1.1 Systems of Linear equations

A linear equation is an equation that follows the form $a_1x_1 + a_2x_2 + a_3x_3...a_nx_n$ where a_n is a constant and x_n is a variable.

multiple linear equations in a system can be represent by a matrix.

$$x_1 + 2x_2 - 5x_3$$
$$2x_1 + 3x_2x_3$$
$$-x_1 + 0x_2x_3$$
$$= \begin{matrix} 1 & 2 & -5 \\ 2 & 3 & 1 \\ -1 & 0 & 1 \end{matrix}$$

There is a row for each each equation and a column for each variable. If $m =$ number of equations and $n =$ number of variables then is matrix is of size $m * n$.

A system is has either
1. no solution (inconsistent)
2. one solution (consistent)
3. infinite solutions (consistent)

the coefficient matrix is the matrix that contains only the coefficients. The augmented matrix contains coefficients and solutions.

Row Operations
1. Replacement: replace a row by the sum of itself and the multiple of another row.
2. Interchange: swap two rows
3. Scaling: multiple all entries of a row by a nonzero constant.

Uniqueness of a solution: if a variable has only one possible value (the equation has one solution) then there is a unique solution. If there is a row of all zeros in a matrix it means theres is a solution for all x thus there are infinite solutions.

1.1 Echelon Forms

Properties of echelon form

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below entry are zeros.

A matrix is in reduced echelon form if the it has properties 4 and 5.

4. leading entry in each non-zero row is one.
5. leading 1 is the only non-zero entry.

There are many echelons forms of a matrix, but there is only one reduced echelon form. The reduced form is obtained from elementary row operations.

Pivot Positions

A pivot position in a matrix is a location that corresponds to a leading one in the reduced echelon form.

Pivot columns are comlumns that contain a pivot position.

Row reduction algorithm

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in
echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic

variable is expressed in terms of any free variables appearing in the equation.

1.3 Vector equations

Vectors can be notatated by $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$.

$\mathbb{R}^2$ denotes all vectors of real numbers with two entries.

Vectors are added by adding corresponding entries.
$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} w_1 + u_1 \\ w_2 + u_2 \end{bmatrix}$.

Multiplying a vector a realnumber "scaler" means multiplying each entry by that real number.

Vectors in $\mathbb{R}^n$

$\mathbb{R}^n$ denotes the collections of all ordered n real numbers.

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

for vector addition multiplication etc just reference the calc 4 notes.

1.2 Linear Combinations

A linear combination y is defined as follows.

$y = c_1v_1 + c_2v_2 + \cdots + c_pv_p$

$x_1a_1 + x_2a_2 + ... + x_na_n = b$

Has the same solution set of the matrix of its coefficients with a solution b.

b can be generated from a linear combination of $a_1 \cdots a_n$ only if there exists a solution to the matrix.

The **weights** denoted by c_p can be any real numbers.

A linear combination can be viewed as the result of a vector equation (the sum of 2 vectors possibly scaled by some factors).

Span $v_1, \cdots v_p$ is the collection of all vectors that can be written as $c_1v1 + \cdots + c_pv_p$.

The zero vector must be present in the set of values so all spans contain a line through the origin.
A vector b is in span v if there is a solution to $x_1v_1 + \cdots x_pv_b = b$

1.3 Matrix Equation

$$Ax = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} [v_1v_2v_3] = c_1v_1 + c_2v_2 + c_3v_3$$

Ax is a matrix equation v_n is a vector and c_n are weights. This is yet another way to visualize linear combinations.

The equation $Ax = b$ has a solution if any only if b is a linear combination of the columns of A.

theorem 4
For each b in $\mathbb{R}^m$ the equation $Ax = b$ has a solution.

Each b in $\mathbb{R}^m$ is a linear combination of the columns of A.

The columns of A span $\mathbb{R}^m$.

A has a pivot position in every row.

1.4 Solution Sets of Linear Equations

Definition: a linear system is called homogenous if it can be written in the form $Ax = 0$.

