Option Pricing

Luis A. Seco

Univ. of Toronto

Financial Instruments – Equity

European Options Expire at a preset future time. Their pay-off f depends on the price of the underlying S_T at expiration.

Call options with strike K have pay-off given by

$$P(S_T) = (S_T - K)_+.$$

Put options have pay–off given by

$$P(S_T) = (K - S_T)_+.$$

American Options Can be exercised at any time prior to expiration T. Their pay-off is a function of the value of the underlying at that time (exercise time t), S_t with $t \leq T$.

Call options with strike K have pay-off given by

$$P(S_t, t) = (S_t - K)_+.$$

Put options have pay–off given by

$$P(S_t, t) = (K - S_t)_+.$$

Asian Options Their payoff depends on the average value of the underlying at certain times prior to expiration. Can be issued with a European or American style.

Example payoffs are:

$$P = \frac{1}{n}\sum_{i=1}^{n}(S_{t_i} - K)_+, \qquad t_i \leq T,$$

$$P = \left(\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - K\right)_+, \qquad t_i \leq T,$$

$$P = \left(K - \frac{1}{n}\sum_{i=1}^{n} S_{t_i}\right)_+, \qquad t_i \leq T,$$

...etc

Bermudan Options They are American options that can be exercised only at prescribed discrete future times.

Fixed Income Derivatives

Bonds They pay a fixed amount (e.g., \$1) at a future time. They are sold at a discount; their price determines interest rates. They usually pay coupons every few months or every year.

Bond Options Bonds can be bought or sold any time before they expire. Their price will fluctuate. As a consequence, they can be used as financial underlying for options. They are quite similar to equity, except for the fact that at the time of expiry of the bond, options make no sense. This in fact have very important implications.

Caps They are contracts that offer protection against time dependent interest rates rising over a certain ceiling, by paying the corresponding exceeding interest on a fixed notional.

Floors They charge the corresponding missing interest on a fixed notional.

They have negative value.

Collars A combination of a cap and a floor. By setting the ceiling and floor appropriately, they can be issued for free.

Swaps They exploit the different interest rates that different parties will be charged for fixed and floating rate loans; a swap is a contract that exchanges future payments at fixed and floating rates.

Swaptions When a swaps is viewed as an underlying, options are issued on them.

Cross Currency swaps Same as swaps, but the exchange is between payments in two currencies.

Many other financial instruments are available for trade. Most of the time, they are designed with the objective of removing risk from uncertain future situations. They also offer risky speculative alternatives.

Continuous time pricing

We think of infinitesimal time intervals dt.

Brownian motion moves up or down with probability $\frac{1}{2}$, by an amount of $\sqrt{dt}$:

$$dW_t = \pm\sqrt{dt}, \qquad \mathbb{E}(dW_t) = 0.$$

It is distributed at time t according to

$$P(x,t) \;=\; \frac{1}{\sqrt{2\pi t}}\,\exp\left(-\frac{x^2}{2t}\right).$$

Infinitesimal stock movements will be

$$dS_t = S_t \cdot (\mu\, dt + \sigma\, dW_t)\,.$$

Ito's Lemma says that

$$d_t f(S_t,t) = \partial_S f(S_t,t)\, dS_t + \partial_t f(S_t,t)\, dt + \tfrac{1}{2}\sigma^2\, S_t^2\, \partial_S^2 f(S_t,t)\, dt.$$

Note that

$$d\log S_t = (\mu - \tfrac{1}{2}\sigma^2)\, dt + \sigma\, dW_t.$$

The Black-Scholes Theory

Assume an option has price $f(t, S)$, at any given point in time, conditional on any possible value $S_t = S$ of the underlying at time t, which is therefore assumed to be known.

Let's set up the following arbitrage free argument:

At time t, build a portfolio Π consisting of

$$a = -\partial_S f(S, t)$$

units of stock, and the option.

Using Ito's formula,

$$\begin{aligned} d_t\Pi &= d_t f + a\, dS \\ &= \left(\tfrac{1}{2}\sigma^2\, S^2\, \partial_S^2 f + \partial_t f\right)\, dt + \partial_S f\, dS + a\, dS \\ &= \left(\tfrac{1}{2}\sigma^2\, S^2\, \partial_S^2 f + \partial_t f\, dt\right). \end{aligned}$$

This is a risk-free investment. Hence, it must earn risk-free interest and we obtain:

$$\begin{cases} \dfrac{\partial f}{\partial t} \;=\; -\dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 f}{\partial S^2} \;-\; r\,S\,\dfrac{\partial f}{\partial S} \;+\; r\,f, \\ f(S,T) = f_0(S). \end{cases}$$

It is a backward parabolic equation.

The solution is given by

$$f(S,t) \;=\; e^{-r(T-t)} \int_{-\infty}^{\infty} f_0\left(S e^{(r-\frac{\sigma^2}{2})(T-t)\,+\,x} \right) P_\sigma(x, T-t)\, dx$$

with

$$P_\sigma(x,t) \;=\; \frac{1}{\sqrt{2\pi t\sigma^2}}\; \exp\left(-\; \frac{x^2}{2t\sigma^2} \right).$$

The Black-Scholes Formulas

The price of a European call option on a stock S, valued today at S_0, maturing at time T with strike K, (constant) volatility σ and interest rate r is given by

$$V(t, K, \sigma, r) = S_0 \cdot N(d_1) - K \cdot e^{-r(T-t)}\, N(d_2),$$

where $N(d)$ is the cumulative normal

$$N(d) = \int_{-\infty}^{d} e^{-x^2/2}\, \frac{dx}{\sqrt{2\pi}},$$

and

$$d_1 = \frac{\ln(S_0/K) + (r + \frac{1}{2}\sigma^2)\,(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln(S_0/K) + (r - \frac{1}{2}\sigma^2)\,(T-t)}{\sigma\sqrt{T-t}}.$$