Computational Medicine
Computational Anatomy
Homework 2

Seraj Grimes, Kavita Krishnan, Chantelle Lim, Joshua Popp, Tony Wei

September 2019

1 Invertible Matrices

Consider a vector $x = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, and a transformation $\varphi(x) = Ax = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, with Jacobian matrix $D\phi(x) = \begin{pmatrix} \frac{\partial \varphi_1(x)}{\partial x} & \frac{\partial \varphi_1(x)}{\partial y} \\ \frac{\partial \varphi_2(x)}{\partial x} & \frac{\partial \varphi_2(x)}{\partial y} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = A$.

Show that if the matrix A is invertible, then the transformation φ is 1 to 1 and onto. That is:

1.1

Show that $\varphi(x) = \varphi(x') \implies x = x'$

Let I be the 2x2 identity matrix.
$\varphi(x) = \varphi(x') = Ax = Ax'$
If matrix A is invertible: $A^{-1}Ax = A^{-1}Ax' = Ix = Ix'$
Then: $x = x'$

1.2

Show that $\forall y \in \mathbb{R}^2$, $\exists x \in \mathbb{R}^2$ such that $\varphi(x) = y$.

$\varphi(x) : \mathbb{R}^{2x1} \implies \mathbb{R}^{2x1}$
$y = \varphi(x) = Ax$
$A^{-1}Ax = A^{-1}y = Ix = A^{-1}y$
$x = A^{-1}y$
$\varphi(A^{-1}y) = Ax = AA^{-1}y = y$
Then: $\varphi(x) = y$

2 Jacobian Chain Rule

2.1 Scalar transformations

Consider a point $x \in \mathbb{R}$ and the transformation $f : x \mapsto f(x) \in \mathbb{R}$. Let the inverse transformation be $f^{-1} : x \mapsto f^{-1}(x) \in \mathbb{R}$

Show that $\frac{d}{dx} f^{-1}(x) = \frac{1}{\frac{d}{dx} f|_{f^{-1}(x)}}$
Hint: prove and use the chain rule $\frac{d}{dx} f \circ f^{-1}(x) = \frac{d}{dx} f|_{f^{-1}(x)} \frac{d}{dx} f^{-1}(x)$.

Let

$$g(x) = f^{-1}x$$

and

$$\frac{d}{dx} f(x) = f'(x)$$

By the definition of an inverse function:

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= x \end{aligned}$$

Then:

$$\begin{aligned} \frac{d}{dx} f \circ g(x) &= \frac{d}{dx} x \\ &= 1 \end{aligned}$$

Using the chain rule:

$$\begin{aligned} \frac{d}{dx} f \circ g(x) &= \frac{d}{dx} f(g(x)) \\ &= f'(g(x)) g'(x) \\ &= \frac{d}{dx} f|_{f^{-1}(x)} \frac{d}{dx} f^{-1}(x) \\ &= 1 \end{aligned}$$

Then it can be shown:

$$\begin{aligned} g'(x) &= \frac{d}{dx} f^{-1}(x) \\ &= \frac{1}{f'(g(x))} \\ &= \frac{1}{\frac{d}{dx} f|_{f^{-1}(x)}} \end{aligned}$$

Proof of chain rule via definition of the derivative:

$$
\begin{aligned}
(f \circ g)'(a) &= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \\
&= \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a} \frac{g(x) - g(a)}{g(x) - g(a)} \\
&= \lim_{x \to a} \frac{f(g(x)) - f(g(a)}{g(x) - g(a)} \lim_{x \to a} \frac{g(x) - g(a)}{x - a} \\
&= f'(g(a))g'(a) \\
&= \frac{d}{dx} f(f^{-1}(a)) \frac{d}{dx} f^{-1}(a) \\
&= \frac{d}{dx} f|_{f^{-1}(a)} \frac{d}{dx} f^{-1}(a)
\end{aligned}
$$

2.2 Vector transformations

Consider a point $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, and the transformation $\varphi : x \mapsto \varphi(x) \in \mathbb{R}^3$. Let the inverse transformation be defined by $\varphi^{-1} : x \mapsto \varphi^{-1}(x) \in \mathbb{R}^3$.

Show that $D\varphi^{-1}(x) = \left(D\varphi|_{\varphi^{-1}(x)} \right)^{-1}$.
Hint: prove and use the chain rule $D(\varphi \circ \varphi^{-1}(x) = D\varphi|_{\varphi^{-1}(x)} D\varphi^{-1}(x)$.
The Jacobian matrix is defined as $D\varphi(x) = \begin{pmatrix} \frac{\partial \varphi_1(x)}{\partial x} & \frac{\partial \varphi_1(x)}{\partial y} & \frac{\partial \varphi_1(x)}{\partial z} \\ \frac{\partial \varphi_2(x)}{\partial x} & \frac{\partial \varphi_2(x)}{\partial y} & \frac{\partial \varphi_2(x)}{\partial z} \\ \frac{\partial \varphi_3(x)}{\partial x} & \frac{\partial \varphi_3(x)}{\partial y} & \frac{\partial \varphi_3(x)}{\partial z} \end{pmatrix}$

Let I be the 3x3 identity matrix. By the definition of an inverse transformation:

$$
\begin{aligned}
\varphi \circ \varphi^{-1}(x) &= \varphi(\varphi^{-1}(x)) \\
&= x
\end{aligned}
$$

Applying the Jacobian with the chain rule results in:

$$
\begin{aligned}
D(\varphi \circ \varphi^{-1}(x)) &= D(\varphi(\varphi^{-1}(x))) \\
&= D\varphi|_{\varphi^{-1}(x)} D\varphi^{-1}(x) \\
&= Dx \\
&= I
\end{aligned}
$$

Solving for $D\varphi^{-1}(x)$:

$$\begin{aligned}(D\varphi|_{\varphi^{-1}(x)})^{-1} D\varphi|_{\varphi^{-1}(x)} D\varphi^{-1}(x) &= (D\varphi|_{\varphi^{-1}(x)})^{-1} I \\ &= D\varphi^{-1}(x) \\ &= (D\varphi|_{\varphi^{-1}(x)})^{-1}\end{aligned}$$

3 Splines on the real line

In this problem, we will be considering one-dimensional "images" that are simply a series of Gaussians centered at "landmark points," t_i, given by $\sum_{i=1} \frac{1}{2\pi} \exp\left(-\frac{1}{2}(t-t_i)^2\right)$. The atlas image, $I_{atlas}(t)$ has landmarks $\{t_1 = 20, t_2 = 40, t_3 = 60, t_4 = 80\}$. The target image (and its landmarks $\{t'_1, t'_2, t'_3, t'_4\}$), is simply a leftward shift of the atlas $I_{target}(t) = I_{atlas}(t+s)$ for some s.

Our goal is to match the landmarks between the two images using splines, in particular, the spline associated with the Green's kernel $k(t-t_0) = \frac{1}{2a} \exp\left(-a * |t-t_0|\right)$, where $a \in \mathbb{R}^+$.

Let the transformation vector field be given by $v(t) = \sum_{j=1}^{n} k(t-t_j)p_j$ where the p_j's are chosen such that $v(t_i) = t'_i - t_i$.

3.1

Plot the images $I_{target}(t)$, and $I_{atlas}(t)$ for $s = 2$. Note that if you are plotting in matlab, you will need to sample the variable `t` at a high resolution, such as `t = 0:0.01:100` (in Python, `t = numpy.linspace(0, 100, num=10000)`).

Please note that all calculations were performed in the code directly. All code is available in the file ICM_hw2_q3.m which is attached with this homework. The code for each section has been headed with the section number and contains comments to describe the calculations occurring.

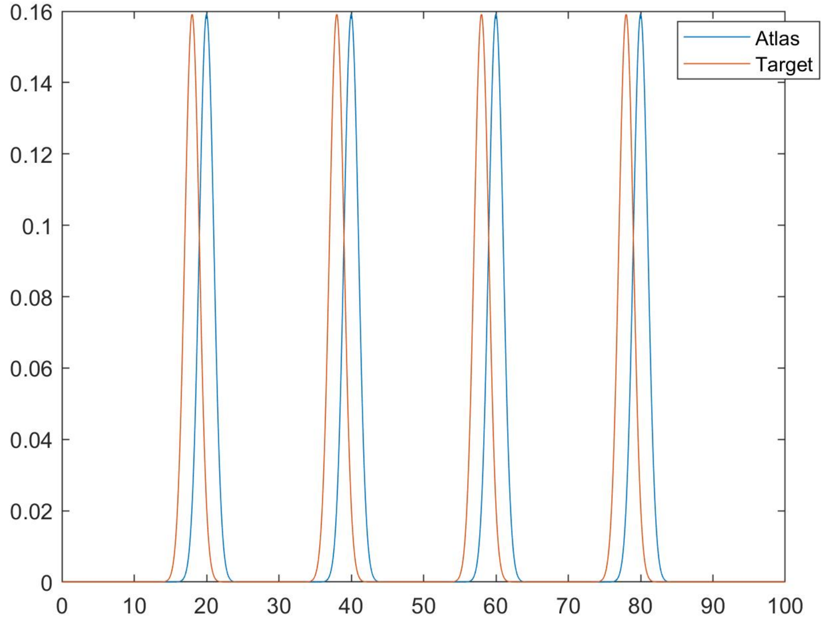


Figure 1: s = 2, a = 0.5

3.2

Calculate the required p_j's for $s = 1$ and $a = 0.5$.

Plot the transformed target image, $I_{target}(x + v(x))$, on top of $I_{target}(t)$ and $I_{atlas}(t)$. The transformed target should take the same values as the atlas at all $t_i s$. Also plot $v(t)$ vs t. Would you say the landmark matching was successful?

