VOL. 9, NO. 3, MARCH 1972 J. AIRCRAFT

Three-Dimensional, Minimum Fuel Turns for a Supersonic Aircraft

J. K. HEDRICK* AND A. E. BRYSON, JR.†
Stanford University, Stanford, Calif.

Using the energy-state approximation, turns are calculated for a particular aircraft where the initial energy, the change in heading angle, and the final energy are specified. Angle-of-attack and thrust are constrained. In general, the optimum turns are composed of variable altitude, variable bank angle programs, which use either maximum thrust or zero thrust. There are three types of minimum fuel turns: a) powered accelerating turns, b) coasting decelerating turns, and c) combinations of powered-coast or coast-powered arcs. Numerical results are presented for a) coasting turns, b) climbing turns from takeoff, and c) general three-dimensional turns.

Nomenclature

c = fuel consumption constant
C_{D_0} = zero lift drag coefficient
C_{L_α} = lift coefficient curve slope $(dC_L/d\alpha)$
D = drag = $D_0 + D_L$
D_L = $\eta(mg)^2/L_\alpha$ = drag due to lift when $\sigma = 0$
D_o = zero lift drag = $C_{D_0}qS$
E = energy = $(V^2/2) + gh$
g = acceleration of gravity
h = altitude
$\mathscr{H}$ = Hamiltonian
L = lift
L_α = lift curve slope = $C_{L_\alpha}qS$
m = mass of aircraft
M = Mach number
q = dynamic pressure = $\frac{1}{2}\rho V^2$
R = turning radius
S = area
T = thrust
$T_{\max}$ = maximum thrust = $T_{\max}(V,h)$
V = velocity
α = angle-of-attack
α_S = angle-of-attack at stall
β = heading angle
η = aerodynamic efficiency factor ($\frac{1}{2} \leq \eta \leq 1$)
σ = bank angle
λ_β = heading angle adjoint
λ_E = energy adjoint
μ_1 = thrust constraint adjoint
μ_2 = angle-of-attack constraint adjoint
μ_3 = altitude constraint adjoint
ρ = density of atmosphere
γ = flightpath angle

Subscripts

$(\ )_o$ = initial value
$(\ )_f$ = final value

Introduction

IN the last few years, optimization techniques have been applied to the turning performance of high-speed aircraft. In Refs. 1–3, minimum fuel and minimum time turns at constant altitude were investigated. Reference 4 discusses three-dimensional minimum time turns. Beebee[5] has used the energy state approximation to determine minimum time three-dimensional turns for a hypersonic rocket-powered aircraft. Kelley and Edelbaum[6] have discussed energy climbs and energy turns in terms of asymptotic expansions. Boyd and Christie[7] have also made contributions to the study of performance optimization using energy methods. In this paper altitude, bank angle and thrust programs are determined which minimize the fuel required to turn through a specified heading angle and reach a specified energy.

Received August 5, 1971. Presented as Paper 71-913 at the AIAA Guidance, Control and Flight Mechanics Conference, Hofstra University, Hempstead, New York, August 16–18, 1971; revision received November 1, 1971. This research was supported by NASA Grant NgL-05-020-007.
Index category: Aircraft Performance.

* Research Assistant; now Assistant Professor, Arizona State University, Tempe, Ariz.
† Professor of Applied Mechanics, Aeronautics and Astronautics. Fellow AIAA.

Equations of Motion and Constraints

The equations of motion and constraints for a minimum fuel turn are (see Fig. 1):

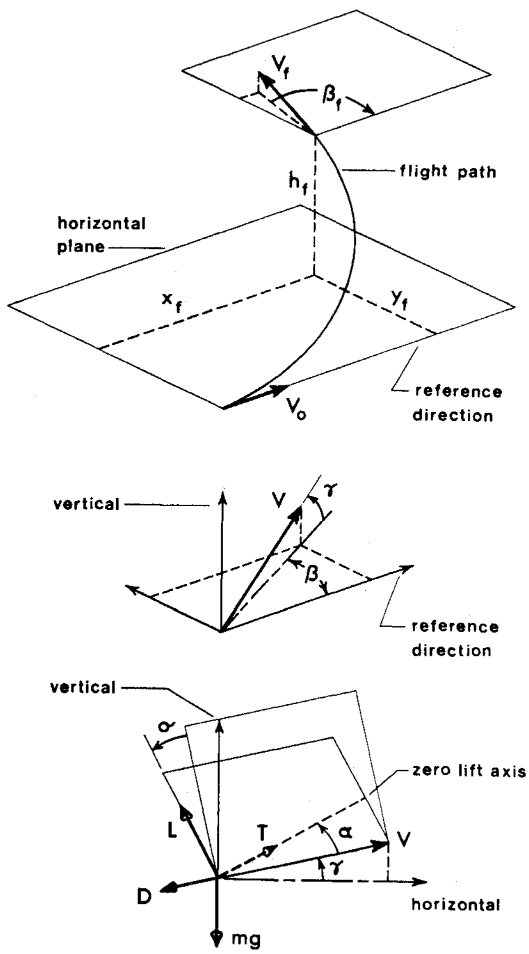

Fig. 1 Nomenclature.

$$\dot{E} = V(T - D)/m \quad (1)$$

$$\beta = g \tan\sigma / V \quad (2)$$

$$\alpha = mg \sec\sigma / L_\alpha \quad (3)$$

$$h = (E - V^2/2)/g \quad (4)$$

$$\left.\begin{array}{l} h \geq 0 \\ mg \sec\sigma \leq L_\alpha \alpha_S \\ 0 \leq T \leq T_{max}(V,E) \end{array}\right\} \quad (5)$$

The change in mass due to the burning of fuel is assumed to be small compared to the total mass of the aircraft, and an average constant mass is used in Eqs. (1) and (3).

