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Model Building in Mathematical Programming

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4

Structured linear programming models

4.1 Multiple plant, product and period models

The purpose of this section is to show how large linear programming (LP) models can arise through the combining of smaller models. Almost all very large models arise in this way. Such models prove to be more powerful as decision making tools than the submodels from which they are constructed. In order to illustrate how a multiplant model can arise in this way, we take a very small illustrative example.

Example 4.1: A Multiplant Model

A company operates in two factories, A and B. Each factory makes two products, *standard* and *deluxe*. A unit of *standard* gives a profit contribution of £10, while a unit of *deluxe* gives a profit contribution of £15.

Each factory uses two processes, grinding and polishing, for producing its products. Factory A has a grinding capacity of 80 hours per week and polishing capacity of 60 hours per week. For factory B, these capacities are 60 and 75 hours per week, respectively.

The grinding and polishing times in hours for a unit of each type of product in each factory are given in the table below.

	Factory A		Factory B	
	Standard	Deluxe	Standard	Deluxe
Grinding	4	2	5	3
Polishing	2	5	5	6

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It is possible, for example, that factory B has older machines than factory A, resulting in higher unit processing times.

In addition, each unit of each product uses 4 kg of a raw material, which we refer to as *raw*. The company has 120 kg of *raw* available per week. To start with, we will assume that factory A is allocated 75 kg of *raw* per week and factory B the remaining 45 kg per week.

Each factory can build a very simple LP model to maximize its profit contribution. This is an obvious example of the product mix application of LP mentioned in Section 1.2. The following are the resultant models:

Factory A's Model

$$\begin{array}{lll}
\text{Maximize} & \text{Profit A} & 10x_1 + 15x_2 \\
\text{subject to} & \text{Raw A} & 4x_1 + 4x_2 \leq 75, \\
 & \text{Grinding A} & 4x_1 + 2x_2 \leq 80, \\
 & \text{Polishing A} & 2x_1 + 5x_2 \leq 60, \\
 & & x_1, x_2 \geq 0,
\end{array}$$

where x_1 is the quantity of standard to be produced in A and x_2 is the quantity of deluxe to be produced in A.

Factory B's Model

$$\begin{array}{lll}
\text{Maximize} & \text{Profit B} & 10x_3 + 15x_4, \\
\text{subject to} & \text{Raw B} & 4x_3 + 4x_4 \leq 45, \\
 & \text{Grinding B} & 5x_3 + 3x_4 \leq 60, \\
 & \text{Polishing B} & 5x_3 + 6x_4 \leq 75, \\
 & & x_3, x_4 \geq 0,
\end{array}$$

where x_3 is the quantity of standard to be produced in B and x_4 is the quantity of deluxe to be produced in B.

These two models can easily be solved graphically. Our purpose is not, however, to concentrate on the mechanics of solving these individual models. We do, however, give the optimal solutions below as these will be discussed later.

Optimal Solution to Factory A's Model

Profit is £225 obtained from making 11.25 of standard and 7.5 of deluxe. There is a surplus grinding capacity of 20 hours.

Optimal Solution to Factory B's Model

Profit is £168.75 obtained from making 11.25 deluxe. There is a surplus grinding capacity of 26.25 hours, and a surplus polishing capacity of 7.5 hours.

Suppose now that a company model is built in order to maximize total profit. We will assume that the factories remain distinct and geographically separated. We will, however, no longer allocate 75 kg of raw to A and 45 kg to B. Instead, we will allow the model to decide this allocation. There will now be a single raw material constraint limiting the company to 120 kg per week. The resultant model is as follows:

$$
\begin{array}{lll}
\text{Maximize} & \text{Profit} & 10x_1 + 15x_2 + 10x_3 + 15x_4 \\
\quad\text{subject to} & \text{Raw} & 4x_1 + 4x_2 + 4x_3 + 4x_4 \le 120, \\
 & \text{Grinding A} & 4x_1 + 2x_2 \le 80, \\
 & \text{Polishing A} & 2x_1 + 5x_2 \le 60, \\
 & \text{Grinding B} & 5x_3 + 3x_4 \le 60, \\
 & \text{Polishing B} & 5x_3 + 6x_4 \le 75, \\
 & & x_1, x_2, x_3, x_4 \ge 0.
\end{array}
$$

The Company Model

The fact that the constraints raw A and raw B of the factory models have been combined into a single constraint for the company model is of crucial significance. We are now asking the model to split the 120 kg of raw optimally between A and B rather than making an arbitrary allocation ourselves. As a consequence, we would expect a more efficient split resulting in a greater overall company profit. This is borne out by the optimal solution.

Optimal Solution to the Company Model

Total profit is £404.15, obtained from making 9.17 of standard in A, 8.33 of deluxe in A and 12.5 of deluxe in B. There is a surplus grinding capacity in A of 26.67 hours, and a surplus grinding capacity in B of 22.5 hours.