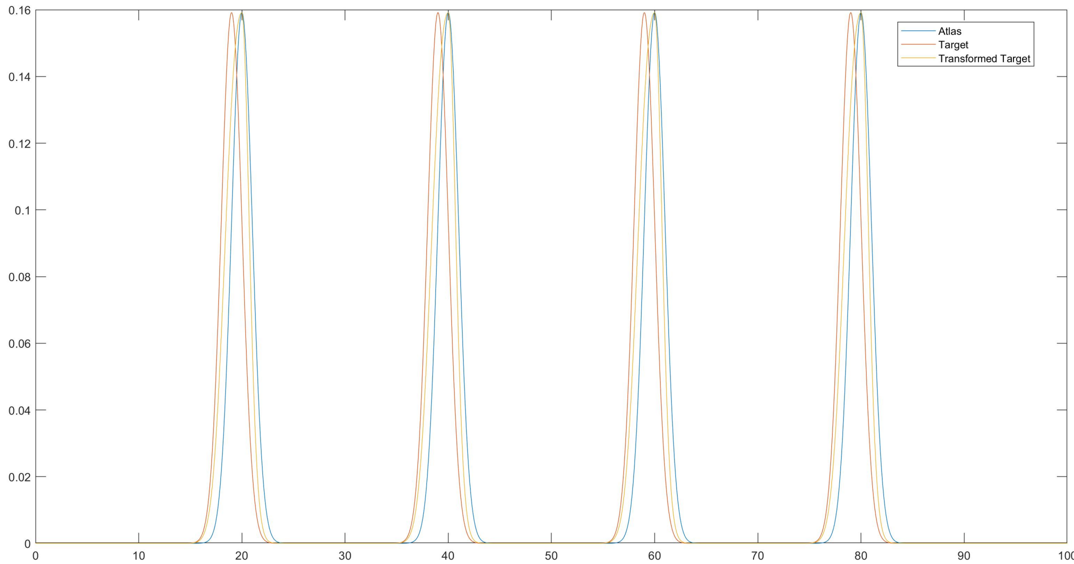


Figure 2: s = 1, a = 0.5

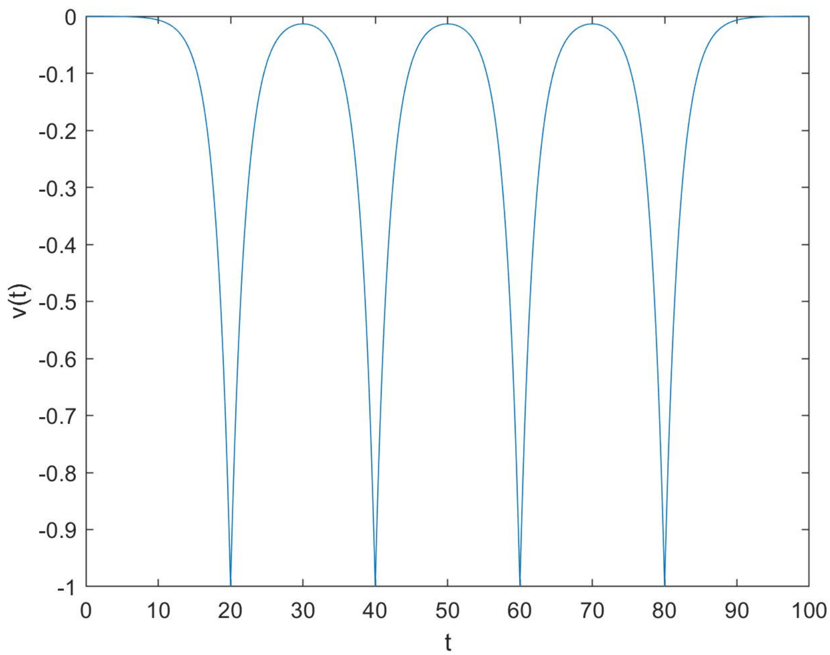


Figure 3: v(t) vs. v graph

The transformed target took the same values as the atlas at all t_is. Although there were slight deviations on both sides of the curve, the landmark matching can be considered successful. Any alterations to the rest of the image can be attributed to the minimization of energy required to change the landmarks.

3.3

Calculate the required p_j's for $s = 10$ and $a = 0.5$.

Plot the transformed target image, $I_{target}(x + v(x))$, on top of $I_{target}(t)$ and $I_{atlas}(t)$. Also plot $v(t)$ vs t. Does the transformed target still resemble the original target?

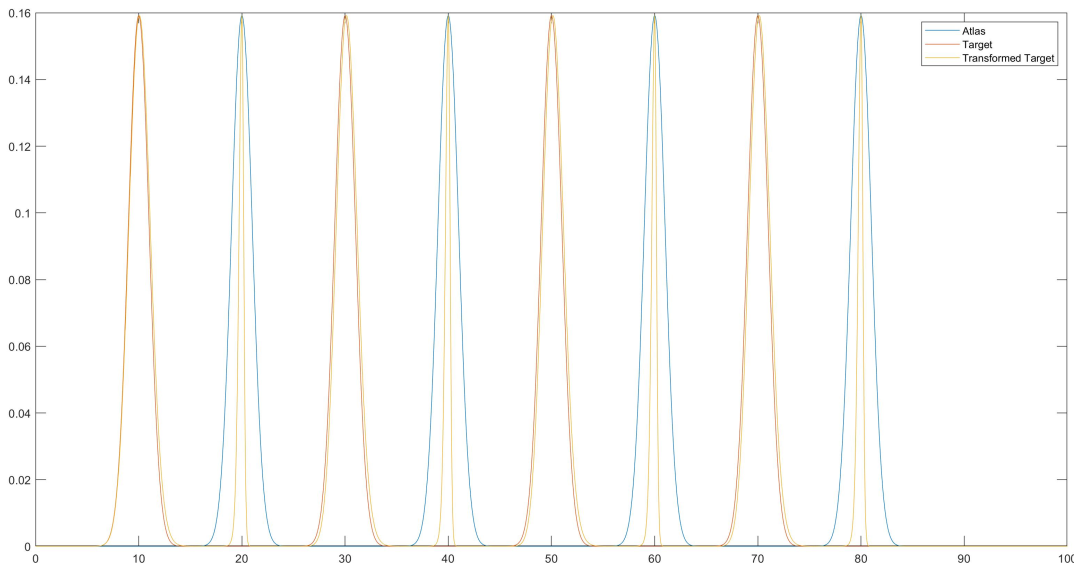


Figure 4: s = 10, a = 0.5

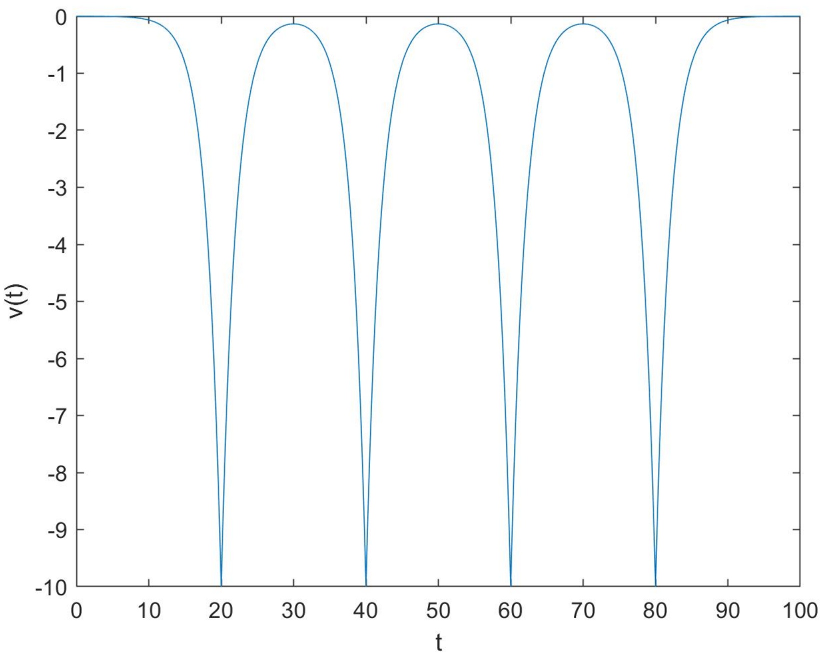


Figure 5: v(t) vs. v graph

The transformed target no longer resembles the original target. Rather than transforming the peaks of the target to resemble the atlas, we see that there are duplicate peaks such that there is a peak in the location for both the atlas and the target. We believe this is due to the nature of the Gaussian curves of the target landmarks in addition to the increase in the shift. To test this, we completed the transformation for a variety of s values and found that the transformed template begins to show multiple maximums for each peak.