The system always has the trivial solution $x = 0$

Theorem 6
Suppose the equation $Ax = b$ is consistent for some given b, and let p be a solution. Then the set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$, where v_h is any solution of the homogenous equation $Ax = 0$.

Writing a vector in parametric form
If $x = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 * \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$
is a solution to $Ax = 0$ then solutions can be expressed as the sum of the constant vector called p and the variable times the second vector denoted as tv then the expression $x = p + tv$ is called the parametric vector form.

1.5 Linear Independence

if $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a unique solution to $Ax = 0$ then the constituent vectors $u_1, u_2, \cdots u_n$ are said to be independent. They are irrevalent to the result of the equation.

If $x_1 \cdots x_n$ are not all zero then $u_1 \cdots u_n$ is a linear dependence relation. This is because u_n must equal $\vec{0}$ in order for the solution to exist.

A set of two vectors v_1, v_2 is called linearly dependent if v_1 is a multiple of v_2.

Theorem 7: A set of two or more vectors $v_1 \cdots v_n$ is linearly dependent if at least one of the vectors in the set is a linear combination of others in the set.

For example set w, v, u with independent w, v will be dependent only if u is a linear combination of w and v.

Theorem 8: If a set contains more vectors then entries in each vector then the set is linearly dependent.

Theorem 9: If a set of vectors contains the zero vector then the set is linearly dependent.

1.6 Linear Transformations

A matrix A can be though of as an object that acts on vector x to produce a vector Ax. From this we can say the A transforms x into b.

This is useful to think about transformations of 4d vectors to 2d vectors by multiplying a 4d by a 2d.

A transformation is a function or mapping of T from $\mathbb{R}^n \to \mathbb{R}^m$. The domain of T is $\mathbb{R}^n$ and the codomain is $\mathbb{R}^m$

for x in $\mathbb{R}^n$ $T(x)$ in $\mathbb{R}^m$ is called the image of x.

$T(x) = Ax$
x transformed by $A = x \mapsto Ax$

asking what the image of u under transformation T is the same as evaluating $T(u)$

asking weather an image of u exists in the domain of T and if u is in the range of T is the same as evaluating $Ax = u$ to see if it has a solution.

Transformations can and should be viewed as somthing that transforms a vector into another vector.

if $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
Then the resulting transformation removes or x_3 producing a 2d vector from a 3d one.

Linear Transformation Properties

$A(u+v) = Au + Av$
$A(cu) = cAu$

This suggests the following.

$T(u+v) = T(u) + T(v)$
$T(cu) = cT(u)$

These propeties are true for all u, v in the domain of T.

1.7 Matrix of a linear transform

Every linear transform is a matrix transform but not all matrix transforms are linear transforms.

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transform. Then there exists a unique matrix A such that $T(x) = Ax$ for all x in $\mathbb{R}^n$

Then A is the mxn matrix whose jth column is the vector $T(e_j)$, where e_j is the jth column of the identity matrix in $\mathbb{R}^n$

$A = [T(e_1) \cdots T(e_n)]$

Theorom 11
A linear transformation T is one to one if and only if the equation $T(x) = 0$ has only the trivial solution.

Theorem 12
Let T be a linear transformation and A be its standard matrix.
a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$

b. T is one-to-one if and only if the columns of A are linearly independent.

Onto Mapping

In order for a mapping to be onto the matrix A must have at least m columns. That is it must be all to accept all components of the vector it is transforming as parameters. Obviously it must also be a valid matrix equation.

In other words in order to be onto it must have at least as many columns as rows $m \leq n$

One-to-One mapping

For a mapping to one to one it must have a unique solution for all inputs. However its range does not have to be the span of $\mathbb{R}^{>}$ thus the columns can be less than m.

In other words in order for it to be one to one it must have at least as many rows as columns $n \leq m$

One-to-one and Onto

A in order for mapping to be one-to-one and onto $m = n$

2.2 Matrix Algebra

Two matrixs can be multiplied if the number of columns in matrix A matches the number of rows in B thats if $A : m$ x p and $B : P$ x n.