Necessary Conditions

The mass flow rate can be approximated as

$$\dot{m} = -T/c \quad (6)$$

where c is assumed to be constant. We wish to minimize the fuel burned (Δm)

$$\Delta m = \int_{t_o}^{t_f} (T/c)dt \quad (7)$$

To do this, we must find the $\sigma(t)$, $T(t)$, and $V(t)$ which minimize Δm, subject to Eqs. (1, 2, 5) and

$$\beta(0) = 0; \beta(t_f) = \beta_f, E(0) = E_0, E(t_f) = E_f \quad (8)$$

The variational Hamiltonian for this problem may be written as

$$H = (T/c) + (\lambda_\beta g \tan\sigma/V) + [\lambda_E V(T - D)/m] + \mu_1 T(T - T_{max}) + \mu_2[(mg \sec\sigma/L_\alpha) - \alpha_S] + \mu_3[V - (2E)^{1/2}] \quad (9)$$

where

$$\mu_1 \begin{cases} > 0 \text{ if } T = 0, \text{ or } T = T_{max} \\ = 0 \text{ if } 0 < T < T_{max} \end{cases} \quad (10)$$

$$\mu_2 \begin{cases} > 0 \text{ if } \alpha = \alpha_S \\ = 0 \text{ if } \alpha < \alpha_S \end{cases} \quad (11)$$

$$\mu_3 \begin{cases} > 0 \text{ if } V = (2E)^{1/2} \\ = 0 \text{ if } V < (2E)^{1/2} \end{cases} \quad (12)$$

Necessary conditions for a minimum fuel turn include Eqs. (1–5, 8, 10–12) and the following equations:

$$\dot{\lambda}_E = -\partial H/\partial E = (\lambda_E V/m)(\partial D/\partial E) + (\mu_1 T \partial T_{max}/\partial E) + (\mu_2 \alpha_S / L_\alpha)(\partial L_\alpha/\partial E) + [\mu_3/(2E)^{1/2}] \quad (13)$$

$$\dot{\lambda}_\beta = -\partial H/\partial\beta = 0 \quad (14)$$

$$\partial H/\partial T = (1/c) + (\lambda_E V/m) + \mu_1(2T - T_{max}) = 0 \quad (15)$$

$$\partial H/\partial\sigma = \sec^2\sigma[(g\lambda_\beta/V) - (2\lambda_E V D_L \tan\sigma/m) + (\mu_2 mg \sin\sigma/L_\alpha)] = 0 \quad (16)$$

$$\partial H/\partial V = (-\lambda_\beta g \tan\sigma/V^2) + \mu_3 - (\lambda_E/m)[\partial(VD)/\partial V] + (\lambda_E T/m) - (\mu_1 T \partial T_{max}/\partial V) - (\mu_2 \alpha_S/L_\alpha)(\partial L_\alpha/\partial V) = 0 \quad (17)$$

$$\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0 \quad (18)$$

$$H(t_f) = 0 \quad (19)$$

Since H is not an explicit function of time, a first integral of the optimal turn is $H = \text{constant}$; from Eq. (19) this requires

$$H \equiv 0, 0 \leq t \leq t_f \quad (20)$$

Equations (10) and (15) require that

$$T = \begin{cases} T_{max} \text{ if } 1/c + \lambda_E V/m < 0 \\ 0 \quad \text{if } 1/c + \lambda_E V/m > 0 \end{cases} \quad (21)$$

It is shown in Ref. 8 that the use of partial thrust ($0 < T < T_{max}$), $\mu_1 = 0$, over a finite period of time is not minimizing. The expression $[(1/c) + (\lambda_E V/m)]$ is often called a "switching function" since a change in its sign determines when to switch off or switch on the thrust.

Arcs Where $T = T_{max}$, $\alpha < \alpha_S$, $h > 0$

The necessary conditions (16) and (17) cannot be satisfied with $T = 0$ when $\mu_2 = \mu_3 = 0$. Hence $T = T_{max}$ when $\alpha < \alpha_S$, $h > 0$. With $\mu_2 = 0$, (16) requires that

$$\lambda_E = mg\lambda_\beta \operatorname{ctn}\sigma / 2V^2 D_L \quad (22)$$

From Eqs. (20) and (22), $\tan\sigma$ is a function of V, D, T_{max}, and λ_β

$$\tan\sigma = (-VT_{max}/g\lambda_\beta c) \pm [(VT_{max}/g\lambda_\beta c)^2 - (T_{max} - D_o - D_L)/D_L]^{1/2} \quad (23)$$

Since, $d(\Delta m) = \lambda_\beta \delta\beta$, it is clear that $\lambda_\beta \leq 0$, since $\delta\beta > 0$ certainly implies $d(\Delta m) \leq 0$. Substituting Eq. (23) into Eq. (1)

$$\dot{E} = \pm(2VD_L \tan\sigma/m)[(VT_{max}/g\lambda_\beta c)^2 - (T_{max} - D_o - D_L)/D_L]^{1/2} \quad (24)$$

The (+) sign in Eq. (24) corresponds to the (−) sign of Eq. (23), and the (−) sign of Eq. (24) corresponds to the (+) sign of Eq. (23).