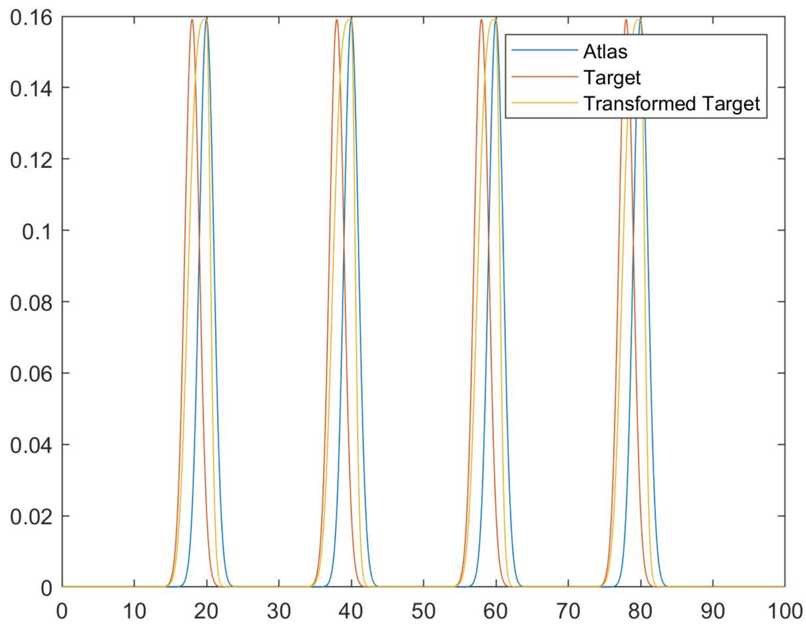


Figure 6: s = 2, a = 0.5

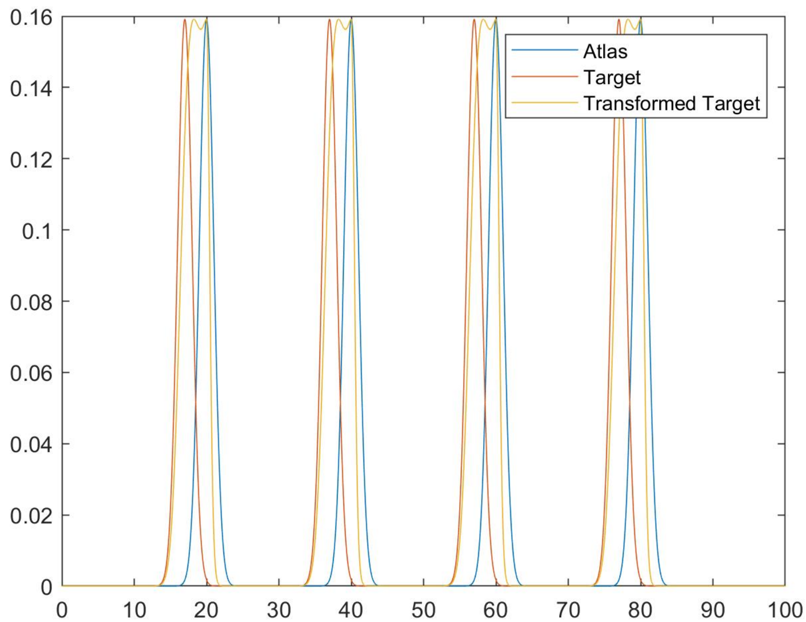


Figure 7: s = 3, a = 0.5

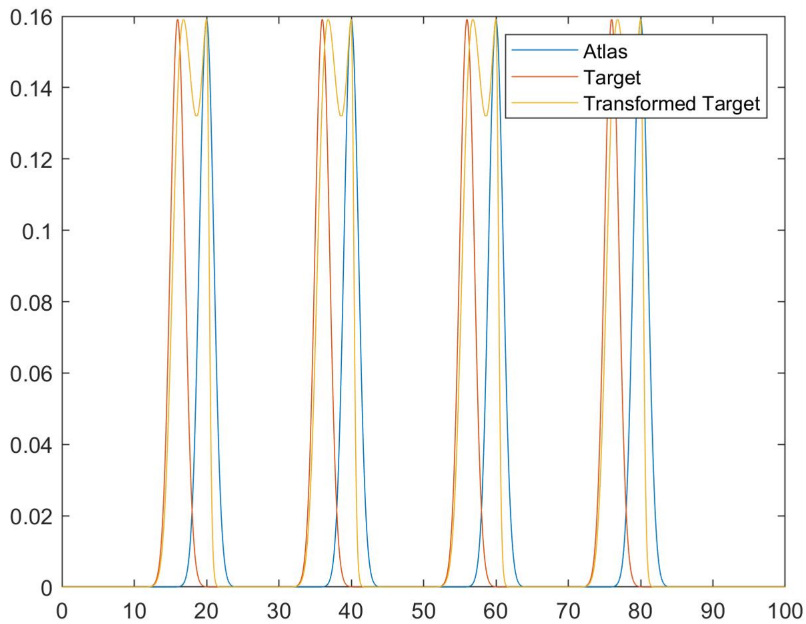


Figure 8: s = 4, a = 0.5

3.4

What is a qualitative difference between the $v(t)$ vs. t plots in Problems 3.2 and 3.3? Repeat problem 3.3 with $a = 0.1$, did things change?

The biggest difference between the two v(t) vs t plots is the scale of the v(t) overall. By performing the transformation with a = 0.1 and s = 10 we are able to fix the errors seen in part 3.3 with the transformations that had multiple peaks. Instead, we are able to show a match of the transformed landmarks that (reasonably) resembles the atlas. By implementing an a value of 0.05, the transformation is improved even further.

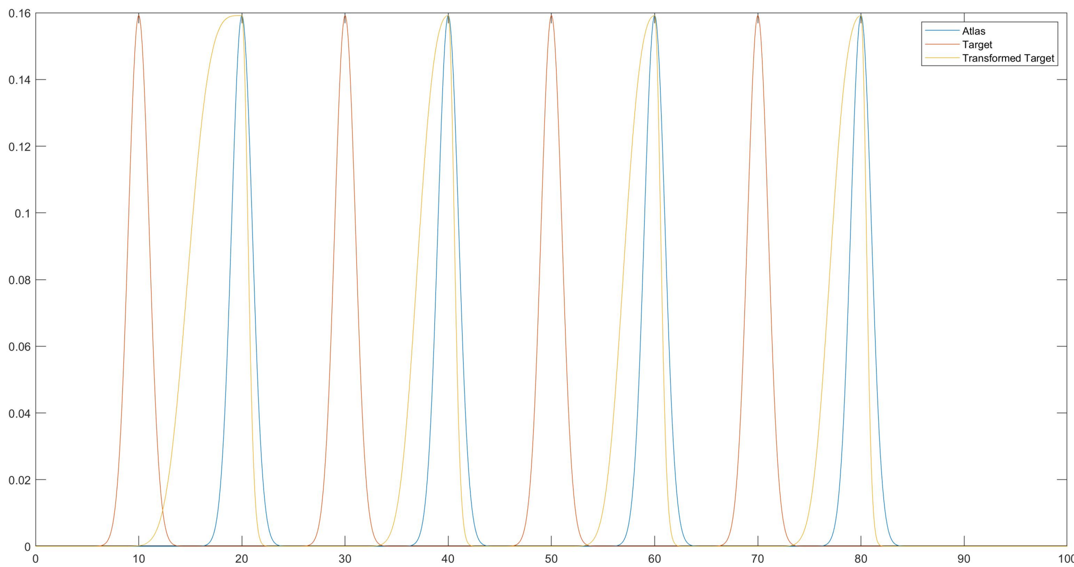


Figure 9: s = 10, a = 0.1

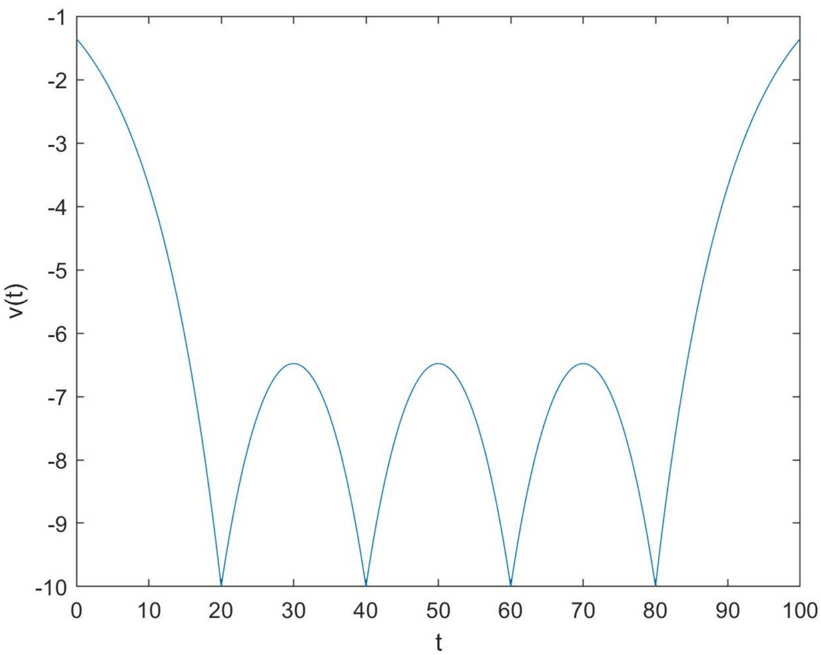


Figure 10: v(t) vs. v graph

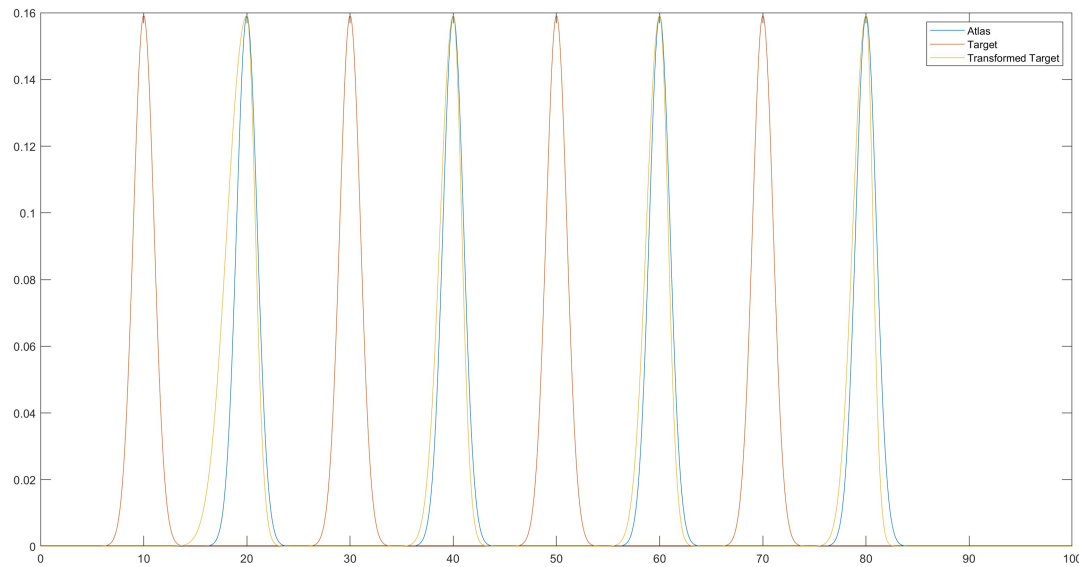


Figure 11: s = 10, a = 0.05

4 Inverses and compatibility

Consider matrices A, B and vector x. From linear algebra we know that $(B \circ A) \cdot x = B \cdot (A \cdot x)$.

In words, "If you transform x with the matrix $B \circ A$, you will get the same result as if you first transform x with A and then transform it with B."

This is called "compatibility" between the group of matrices, and their action on the vector x.

In computational anatomy we consider the action of transformations φ, ψ on images I. A transformation acts on an image according to $\varphi \cdot I = I \circ \varphi^{-1}$, and one transformation combines with another according to $\psi \circ \varphi(x) = \psi(\varphi(x))$.

Prove that $(\psi \circ \varphi) \cdot I = \psi \cdot (\varphi \cdot I)$. Show that this does not hold if the action of the transformation on an image did not include the inverse.

Hint: recall that $(\psi \circ \varphi)^{-1} = \varphi^{-1} \circ \psi^{-1}$.