Example: $A * B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix} * \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$
$Ab_1 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix}$
$Ab_2 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix}$
$Ab_3 = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix}$

The associative and distributive laws hold for matrix multiplication however beware order matters $AB \neq BA$ neither do cancelation laws if $AB = AC$ it does not mean that $B = C$. Finally just because $AB = 0$ does not mean the either A or B is the zero vector.

Powers of a matrix

$A^k = A \cdots A$ that is if $\neq 0$ then A^k denotes the product of k copies of A.

Contrary to integers $A^0 = A$ not 1.

The trans pose of a matrix A is the n x m matrix whos columns are the rows of A. It is denoted as A^T.

if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
then $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$(A^T)^T = A$
$(A + B)^T = A^T + B^T$
$rA^T = (rA)^T$
$(AB)^T = B^T A^T$

The transposes of a product of matrices equals the product of thier transposes on reverse order.

2.3 Inverse Matrix

The inverse of a matrix A is denoted as A^{-1} and if it exists it has the following properties.

$A^{-1}A = I$ and $AA{-}1 = I$ where I is the n x n identity matrix.

$(A^{-1})^{-1} = A$
$(AB)^{-1} = B^{-1}A^{-1}$
$(A^T+)^{-1} = (A^{-1})^T$

Only square matrices are invertable.

A matrix that is not invertible is called a singular matrix and an invertible matrix is called a nonsingular matrix.

if $d(A)$ is the determinate of a square matrix A then a formula for the inverse is as follows.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
$A^{-1} = \frac{1}{d(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
A 2 x 2 matrix A is invertible only if its determinate does not equal zero.

Theorem 5: If A is an invertible n x n matrix then for each b in $\mathbb{R}^n$ the equation $Ax = b$ has the unique solution $x = A^{-1}b$.

Theorem 7: An n x n matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

Find A^{-1} algorithm

Row reduce the augmented matrix $[AI]$. If A is row equivalent to $[AI]$ id row equivalent to $[IA^{-1}]$. Otherwise, A does not have an inverse.

Invertible Matrix Theorem

The following statemnts are all equivalent.
A is a square n by n matrix.

A is Invertible

A is rowequivalent to its n by n identity Matrix.

A has n pivot positions

The equation $Ax = 0$ has only the trivial solution.

The columns of A are linearly independent.

The transfrom $x \mapsto Ax$ is one-to-one.

The equation $Ax = b$ has at least one solution for each b in $\mathbb{R}^n$.

The columns of A span $\mathbb{R}^n$.

The linear transformations $x \mapsto Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

There is an n by n matrix C such that $CA = I$.

There is an n by n matrix D such that $DA = I$.

A^T is invertible.

Theorem 9: If the standard matrix of a linear transformation is invertible then the transformation is invertibles and the matrix of the inverse transform is the inverse of the standard matrix.

$S(T(x)) = x$ if S is the inverser transform of T.

2.4 Partitioned Matrices

A matrix can be cut into paritioned blocks to form a matrix of these blocks.
How exactly thats done would be a pain to type so just reference section 2.4 examples.

if two matrices A, B are partitioned in the same way then their $A + B$ can be obtained from adding the equivalent blocks in each matrix.
That is if $A = [a_1 a_2]$ and $B = [b_1 b_2]$ then $A + B = [a_1 + b_1, a_2 + b_2]$.

Two partitioned matrices A, B can be multiplied AB so long as the column partition of A matches the row partition of B. That is to say they must be conformable for block multiplication.

2.5 Matrix Factorization

A Factorization of a matrix A is a equation that expresses A as the product of two or more matrices.

Assume A is a n m by n matrix that can be row reduced to echelon form. Then $A = LU$ where L is an m by m lower triangular matrix with 1s on the diagonal and U is an m by n echelon form of A. This factorization is called the LU factorization.

This leads the to the expression $Ax = L(Ux) = b$.

and further.
$Ly = b$
$Ux = y$

LU Factorization Algorithm

1. Reduce A to an echelon form U by a sequence of row operations.
2. Place entries in L such that the same sequences of row operations reduces L to I.

