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Optimal projection for parametric importance sampling in high dimension

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Abstract

In this paper we propose a dimension-reduction strategy in order to improve the performance of importance sampling in high dimension. The idea is to estimate variance terms in a small number of suitably chosen directions. We first prove that the optimal directions, i.e., the ones that minimize the Kullback–Leibler divergence with the optimal auxiliary density, are the eigenvectors associated to extreme (small or large) eigenvalues of the optimal covariance matrix. We then perform extensive numerical experiments that show that as dimension increases, these directions give estimations which are very close to optimal. Moreover, we show that the estimation remains accurate even when a simple empirical estimator of the covariance matrix is used to estimate these directions. These theoretical and numerical results open the way for different generalizations, in particular the incorporation of such ideas in adaptive importance sampling schemes.

Keywords: Importance sampling, High dimension, Gaussian covariance matrix, Kullback-Leibler divergence, Projection

Contents

```
########################################################################
# Figure 1. Plot of the function "l"
########################################################################
import numpy as np
import scipy as sp
import scipy.stats
import matplotlib.pyplot as plt
from CEIS_vMFNM import *
from IPython.display import display, Math, Latex
from IPython.display import Markdown
from tabulate import tabulate
np.random.seed(10)

x = np.linspace(np.finfo(float).eps,4.0,100)
y = -np.log(x) + x -1
```

```
# plot
fig, ax = plt.subplots()
ax.plot(x, y, linewidth=2.0)
ax.set(xlim=(0, 4), xticks=[0,1,2,3],
       ylim=(0, 0.5), yticks=[0,0.5,1,1.5])
plt.grid()
plt.xlabel(r"$x$",fontsize=16)
plt.ylabel(r"$\ell(x)$",fontsize=16)
for tickLabel in plt.gca().get_xticklabels() + plt.gca().get_yticklabels():
    tickLabel.set_fontsize(16)
plt.show()
```

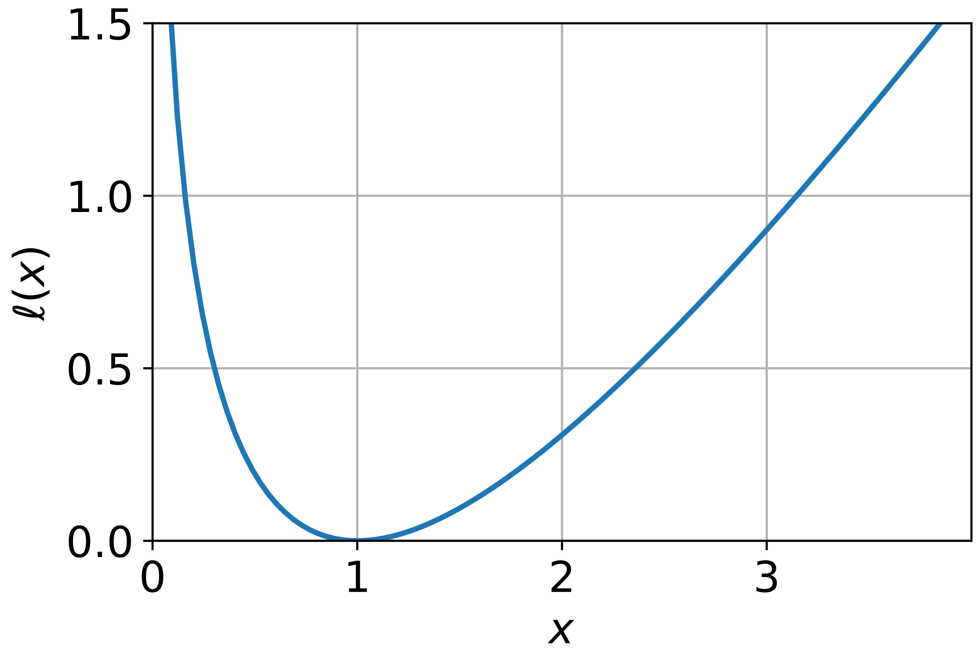


Figure 1: Plot of the function $\ell = -\log(x) + x - 1$ given by **?@eq-l**

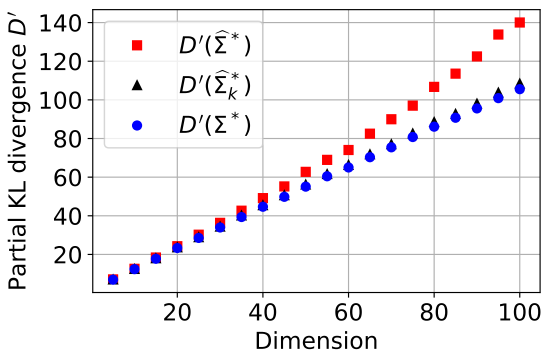
```
###########################################################################
# Figure 5. Evolution of the partial KL divergence and spectrum of the
# eigenvalues for the large portfolio loss application
###########################################################################

def Portfolio(X):
    N=np.shape(X)[0]
```

```
    nn=np.shape(X)[1]
    n=nn-2
    lamb=np.array(sp.stats.gamma.ppf(sp.stats.norm.cdf(X[:,0]),6,scale=1/6)\
                  ,ndmin=2).T
    eta=3*X[:,2:]
    ZZ=np.array(X[:,1],ndmin=2).T

    XX=(1/4*ZZ+np.sqrt(1-1/4**2)*eta)/np.sqrt(lamb)
    IndX=(XX>0.5*np.sqrt(n))*1
    PF=np.sum(IndX,axis=1)
    return(PF-0.25*n-0.1)

def Portfolio_md(X):
    N=np.shape(X)[0]
    nn=np.shape(X)[1]
    n=nn-2
    lamb=np.array( sp.stats.gamma.ppf(sp.stats.norm.cdf(X[:,0]),6,scale=1/6)\
                  ,ndmin=2).T
    eta=3*X[:,2:]
    ZZ=np.array(X[:,1],ndmin=2).T

    XX=(1/4*ZZ+np.sqrt(1-1/4**2)*eta)/np.sqrt(lamb)
    IndX=(XX>0.5*np.sqrt(n))*1
    PF=np.sum(IndX,axis=1)
    return(PF-0.3*n-0.1)

def Portfolio_ld(X):
    N=np.shape(X)[0]
    nn=np.shape(X)[1]
    n=nn-2
    lamb=np.array(sp.stats.gamma.ppf(sp.stats.norm.cdf(X[:,0]),6,scale=1/6)\
                  ,ndmin=2).T
    eta=3*X[:,2:]
    ZZ=np.array(X[:,1],ndmin=2).T

    XX=(1/4*ZZ+np.sqrt(1-1/4**2)*eta)/np.sqrt(lamb)
    IndX=(XX>0.5*np.sqrt(n))*1
    PF=np.sum(IndX,axis=1)
    return(PF-0.45*n-0.1)


DKL=np.zeros(20)
```