Treating the composition of the two transformations as a transformation in itself

$$(\psi \circ \varphi) \cdot I = I \circ (\psi \circ \varphi)^{-1}$$

Using the hint, this becomes

$$(\psi \circ \varphi) \cdot I = I \circ (\varphi^{-1} \circ \psi^{-1})$$

The composition operation is associative, therefore

$$(\psi \circ \varphi) \cdot I = (I \circ \varphi^{-1}) \circ \psi^{-1}$$

From here, we can use the equation for the action of a transformation on an image (twice)

$$\begin{aligned}(\psi \circ \varphi) \cdot I &= (\varphi \cdot I) \circ \psi^{-1} \\ (\psi \circ \varphi) \cdot I &= \psi \cdot (\varphi \cdot I)\end{aligned}$$

If the action of the transformation on an image did not include the inverse (in other words, if $\varphi \cdot I = I \circ \varphi$), then we would have

$$(\psi \circ \varphi) \cdot I = I \circ (\psi \circ \varphi)$$

We can turn again to the associativity of composition

$$\begin{aligned}(\psi \circ \varphi) \cdot I &= (I \circ \psi) \circ \varphi \\ (\psi \circ \varphi) \cdot I &= (\psi \cdot I) \circ \varphi \\ (\psi \circ \varphi) \cdot I &= \varphi \cdot (\psi \cdot I)\end{aligned}$$

Note that matrix multiplication is not generally commutative, so this does **not** imply that $\psi \cdot (\varphi \cdot I) = (\psi \circ \varphi) \cdot I$, and our original identity therefore would not hold if the action of the transformation did not include the inverse.

5 Functions on the real line

5.1 Fourier transform of Convolution

The Fourier transform of a continuous function $f(t)$ is defined as $F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega t}dx$. The convolution of two functions $f(x)$, $g(x)$ is defined as $f(x) * g(x) = \int_{-\infty}^{\infty} f(\tau)g(x-\tau)d\tau$. Write the Fourier transform of $f(x) * g(x)$ in terms of the Fourier transforms of $f(x)$ and $g(x)$. Derive this result.

$$\begin{aligned}\mathcal{F}(f(x) * g(x)) &= F(\omega)G(\omega) \\ f(x) * g(x) &= \int_{-\infty}^{\infty} f(\tau)g(x-\tau)d\tau \\ \mathcal{F}(f(x) * g(x)) &= \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} f(\tau)g(x-\tau)d\tau)e^{-i\omega x}dx \\ &= \int_{-\infty}^{\infty} f(\tau) \int_{-\infty}^{\infty} g(x-\tau)e^{-i\omega x}dxd\tau \\ &= \int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}G(\omega)d\tau \\ &= G(\omega) \int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}d\tau \\ &= F(\omega)G(\omega)\end{aligned}$$

5.2 Convolution of Fourier transforms

Write the convolution of the Fourier transforms ($F(\omega) * G(\omega)$) in terms of the original functions, $f(x)$, and $g(x)$. Derive this result.

$$F(\omega) * G(\omega) = \mathcal{F}(f(x)g(x))$$

$$\begin{aligned}
F(\omega) * G(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\gamma)G(\omega-\gamma)d\gamma \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\gamma)\int_{-\infty}^{\infty} g(x)e^{-i(\omega-\gamma)x}dxd\gamma \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\gamma)\int_{-\infty}^{\infty} g(x)e^{i\gamma x}e^{-i\omega x}dxd\gamma \\
&= \int_{-\infty}^{\infty}(\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\gamma)e^{i\gamma x}d\gamma)g(x)e^{-i\omega x}dx \\
&= \int_{-\infty}^{\infty} f(x)g(x)e^{-i\omega x}dx \\
&= \mathcal{F}(f(x)g(x))
\end{aligned}$$

5.3 Convolution

Prove using convolution $\frac{1}{2}\exp(-|x|) = \exp(-x)u_s(x) * \exp(x)u_s(-x)$. Here u_s is the step function, $u_s(x) = 0$ for $x < 0$ and $u_s(x) = 1$ for $x > 0$.

Let $f(x) = e^{-x}u_s(x)$ and $g(x) = e^{x}u_s(-x)$.

$$\begin{aligned}
f(x) * g(x) &= \int_{-\infty}^{\infty} f(\tau)g(x-\tau)d\tau \\
&= \int_{-\infty}^{\infty} e^{-\tau}u_s(\tau)e^{x-\tau}u_s(-x+\tau)d\tau \\
&= \int_{-\infty}^{\infty} e^{-\tau}e^{x-\tau}u_s(\tau)u_s(-x+\tau)d\tau \\
&= \int_{-\infty}^{\infty} e^{-\tau+x-\tau}u_s(\tau)u_s(-x+\tau)d\tau
\end{aligned}$$

The multiplied step functions can be used to refine the bounds of integration for this problem. Note that $u_s(\tau)u_s(-x+\tau)$ is equal to 1 only when $\tau > 0$ and $\tau > x$.

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\tau+x-\tau}u_s(\tau)u_s(-x+\tau)d\tau &= \int_{x}^{\infty} e^{x-2\tau}d\tau \\
&= e^{x}\int_{x}^{\infty} e^{-2\tau}d\tau \\
&= e^{x}\cdot(\frac{1}{2}e^{-2x})
\end{aligned}$$

Since x must be greater than 0 from the step function

$$e^{-x}u_s(x) * e^{x}u_s(-x) = \frac{1}{2}e^{-|x|}$$

5.4 Convolution via Fourier transforms

Prove using Fourier transforms $\frac{1}{2}\exp(-|x|) = \exp(-x)u_s(x) * \exp(x)u_s(-x)$.

Hints: Write $\frac{1}{2}\exp(-|x|) = \frac{1}{2}\exp(x)u_s(-x) + \frac{1}{2}\exp(-x)u_s(x)$.

Recall that if $u(x)$ and $U(f)$ are Fourier transform pairs, then $u(-x)$ and $U(-f)$ are also Fourier transform pairs.

Last note that $\frac{1}{2}(\frac{1}{1+i2\pi f}) + \frac{1}{2}(\frac{1}{1-i2\pi f}) = \frac{1}{1+(2\pi f)^2}$.

Let $f(x) = e^{-x}u_s(x)$ and $g(x) = e^x u_s(-x)$.

$$\begin{aligned}
\mathcal{F}(f(x) * g(x) &= F(\omega)G(\omega) \\
\mathcal{F}(f(x)) &= \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \\
&= \int_{-\infty}^{\infty} e^{-x}u_s(x)e^{-i\omega x}dx \\
&= \int_{0}^{\infty} e^{-x}e^{-i\omega x}dx \\
&= \int_{0}^{\infty} e^{-x(1+i\omega)}dx \\
F(\omega) &= \frac{1}{1+i\omega} \\
\mathcal{F}(g(x)) &= \int_{-\infty}^{\infty} g(x)e^{-i\omega x}dx \\
&= \int_{-\infty}^{\infty} e^{x}u_s(-x)e^{-i\omega x}dx \\
&= \int_{-\infty}^{0} e^{x}e^{-i\omega x}dx \\
&= \int_{-\infty}^{0} e^{x(1-i\omega)}dx \\
G(\omega) &= \frac{1}{1-i\omega} \\
F(\omega)G(\omega) &= \frac{1}{1+i\omega}\frac{1}{1-i\omega} \\
&= \frac{1}{1+\omega^2} \\
&= \frac{1}{2}\frac{1-i\omega+1+i\omega}{1+\omega^2} \\
&= \frac{1}{2}(\frac{1}{1+i\omega}+\frac{1}{1-i\omega}) \\
\mathcal{F}^{-1}(F(\omega)G(\omega)) &= \mathcal{F}^{-1}(\frac{1}{2}(\frac{1}{1+i\omega}+\frac{1}{1-i\omega})) \\
&= \int_{-\infty}^{\infty} \frac{1}{2}(\frac{1}{1+i\omega}+\frac{1}{1-i\omega})e^{i\omega x}d\omega \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1+i\omega}d\omega + \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{1-i\omega}d\omega \\
&= \frac{1}{2}e^{-x}u_s(x) + \frac{1}{2}e^{x}u_s(-x) \\
&= \frac{1}{2}e^{-|x|}
\end{aligned}$$

6 Differential operators

Define $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)dx$.

Let $A = -\frac{\partial^2}{\partial x^2} + id$ and let $L = \frac{\partial}{\partial x} + id$.

Prove that $\langle Aw, w\rangle = \langle Lw, Lw\rangle$ for a function w that vanishes at infinity (i.e. $\lim_{x\to-\infty}\omega(x) = \lim_{x\to\infty}\omega(x) = 0$).

$$\begin{aligned}
\langle Aw, w\rangle &= \int_{-\infty}^{\infty}\Big[(-\frac{\partial^2}{\partial x^2} + id)w\Big]wdx \\
&= \int_{-\infty}^{\infty}\Big(-\frac{\partial^2 w}{\partial x^2} + w\Big)wdx \\
&= \int_{-\infty}^{\infty} -w\frac{\partial^2 w}{\partial x^2} + w^2 dx \\
\langle Lw, Lw\rangle &= \int_{-\infty}^{\infty}\Big(\frac{\partial}{\partial x} + id\Big)w\Big(\frac{\partial}{\partial x} + id\Big)wdx \\
&= \int_{-\infty}^{\infty}\Big(\frac{\partial w}{\partial x} + w\Big)\Big(\frac{\partial w}{\partial x} + w\Big) \\
&= \int_{-\infty}^{\infty}\Big(\frac{\partial w}{\partial x}\Big)^2 + 2w\frac{\partial w}{\partial x} + w^2 \\
&= \int_{-\infty}^{\infty}\Big(\frac{\partial w}{\partial x}\Big)\Big(\frac{\partial w}{\partial x}\Big) + 2w\frac{\partial w}{\partial x} + w^2
\end{aligned}$$

Subtract $\int_{-\infty}^{\infty} w^2 dx$ from both of these terms.
Use integration by parts to simplify $\langle Lw, Lw\rangle$.

$$\begin{aligned}
u &= \frac{\partial w}{\partial x} \quad \text{and} \quad dv = \frac{\partial w}{\partial x}dx \\
du &= \frac{\partial^2 w}{\partial x^2}dx \quad \text{and} \quad v = w
\end{aligned}$$

$$\langle Lw, Lw\rangle = w\frac{\partial w}{\partial x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} w\frac{\partial^2 w}{\partial x^2}dx + \int_{-\infty}^{\infty} 2w\frac{\partial w}{\partial x}dx$$

Note that $\lim_{x\to-\infty}\omega(x) = \lim_{x\to\infty}\omega(x) = 0)$ so $w\frac{\partial w}{\partial x}\Big|_{-infty}^{infty}$ goes to 0. Then, as $\langle Aw, w\rangle = \langle Lw, Lw\rangle$

$$\int_{-\infty}^{\infty} -w\frac{\partial^2 w}{\partial x^2}dx = -\int_{-\infty}^{\infty} w\frac{\partial^2 w}{\partial x^2}dx + \int_{-\infty}^{\infty} 2w\frac{\partial w}{\partial x}dx$$

Add $\int_{-\infty}^{\infty} w\frac{\partial^2 w}{\partial x^2}dx$ to both sides.

$$0 = \int_{-\infty}^{\infty} 2w\frac{\partial w}{\partial x}dx$$

Divide by 2. Use integration by parts to rewrite the right hand side:

$$u = w \quad \text{and} \quad dv = \frac{\partial w}{\partial x} dx$$
$$du = \frac{\partial w}{\partial x} dx \quad \text{and} \quad v = w$$

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} 2w \frac{\partial w}{\partial x} dx \\ &= w^2|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} w \frac{\partial w}{\partial x} dx \end{aligned}$$

Add $\int_{-\infty}^{\infty} w \frac{\partial w}{\partial x} dx$ to both sides:

$$\begin{aligned} w^2|_{-\infty}^{\infty} &= \int_{-\infty}^{\infty} w \frac{\partial w}{\partial x} dx \\ &= w(\infty)^2 - w(-\infty)^2 = 0 \end{aligned}$$

As w vanishes, the integral of the derivative of w will vanish at infinity and negative infinity as well, and therefore we end up with $0 = 0$, which is a valid equivalence. We know this because we are given $\lim_{x \to -\infty} \omega(x) = \lim_{x \to \infty} \omega(x) = 0$).