A number of points are worth noting in comparing this solution with those for factories A and B individually:

1. The total profit is £404.14, which is greater than the combined profit £393.75 from A and B acting independently.
2. Factory A only contributes £216.65 to the new total profit, whereas it produced a profit of £225 before. Factory B, however, now contributes £187.5 to total profit, whereas it only produced £168.75 before.
3. Factory A now uses 70 kg of raw and factory B uses 50 kg.

It is clear that the company model has biased production more toward factory B than before. This has been done by allocating B 50 kg of raw instead of 45 kg and so depriving A of 5 kg. If it had been possible to decide this 70/50 split before, it would not have been necessary to build a company model. This argument also applies to much larger, more realistic, multiplant models. Normally, however, there will be a number of scarce resources that must be shared between plants rather than the single resource *raw*, which we consider here. An optimal split would have to be found for each of these resources. Determining such splits would obviously be complex. The needs of each plant have to be balanced against how efficiently they use the scarce resources. In our example, factory B's older machinery results in it being allocated less of raw than A. To start with, however, our 75/45 split was over biased in A's favour.

The above example is intended to show how a multiplant model can arise. It is a method of using LP to cope with allocation problems *between* plants as well as help with decision making *within* plants. The model that we built was a very simple example of a common sort of structure, which arises in multiplant models. This is known as a *block angular* structure. If we detach the coefficients in the company model and present the problem in a diagrammatic form, we obtain Figure 4.1.

10	15	10	15		
4	4	4	4	≤	120
4	2			≤	80
2	5			≤	60
		5	3	≤	60
		5	6	≤	75

Figure 4.1

The first two rows are known as *common rows*. Obviously, one of the common rows will always be the objective row. The two diagonally placed blocks of coefficients are known as *submodels*. For a more general problem with a number of shared resources and *n* plants, we would obtain the general block angular structure shown in Figure 4.2.

The cost of extraction increases with depth. At successive levels, the cost of extracting a block is as follows:

Level 1	£3000
Level 2	£6000
Level 3	£8000
Level 4	£10 000

The revenue obtained from a '100% value block' would be £200 000. For each block here, the revenue is proportional to ore value.

Build a model to help decide the best blocks to extract. The objective is to maximise revenue−cost.

The larger version of this problem arose with open-cast iron mining in South Africa.

12.15 Tariff rates (power generation)

A number of power stations are committed to meeting the following electricity load demands over a day:

12 p.m. to 6 a.m.	15 000 MW
6 a.m. to 9 a.m.	30 000 MW
9 a.m. to 3 p.m.	25 000 MW
3 p.m. to 6 p.m.	40 000 MW
6 p.m. to 12 p.m.	27 000 MW

There are three types of generating unit available: 12 of type 1, 10 of type 2 and five of type 3. Each generator has to work between a minimum and a maximum level. There is an hourly cost of running each generator at minimum level. In addition, there is an extra hourly cost for each megawatt at which a unit is operated above the minimum level. Starting up a generator also involves a cost. All this information is given in Table 12.6 (with costs in £).

In addition to meeting the estimated load demands there must be sufficient generators working at any time to make it possible to meet an increase in load of up to 15%. This increase would have to be accomplished by adjusting the output of generators already operating within their permitted limits.

Table 12.6

	Minimum level	Maximum level	Cost per hour at minimum	Cost per hour per megawatt above minimum	Cost
Type 1	850 MW	2000 MW	1000	2	2000
Type 2	1250 MW	1750 MW	2600	1.30	1000
Type 3	1500 MW	4000 MW	3000	3	500

Which generators should be working in which periods of the day to minimise total cost?

What is the marginal cost of production of electricity in each period of the day; that is, what tariffs should be charged?

What would be the saving of lowering the 15% reserve output guarantee; that is, what does this security of supply guarantee cost?

12.16 Hydro power

This is an extension of the Tariff Rates (Power Generation) problem of Section 12.15. In addition to the thermal generators, a reservoir powers two hydro generators: one of type A and one of type B. When a hydro generator is running, it operates at a fixed level and the depth of the reservoir decreases. The costs associated with each hydro generator are a fixed start-up cost and a running cost per hour. The characteristics of each type of generator are shown in Table 12.7.

For environmental reasons, the reservoir must be maintained at a depth of between 15 and 20 m. Also, at midnight each night, the reservoir must be 16 m deep. Thermal generators can be used to pump water into the reservoir. To increase the level of the reservoir by 1 m, it requires 3000 MWh of electricity. You may assume that rainfall does not affect the reservoir level.

At any time, it must be possible to meet an increase in demand for electricity of up to 15%. This can be achieved by any combination of the following: switching on a hydro generator (even if this would cause the reservoir depth to fall below 15 m); using the output of a thermal generator, which is used for pumping water into the reservoir; and increasing the operating level of a thermal generator to its maximum. Thermal generators cannot be switched on instantaneously to meet increased demand (although hydro generators can be).