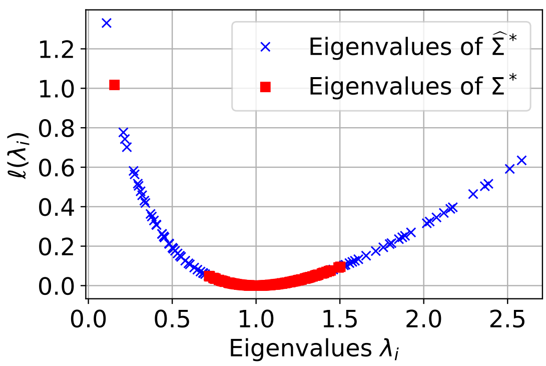
```
DKLp=np.zeros(20)
DKLm=np.zeros(20)
DKLstar=np.zeros(20)

n=100
bigsample=20*10**5
M=300

for d in range(5,n+1,5):

    if d<=30:
        phi=Portfolio_ld
    if d>70:
        phi=Portfolio
    else:
        phi=Portfolio_md

    VA=sp.stats.multivariate_normal(mean=np.zeros(d+2),cov=np.eye(d+2))
    X01=VA.rvs(size=bigsample)
    ind1=(phi(X01)>0)
    X1=X01[ind1,:]
    X1=X1[:M*10,:]
    #Mstar
    Mstar=np.mean(X1.T,axis=1)
    #Sigmastar
    X1c=(X1-Mstar).T
    Sigstar=X1c.dot(X1c.T)/np.shape(X1c)[1]


    ## g*-sample
    VA0=sp.stats.multivariate_normal(mean=np.zeros(d+2),cov=np.eye(d+2))
    X0=VA0.rvs(size=M*1000)

    ind=(phi(X0)>0)
    X=X0[ind,:]
    X=X[:M,:]             # g*-sample of size M

    ## estimated mean and covariance
    mm=np.mean(X,axis=0)

    Xc=(X-mm).T
    sigma =Xc @ Xc.T/np.shape(Xc)[1]
```

```
    ## projection with the eigenvalues of sigma
    Eig=np.linalg.eigh(sigma)
    logeig=np.sort(np.log(Eig[0])-Eig[0])
    delta=np.zeros(len(logeig)-1)
    for j in range(len(logeig)-1):
        delta[j]=abs(logeig[j]-logeig[j+1])

    k=np.argmax(delta)+1          # biggest gap between the l(lambda_i)

    indi=[]
    for l in range(k):
        indi.append(np.where(np.log(Eig[0])-Eig[0]==logeig[l])[0][0])

    P1=np.array(Eig[1][:,indi[0]],ndmin=2).T          # projection matrix
    for l in range(1,k):
        P1=np.concatenate((P1,np.array(Eig[1][:,indi[l]],ndmin=2).T),axis=1)

    diagsi=np.diag(Eig[0][indi])
    sig_opt_d=P1.dot((diagsi-np.eye(k))).dot(P1.T)+np.eye(d+2)

    DKL[int((d-5)/5)]=np.log(np.linalg.det(sigma))+np.sum(np.diag(\
                                Sigstar.dot(np.linalg.inv(sigma))))
    DKLp[int((d-5)/5)]=np.log(np.linalg.det(sig_opt_d))+np.sum(np.diag(\
                                Sigstar.dot(np.linalg.inv(sig_opt_d))))
    DKLstar[int((d-5)/5)]=np.log(np.linalg.det(Sigstar))+d+2

#### plot of partial KL divergence
plt.plot(range(5,n+1,5),DKL,'rs',label=r"$D'(\widehat{\Sigma}^*)$")
plt.plot(range(5,n+1,5),DKLp,'k^',label=r"$D'(\widehat{\Sigma}^*_k)$")
plt.plot(range(5,n+1,5),DKLstar,'bo',label=r"$D'(\Sigma^*)$")

plt.grid()
plt.xlabel('Dimension',fontsize=16)
plt.ylabel(r"Partial KL divergence $D'$",fontsize=16)
plt.legend(fontsize=16)
for tickLabel in plt.gca().get_xticklabels() + plt.gca().get_yticklabels():
    tickLabel.set_fontsize(16)
plt.show()

#### plot of the eigenvalues
Eig1=np.linalg.eigh(sigma)
logeig1=np.log(Eig1[0])-Eig1[0]+1
```

```
Table_eigv=np.zeros((n+2,2))
Table_eigv[:,0]=Eig1[0]
Table_eigv[:,1]=-logeig1

Eigst=np.linalg.eigh(Sigstar)
logeigst=np.log(Eigst[0])-Eigst[0]+1
Table_eigv_st=np.zeros((n+2,2))
Table_eigv_st[:,0]=Eigst[0]
Table_eigv_st[:,1]=-logeigst

plt.grid()
plt.xlabel(r"Eigenvalues $\lambda_i$",fontsize=16)
plt.ylabel(r"$\ell(\lambda_i)$",fontsize=16)
for tickLabel in plt.gca().get_xticklabels() + plt.gca().get_yticklabels():
    tickLabel.set_fontsize(16)

plt.plot(Table_eigv[:,0],Table_eigv[:,1],'bx',\
         label=r"Eigenvalues of $\widehat{\Sigma}^*$")
plt.plot(Table_eigv_st[:,0],Table_eigv_st[:,1],'rs',\
         label=r"Eigenvalues of $\Sigma^*$")
plt.legend(fontsize=16)
plt.show()
```