7 Calculating Linear Transformations and Jacobians

In this set of exercises you will calculate transformations to match one pair of landmarks to another, visualize the transformation as a grid, and calculate the Jacobian. You will demonstrate that the model $\phi(x) = x + v(x)$ may result in transformations which do not have an inverse.

The assignment can be performed in MATLAB or Python. Python boilerplate code has been provided in the appendix. Please hand in any code you write, in addition to anything else the problems ask for.

7.1 Generate a grid

We will calculate the value of our transformations φ at each point on a grid. Generate a grid as in the previous homework.

7.2 Generate the landmarks

Start with a pair of landmarks at locations (125,100) and (150,50). Store each landmark as a column in the variable **X**.

For $\theta = 30$ degrees, generate a counterclockwise rotation matrix stored in the variable **R**.

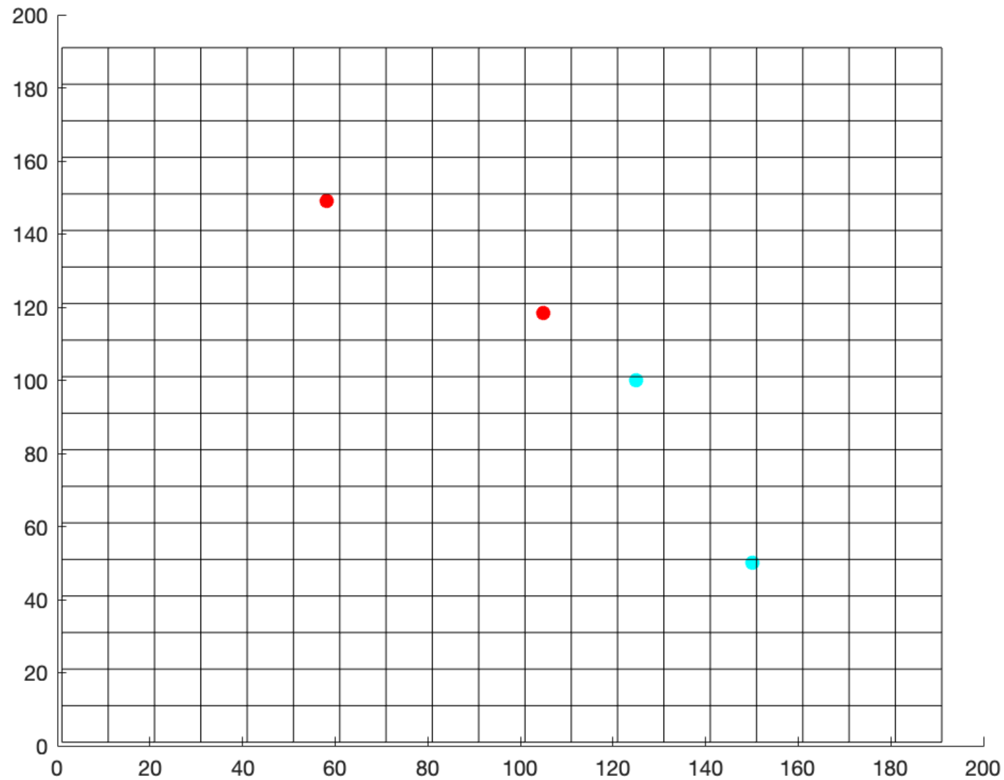
Define a pair of target landmarks **Y** by rotating **(**X) by 30 degrees.

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{pmatrix} \approx \begin{pmatrix} .8660 & -.5000 \\ .5000 & .8660 \end{pmatrix}$$

$$Y = RX = \begin{pmatrix} .8660 & -.5000 \\ .5000 & .8660 \end{pmatrix} \begin{pmatrix} 125 & 150 \\ 100 & 50 \end{pmatrix} \approx \begin{pmatrix} 58.2532 & 104.9038 \\ 149.1025 & 118.3013 \end{pmatrix}$$

7.3 Plot the landmarks and grid

Show the landmarks $\mathbf{X}$ as a scatterplot in cyan. Show the landmarks $\mathbf{Y}$ in the same scatterplot in red. Draw an untransformed grid in the same plot as follows.



7.4 Calculate an optimal 2 x 2 matrix transformation

Find the $A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$, a 2×2 matrix, which brings $\mathbf{X}$ to $\mathbf{Y}$ by minimizing the error

$$E = \sum_{i=1}^{2} |AX(i) - Y(i)|^2 = \sum_{i=1}^{2} |\sum_{j=1}^{2} \sum_{k=1}^{2} a_{jk} X_k(i) - Y_j(i)|^2$$

Minimize this error analytically. You should be able to derive a matrix equation whose solution gives the optimal A.

In MATLAB, numerically calculate A for these landmarks.

Note that you should already know what optimal A is for this example (what is it?), use this to check your work.

Hint: Do you recognize this problem?

$$\begin{aligned}
\frac{\partial E}{\partial A} &= \frac{\partial}{\partial A}\sum_{i=1}^{2}|AX(i) - Y(i)|^2 \\
0 &= \frac{\partial}{\partial A}\sum_{i=1}^{2}(AX(i) - Y(i))^T(AX(i) - Y(i)) \\
&= \frac{\partial}{\partial A}\sum_{i=1}^{2}((AX(i))^T - (Y(i))^T)(AX(i) - Y(i)) \\
&= \frac{\partial}{\partial A}\sum_{i=1}^{2}((X(i))^T A^T - (Y(i))^T)(AX(i) - Y(i)) \\
&= \frac{\partial}{\partial A}\sum_{i=1}^{2} X(i)^T A^T AX(i) - X(i)^T A^T Y(i) - Y(i)^T AX(i) + Y(i)^T Y(i)
\end{aligned}$$

to simplify notation, $X(i) \to X_i$

$$\begin{aligned}
0 &= \frac{\partial}{\partial A}\sum_{i=1}^{2} X_i^T A^T AX_i - X_i^T A^T Y_i - Y_i^T AX_i + Y_i^T Y_i \\
&= \sum_{i=1}^{2} \frac{\partial}{\partial A}X_i^T A^T AX_i - \frac{\partial}{\partial A}X_i^T A^T Y_i - \frac{\partial}{\partial A}Y_i^T AX_i + \cancelto{0}{\frac{\partial}{\partial A}Y_i^T Y_i}
\end{aligned}$$

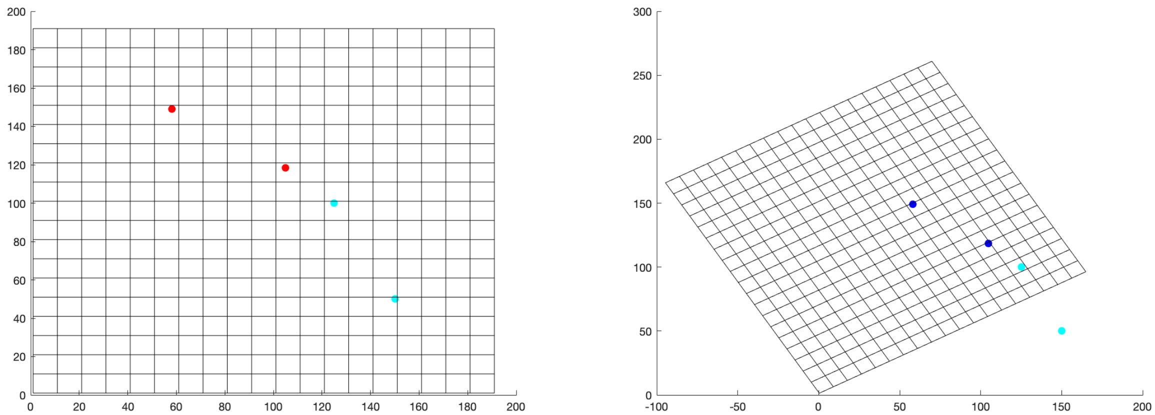
Thank god for matrix cookbook. See equations 70, 71, and 77

$$\begin{aligned}
0 &= \sum_{i=1}^{2} 2AX_iX_i^T - 2Y_iX_i^T \\
\cancel{2}A\sum_{i=1}^{2} X_iX_i^T &= \cancel{2}\sum_{i=1}^{2} Y_iX_i^T \\
A(XX) &= YX \\
A &= (YX)(XX)^{-1}
\end{aligned}$$

A quick check in MATLAB confirms that $A = R$, as we would expect

7.5 Plot the transformed landmarks and grid

Transform your landmarks $\mathbf{X}$ by left multiplying with $\mathbf{A}$. Show the transformed landmarks, $\mathbf{AX}$, as a scatterplot in blue. Show the landmarks $\mathbf{Y}$ in the same scatter plot in red. Draw a transformed grid in the same plot.



At left, the same plot as shown in section 7.3. At right, the points transformed by our optimal matrix A and the transformed grid. Note that the mapping A is so precise that the transformed points AX in blue completely eclipse the points Y in red. Further, note that on the right, the blue points AX are in the same position with respect to the *transformed* grid as the original cyan points X are with respect to the untransformed grid on the left.

7.6 Calculate the Jacobian

Use MATLAB's **gradient** function (or Python's **numpy.gradient**) to calculate the Jacobian of this transformation (**Axij** and **Ayij**) and its determinant everywhere on the 200 x 200 grid. Visualize the Jacobian determinant as an image with a colorbar.