Given values of σ and V, Eqs. (24) and (2) can be integrated numerically to give a one parameter family of optimal paths in the β,E space, with λ_β as the parameter. The values of σ and V may be determined from Eq. (23) and the following equation:

$$0 = \partial H/\partial V = (-\lambda_\beta g \tan\sigma)V^2) - (\lambda_\beta g \operatorname{ctn}\sigma/2V^2 D_L) \times \{D_o + D_L \sec^2\sigma - T_{max} + V[(\partial D_o/\partial V) + (\partial D_L/\partial V)\sec^2\sigma]\} + (\partial T_{max}/\partial V) \times [(1/c) + (\lambda_\beta g \operatorname{ctn}\sigma/2VD_L)] \quad (25)$$

Arcs Where $T = 0$, $= \alpha_S$, $h > 0$ (Gliding Turns)

For our example airplane, optimizing arcs with $T = T_{max}$ cannot occur with $\alpha = \alpha_S$, $h > 0$. With $T = 0$, $\alpha = \alpha_S$, $h > 0$, the necessary conditions are identical to those for maximum turn glides. Reference 8 treated several aspects of gliding turns, i.e., turns with zero thrust and with final energy less than initial energy ($E_f < E_o$). In particular, the problem of maximizing final turning angle, β_f, for a given final energy, E_f, was treated. These maximum turn glides constitute coasting arcs of minimum fuel turns. If the required turn can be completed with zero thrust, no fuel will be used; this is clearly the minimum amount of fuel. If the maximum turn glide for specified E_f gives a β_f greater than the required β_f, the minimum fuel turn is not defined uniquely, but clearly can be accomplished with $T = 0$, e.g., by ceasing to turn when the required β_f has been reached, and coasting to the desired E_f. However, if the maximum turn glide for a specified E_f gives less than the required β_f, or if $E_f > E_o$, the minimum fuel path obviously must include a powered flight arc.

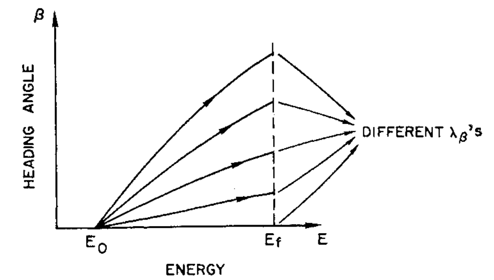

Fig. 2 A one parameter family sketch in the β,E space.

Given $\beta_o = 0$ and E_o, we may generate a field of extremals by varying λ_β continuously. Figure 2 illustrates such a field qualitatively. Given E_o, E_f, and $\Delta\beta$, we must find the λ_β that generates the path which satisfies our end conditions.

Minimum Fuel, Increasing Energy Turns with Final Heading Angle Specified

This case includes the case of a climbing turn from takeoff. After takeoff, the airplane accelerates at sea level as long as $\mu_3 > 0$. Assuming $T = T_{max}(\mu_1 > 0)$, $\alpha < \alpha_S(\mu_2 = 0)$ and $h = 0[V = (2E)^{1/2}]$, μ_3 is found by eliminating μ_1 and λ_E from Eqs. (15, 17, and 22)

$$\mu_3 = (g\lambda_\beta/V^2)\{\tan\sigma - (\text{ctn}\sigma/2D_L)(\partial/\partial V)[V(T_{max} - D)]\} - (1/c)(\partial T_{max}/\partial V) \quad (26)$$

When $\mu_3 = 0$, the airplane should begin to climb. The "climb Mach number" and the corresponding bank angle can be found for each λ_β by setting Eq. (26) equal to zero, and using the $(-)$ root of Eq. (23). The numerical results that follow are for the F4H aircraft (a supersonic interceptor) with $mg = 35{,}000$ lb, whose data are given in Ref. 9.

In Fig. 3 the dashed curve sloping upward to the left represents $\mu_3 = 0$ in the $\sigma,E/g$ space. The $\mu_3 = 0$ curve intersects $\sigma = 0$ at $E/g \simeq 13$ kft ($M \simeq 0.83$, $h = 0$); this agrees with the begin climb point found in Ref. 9 for the case not involving turns. The dash-dot line in Fig. 3 is the curve at which the switching function equals zero, i.e., $1/c + \lambda_E V/m = 0$. Below this curve the minimum fuel trajectories require $T = T_{max}$. The cross-hatched curve in Fig. 3 represents the stall boundary at sea level. The solid lines represent minimum fuel paths, with λ_β varying between -0.25 and -9.0. Bank angle increases until $E/g = 12.5$ kft and then begins to decrease; at $E/g = 59$ kft there is a jump discontinuity after which the bank angle decreases gradually.

Figure 4 shows that minimum fuel paths remain at sea level until $E/g \simeq 13$ kilofeet ($h = 0$, $M = 0.83$), and then climb with increasing energy to $E/g \simeq 59$ kft ($h = 46{,}000$ ft, $M = 0.87$). At this point there is a "corner" in the controls representing a zoom dive from 46,000 ft to 33,000 ft

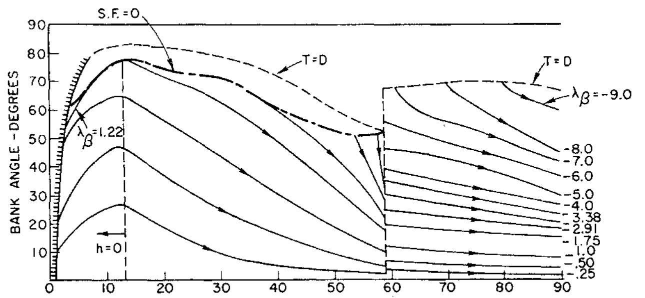

Fig. 3 Bank angle as a function of energy-height for minimum fuel increasing energy turns.

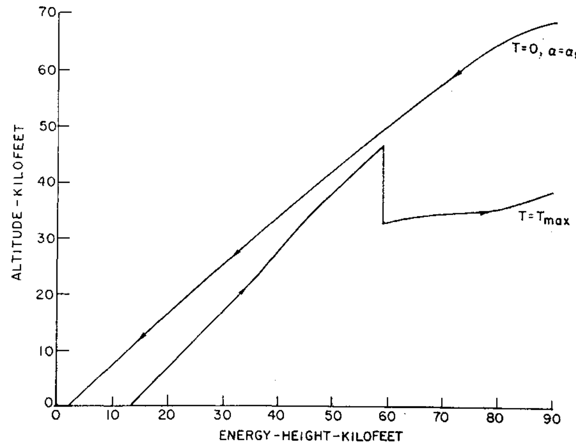

Fig. 4 Altitude as a function of energy-height for minimum fuel turns.

