RLC Analytical Responses

John Morrow

Equation 1 is the differential equation for the series-connected RLC circuit (Figure 1) in terms of i, the current flowing in the circuit. R, L, and C represent the resistance, inductance, and capacitance.

$$L\frac{di^2}{dt^2} + R\frac{di}{dt} + \frac{1}{C} = 0 \tag{1}$$

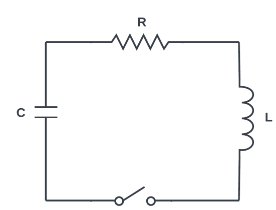

Figure 1: **Series-connected RLC circuit**

Taking the Laplace transform of each term in Equation 1:

$$\mathcal{L}\left\{L\frac{di^2}{dt^2}\right\} = L\left(s^2F(s) - si(0) - i'(0)\right) \tag{2}$$

$$\mathcal{L}\left\{R\frac{di}{dt}\right\} = R\left(sF(s) - i(0)\right) \tag{3}$$

$$\mathcal{L}\left\{\frac{1}{C}i\right\} = \frac{1}{C}F(s) \tag{4}$$

Combining and collecting terms, the Laplace transform of Equation 1 is:

$$(Ls^2 + Rs + \frac{1}{C})F(s) - (s+1)i(0) - i'(0) = 0 \tag{5}$$

Solving for $F(s)$:

$$F(s) = \frac{(s+1)i(0) + i'(0)}{Ls^2 + Rs + \frac{1}{C}} \tag{6}$$

Applying the initial conditions, $i(0) = 0$ and $i'(0) = \frac{V(0)}{L}$

where $V(0)$ is the initial capacitor voltage at $t = 0$:

$$F(s) = \frac{\frac{V(0)}{L}}{Ls^2 + Rs + \frac{1}{C}} \tag{7}$$

Factoring the denominator of Equation 7 and substituting $\alpha = \frac{R}{2L}$:

$$\left(s + \alpha + \sqrt{\alpha^2 - \frac{1}{LC}}\right)\left(s + \alpha - \sqrt{\alpha^2 - \frac{1}{LC}}\right) \tag{8}$$

Simplifying further by substituting $z = \sqrt{\alpha^2 - \frac{1}{LC}}$, the roots of $F(s)$ are:

$$(s + \alpha - z)(s + \alpha + z) \tag{9}$$

under-damped response

For the under-damped case, where $\alpha < \sqrt{\frac{1}{LC}}$, z becomes imaginary (denoted here by j):

$$(s + \alpha - zj)(s + \alpha + zj) \tag{10}$$

Expressing $F(s)$ from Equation 7 with partial fractions:

$$F(s) = \left(\frac{\frac{V(0)}{L}}{(s + \alpha - zj)(s + \alpha + zj)}\right) = \frac{A}{s + \alpha + zj} + \frac{B}{s + \alpha - zj} \tag{11}$$

$$F(s) = \frac{\frac{V(0)}{L}}{(s + \alpha - zj)(s + \alpha + zj)} = \frac{A(s + \alpha - zj) + B(s + \alpha + zj)}{(s + \alpha - zj)(s + \alpha + zj)} \tag{12}$$

Equating the numerators in Equation 12:

$$\frac{V(0)}{L} = A(s + \alpha - zj) + B(s + \alpha + zj) \tag{13}$$

When $s = -\alpha - zj$, Equation 13 becomes:

$$\frac{V(0)}{L} = A\left(-2zj\right) \tag{14}$$

Solving for A:

$$A = \frac{V(0)}{(-2zj)L}\left(\frac{j}{j}\right) = \frac{V(0)j}{2zL} \tag{15}$$

When $s = -\alpha + zj$, Equation 13 becomes:

$$\frac{V(0)}{L} = B\,(2zj) \tag{16}$$

Solving for B:

$$B = \frac{V(0)}{(2zj)L}\left(\frac{j}{j}\right) = \frac{-V(0)j}{2zL} \tag{17}$$