What do you expect the value of the Jacobian determinant to be? You should use this to check your work.

$$v = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$A = \begin{pmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{pmatrix}$$
$$Av = \begin{pmatrix} 0.866x - 0.5y \\ 0.5x + 0.866y \end{pmatrix}$$

To calculate the Jacobian, we can define ϕ as the transformation described by A

$$\phi(v) = Av = \begin{pmatrix} 0.866x - 0.5y \\ 0.866x + 0.5y \end{pmatrix}$$

So now we have

$$\phi_x(v) = 0.866x - 0.5y$$
$$\phi_y(v) = 0.5x + 0.866y$$

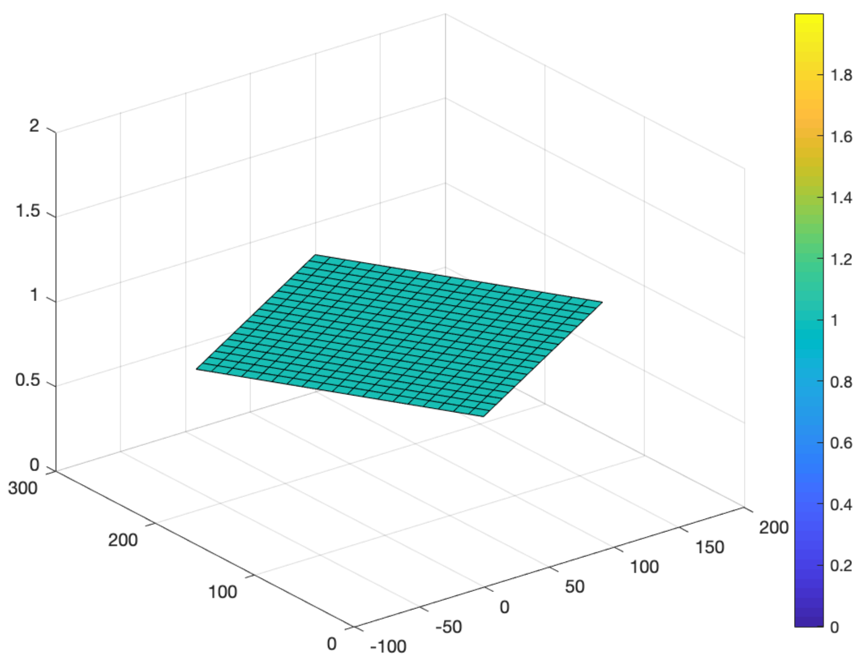
With this format, we can more easily define the Jacobian

$$J = \begin{pmatrix} \frac{\partial \phi_x}{\partial x} & \frac{\partial \phi_x}{\partial y} \\ \frac{\partial \phi_y}{\partial x} & \frac{\partial \phi_y}{\partial y} \end{pmatrix}$$

Which actually just gives us back the matrix A!

$$J = \begin{pmatrix} 0.866 & -0.5 \\ 0.5 & 0.866 \end{pmatrix}$$

Considering we got these values from the rotation matrix, we've seen this determinant before. We expect the determinant to be equal to $\sin^2(30) + \cos^2(30) = 1$ (note that the determinant is independent of x and y, the coordinates of the point at which the transformation is being applied). Additionally, as the transformation is linear and introduces no change in the area of any infinitesimal portion of the image, we know that the determinant is 1. This gives us a very boring distribution over our domain:



7.7 Calculate an optimal Gaussian kernel transformation

Choose the standard deviation $\sigma = 50$ for this exercise.

We calculate a displacement vector field of the form

$$
\begin{aligned}
v_1(x) &= \sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|x - X(i)|^2\big) p_1(i) \\
v_2(x) &= \sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|x - X(i)|^2\big) p_2(i)
\end{aligned}
$$

while satisfying boundary conditions.

The boundary conditions

$$
\begin{aligned}
v_1(X(1)) = Y_1(1) - X_1(1) &= \sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|X(1) - X(i)|^2\big) p_1(i) \\
v_1(X(2)) = Y_1(2) - X_1(2) &= \sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|X(2) - X(i)|^2\big) p_1(i)
\end{aligned}
$$

can be written as a 2×2 matrix vector equation for the x component of the transformation, $V_1 = \hat{K} P_1$, where $\hat{K}$ is a 2×2 matrix, and V_1 and P_1 are 2×1 vectors storing x components of v and p respectively. Write out this equation for the x component of the $p(i)$. You should solve it analytically, and computationally in matlab.

Analytical solution:

$$
\begin{aligned}
V_1 &= \hat{K} P_1 \\
V_1 &= \begin{pmatrix} \exp\big(-\frac{1}{2\sigma^2}|x - X(1)|^2\big) & \exp\big(-\frac{1}{2\sigma^2}|x - X(2)|^2\big) \\ \exp\big(-\frac{1}{2\sigma^2}|x - X(1)|^2\big) & \exp\big(-\frac{1}{2\sigma^2}|x - X(2)|^2\big) \end{pmatrix} P_1 \\
\begin{pmatrix} Y_1(1) - X_1(1) \\ Y_1(2) - X_1(2) \end{pmatrix} &= \begin{pmatrix} \exp\big(-\frac{1}{2\sigma^2}|X(1) - X(1)|^2\big) & \exp\big(-\frac{1}{2\sigma^2}|X(1) - X(2)|^2\big) \\ \exp\big(-\frac{1}{2\sigma^2}|X(2) - X(1)|^2\big) & \exp\big(-\frac{1}{2\sigma^2}|X(2) - X(2)|^2\big) \end{pmatrix} P_1 \\
\begin{pmatrix} Y_1(1) - X_1(1) \\ Y_1(2) - X_1(2) \end{pmatrix} &= \begin{pmatrix} 1 & \exp\big(-\frac{1}{2\sigma^2}|X(1) - X(2)|^2\big) \\ \exp\big(-\frac{1}{2\sigma^2}|X(2) - X(1)|^2\big) & 1 \end{pmatrix} P_1 \\
P_1 &= \begin{pmatrix} 1 & \exp\big(-\frac{1}{2\sigma^2}|X(1) - X(2)|^2\big) \\ \exp\big(-\frac{1}{2\sigma^2}|X(2) - X(1)|^2\big) & 1 \end{pmatrix}^{-1} \begin{pmatrix} Y_1(1) - X_1(1) \\ Y_1(2) - X_1(2) \end{pmatrix} \\
P_1 &= \frac{1}{\det \hat{K}} \begin{pmatrix} 1 & -\exp\big(-\frac{1}{2\sigma^2}|X(1) - X(2)|^2\big) \\ -\exp\big(-\frac{1}{2\sigma^2}|X(2) - X(1)|^2\big) & 1 \end{pmatrix} \begin{pmatrix} Y_1(1) - X_1(1) \\ Y_1(2) - X_1(2) \end{pmatrix}
\end{aligned}
$$

To calculate the inverse of $\hat{K}$ we need to calculate its determinant

$$
\begin{aligned}
\det \hat{K} &= (1)(1) - [\exp\big(-\frac{1}{2\sigma^2}|X(1) - X(2)|^2\big)][\exp\big(-\frac{1}{2\sigma^2}|X(2) - X(1)|^2\big)] \\
\det \hat{K} &= 1 - (\exp\big(-\frac{1}{2\sigma^2}|X(1) - X(2)|^2\big))^2 \\
\det \hat{K} &= 1 - \exp\big(-\frac{1}{\sigma^2}|X(1) - X(2)|^2\big)
\end{aligned}
$$

From here we plug and chug

$$
\begin{aligned}
&|X(1) - X(2)|^2 = |\begin{pmatrix}125\\100\end{pmatrix} - \begin{pmatrix}150\\50\end{pmatrix}|^2 = |\begin{pmatrix}-25\\50\end{pmatrix}|^2 = 25^2 + 50^2 = 3125\\
&\exp\big(-\frac{1}{2\sigma^2}|X(1) - X(2)|^2\big) = \exp\big(-\frac{3125}{2(50)^2}\big) = \exp(-0.625)\\
&Y_1(1) - X_1(1) = 125\cos(30) - 100\sin(30) - 125\\
&Y_1(2) - X_1(2) = 150\cos(30) - 50\sin(30) - 150
\end{aligned}
$$

Plugging this all back in we find...

$$
\begin{aligned}
P_1 &= \frac{1}{1 - \exp(-1.25)}\begin{pmatrix}1 & -\exp(-0.625)\\ -\exp(-0.625) & 1\end{pmatrix}\begin{pmatrix}125\cos(30) - 100\sin(30) - 125\\ 150\cos(30) - 50\sin(30) - 150\end{pmatrix}\\
&\approx \frac{1}{1 - 0.2865}\begin{pmatrix}1 & -0.5353\\ -0.5353 & 1\end{pmatrix}\begin{pmatrix}-66.7468\\ -45.0962\end{pmatrix}\\
&\approx 1.4015\begin{pmatrix}-42.6068\\ -9.3666\end{pmatrix}\\
&\approx \begin{pmatrix}-59.7134\\ -13.1273\end{pmatrix}
\end{aligned}
$$

This lines up very well with the results from Matlab, which calculates $P_1 = \begin{pmatrix}-59.7181\\ -13.1314\end{pmatrix}$. The difference can be attributed to rounding.

Do the same for the y component of $p(i)$ by writing the boundary conditions

$$
\begin{aligned}
v_2(X(1)) = Y_2(1) - X_2(1) = \sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|X(1) - X(i)|^2\big)p_2(i)\\
v_2(X(2)) = Y_2(2) - X_2(2) = \sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|X(2) - X(i)|^2\big)p_2(i)
\end{aligned}
$$

as a matrix equation.