($M = 1.33$) at constant energy, after which energy and altitude continue to increase gradually.

Figure 5 shows β vs E/g for the initial conditions $h = 0$, $M = 0.4$ as λ_β varies between -0.25 and -1.75. The trajectory to the far left represents $\alpha = \alpha_S$, $h = 0$. Given the final heading angle and energy height, this chart can be used to find the value of λ_β for a minimum fuel turn from $h = 0$, $M = 0.4$. For example, suppose we wish to reach a final energy height of 50 kilofeet while turning through 100°, starting at $h = 0$, $M = 0.4$. From Fig. 5, the appropriate value of λ_β lies between -0.25 and -0.60; interpolating, $\lambda_\beta \simeq -0.37$. We can now find the control histories (σ,h vs E/g) from Figs. 3 and 4.

General Three-Dimensional Turns

There are several major differences between minimum time and minimum fuel turns. For minimum time turns (see Ref. 4), the condition for maximum thrust ($\lambda_E < 0$) is satisfied for all trajectories with $\alpha < \alpha_S$. The condition for maximum thrust in the minimum fuel problem ($\lambda_E < -m/Vc$) is not satisfied by all trajectories with $\alpha < \alpha_S$.

From Eq. (20), for $T = T_{max}$, we have

$$(1/c) + (\lambda_E V/m) = (1/T_{max})[(\lambda_E VD/m) - (g\lambda_\beta \tan\sigma/V)] \quad (27)$$

From Eq. (21), this must be negative for maximum thrust to be optimal. For $\alpha < \alpha_S$ Eq. (16) becomes

$$\lambda_E V/m = g\lambda_\beta/2VD_L \tan\sigma \quad (28)$$

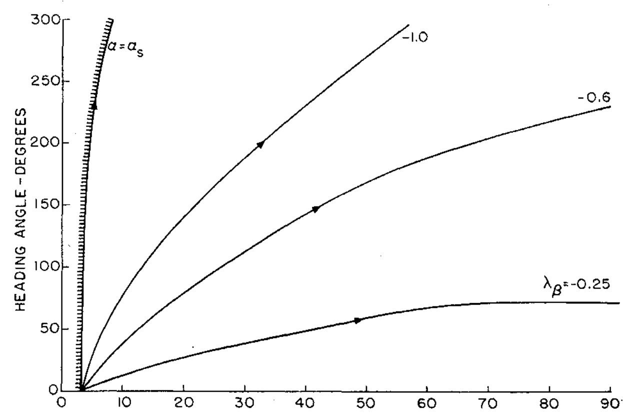

Fig. 5 Heading angle as a function of energy-height for minimum fuel turns starting from take off.

Substituting into Eq. (27) and rearranging terms

$$(1/c) + (\lambda_E V/m) = (g\lambda_\beta/2VD_L \tan\sigma T_{max}) \times (D_o + D_L - D_L \tan^2\sigma) \quad (29)$$

Since λ_β is negative, the right factor of Eq. (29) must be positive if the switching function is to be negative. Rearranging,

$$\tan^2\sigma < 1 + (D_o/D_L) \quad (30)$$

is the condition which must be satisfied for $T = T_{max}$ and $\alpha < \alpha_S$ to be optimal.

Equation (30) may be written as

$$\tan^2\sigma < f(h,E) \quad (31)$$

From Fig. 4 we can see that all the accelerating paths have nearly the same altitude vs energy height histories; therefore, Eq. (31) can be expressed by the approximation

$$\tan^2\sigma < f(E) \quad (32)$$

This curve, labelled S.F. $= 0$ (switching function $= 0$), is represented in Fig. 3 by a dash-dot curve. For energies less than 59 kilofeet, it lies below the $T = D$ curve. All the trajectories below this curve use maximum thrust. As can be seen from Fig. 3, there are very few trajectories which intersect the S.F. $= 0$ curve; however, there are several trajectories that begin at the stall boundary and do intersect this curve, e.g., $\lambda_\beta = -1.22$. These trajectores are discussed in more detail in the next section. For energies greater than 10 kft, the trajectories either begin on the switching function curve or never intersect it. This means that the only trajectories consisting of a powered arc followed by a coasting arc occur at very low energies; all other trajectories are purely accelerating, or are composed of a coasting arc followed by a powered arc.

In Fig. 3, there are no decelerating trajectories. For the aircraft described in Ref. 9, it is never optimal to use maximum thrust while decelerating; i.e., Eqs. (13–19) cannot be satisfied for $T = T_{max}$ when the drag is gretaer than T_{max}. Therefore, $T = 0$ whenever $\dot{E} < 0$. This, however, is a numerical result for a particular aircraft, and not necessarily true for all aircraft.

From Figs. 3 and 4 we see that there are, in general, three types of minimum fuel turns: 1) accelerating turns with $T = T_{max}$, 2) gliding turns with $T = 0$, or 3) a combination of gliding and powered turns. Note that when $E/g > 59$ kft, all trajectories beneath the thrust equal drag curve satisfy Eq. (30).

In Ref. 8, it is shown that to maximize $\Delta\beta$ when $T = 0$, we must choose V,σ such that

$$\max_{V,\sigma}(\tan\sigma/V^2D)|_E \quad (33)$$

For the aircraft whose data are given in Ref. 9, the solution to Eq. (33) occurs when $\alpha = \alpha_S$, so that

$$\sigma = \cos^{-1}(mg/L_\alpha\alpha_S)$$

and Eq. (33) becomes

$$\max\{(\alpha_S{}^2L_\alpha{}^2 - 1)^{1/2}/V^2[D_o + (D_L\alpha_S{}^2L_\alpha{}^2)/m^2g^2]\}|_E \quad (34)$$

Figures 6 and 7 show the numerical solution to Eq. (34). For a maximum turn glide, Fig. 6 shows altitude as a function of Mach number; Fig. 7 shows the bank angle as a function of Mach number.