Substituting these values for A and B into Equation 11:

$$F(s) = \frac{\frac{V(0)j}{2zL}}{s+\alpha+zj} + \frac{\frac{-V(0)j}{2zL}}{s+\alpha-zj} \tag{18}$$

Taking the inverse Laplace transform of $F(s)$:

$$i(t) = \mathcal{L}^{-1}F(s) \tag{19}$$

$$i(t) = \frac{V(0)}{2zL}\left(je^{-(\alpha+zj)t} - je^{-(\alpha-zj)t}\right) \tag{20}$$

$$i(t) = \frac{V(0)}{2zL}e^{-\alpha t}j\left(\frac{j}{j}\right)\left(e^{-zjt} - e^{zjt}\right) \tag{21}$$

$$i(t) = \frac{V(0)}{zL}e^{-\alpha t}\frac{\left(e^{zjt} - e^{-zjt}\right)}{2j} \tag{22}$$

Substituting the identity $sin(x) = \frac{\left(e^{jx}-e^{-jx}\right)}{2j}$:

$$i(t) = \frac{V(0)}{zL}e^{-\alpha t}sin(zt) \tag{23}$$

Figure 2 is the response of Equation 23, with $R = 1.2$ (ohms), $L = 1.5$ (henries), $C = 0.3$(farads), and $V(0) = 12$ (volts). With these component values, $\alpha = 0.4$ and $z = 1.44$.

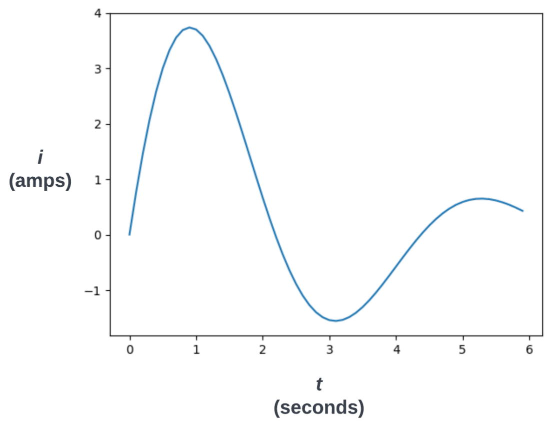

Figure 2: **Under-damped response**

critically-damped response

For the critically-damped case, $\alpha = \sqrt{\frac{1}{LC}}$. This results in $z = 0$, and both roots of $F(s)$ in Equation 9 are identical.

$$(s+\alpha)(s+\alpha) \tag{24}$$

Expressing F(s) from Equation 7 with partial fractions and two identical roots:

$$F(s) = \left(\frac{\frac{V(0)}{L}}{(s+\alpha)(s+\alpha)}\right) = \frac{A}{s+\alpha} + \frac{B}{(s+\alpha)^2} \tag{25}$$

$$F(s) = \frac{\frac{V(0)}{L}}{(s+\alpha)(s+\alpha)} = \frac{A(s+\alpha)+B}{(s+\alpha)^2} \tag{26}$$

Equating the numerators in Equation 26:

$$\frac{V(0)}{L} = A(s+\alpha) + B \tag{27}$$

When $s = -\alpha$, Equation 27 becomes:

$$\frac{V(0)}{L} = A(-\alpha+\alpha) + B \tag{28}$$

Solving for B and A:

$$B = \frac{V(0)}{L}, \quad A = 0 \tag{29}$$

Substituting the values for A and B into Equation 25:

$$F(s) = \frac{\frac{V(0)}{L}}{(s+\alpha)^2} \tag{30}$$

Taking the inverse Laplace transform of $F(s)$:

$$i(t) = \mathcal{L}^{-1}F(s) \tag{31}$$

$$i(t) = \frac{V(0)}{L}te^{-\alpha t} \tag{32}$$

Figure 3 is the response of Equation 32, with $R = 4.47$ (ohms), $L = 1.5$ (henries), $C = 0.3$(farads), and $V(0) = 12$ (volts). With these component values, $\alpha = 1.49$.

