Option Pricing

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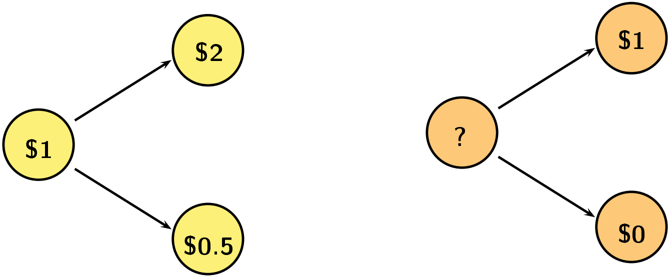
Example

Example-Pricing a Call Option

- In the following example we will price a call option.
- For the moment ignore interest rates.
- A call option has the payoff function:

$$f_0(S) = (S - \$1)_+.$$



Example-Pricing a Call Option Continued...

- Can we assume $p = 50\%$. Is $V = 0.50$?... NO!
- The actual price of the option is

$$V = 1/3\$$$

but why?

- The price of the option is $V = 1/3\$$, how do we get this number?
- Let's construct a replicating portfolio:
  1. We borrow $\$1/3$
  2. We buy $\$2/3$ of S,

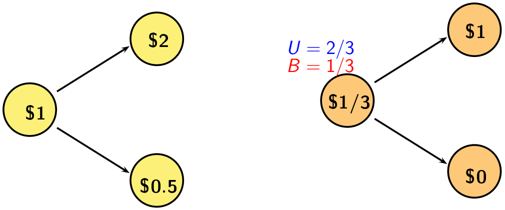
  then we will exactly cover (or hedge) our payoff.
- Since it costs $\$1/3$ to purchase this portfolio, the price should be the same.

Discounted Values

Time is money.

- Assume the existence of a bond with constant interest rate r.
- We build the following portfolio Π:

$$\Pi = \left(\frac{2}{3}\right) \text{ Stock units } + \left(-\frac{1}{3}\right) \text{Bonds}$$



Time is Money Continued...

- No matter what p is, absence of arbitrage implies:

$$\begin{aligned}\text{Option Price} &= \frac{2}{3} - \frac{1}{3}B \\ &= \frac{2}{3} - \frac{1}{3}e^{-rT}.\end{aligned}$$

where T is the time to expiration and r is the (constant) interest rate.

Implied Probabilities

- We can still achieve:

$$\begin{aligned}\text{Option Price} &= \mathbb{E}\left(e^{-rT}\, f_0\right) \\ &= p\, e^{-rT}\end{aligned}$$

by selecting

$$p = \frac{2}{3}\, e^{rT} - \frac{1}{3}$$

- In other words, we can construct a probability measure $\mathbb{P}$ for the stock process, such that

$$\text{Option Price} = \mathbb{E}_{\mathbb{P}}\left(B_T^{-1}\, f_0\right).$$

Implied Probabilites Continued...

- More generally, if we define the (arbitrage-free) price to equal the discounted pay-off

$$V = B_T^{-1} f_0,$$

then, there exists a measure $\mathbb{P}$ under which V is a martingale: its value today is its expected future value.

Markets

Implied Market Data

Example (Implied Market Data)

Assume the call option in the previous example is sold for $0.50.

$$\frac{2}{3} - \frac{1}{3}\, e^{-r} = 0.5.$$

Hence, the risk-free rate must equal

$$r = -\ln 2.$$

Incomplete Markets

Example (Incomplete Markets)

Assume the stock valued at \$1 today, can be worth

$$S = \begin{cases} \$2 \\ \$1 \\ \$0.5 \end{cases}$$

after a year. How can we price the call option with strike 1?.

Solution (Incomplete Markets)

Two possibilities:

1. *Another derivative price is known*
2. *We can re-balance our hedge once before maturity.*

Pricing Theory

Binomial Pricing Theory

- Pay-off matrix:

$$D = \begin{bmatrix} 1 & 2 \\ 1 & 0.5 \end{bmatrix}$$

- The replicating strategy is given by:

$$D \cdot \begin{bmatrix} x = \text{bond units} \\ y = \text{stock units} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- Cost vector:

$$q = (0.9, 1).$$

- Price:

$$\begin{aligned} \text{Price} &= q \cdot x \\ &= q \cdot D^{-1} \cdot (\text{Pay-off vector}) \\ &= \text{Expected Pay-off} \end{aligned}$$

Multiperiod Pricing

- Assume a call option with strike \$75 can be priced as follows (r=0):
- So its value today is \$15.
- This is the arbitrage-free price. Implied probabilities can be obtained as usual.

Pricing Theory (One Period)

- Implied probabilities can be obtained, not only from prices dictated by arbitrage arguments, but also from market prices.
- The implications of this is that a probabilistic approach to pricing is more useful than might have seemed from the considerations above.