If $E_f < E_o$ and if between E_o and E_f we can make a gliding turn with $\Delta\beta$ greater than or equal to the specified $\Delta\beta$, then clearly this is a minimum fuel path. If the specified $\Delta\beta$ is greater than the maximum gliding turn $\Delta\beta$, or if $E_f > E_o$, the trajectories which minimize fuel must involve a powered arc.

As shown in Fig. 3, the only possible trajectories for energies greater than 10 kilofeet are either purley accelerating

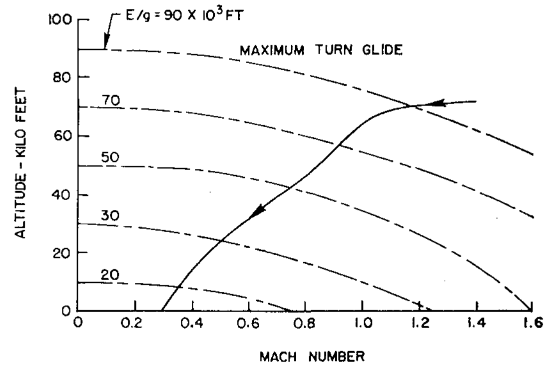

Fig. 6 Altitude for maximum $\Delta\beta$ as a function of Mach number.

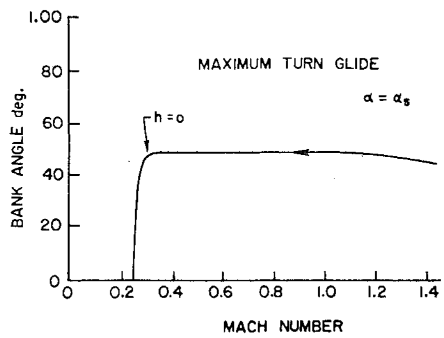

Fig. 7 Bank angle for maximum $\Delta\beta$ as a function of mach number.

trajectories with $T = T_{max}$, or trajectories including a coasting arc followed by a powered arc. The point at which the thrust should be turned on is determined by the point at which the switching function, $(1/c) + (\lambda_E V/m)$, changes sign for a particular λ_β.

Figure 8 is a state space chart for a final energy height of 35 kft. Heading angle to go, $\Delta\beta$, is plotted as a function of energy-height for several minimum fuel turns. The stall boundary is represented by the cross-hatched curve. The dashed curve shows where the switching function is equal to zero, i.e., the curve where T switches from $T = 0$ to $T = T_{max}$. Figure 8 also contains contours of constant fuel expenditure to go. Contours of Δm equal to 30, 20, and 10 slugs to go are indicated by the dashed curves sloping upward toward the right.

Those trajectories whose initial conditions lie within the area circumscribed by the $\alpha = \alpha_S$ curve and S.F. $= 0$ curve are accelerating paths with $T = T_{max}$. Those trajectories which lie to the right of the coasting trajectory ($T = 0$) with the final conditions $E/g = 35$ kft, $\Delta\beta = 0$, may all be accomplished with $T = 0$. Those trajectories whose initial conditions lie in the region above the *S.F.* $= 0$ curve and to the eft of the coasting curve with the final conditions $E/g = 35$ kft, $\Delta\beta = 0$, consist of a gliding arc at $\alpha = \alpha_S$ and a powered arc at $T = T_{max}$. For the gliding arc, h and σ can be found from Figs. 6 and 7. When the path reaches the switching curve, the thrust is switched from $T = 0$ to $T = T_{max}$, a zoom dive at constant energy occurs, and there is an instantaneous increase in bank angle. Given the energy at the point where the discontinuity occurs, the discontinuity in altitude can be found from Fig. 4: the trajectory drops from the $T = 0$ curve to the $T = T_{max}$ curve at a constant energy. Given $E(t_1)$ we can find the value of σ before the discontinuity from Fig. 7; from Fig. 8, we can find $\lambda_\beta(t_1)$, and, knowing this, we can then find the value of σ after the discontinuity from Fig. 3.

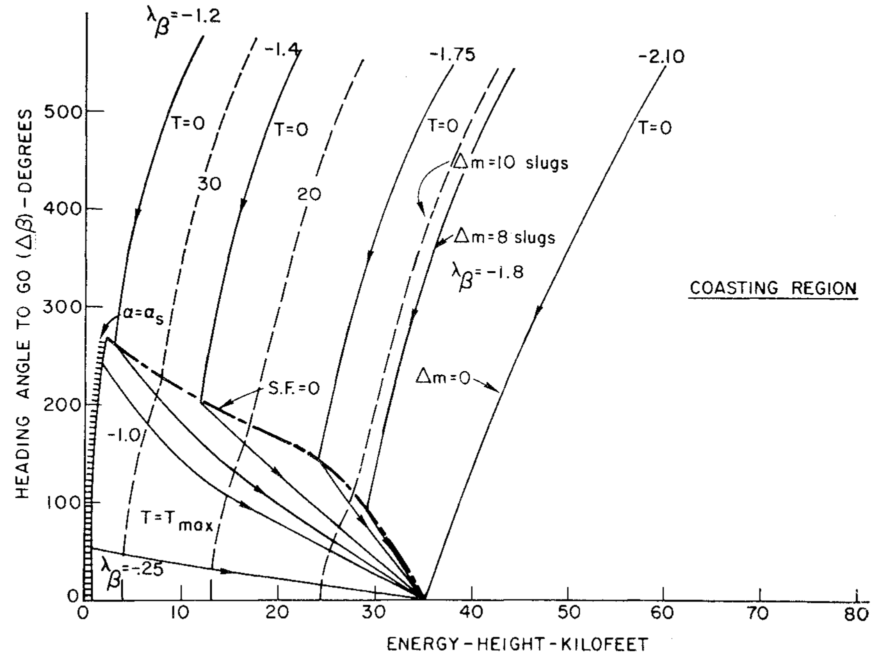

Fig. 8 Heading angle to go as a function of energy-height for a final energy-height of 35 kilofeet with contours of const Δm.