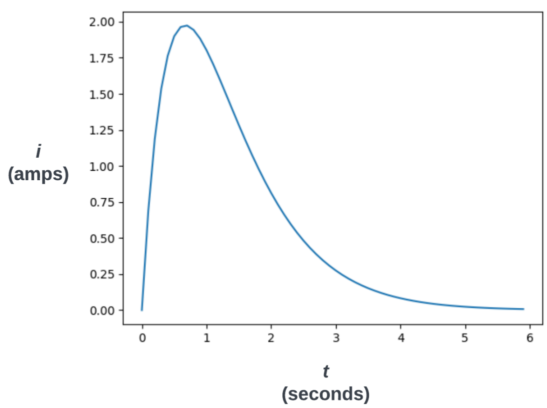

Figure 3: **Critically-damped response**

over-damped response

For the over-damped case, $\alpha > \sqrt{\frac{1}{LC}}$. Expressing $F(s)$ from Equation 7 with partial fractions:

$$F(s) = \left(\frac{\frac{V(0)}{L}}{(s+\alpha-z)(s+\alpha+z)}\right) = \frac{A}{s+\alpha+z} + \frac{B}{s+\alpha-z} \tag{33}$$

$$F(s) = \frac{\frac{V(0)}{L}}{(s+\alpha-zj)(s+\alpha+zj)} = \frac{A(s+\alpha-z)+B(s+\alpha+z)}{(s+\alpha-z)(s+\alpha+z)} \tag{34}$$

Equating the numerators in Equation 34:

$$\frac{V(0)}{L} = A(s + \alpha - z) + B(s + \alpha + z) \tag{35}$$

When $s = -\alpha - z$, Equation 35 becomes:

$$\frac{V(0)}{L} = A\,(-2z) \tag{36}$$

Solving for A:

$$A = \frac{V(0)}{-2zL} \tag{37}$$

When $s = -\alpha + z$, Equation 35 becomes:

$$\frac{V(0)}{L} = B\,(2z) \tag{38}$$

Solving for B:

$$B = \frac{V(0)}{2zL} \tag{39}$$

Substituting the values for A and B into Equation 33:

$$F(s) = \frac{V(0)}{2zL}\left(\frac{1}{s + \alpha - z} - \frac{1}{s + \alpha + z}\right) \tag{40}$$

Taking the inverse Laplace transform of $F(s)$:

$$i(t) = \mathcal{L}^{-1}F(s) \tag{41}$$

$$i(t) = \frac{V(0)}{2zL}\left(e^{(-\alpha+z)t} - e^{(-\alpha-z)t}\right) \tag{42}$$

Figure 4 is the response of Equation 42, with $R = 6.0$ (ohms), $L = 1.5$ (henries), $C = 0.3$(farads), and $V(0) = 12$ (volts). With these component values, $\alpha = 2.0$ and $z = 1.33$.

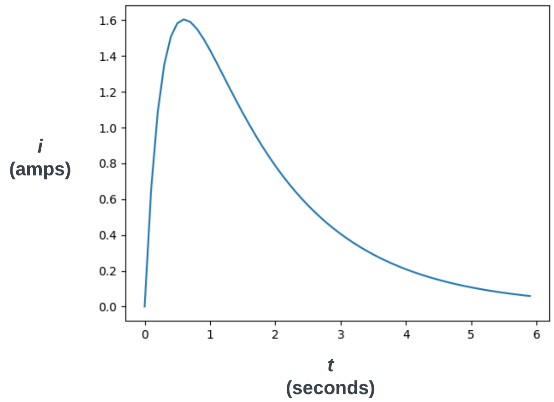

Figure 4: **Over-damped response**