(a) Evolution of the partial KL divergence as the dimension increases, with the optimal covariance matrix Σ^* (blue circles), the sample covariance $\widehat{\Sigma}^*$ (red squares), and the projected covariance $\widehat{\Sigma}_k^*$ (black triangles).



(b) Computation of $\ell(\lambda_i)$ for the eigenvalues of Σ^* (red squares) and $\widehat{\Sigma}^*$ (blue crosses) in dimension $n = 100$ for the large portfolio losses of **?@eq-portfolio**.

Figure 2: Partial KL divergence and spectrum for the function $\phi = \mathbb{I}_{\varphi \geq 0}$ with φ the function given by **?@eq-portfolio**.

```
###############################################################################
# Table 5. Numerical comparison on the large portfolio loss application
###############################################################################

n=100         # dimension
phi=Portfolio
E=1.82*10**(-3)

def mypi(X):
    nn=np.shape(X)[1]
    n=nn-2
    f0=sp.stats.multivariate_normal.pdf(X,mean=np.zeros(nn),cov=np.eye(nn))
    return((phi(X)>0)*f0)


N=2000
M=500
B=500    # number of runs

Eopt=np.zeros(B)
EIS=np.zeros(B)
Eprj=np.zeros(B)
Eprm=np.zeros(B)
Eprjst=np.zeros(B)
Eprmst=np.zeros(B)
Evmfn=np.zeros(B)

SI=[]
SIP=[]
SIPst=[]
SIM=[]
SIMst=[]

bigsample=2*10**6

VA=sp.stats.multivariate_normal(mean=np.zeros(n+2),cov=np.eye(n+2))
X01=VA.rvs(size=bigsample)
ind1=(phi(X01)>0)
X1=X01[ind1,:]

#Mstar
Mstar=np.mean(X1.T,axis=1)
```

```
#Sigmastar
X1c=(X1-Mstar).T
Sigstar=X1c.dot(X1c.T)/np.shape(X1c)[1]

Eigst=np.linalg.eigh(Sigstar)
logeigst=np.sort(np.log(Eigst[0])-Eigst[0])
deltast=np.zeros(len(logeigst)-1)

for i in range(len(logeigst)-1):
    deltast[i]=abs(logeigst[i]-logeigst[i+1])

## choice of the number of dimension
k_st=np.argmax(deltast)+1

indist=[]
for i in range(k_st):
    indist.append(np.where(np.log(Eigst[0])-Eigst[0]==logeigst[i])[0][0])

P1st=np.array(Eigst[1][:,indist[0]],ndmin=2).T
for i in range(1,k_st):
    # matrix of influential directions
    P1st=np.concatenate((P1st,np.array(Eigst[1][:,indist[i]],ndmin=2).T),\
                        axis=1)

#np.random.seed(0)
for i in range(B):
############################# Estimation of the matrices

   ## g*-sample of size M
    VA=sp.stats.multivariate_normal(np.zeros(n+2),np.eye(n+2))
    X0=VA.rvs(size=M*1000)
    ind=(phi(X0)>0)
    X=X0[ind,:]
    X=X[:M,:]

    R=np.sqrt(np.sum(X**2,axis=1))
    Xu=(X.T/R).T

   ## estimated gaussian mean and covariance
    mm=np.mean(X,axis=0)
    Xc=(X-mm).T
    sigma =Xc @ Xc.T/np.shape(Xc)[1]
```

```
 SI.append(sigma)

## von Mises Fisher parameters
 normu=np.sqrt(np.mean(Xu,axis=0).dot(np.mean(Xu,axis=0).T))
 mu=np.mean(Xu,axis=0)/normu
 mu=np.array(mu,ndmin=2)
 chi=min(normu,0.95)
 kappa=(chi*n-chi**3)/(1-chi**2)

## Nakagami parameters
 omega=np.mean(R**2)
 tau4=np.mean(R**4)
 pp=omega**2/(tau4-omega**2)

###
 Eig=np.linalg.eigh(sigma)
 logeig=np.sort(np.log(Eig[0])-Eig[0])
 delta=np.zeros(len(logeig)-1)
 for j in range(len(logeig)-1):
     delta[j]=abs(logeig[j]-logeig[j+1])

 k=np.argmax(delta)+1

 indi=[]
 for l in range(k):
     indi.append(np.where(np.log(Eig[0])-Eig[0]==logeig[l])[0][0])

 P1=np.array(Eig[1][:,indi[0]],ndmin=2).T
 for l in range(1,k):
     P1=np.concatenate((P1,np.array(Eig[1][:,indi[l]],ndmin=2).T),axis=1)

 diagsi=np.diag(Eig[0][indi])
 sig_opt_d=P1.dot((diagsi-np.eye(k))).dot(P1.T)+np.eye(n+2)
 SIP.append(sig_opt_d)

###
 diagsist=P1st.T.dot(sigma).dot(P1st)
 sig_opt=P1st.dot(diagsist-np.eye(k_st)).dot(P1st.T)+np.eye(n+2)
 SIPst.append(sig_opt)

###
 Norm_mm=np.linalg.norm(mm)
```

```
    normalised_mm=np.array(mm,ndmin=2).T/Norm_mm
    vhat=normalised_mm.T.dot(sigma).dot(normalised_mm)
    sig_mean_d=(vhat-1)*normalised_mm.dot(normalised_mm.T)+np.eye(n+2)
    SIM.append(sig_mean_d)

   ###
    Norm_Mstar=np.linalg.norm(Mstar)
    normalised_Mstar=np.array(Mstar,ndmin=2).T/Norm_Mstar
    vhatst=normalised_Mstar.T.dot(sigma).dot(normalised_Mstar)

    sig_mean=(vhatst-1)*normalised_Mstar.dot(normalised_Mstar.T)+np.eye(n+2)
    SIMst.append(sig_mean)

############################################# Estimation of the integral
   ###
    Xop=sp.stats.multivariate_normal.rvs(mean=mm, cov=Sigstar,size=N)