Analytical solution:

$$
\begin{aligned}
V_2 &= \hat{K} P_2 \\
V_2 &= \begin{pmatrix} \exp\big(-\frac{1}{2\sigma^2}|X(1)-x|^2\big) & \exp\big(-\frac{1}{2\sigma^2}|X(2)-x|^2\big) \\ \exp\big(-\frac{1}{2\sigma^2}|X(1)-x|^2\big) & \exp\big(-\frac{1}{2\sigma^2}|X(2)-x|^2\big) \end{pmatrix} P_2 \\
\begin{pmatrix} Y_2(1)-X_2(1) \\ Y_2(2)-X_2(2) \end{pmatrix} &= \begin{pmatrix} \exp\big(-\frac{1}{2\sigma^2}|X(1)-X(1)|^2\big) & \exp\big(-\frac{1}{2\sigma^2}|X(2)-X(1)|^2\big) \\ \exp\big(-\frac{1}{2\sigma^2}|X(1)-X(2)|^2\big) & \exp\big(-\frac{1}{2\sigma^2}|X(2)-X(2)|^2\big) \end{pmatrix} P_2 \\
\begin{pmatrix} Y_2(1)-X_2(1) \\ Y_2(2)-X_2(2) \end{pmatrix} &= \begin{pmatrix} 1 & \exp\big(-\frac{1}{2\sigma^2}|X(2)-X(1)|^2\big) \\ \exp\big(-\frac{1}{2\sigma^2}|X(1)-X(2)|^2\big) & 1 \end{pmatrix} P_2 \\
P_2 &= \begin{pmatrix} 1 & \exp\big(-\frac{1}{2\sigma^2}|X(2)-X(1)|^2\big) \\ \exp\big(-\frac{1}{2\sigma^2}|X(1)-X(2)|^2\big) & 1 \end{pmatrix}^{-1} \begin{pmatrix} Y_2(1)-X_2(1) \\ Y_2(2)-X_2(2) \end{pmatrix} \\
P_2 &= \frac{1}{\det \hat{K}} \begin{pmatrix} 1 & -\exp\big(-\frac{1}{2\sigma^2}|X(2)-X(1)|^2\big) \\ -\exp\big(-\frac{1}{2\sigma^2}|X(1)-X(2)|^2\big) & 1 \end{pmatrix} \begin{pmatrix} Y_2(1)-X_2(1) \\ Y_2(2)-X_2(2) \end{pmatrix}
\end{aligned}
$$

To calculate the inverse of $\hat{K}$ we need to calculate its determinant

$$
\begin{aligned}
\det \hat{K} &= (1)(1) - [\exp\big(-\frac{1}{2\sigma^2}|X(1)-X(2)|^2\big)][\exp\big(-\frac{1}{2\sigma^2}|X(2)-X(1)|^2\big)] \\
\det \hat{K} &= 1 - (\exp\big(-\frac{1}{2\sigma^2}|X(1)-X(2)|^2\big))^2 \\
\det \hat{K} &= 1 - \exp\big(-\frac{1}{\sigma^2}|X(1)-X(2)|^2\big)
\end{aligned}
$$

From here we plug and chug

$$
\begin{aligned}
&|X(1)-X(2)|^2 = |\begin{pmatrix} 125 \\ 100 \end{pmatrix} - \begin{pmatrix} 150 \\ 50 \end{pmatrix}|^2 = |\begin{pmatrix} -25 \\ 50 \end{pmatrix}|^2 = 25^2 + 50^2 = 3125 \\
&\exp\big(-\frac{1}{2\sigma^2}|X(1)-X(2)|^2\big) = \exp\big(-\frac{3125}{2(50)^2}\big) = \exp(-0.625) \\
&Y_2(1) - X_2(1) = 125\sin(30) + 100\cos(30) - 100 \\
&Y_2(2) - X_2(2) = 150\sin(30) + 50\cos(30) - 50
\end{aligned}
$$

Plugging this all back in we find...

$$
\begin{aligned}
P_2 &= \frac{1}{1-\exp(-1.25)} \begin{pmatrix} 1 & -\exp(-0.625) \\ -\exp(-0.625) & 1 \end{pmatrix} \begin{pmatrix} 125\sin(30) + 100\cos(30) - 100 \\ 150\sin(30) + 50\cos(30) - 50 \end{pmatrix} \\
&\approx \frac{1}{1-0.2865} \begin{pmatrix} 1 & -0.5353 \\ -0.5353 & 1 \end{pmatrix} \begin{pmatrix} 49.1025 \\ 68.3013 \end{pmatrix} \\
&\approx 1.4015 \begin{pmatrix} 12.5408 \\ 42.0167 \end{pmatrix} \\
&\approx \begin{pmatrix} 17.5759 \\ 58.8864 \end{pmatrix}
\end{aligned}
$$

This lines up pretty well with the results from Matlab, which calculates $P_2 = \begin{pmatrix} 17.5804 \\ 58.8912 \end{pmatrix}$. The difference can be attributed to rounding.

7.8 Plot the transformed landmarks and grid

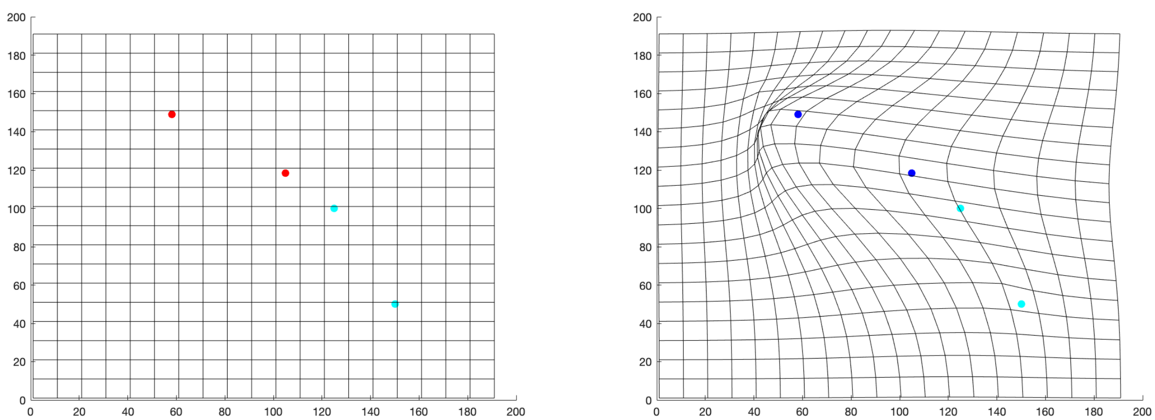
Transform your landmarks by adding $v(X(i))$ to them. Plot these in blue as a scatterplot. On the same plot, show Y as a scatterplot in red.

Calculate the transformation at every point on your grid.

```
% initialize to identity, we will add the displacment v
phix = xij;
phiy = yij;
for i = 1 : nY
    for j = 1 : nX
        % add the displacement for each p(k) in the sum
        for k = 1 : size(X,2) % number of landmarks
            Kij = % ... implement this, the kernel evaluated at (j,i) - X(k)
            phix(i,j) = phix(i,j) + % ... add the x component for p(k)
            phiy(i,j) = phiy(i,j) + % ... add the y component for p(k)
        end
    end
end
```

Visualize the deformed grid as above.

```
phixdown = phix(1:down:end,1:down:end);
phiydown = phiy(1:down:end,1:down:end);
surf(phixdown,phiydown,ones(size(phixdown)),'facecolor','none','edgecolor','k');
```



At left, the same image from 7.3. At right, the transformed points as well as the transformed grid. Note that the transformed points in blue once again eclipse the original transformed points Y in red.

7.9 Calculate the Jacobian

Calculate the Jacobian of `phix` and `phiy` and its determinant as above. Visualize it as an image with a colorbar.

$$\phi_x(w) = w + v(w)$$

Let $w = \begin{pmatrix} x \\ y \end{pmatrix}$ where w is the point being translated

$$\begin{aligned}
\phi_x &= w + \sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|w - X(i)|^2\big)p_1(i) \\
\frac{\partial \phi_x}{\partial x} &= 1 + \sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|w - X(i)|^2\big)p_1(i)(-\frac{1}{\sigma^2}|w - X(i)|) \\
\frac{\partial \phi_x}{\partial y} &= \sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|w - X(i)|^2\big)p_1(i)(-\frac{1}{\sigma^2}|w - X(i)|)
\end{aligned}$$

Analogously

$$\begin{aligned}
\frac{\partial \phi_y}{\partial x} &= \sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|w - X(i)|^2\big)p_2(i)(-\frac{1}{\sigma^2}|w - X(i)|) \\
\frac{\partial \phi_y}{\partial y} &= 1 + \sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|w - X(i)|^2\big)p_2(i)(-\frac{1}{\sigma^2}|w - X(i)|)
\end{aligned}$$

Recall that

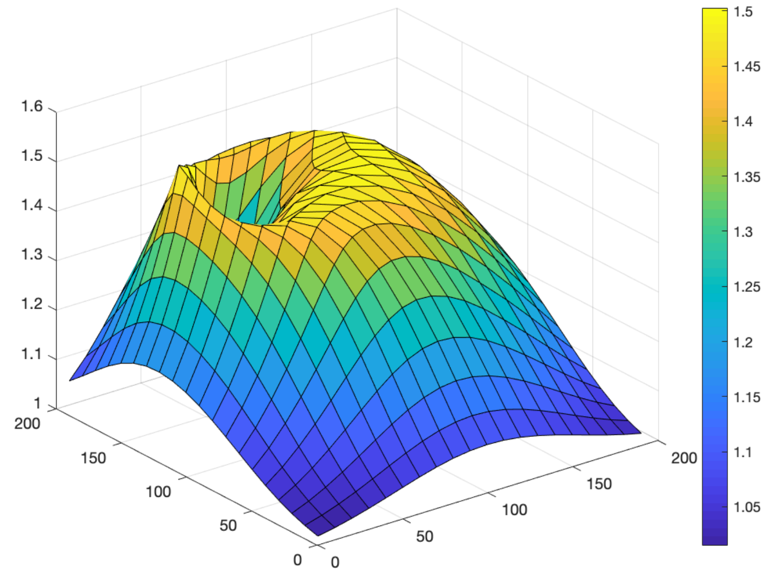
$$J = \begin{pmatrix} \frac{\partial \phi_x}{\partial x} & \frac{\partial \phi_x}{\partial y} \\ \frac{\partial \phi_y}{\partial x} & \frac{\partial \phi_y}{\partial y} \end{pmatrix}$$

And so the Jacobian is

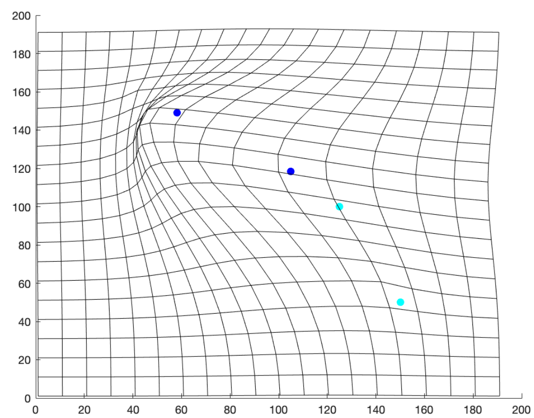
$$J = \begin{pmatrix} 1 + \sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|w - X(i)|^2\big)p_1(i)(-\frac{1}{\sigma^2}|w - X(i)|) & \sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|w - X(i)|^2\big)p_1(i)(-\frac{1}{\sigma^2}|w - X(i)|) \\ \sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|w - X(i)|^2\big)p_2(i)(-\frac{1}{\sigma^2}|w - X(i)|) & 1 + \sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|w - X(i)|^2\big)p_2(i)(-\frac{1}{\sigma^2}|w - X(i)|) \end{pmatrix}$$