The example trajectory given in Fig. 8 has the initial conditions $E/g = 35$ kft, $\Delta\beta = 360°$. A coasting trajectory is followed until $\Delta\beta = 90°$ and $E/g = 29.0$ kft. At this point there is a discontinuity in altitude and bank angle, and the accelerating trajectory indexed by $\lambda_\beta = -1.8$ is followed until $E/g = 35$ kft. From the contours of constant Δm we can see that this path requires a Δm of 8 slugs.

Figure 9 is a feedback control chart for a final energy height of 35 kilofeet; it combines the information contained in Figs. 3, 4, and 8. Given the initial energy, we can find the thrust, altitude and bank angle programs from Fig. 9 which will minimize the fuel required to turn through a specified $\Delta\beta$ ending at $E/g = 35$ kilofeet. The vertical dashed lines inside the S.F. $= 0$ curve are contours of constant altitude

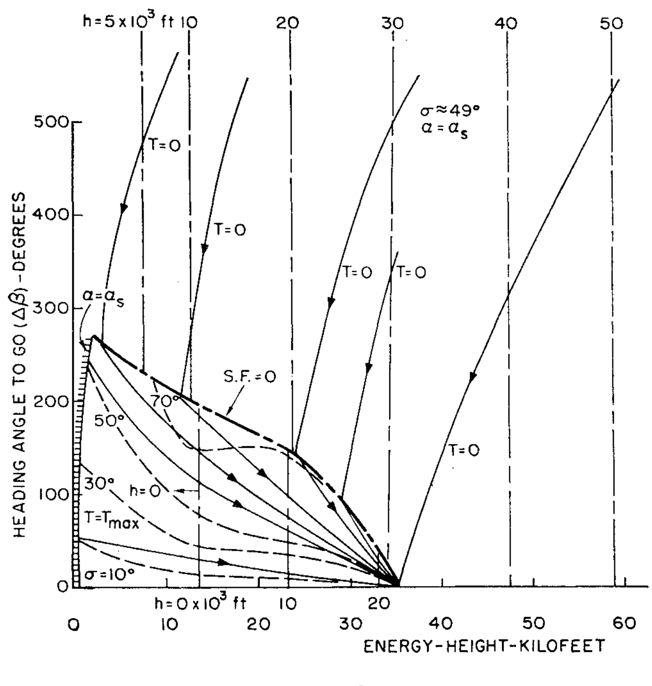

Fig. 9 Feedback control chart for a final energy-height of 35 kft.

for accelerating paths ($T = T_{max}$). The vertical dashed lines outside the S.F. $= 0$ curve are contours of constant altitude for gliding trajectories. The dashed curves within the S.F. $= 0$ and $\alpha = \alpha_S$ boundaries are contours of constant bank angle and are shown for $\sigma = 10°$ to $\sigma = 70°$. No contours of constant bank angle are given in the coasting region, as the bank angle is very nearly constant at 49° for all $E/g > 5$ kft (see Fig. 6).

Let us consider a couple of examples. Suppose we wish to turn through 150°, with $(E/g)_o = 9$ kft and $(E/g)_f = 35$ kft. From Fig. 9, we see that the optimal trajectory requires $T = T_{max}$ for the entire turn, has an initial altitude of 9 kft and an initial bank angle of 60°. The bank angle increases to 65° and the energy height to 12.5 kft. At this point, σ begins to decrease steadily, until, at $E/g = 35$ kft, it is $= 10°$. $h = 0$ until $E/g = 13.5$ kft, and then steadily increases until, at $E/g = 35$ kilofeet, $h = 23{,}000$ ft.

Given initial conditions in the coasting region, for example, $(E/g)_o = (E/g)_f = 35$ kilofeet and $\Delta\beta = 360°$, the minimum fuel trajectory begins with $T = 0$, $\alpha = \alpha_S$, $\sigma \simeq 49°$, $h = 30{,}000$ ft. At a time t_1, when the energy-height has decreased to 29 kft, there is the following discontinuity in the controls:

$$
\begin{aligned}
h(t_1^-) &= 24\,000 \text{ ft} & h(t_1^+) &= 16{,}000 \text{ ft} \\
\sigma(t_1^-) &= 49° & \sigma(t_1^+) &= 68° \\
T(t_1^-) &= 0 & T(t_1^+) &= T_{max}
\end{aligned}
$$

$T = T_{max}$ from $E/g = 29$ kft to $E/g =$ kft; σ decreases steadily as altitude increases until, at $E/g = 35$ kft, $\sigma = 60°$, and $h = 23{,}000$ ft.

Figure 10 is a state space chart for $(E/g)_f = 65$ kft. The stall boundary is indicated by the cross-hatched curve and the S.F. $= 0$ curve is represented by the dash-dot curve. As shown in Figs. 3 and 4, at $E/g = 59$ kft there is a discontinuity in the accelerating curves.

The purely coasting region at the right of the graph lies beneath the gliding path which ends at $(E/g)_f = 65$ kft, $\Delta\beta$ to go $= 0$. Trajectories whose initial conditions lie within this area may be completed with $T = 0$. For trajectories whose initial conditions lie within the area circumscribed by the S.F. $= 0$, and $\alpha = \alpha_S$ curves, $T = T_{max}$ at all times. For trajectories whose initial conditions lie above both of these areas, a minimum fuel path will include a gliding arc and a powered arc.