    wop=mypi(Xop)/sp.stats.multivariate_normal.pdf(Xop,mean=mm, cov=Sigstar)
    Eopt[i]=np.mean(wop)

   ###
    Xis=sp.stats.multivariate_normal.rvs(mean=mm, cov=sigma,size=N)
    wis=mypi(Xis)/sp.stats.multivariate_normal.pdf(Xis,mean=mm, cov=sigma)
    EIS[i]=np.mean(wis)

   ###
    Xpr=sp.stats.multivariate_normal.rvs(mean=mm, cov=sig_opt_d,size=N)
    wpr=mypi(Xpr)/sp.stats.multivariate_normal.pdf(Xpr,mean=mm, \
                                                   cov=sig_opt_d)
    Eprj[i]=np.mean(wpr)

   ###
    Xpm=sp.stats.multivariate_normal.rvs(mean=mm, cov=sig_mean_d,size=N)
    wpm=mypi(Xpm)/sp.stats.multivariate_normal.pdf(Xpm,mean=mm, \
                                                   cov=sig_mean_d)
    Eprm[i]=np.mean(wpm)

   ###
    Xprst=sp.stats.multivariate_normal.rvs(mean=mm, cov=sig_opt,size=N)
    wprst=mypi(Xprst)/sp.stats.multivariate_normal.pdf(Xprst,mean=mm, \
                                                       cov=sig_opt)
    Eprjst[i]=np.mean(wprst)
```

```
    ###
     Xpmst=sp.stats.multivariate_normal.rvs(mean=mm, cov=sig_mean,size=N)
     wpmst=mypi(Xpmst)/sp.stats.multivariate_normal.pdf(Xpmst,mean=mm, \
                                                         cov=sig_mean)
     Eprmst[i]=np.mean(wpmst)

    ###
     Xvmfn = vMFNM_sample(mu, kappa, omega, pp, 1, N)
     Rvn=np.sqrt(np.sum(Xvmfn**2,axis=1))
     Xvnu=Xvmfn.T/Rvn

     h_log=vMF_logpdf(Xvnu,mu.T,kappa)+nakagami_logpdf(Rvn,pp,omega)
     A = np.log(n+2) + np.log(np.pi ** ((n+2) / 2)) - sp.special.gammaln((n+2) / 2 + 1)
     f_u = -A
     f_chi = (np.log(2) * (1 - (n+2) / 2) + np.log(Rvn) * ((n+2) - 1)\
              - 0.5 * Rvn ** 2 - sp.special.gammaln((n+2) / 2))
     f_log = f_u + f_chi
     W_log = f_log - h_log

     wvmfn=(phi(Xvmfn)>0)*np.exp(W_log)
     Evmfn[i]=np.mean(wvmfn)

### KL divergences
dkli=np.zeros(B)
dklp=np.zeros(B)
dklm=np.zeros(B)
dklpst=np.zeros(B)
dklmst=np.zeros(B)
dklpca=np.zeros(B)

for i in range(B):
    dkli[i]=np.log(np.linalg.det(SI[i]))+sum(np.diag(\
                        Sigstar.dot(np.linalg.inv(SI[i]))))
    dklp[i]=np.log(np.linalg.det(SIP[i]))+sum(np.diag(\
                        Sigstar.dot(np.linalg.inv(SIP[i]))))
    dklm[i]=np.log(np.linalg.det(SIM[i]))+sum(np.diag(\
                        Sigstar.dot(np.linalg.inv(SIM[i]))))
    dklpst[i]=np.log(np.linalg.det(SIPst[i]))+sum(np.diag(\
                        Sigstar.dot(np.linalg.inv(SIPst[i]))))
    dklmst[i]=np.log(np.linalg.det(SIMst[i]))+sum(np.diag(\
                        Sigstar.dot(np.linalg.inv(SIMst[i]))))
```

```python
Tabresult=np.zeros((3,7)) # table of results

Tabresult[0,0]=np.log(np.linalg.det(Sigstar))+n+2
Tabresult[0,1]=np.mean(dkli)
Tabresult[0,2]=np.mean(dklpst)
Tabresult[0,3]=np.mean(dklmst)
Tabresult[0,4]=np.mean(dklp)
Tabresult[0,5]=np.mean(dklm)
Tabresult[0,6]=None

Tabresult[1,0]=np.mean(Eopt-E)/E*100
Tabresult[1,1]=np.mean(EIS-E)/E*100
Tabresult[1,2]=np.mean(Eprjst-E)/E*100
Tabresult[1,3]=np.mean(Eprmst-E)/E*100
Tabresult[1,4]=np.mean(Eprj-E)/E*100
Tabresult[1,5]=np.mean(Eprm-E)/E*100
Tabresult[1,6]=np.mean(Evmfn-E)/E*100

Tabresult[2,0]=np.sqrt(np.mean((Eopt-E)**2))/E*100
Tabresult[2,1]=np.sqrt(np.mean((EIS-E)**2))/E*100
Tabresult[2,2]=np.sqrt(np.mean((Eprjst-E)**2))/E*100
Tabresult[2,3]=np.sqrt(np.mean((Eprmst-E)**2))/E*100
Tabresult[2,4]=np.sqrt(np.mean((Eprj-E)**2))/E*100
Tabresult[2,5]=np.sqrt(np.mean((Eprm-E)**2))/E*100
Tabresult[2,6]=np.sqrt(np.mean((Evmfn-E)**2))/E*100

Tabresult=np.round(Tabresult,1)

table=[["D'",Tabresult[0,0],Tabresult[0,1],Tabresult[0,2],Tabresult[0,3],
        Tabresult[0,4],Tabresult[0,5],Tabresult[0,6]],
      ["Relative error (\%)",Tabresult[1,0],Tabresult[1,1],Tabresult[1,2],
       Tabresult[1,3],Tabresult[1,4],Tabresult[1,5],Tabresult[1,6]],
    ["Coefficient of variation (\%)",Tabresult[2,0],Tabresult[2,1],
     Tabresult[2,2],Tabresult[2,3],Tabresult[2,4],Tabresult[2,5],
     Tabresult[2,6]]]
Markdown(tabulate(
  table,
  headers=["","$\Sigma^*$", "$\widehat{\Sigma}^*$", "$\widehat{\Sigma}_{opt}$", \
           "$\widehat{\Sigma}_{mean}$", "${\widehat{\Sigma}^{+d}_{opt}}$",\
           "$\widehat{\Sigma}^{+d}_{mean}$", "vMFN"],
    tablefmt="pipe"))
```