This basically means the columns in L will the reduced forms of the columns in U before you zero the column to create your echelon form.

So to create L you first fill 1s on its diagonal zero everything above the diagonal, then insert the reduced first column of A, then repeat with columns of A as you reduce A to echelon form.

2.6 Vector Space and Row/Col Space

1. The sum of u and v, denoted by $u+v$, is in V.

2. $u+v=v+u$.

3. $(u+v)+w=u+(w+v)$.

4. There is a zero vector in V such that $u+0=u$.

5. For each u in V, there is a vector $-u$ in V such that $u+(-u)=0$.

6. The multiple of u by c is in V.

7. $c(u+v)=cu+cv$.

8. $(c+d)u=cu+du$.

9. $c(du)=(cd)u$.

10. $1u=u$.

If the above axioms are satisfied by a non-empty set V then that set can be called a vector space.

Properties of a vector subspace.

H is a subspace of V if

1. The zero vector of V is in H^2
2. H is closed under vector addition.
3. H is closed under scaler multiplication.

if $v_1 \cdots v_p$ are in vector space V then $\text{Span} v_1, \cdots, v_p$ is a subspace of V.

Null space

The null space of an m by n matrix A, written as Nul A, is the set of all solutions of the homogeneous equation $Ax = 0$. In set notation,

$$NullA = x : xin\mathbb{R}^n, Ax = 0$$

Also theorem that states same thing just cause.

The null space of an m x n matrix A is a subsapce of R^n. Equivalently, the set of all solution to a system $Ax = 0$ of m homogeneous linear equations in n unknowns is subspace of $\mathbb{R}^n$.

Column Space

The column space of an m x n matrix A, written as col A, is the set of all linear combinations of the columns of A. If $A = [a_1 \cdots a_n]$, then

Col $A = Spana_1, \cdots, a_n$

Theorem: The column space of an m x n matrix A is a subspace of $\mathbb{R}^m$.

2.7 linearly independent Sets, and Bases

Recall the theorem of linear dependence. An indexed set $v_1, \cdots, v_p$ of two or more vectors, with $v_1 \neq 0$, is linearly dependent iff some v_j for $(j > 1)$ is linear combination of the preceding vectors, $v_1, \cdots, v_{j-1}$.

Basis Vectors

Let H be a subspace of V. An indexed set of vectors $\mathcal{B} = b_1, \cdots, b_p$ in V is a basis for H if

1. $\mathcal{B}$ is linearly independent.
2. The subspace spanned by $\mathcal{B}$ conicides with H; that is, $H = span b_1, \cdots, b_p$

The standard basis for $\mathbb{R}^n$ can be described by $e_1, \cdots, e_n$ where e is a unit vector in the identity matrix of $\mathbb{R}^n$

A fundemental theorem in algebra states that the only polynomial with more that n zeros is the zero polynomial.

Spanning Set Theorem: $S = v_1, \cdots, v_p$ a set in V, and let $H = \text{span} v_1, \cdots, v_p$

1. If one of the vectors in S such as v_k is a linear combination of the remaining vectors, then the set formed by removing v_k still spans H.

2. If $H \neq 0$ (null set), some subset of S is a basis for H.

This implies the following. given a matrix $B = [b_1 b_2 \cdots b_5]$ we can find the basis of B by discarding all non-pivot columns as they are linear combinations of the remaining.
so if b_2, b_4 do not contain a pivot position the the set b_1, b_3, b_5 still spans column space of B.

Theorem: The pivot columns of matrix A form a basis for Col(A).