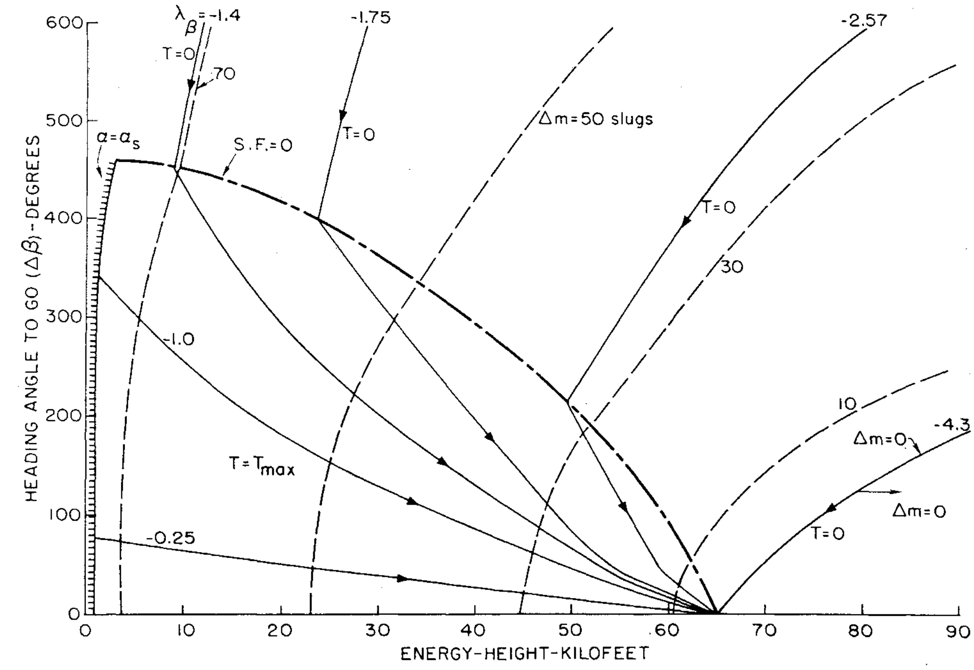

Fig. 10 Heading angle to go as a function of energy-height for a final energy-height of 65 kft with contours of const Δm.

Contours of constant fuel expenditure to go are shown for Δm equal to 10, 30, 50, and 70 slugs.

Figure 11 is a feedback control chart for a final energy-height of 65 kft. Given an initial energy-height, we can find the thrust, altitude and bank angle programs for minimum fuel trajectories which turn through a specified $\Delta\beta$ and end at $E/g = 65$ kft.

Three-dimensional minimum fuel trajectories offer a significant savings in fuel. For example, the minimum fuel trajectory for $(E/g)_o = (E/g)_f = 35$ kft, $\Delta\beta = 360°$ requires 8 slugs of fuel to perform. However, the corresponding constant altitude, constant velocity, constant bank angle maneuver requires 24 slugs of fuel to perform.

Minimum Fuel Turns with only Final Heading Angle Specified

In this section we investigate minimum fuel paths with β_f specified and E_f unspecified. If the required turn can be accomplished with $T = 0$, $h \geq 0$, and $\alpha \leq \alpha_S$, the cost will be zero. Assuming a change in heading angle of 360° or less, for nearly the entire range of energies, the turn can be completed with $T = 0$. However, if we reach an energy height of approximately 1.4 kft, corresponding to $h = 0$, $\sigma = 0$, $\alpha = \alpha_S$, before the required turn is completed, then the minimum fuel turn must include a powered flight arc. This is the lowest energy we can have and still satisfy the requirement $mg = L$. In effect, this puts a constraint on our final energy:

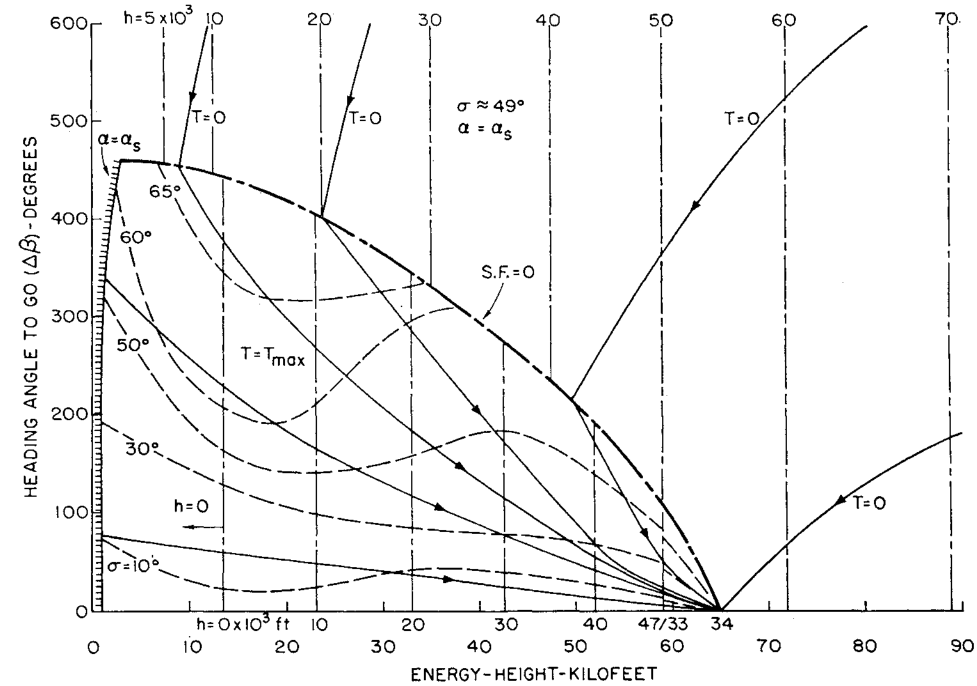

Fig. 11 Feedback control chart for a final energy-height of 65 kft.