Table 1: Numerical comparison of the estimation of $E \approx 1.82 \cdot 10^{-3}$ considering the Gaussian density with the six covariance matrices defined in **?@sec-def_cov** and the vFMN model,$\phi = \mathbb{I}_{\varphi \geq 0}$ with φ given by **?@eq-portfolio**.

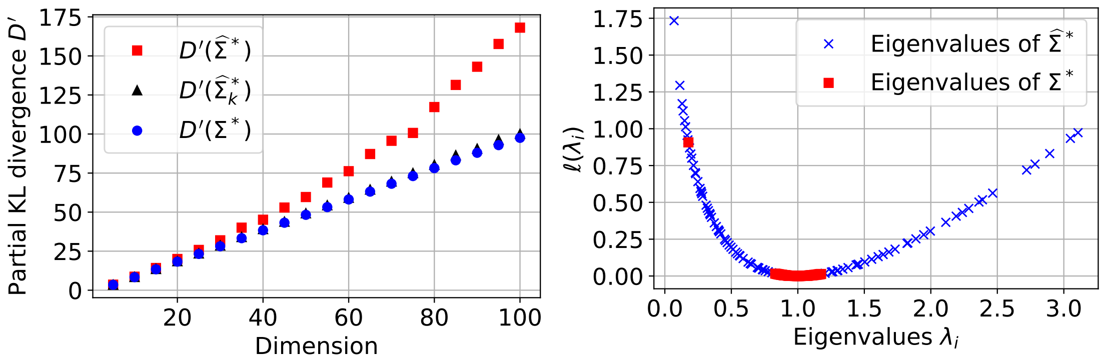
	Σ^*	$\widehat{\Sigma}^*$	$\widehat{\Sigma}_{opt}$	$\widehat{\Sigma}_{mean}$	$\widehat{\Sigma}_{opt}^{+d}$	$\widehat{\Sigma}_{mean}^{+d}$	vMFN
D'	106.1	122.9	107.9	107.9	108.3	108	nan
Relative error (%)	-0.5	-13.7	-0.3	-0.1	1.9	0.3	0.3
Coefficient of variation (%)	12.2	319.8	8	7.5	42	11.6	6.4

```
###########################################################################
# Figure 6. Evolution of the partial KL divergence and spectrum of the
# eigenvalues for the asian payoff application
###########################################################################

def payoff(X):
    d=np.shape(X)[1]
    S0=50
    r=0.05
    T=0.5
    sig2=0.01
    K=55
    uk=(r-sig2/2)*T/d+np.sqrt(T*sig2/d)*X
    cumuk=np.cumsum(uk,axis=1)
    en=S0*np.exp(cumuk)
    FK=np.exp(-r*T)*(1/d*np.sum(en,axis=1)-K)
    return(FK*(FK>0))


DKL=np.zeros(20)
DKLp=np.zeros(20)
DKLm=np.zeros(20)
DKLstar=np.zeros(20)

n=100
M=300
bigsample=10*10**5
phi=payoff
```

```
for d in range(5,n+1,5):

    VA=sp.stats.multivariate_normal(mean=np.zeros(d),cov=np.eye(d))
    X1=VA.rvs(size=bigsample)
    W1=phi(X1)
    W=W1[(W1>0)]
    X=X1[(W1>0),:]
#     W=W[:10*M]
#     X=X[:10*M,:]

    ## Mstar
    Mstar = np.divide((W.T @ X), sum(W))
    ## Sigmastar
    Xc = np.multiply((X - Mstar).T, np.sqrt(W))
    Sigstar = np.divide((Xc @ Xc.T), sum(W))

    ##
    VA0=sp.stats.multivariate_normal(np.zeros(d),np.eye(d))
    X0=VA0.rvs(size=M*100)
    W0=phi(X0)
    Wf=W0[(W0>0)]
    Xf=X0[(W0>0),:]
    Wf=Wf[:M]
    Xf=Xf[:M,:]

    ## estimated mean and covariance
    mm=np.divide((Wf.T @ Xf), sum(Wf))
    Xcf=np.multiply((Xf - mm).T, np.sqrt(Wf))
    sigma =np.divide((Xcf @ Xcf.T), sum(Wf))


    ## projection with the eigenvalues of sigma
    Eig=np.linalg.eigh(sigma)
    logeig=np.sort(np.log(Eig[0])-Eig[0])
    delta=np.zeros(len(logeig)-1)
    for j in range(len(logeig)-1):
        delta[j]=abs(logeig[j]-logeig[j+1])

    k=np.argmax(delta)+1         # biggest gap between the l(lambda_i)

    indi=[]
    for l in range(k):
```

```
