integral using the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule with $n = 12$.

138.

$$\int_0^2 2xe^{-x}dx,\ n = 12 \tag{1}$$
$$\text{Midpoint Rule} = 1.1906 \tag{2}$$
$$\text{Trapezoidal Rule} = 1.1827 \tag{3}$$
$$\text{Simpson's Rule} = 1.1880 \tag{4}$$
$$\text{Calculator} = 1.18799 \tag{5}$$

19 Word problems

140. The median waiting time (in minutes) for people waiting for service in a convenience store is given by the solution of the equation $\int_0^x 0.3e^{-0.3}dt = \frac{1}{2}$. Solve the equation.

$$\int_0^x 0.3e^{-0.3}dt = \frac{1}{2} = \frac{1}{2} \tag{1}$$
$$[-e^{-0.3t}]_0^x = \frac{1}{2} \tag{2}$$
$$-e{-}0.3x + 1 = \frac{1}{2} \tag{3}$$
$$e^{-0.3x} = \frac{1}{2} \tag{4}$$
$$-0.3x = \ln\frac{1}{2} = -\ln 2 \tag{5}$$
$$x = \frac{\ln 2}{0.3} \approx 2.31 \text{ minutes} \tag{6}$$

20 Capstone

148. Describe the relationship between the graphs of $f(x) = \ln x$ and $g(x) = e^x$.
Both graphs mirror each other across the line $y = x$.