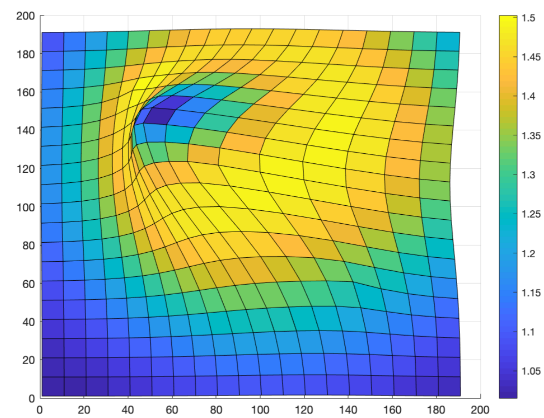
The determinant of this 2 by 2 Jacobian is

$$\begin{aligned}
|J| &= \frac{\partial \phi_x}{\partial x}\frac{\partial \phi_y}{\partial y} - \frac{\partial \phi_x}{\partial y}\frac{\partial \phi_y}{\partial x} \\
&= \big(1 + \sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|w - X(i)|^2\big)p_1(i)(-\frac{1}{\sigma^2}|w - X(i)|)\big)\big(1 + \sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|w - X(i)|^2\big)p_2(i)(-\frac{1}{\sigma^2}|w - X(i)|)\big) \\
&\quad - \big(\sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|w - X(i)|^2\big)p_1(i)(-\frac{1}{\sigma^2}|w - X(i)|)\big)\big(\sum_{i=1}^{2} \exp\big(-\frac{1}{2\sigma^2}|w - X(i)|^2\big)p_2(i)(-\frac{1}{\sigma^2}|w - X(i)|)\big)
\end{aligned}$$



Viewing this from above, we observe that the "hole" aligns well with the 2D plot we produced from this transformation previously.

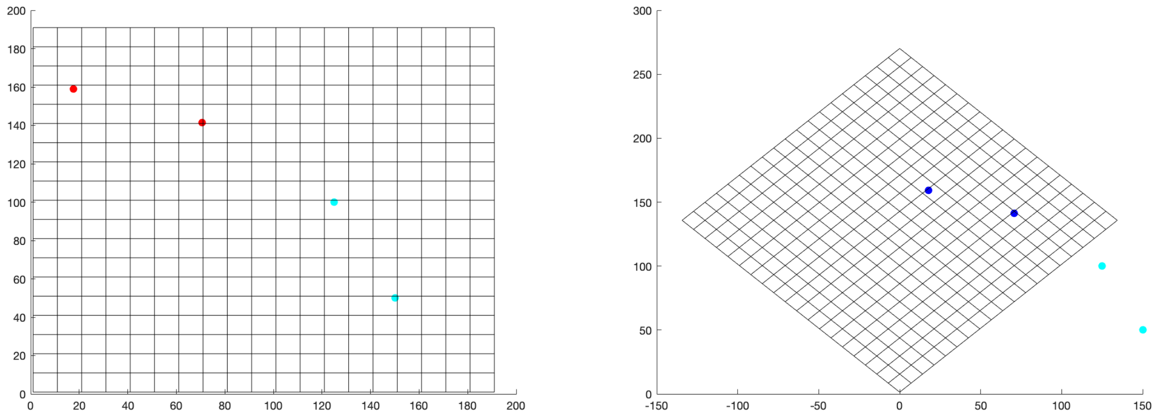




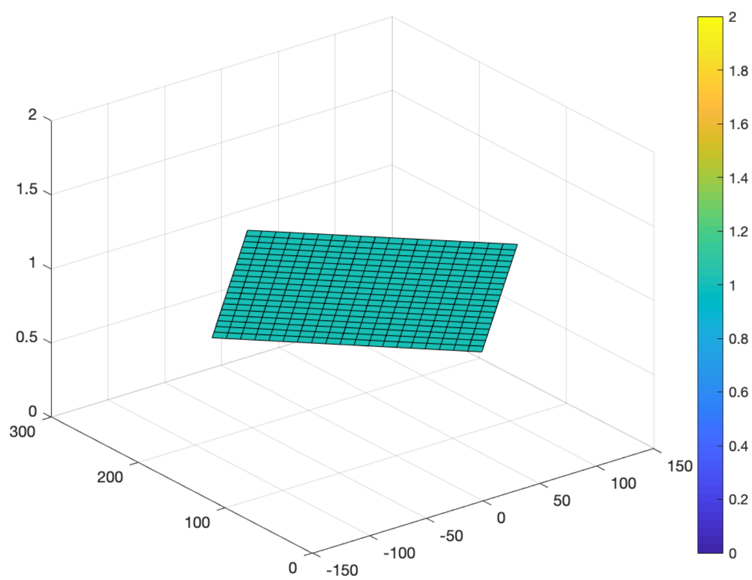
7.10 Repeat the exercise for $\theta = 45$ degrees

Describe what you notice about the deformed grid, the determinant of the Jacobian, and the invertibility of the transformation.

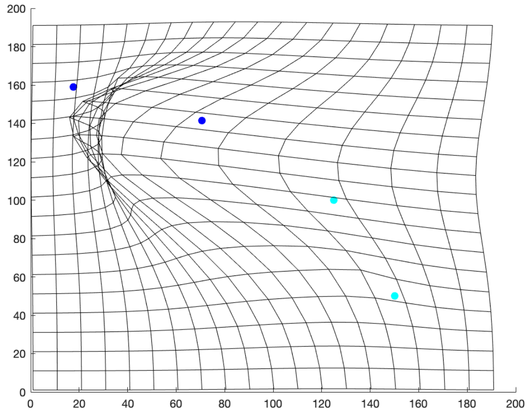
As we would expect, the simple rotation transformation is just a slightly more extreme rotation of the entire axis, this time up to a 45 degree angle:

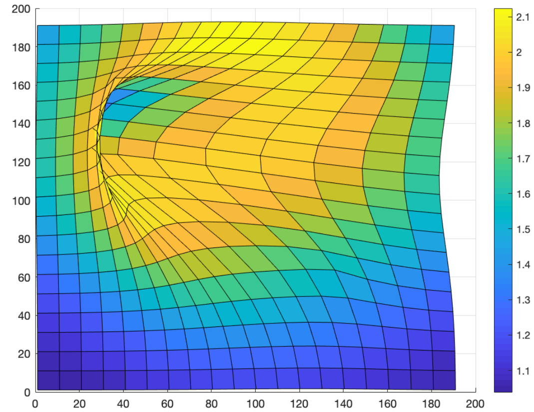


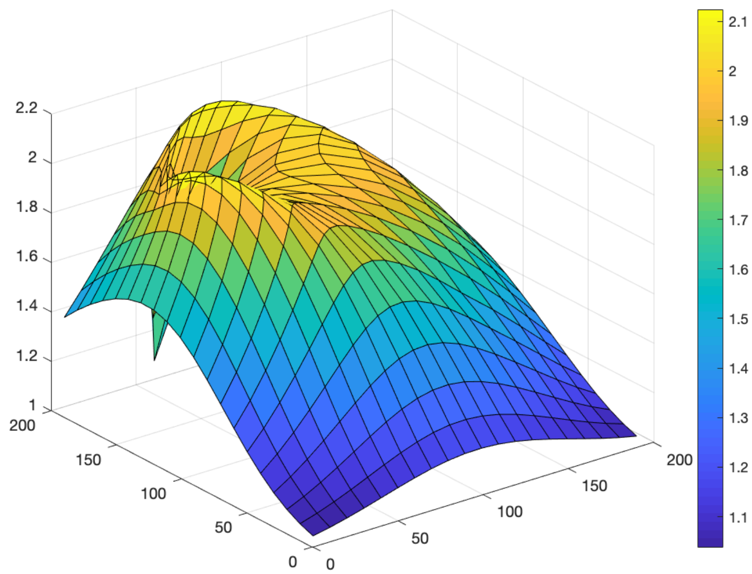
The sum of sine and cosine squared remains the same independent of the angle theta, so we maintain the same boring Jacobian for the rotation matrix:



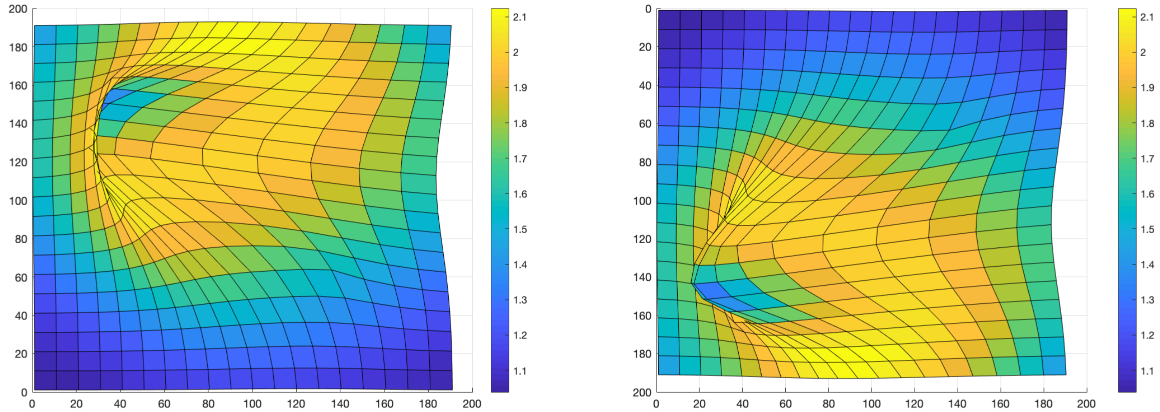
Looking at the Gaussian kernel, however, this rotation is a bit more interesting:







We observe a similar effect to what we saw in the 30 degree rotation, but with a much more extreme "hole" which in fact appears to drop beneath the upper surface (we can see in the grid transformation as well a sort of "overlap"). This is confirmed when we view the color map of the determinant of the Jacobian from below:



Note that the y-axis is flipped in the plot on the right. These figures demonstrate that for a given point $\phi(x)$, the determinant of the Jacobian is not uniquely defined. Though the determinant of the Jacobian does not drop to zero, this still means that the Jacobian matrix of $\phi(x)$ is not invertible since its determinant is not uniquely defined.