        indi.append(np.where(np.log(Eig[0])-Eig[0]==logeig[l])[0][0])

    P1=np.array(Eig[1][:,indi[0]],ndmin=2).T         # projection matrix
    for l in range(1,k):
        P1=np.concatenate((P1,np.array(Eig[1][:,indi[l]],ndmin=2).T),axis=1)

    diagsi=np.diag(Eig[0][indi])
    sig_opt_d=P1.dot((diagsi-np.eye(k))).dot(P1.T)+np.eye(d)

    DKL[int((d-5)/5)]=np.log(np.linalg.det(sigma))+np.sum(\
                            np.diag(Sigstar.dot(np.linalg.inv(sigma))))
    DKLp[int((d-5)/5)]=np.log(np.linalg.det(sig_opt_d))+np.sum(\
                            np.diag(Sigstar.dot(np.linalg.inv(sig_opt_d))))
    DKLstar[int((d-5)/5)]=np.log(np.linalg.det(Sigstar))+d

#### plot of partial KL divergence
plt.plot(range(5,n+1,5),DKL,'rs',label=r"$D'(\widehat{\Sigma}^*)$")
plt.plot(range(5,n+1,5),DKLp,'k^',label=r"$D'(\widehat{\Sigma}^*_k)$")
plt.plot(range(5,n+1,5),DKLstar,'bo',label=r"$D'(\Sigma^*)$")

plt.grid()
plt.xlabel('Dimension',fontsize=16)
plt.ylabel(r"Partial KL divergence $D'$",fontsize=16)
plt.legend(fontsize=16)
for tickLabel in plt.gca().get_xticklabels() + plt.gca().get_yticklabels():
    tickLabel.set_fontsize(16)
plt.show()

#### plot of the eigenvalues
Eig1=np.linalg.eigh(sigma)
logeig1=np.log(Eig1[0])-Eig1[0]+1
Table_eigv=np.zeros((n,2))
Table_eigv[:,0]=Eig1[0]
Table_eigv[:,1]=-logeig1

Eigst=np.linalg.eigh(Sigstar)
logeigst=np.log(Eigst[0])-Eigst[0]+1
Table_eigv_st=np.zeros((n,2))
Table_eigv_st[:,0]=Eigst[0]
Table_eigv_st[:,1]=-logeigst

plt.grid()
```

```python
plt.xlabel(r"Eigenvalues $\lambda_i$",fontsize=16)
plt.ylabel(r"$\ell(\lambda_i)$",fontsize=16)
for tickLabel in plt.gca().get_xticklabels() + plt.gca().get_yticklabels():
    tickLabel.set_fontsize(16)

plt.plot(Table_eigv[:,0],Table_eigv[:,1],'bx',\
         label=r"Eigenvalues of $\widehat{\Sigma}^*$")
plt.plot(Table_eigv_st[:,0],Table_eigv_st[:,1],'rs',\
         label=r"Eigenvalues of $\Sigma^*$")
plt.legend(fontsize=16)
plt.show()
```



(a) Evolution of the partial KL divergence as the dimension increases, with the optimal covariance matrix Σ^* (blue circles), the sample covariance $\widehat{\Sigma}^*$ (red squares), and the projected covariance $\widehat{\Sigma}_k^*$ (black triangles).

(b) Computation of $\ell(\lambda_i)$ for the eigenvalues of Σ^* (red squares) and $\widehat{\Sigma}^*$ (blue crosses) in dimension $n = 100$ for the Asian payoff example of **?@eq-payoff**