Defintions: Pricing Theory I

- In this section we assume there is a probability space for the payoffs of N securities available for trading,
  - A security is characterized by its cost now, and its payoff after one unit of time.
  - The cost of the i-th security, $i = 1, ..., N$, is q_i.
  - The payoff is given by the random variable $D_i(\omega)$.
  - The expected payoff of a security is $E(D_i(\omega))$.
  - A portfolio is a vector $\theta = (\theta_1, \ldots \theta_N) \in \mathbb{R}^N$, which represents the holdings of each security. θ_i can be positive or negative.
    1. If θ_i is positive, our position is said to be long.
    2. If θ_i is negative, our position is said to be short.
  - The payoff of the portfolio θ is $\theta \cdot D(\omega)$.
  - A market is said complete if

$$\text{Span}\{\theta \cdot D(\omega)\ ,\ \theta \in \mathbb{R}^N\} \ = \ L^2(\mu).$$

and markets are usually assumed to be complete. In a complete market,for any payoff there is a portfolio with that payoff.

Definitions: Pricing Theory II

- Continuing...
  - The cost of a portfolio θ is $q \cdot \theta$.
  - If a portfolio has nonzero cost, i.e. $q \cdot \theta \neq 0$, one defines its return to be

$$R_\theta(\omega) = \frac{\theta \cdot D(\omega)}{q \cdot \theta}.$$

Efficient Market Hypothesis

- In a real market, there are hedgers (people trying to minimize risk), speculators (people trying to maximize return) and arbitrageurs (people detecting market inefficiencies).
- We say that there is an **arbitrage opportunity** if there is a portfolio θ such that

$$q \cdot \theta \leq 0, \text{ and } D \cdot \theta \geq 0 \quad a.e.,$$

and $D \cdot \theta > 0$ with non-zero probability.

Efficient Market Hypothesis (EMH)

The **Efficient Market Hypothesis (EMH)** states that there is no arbitrage and there are no transaction costs.

Riesz representation

Theorem (Riesz representation)

If p_i are linear functionals of the payoffs $L^2(\mu)$, then there exists a random variable $\pi(\omega)$ such that

$$p \cdot \theta \;=\; E(\theta\pi \cdot D), \qquad \text{all } \theta \in \mathbb{R}^N. \tag{1}$$

If markets are complete, π is unique. If there are no arbitrage opportunities, $\pi > 0$.

State-Price Deflator and Riskless

- In the case that we consider the cost as that linear functional, we obtain that the cost of a portfolio is the expectation of its payoff with probabilistic weight $\pi(\omega)$, which is called the state-price deflator.
- The name comes from the fact that

$$E(R_\theta \pi) \;=\; 1 \tag{2}$$

  for all portfolios θ.
- We always assume that $D_0(\omega)$ is constant for all $\omega \in \Omega$. This is a savings account.
- A riskless bond is a portfolio θ_0 of constant payoff i.e. such that $\theta \cdot D(\omega) \;=\; \theta \cdot D(\omega')$ for all $\omega, \omega' \in \Omega$.
  - It always exists: put $\theta = (1, 0, \ldots, 0)$.
  - Then from (2) we find

$$R^0 \;\equiv\; E(R_{\theta_0}) \;=\; \frac{1}{E(\pi)}.$$

Riskless Interest Rate

- The riskless interest rate is given by

$$r = -\frac{1}{T} \ln \mathbb{E} \left(R_{\theta_0} \right).$$

Theorem: Price Deflator and Arbitrage

Theorem

Price Deflator and Arbitrage A price deflator exists if and only if there is no arbitrage.

Proof: Price Deflator $\rightarrow$ No Arbitrage

Proof.

1. If a price deflator exists, then $\Pi(0) = E(\pi\,\Pi(T))$.
2. Since π is positive as a functional on L, if $\Pi(T) > 0$ then $\Pi(0) > 0$ and if $\Pi(T) = 0$ then $\Pi(0) = 0$.
3. On the other hand, let us suppose that there is no arbitrage. Let us consider the price-payoff vector space $V = \mathbb{R} \times L$.
   - The (cost , pay-off) hyperplane is

$$M = \{(-\theta \cdot q, \theta \cdot P) : \theta \in \mathbb{R}^N\}.$$

   - The cone $K = \mathbb{R}_+ \times L_+$ contains all securities of non-positive price and non-negative payoff.
   - If there is no arbitrage, then $K \cap M = \{0\}$.

$\square$

Proof: Price Deflator $\rightarrow$ No Arbitrage Continued...

Proof.

Continuing...

$\rightarrow$ By the separating hyperplane theorem, there exists a functional

$$F : V \rightarrow \mathbb{R}$$

such that $F(x) = 0$ for all $x \in M$ and $F(x) > 0$ for all $x \in K \setminus \{0\}$.

$\rightarrow$ The Riesz representation of $F(x)$ is

$$F(v, c) \; = \; \alpha \, v \; + \; E(\phi \cdot c).$$

$\rightarrow$ In terms of α and ϕ,we have that

$$-\alpha \, \theta \cdot q \; + \; \mathbb{E}(\phi \cdot (\theta \cdot P)) \; = \; 0$$

for all $\theta \in \mathbb{R}^N$.

$\therefore$ Hence $\pi \; \equiv \; \dfrac{\phi}{\alpha}$ is a price deflator.

$\square$