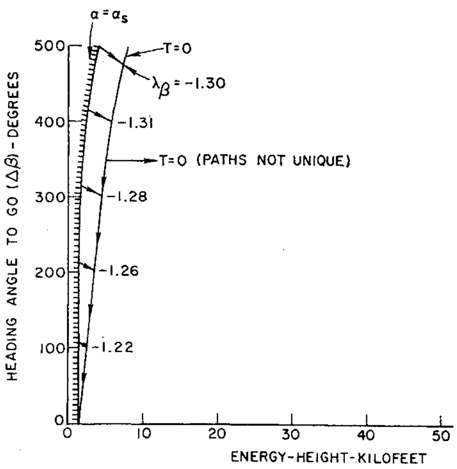

Fig. 12 Heading angle to go as a function of energy-height for minimum fuel turns with E_f unspecified.

$E(t_f)/g \geq 1.4$ kft. Since $E(t_f)$ is unspecified, $\lambda_E(t_f)=0$ or $E(t_f)/g = 1.4$ kft. It is clear that for $E_f/g > 1.4$ kft we could increase $\Delta\beta$ without increasing fuel comsumption; therefore, all minimum fuel paths with an unspecified final energy that include powered arcs must end with $E_f/g = 1.4$ kft.

Figure 12 shows heading angle to go as a function of energy height for a final energy height of 1.4 kft. For initial points to the right of the $T=0$ trajectory, minimum fuel turns with final energy unspecified may be accomplished by coasting. For initial points which lie between the stall boundary curve and the coasting arc, minimum fuel paths involve a powered arc followed by a coasting arc.

For example, suppose we have an initial energy height of 2 kft and wish to turn through 315° using minimum fuel. The trajectory indexed by $\lambda_\beta = -1.28$ satisfies these conditions. We begin with $T = T_{\max}$, $\sigma \simeq 60°$ and $h=0$ and accelerate to $E/g = 4.5$ kft, $\sigma \simeq 65°$, and $h=0$. At this point we switch to $T=0$ and zoom climb to about 2,000 ft and $\sigma \simeq 49°$. From this point on, we follow the trajectories of Figs. 6 and 7 until $(E/g)_f = 1.4$ kft.

Conclusions

In general minimum fuel, three-dimensional turns require variable altitude, variable bank angle programs. Thrust is either zero or $T_{\max}$. There are three characteristic types of minimum fuel turns: a) powered accelerating turns, b) coasting decelerating turns, or c) combined powered-coasting or coasting-powered turns.

The coasting arcs of minimum fuel turns are maximum $\Delta\beta$ gliding turns. If the maximum gliding $\Delta\beta$ for a given E_f is greater than the required $\Delta\beta$, then any gliding maneuver which satisfies the $\Delta\beta$, ΔE requirements may be used.

The altitude-Mach number profile for powered flight shown in Fig. 4 is practically identical in every case despite the different turns required. This trajectory follows the path of maximum increase in energy per pound of fuel burned for wings level flight (see Ref. 9). If we compare bank angle profiles with altitude-Mach number profiles, we find that when a powered arc is required, it occurs at low altitudes. Maximum bank angle occurs at the lowest altitude and the bank angle decreases as altitude increases. Physically, this may be explained by the greater value of $\Delta E/\Delta m|_{\sigma=0}$ (increase in energy per pound of fuel burned) at lower altitudes. Although the value of $\Delta E/\Delta m$ decreases with an increase in bank angle, it is still of considerable magnitude, and permits a high-bank angle and turning rate. Similarly, at high-initial altitudes, the minimum fuel solution consists of a coasting arc to a lower altitude followed by a powered arc at a high-bank angle back up to the higher altitude.

References

[1] Connor, M. A., "Optimization of a Lateral Turn at Constant Height," *AIAA Journal*, Vol. 5, No. 7, Feb. 1967, pp. 335–338.

[2] Bryson, A. E. and Lele, M. L., "Minimum Fuel Lateral Turns at Constant Altitude," *AIAA Journal*, Vol. 7, No. 3, March 1969, pp. 559–560.

[3] Bryson, A. E. and Hedrick, J. K., "Minimum Time Turns for a Supersonic Aircraft at Constant Altitude," *Journal of Aircraft*, Vol. 8, No. 3, March 1971, pp. 182–187.

[4] Bryson, A. E. and Hedrick, J. K. "Three-Dimensional, Minimum-Time Turns for a Supersonic Aircraft," AIAA Paper 71-796, Seattle, Wash., 1971.

Beebee, W., "Time Optimal Co-ordinated Turns Using Energy Methods," *Measurement Systems Laboratory Report*, MIT, Cambridge, Mass., May 1970.

[6] Kelley, H. J. and Edelbaum, T. N., "Energy Climbs, Energy Turns, and Asymptotic Expansions," *Journal of Aircraft*, Vol. 7, No. 1, Jan.-Feb. 1970, pp. 93–95.

[7] Boyd, J. R. and Christie, T. P., "Energy-Maneuverability Theory and Applications," 12th Annual Air Force Science and Engineering Symposium, Oct. 1965.

[8] Hedrick, J. K., *Optimal Three-Dimensional Turning Maneuvers for Supersonic Aircraft*, Ph.D. dissertation, 1970, Dept. of Aeronautics and Astronautics, Stanford Univ., Stanford, Calif.

[9] Bryson, A. E., Desai, M. N., and Hoffman, W. L., "The Energy State Approximation in Performance Optimization of Supersonic Aircraft," *Journal of Aircraft*, Vol. 6, Nov.-Dec. 1969, pp. 481–487.