Figure 3: Partial KL divergence and spectrum for the function ϕ given in **?@eq-payoff**.

```python
#########################################################################
# Table 6. Numerical comparison on the Asian payoff application
########################################################################

n=100         # dimension

bigsample=2*10**6
phi=payoff
E=0.0187

def mypi(X):
    n=np.shape(X)[1]
```

```
    return(sp.stats.multivariate_normal.pdf(X,mean=np.zeros(n),\
                                            cov=np.eye(n))*phi(X))

N=2000
M=500
B=500    # number of runs

VA=sp.stats.multivariate_normal(mean=np.zeros(n),cov=np.eye(n))
X1=VA.rvs(size=bigsample)
W1=phi(X1)
W=np.copy(W1[(W1>0)])
X=np.copy(X1[(W1>0),:])

## Mstar
Mstar = np.divide((W.T @ X), sum(W))
## Sigmastar
Xc = np.multiply((X - Mstar).T, np.sqrt(W))
Sigstar = np.divide((Xc @ Xc.T), sum(W))

Eigst=np.linalg.eigh(Sigstar)
logeigst=np.sort(np.log(Eigst[0])-Eigst[0])
deltast=np.zeros(len(logeigst)-1)

for l in range(len(logeigst)-1):
    deltast[l]=abs(logeigst[l]-logeigst[l+1])

## choice of the number of dimension
k_st=np.argmax(deltast)+1

indist=[]
for j in range(k_st):
    indist.append(np.where(np.log(Eigst[0])-Eigst[0]==logeigst[j])[0][0])

P1st=np.array(Eigst[1][:,indist[0]],ndmin=2).T
for jj in range(1,k_st):
    # matrix of influential directions
    P1st=np.concatenate((P1st,np.array(Eigst[1][:,\
        indist[jj]],ndmin=2).T),axis=1)

Eopt=np.zeros(B)
EIS=np.zeros(B)
Eprj=np.zeros(B)
```

```
Eprm=np.zeros(B)
Eprjst=np.zeros(B)
Eprmst=np.zeros(B)
Evmfn=np.zeros(B)

SI=[]
SIP=[]
SIPst=[]
SIM=[]
SIMst=[]

#np.random.seed(0)
for i in range(B):
############################## Estimation of the matrices

   ##
    VA0=sp.stats.multivariate_normal(mean=np.zeros(n),cov=np.eye(n))
    X0=VA.rvs(size=100*M)
    W0=phi(X0)
    Wf=W0[(W0>0)]
    Xf=X0[(W0>0),:]
    Wf=Wf[:M]
    Xf=Xf[:M,:]

   ## estimated mean and covariance
    mm=np.divide((Wf.T @ Xf), sum(Wf))
    Xcf=np.multiply((Xf - mm).T, np.sqrt(Wf))
    sigma=np.divide((Xcf @ Xcf.T), sum(Wf))
    SI.append(sigma)

    R=np.sqrt(np.sum(Xf**2,axis=1))
    Xu=(Xf.T/R).T

   ## von Mises Fisher parameters
    normu=np.sqrt(np.mean(Xu,axis=0).dot(np.mean(Xu,axis=0).T))
    mu=np.mean(Xu,axis=0)/normu
    mu=np.array(mu,ndmin=2)
    chi=min(normu,0.95)
    kappa=(chi*n-chi**3)/(1-chi**2)

   ## Nakagami parameters
    omega=np.mean(R**2)
```

```
 tau4=np.mean(R**4)
 pp=omega**2/(tau4-omega**2)

###
 Eig=np.linalg.eigh(sigma)
 logeig=np.sort(np.log(Eig[0])-Eig[0])
 delta=np.zeros(len(logeig)-1)
 for j in range(len(logeig)-1):
     delta[j]=abs(logeig[j]-logeig[j+1])

 k=np.argmax(delta)+1

 indi=[]
 for l in range(k):
     indi.append(np.where(np.log(Eig[0])-Eig[0]==logeig[l])[0][0])

 P1=np.array(Eig[1][:,indi[0]],ndmin=2).T
 for l in range(1,k):
     P1=np.concatenate((P1,np.array(Eig[1][:,indi[l]],ndmin=2).T),axis=1)

 diagsi=np.diag(Eig[0][indi])
 sig_opt_d=P1.dot((diagsi-np.eye(k))).dot(P1.T)+np.eye(n)
 SIP.append(sig_opt_d)

###
 diagsist=P1st.T.dot(sigma).dot(P1st)
 sig_opt=P1st.dot(diagsist-np.eye(k_st)).dot(P1st.T)+np.eye(n)
 SIPst.append(sig_opt)

###
 Norm_mm=np.linalg.norm(mm)
 normalised_mm=np.array(mm,ndmin=2).T/Norm_mm
 vhat=normalised_mm.T.dot(sigma).dot(normalised_mm)
 sig_mean_d=(vhat-1)*normalised_mm.dot(normalised_mm.T)+np.eye(n)
 SIM.append(sig_mean_d)

###
 Norm_Mstar=np.linalg.norm(Mstar)
 normalised_Mstar=np.array(Mstar,ndmin=2).T/Norm_Mstar
 vhatst=normalised_Mstar.T.dot(sigma).dot(normalised_Mstar)

 sig_mean=(vhatst-1)*normalised_Mstar.dot(normalised_Mstar.T)+np.eye(n)
```

```
    SIMst.append(sig_mean)

############################################## Estimation of the integral
   ###
    Xop=sp.stats.multivariate_normal.rvs(mean=mm, cov=Sigstar,size=N)
    wop=mypi(Xop)/sp.stats.multivariate_normal.pdf(Xop,mean=mm, cov=Sigstar)
    Eopt[i]=np.mean(wop)

   ###
    Xis=sp.stats.multivariate_normal.rvs(mean=mm, cov=sigma,size=N)
    wis=mypi(Xis)/sp.stats.multivariate_normal.pdf(Xis,mean=mm, cov=sigma)
    EIS[i]=np.mean(wis)

   ###
    Xpr=sp.stats.multivariate_normal.rvs(mean=mm, cov=sig_opt_d,size=N)
    wpr=mypi(Xpr)/sp.stats.multivariate_normal.pdf(Xpr,mean=mm, \
                                               cov=sig_opt_d)
    Eprj[i]=np.mean(wpr)

   ###
    Xpm=sp.stats.multivariate_normal.rvs(mean=mm, cov=sig_mean_d,size=N)
    wpm=mypi(Xpm)/sp.stats.multivariate_normal.pdf(Xpm,mean=mm, \
                                               cov=sig_mean_d)
    Eprm[i]=np.mean(wpm)

   ###
    Xprst=sp.stats.multivariate_normal.rvs(mean=mm, cov=sig_opt,size=N)
    wprst=mypi(Xprst)/sp.stats.multivariate_normal.pdf(Xprst,mean=mm, \
                                                   cov=sig_opt)
    Eprjst[i]=np.mean(wprst)

   ###
    Xpmst=sp.stats.multivariate_normal.rvs(mean=mm, cov=sig_mean,size=N)
    wpmst=mypi(Xpmst)/sp.stats.multivariate_normal.pdf(Xpmst,mean=mm, \
                                                   cov=sig_mean)
    Eprmst[i]=np.mean(wpmst)

   ###
    Xvmfn = vMFNM_sample(mu, kappa, omega, pp, 1, N)
    Rvn=np.sqrt(np.sum(Xvmfn**2,axis=1))
    Xvnu=Xvmfn.T/Rvn
```

```
    h_log=vMF_logpdf(Xvnu,mu.T,kappa)+nakagami_logpdf(Rvn,pp,omega)
    A = np.log(n) + np.log(np.pi ** (n / 2)) - sp.special.gammaln(n / 2 + 1)
    f_u = -A
    f_chi = (np.log(2) * (1 - n / 2) + np.log(Rvn) * (n - 1) - 0.5\
             * Rvn ** 2 - sp.special.gammaln(n / 2))
    f_log = f_u + f_chi
    W_log = f_log - h_log

    wvmfn=(phi(Xvmfn)>0)*np.exp(W_log)
    Evmfn[i]=np.mean(wvmfn)

### KL divergences
dkli=np.zeros(B)
dklp=np.zeros(B)
dklm=np.zeros(B)
dklpst=np.zeros(B)
dklmst=np.zeros(B)
dklpca=np.zeros(B)

for i in range(B):
    dkli[i]=np.log(np.linalg.det(SI[i]))+sum(np.diag(\
        Sigstar.dot(np.linalg.inv(SI[i]))))
    dklp[i]=np.log(np.linalg.det(SIP[i]))+sum(np.diag(\
        Sigstar.dot(np.linalg.inv(SIP[i]))))
    dklm[i]=np.log(np.linalg.det(SIM[i]))+sum(np.diag(\
        Sigstar.dot(np.linalg.inv(SIM[i]))))
    dklpst[i]=np.log(np.linalg.det(SIPst[i]))+sum(np.diag(\
        Sigstar.dot(np.linalg.inv(SIPst[i]))))
    dklmst[i]=np.log(np.linalg.det(SIMst[i]))+sum(np.diag(\
        Sigstar.dot(np.linalg.inv(SIMst[i]))))

Tabresult=np.zeros((3,7)) # table of results

Tabresult[0,0]=np.log(np.linalg.det(Sigstar))+n
Tabresult[0,1]=np.mean(dkli)
Tabresult[0,2]=np.mean(dklpst)
Tabresult[0,3]=np.mean(dklmst)
Tabresult[0,4]=np.mean(dklp)
Tabresult[0,5]=np.mean(dklm)
Tabresult[0,6]=None

Tabresult[1,0]=np.mean(Eopt-E)/E*100
```

```
Tabresult[1,1]=np.mean(EIS-E)/E*100
Tabresult[1,2]=np.mean(Eprjst-E)/E*100
Tabresult[1,3]=np.mean(Eprmst-E)/E*100
Tabresult[1,4]=np.mean(Eprj-E)/E*100
Tabresult[1,5]=np.mean(Eprm-E)/E*100
Tabresult[1,6]=np.mean(Evmfn-E)/E*100

Tabresult[2,0]=np.sqrt(np.mean((Eopt-E)**2))/E*100
Tabresult[2,1]=np.sqrt(np.mean((EIS-E)**2))/E*100
Tabresult[2,2]=np.sqrt(np.mean((Eprjst-E)**2))/E*100
Tabresult[2,3]=np.sqrt(np.mean((Eprmst-E)**2))/E*100
Tabresult[2,4]=np.sqrt(np.mean((Eprj-E)**2))/E*100
Tabresult[2,5]=np.sqrt(np.mean((Eprm-E)**2))/E*100
Tabresult[2,6]=np.sqrt(np.mean((Evmfn-E)**2))/E*100

Tabresult=np.round(Tabresult,1)


table=[["D'",Tabresult[0,0],Tabresult[0,1],Tabresult[0,2],Tabresult[0,3],
        Tabresult[0,4],Tabresult[0,5],Tabresult[0,6]],
      ["Relative error (\%)",Tabresult[1,0],Tabresult[1,1],Tabresult[1,2],
       Tabresult[1,3],Tabresult[1,4],Tabresult[1,5],Tabresult[1,6]],
    ["Coefficient of variation (\%)",Tabresult[2,0],Tabresult[2,1],Tabresult[2,2],
     Tabresult[2,3],Tabresult[2,4],Tabresult[2,5],Tabresult[2,6]]]
Markdown(tabulate(
  table,
  headers=["","$\Sigma^*$", "$\widehat{\Sigma}^*$", "$\widehat{\Sigma}_{opt}$",\
           "$\widehat{\Sigma}_{mean}$", "${\widehat{\Sigma}^{+d}_{opt}}$",\
           "$\widehat{\Sigma}^{+d}_{mean}$", "vMFN"],
    tablefmt="pipe"))
```

Table 2: Numerical comparison of the estimation of $E \approx 18.7 \times 10^{-3}$ considering the Gaussian density with the six covariance matrices defined in **?@sec-def_cov** and the vFMN model, when ϕ is given by **?@eq-payoff**.

	Σ^*	$\widehat{\Sigma}^*$	$\widehat{\Sigma}_{opt}$	$\widehat{\Sigma}_{mean}$	$\widehat{\Sigma}^{+d}_{opt}$	$\widehat{\Sigma}^{+d}_{mean}$	vMFN
D'	98.2	128.1	98.4	98.4	99.6	98.6	nan
Relative error (%)	0.3	-72.9	0.3	0.3	1.1	0.4	18.2

	Σ^*	$\widehat{\Sigma}^*$	$\widehat{\Sigma}_{opt}$	$\widehat{\Sigma}_{mean}$	$\widehat{\Sigma}_{opt}^{+d}$	$\widehat{\Sigma}_{mean}^{+d}$	vMFN
Coefficient of variation (%)	2.7	98.1	2.2	4.7	50.5	2.6	18.7