2.8 Coordinate Systems

Suppose $\mathcal{B} = b_1, \cdots b_n$ is a basis for V and x is in V. The coordinates of x relative to basis $\mathcal{B}$ or the$\mathcal{B}$ coordinates of x are the weights $c_1, \cdots, c_n$ such that $x = c_1 b_1 + \cdots + c_n b_n$.

$$[X]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Let $P_{\mathcal{B}} =$ the standard matrix of a basis $\mathcal{B}$.
If the coordinates of a vector x in $\mathcal{B}$ is multiplied by $P_{\mathcal{B}}$ then we get the original x in $\mathbb{R}^n$.

$$x = P_{\mathcal{B}}[x]_{\mathcal{B}}.$$

Theorem 8: let $\mathcal{B} = b_1, \cdots, b_n$ be a basis for vector space V. Then the coordinate $x \mapsto [x]_{\mathcal{B}}$ is a one to one linear transform from V onto $\mathbb{R}^n$.

Reducing the columns of transform $\mathcal{C}$ to the left of the columns of transform $\mathcal{B}$ yields the change of bases tranformations from $\mathcal{B} \to \mathcal{C}$
$[c_1 c_2 : b_1 b_2] \to [I : P_{\mathcal{C} \leftarrow \mathcal{B}}]$

2.9 Dimensions of Vector Space

Theorem 9: If a vector space V has a basis $\mathcal{B} = b_1, \cdots, b_n$ Then any set in V containing more than n vectors must be linearly dependent.

Theorem 10: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

If V is spanned by a finite set it is said to be finite-dimensional and the dimension of V, written as dim V is the number of vectors in a basos for V. The dimension of the zero vector space 0 is defined to be zero. if V is not spanned by a finite set then V is infinite-dimensional.

A few definitions.
0-dimensional subspace: only the zero subspace
1-dimensional: any space spanned by a single nonzero vector. A line through the origin.
2-dimensional: any subspace spanned by two linearly independent vectors. A plane through the origin.
3-dimensional: only $\mathbb{R}^3$. Any 3 independent vectors in $\mathbb{B}^3$ span all of the space by the invertible matrix theorem.

Theorem 11: Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and
dim $H \leq$ dim V

Theorem 12: Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

3.0 Rank

Theorem 13: If two matrices A and$B$ are row equavalent, then thier row spaces are the same.

If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

The rank of A is the dimension of the column space of A

Theorem 14: The dimensions of the column and row space of an m x n matrix A are equal.
This common dimension, the **rank** of A, also equals the number of pivot positions in A and satifies the below equation.

rank A + dim Nul $A = n$

IVT Continued

let A be a n x n matrix. Then the following statements are equivalent to the statement that A is invertible

1. The columns of A form a basis of $\mathbb{R}^n$
2. Col $A = \mathbb{R}^n$
3. Dim Col $A = n$
4. rank $A = n$
5. Nul $A = 0$ 6. dim Nul $A = 0$

3.1 Determinates

A 3x3 by determinate can be found by multiplying A_1i by the the determinate formed by deleteing the ith row and the jth column.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \text{... etc}$$

For determinates higher than 3 the determinate can be recursively defined as the sum of determinates formed by deleteing the ith row and jth column of a matrix A.

Definition of the determinate
For $n \geq 2$ the determinant of an n x n matrix $A[aij]$ is the sum of the n terms of the form $+/-a1j$ det A_{1j} with + and minus signs alternating, with the entries $a_1 1, a_1 2, \cdots, a_1 n$ as the first row of A.

$detA = a_1 1 detA_1 1 - a_1 2 detA_1 2 + \cdots + (-1)^{1+n} a_1 n detA_1 n = \sum_{j=1}^{n} (-1)^{1+j} a_1 j detA_1 j$

Cofactor Expansion

The determinate of a n x n matrix A can be computed by a cofactor expansion across any row or down any column. The column ith row is

$detA = a_i 1 C_i 1 + \cdots + a_i n C_i n$

Expansion down the jth comlumn is
$detA = a_1 j C_1 j + \cdots + a_i n C_i n$

Theorem 2: If A is a triangular matrix (a matrix with all zeros below its diagonal), then det A is the product of the entries on the main diagonal.

Theorem 3:
Let A be a n x n matrix
a. if a multiple of one of A is added to another row to prouce a

matrix B, then $detB = detA$
b. If two rows of A are interchanged to produce B, then $detB = -detA$
c. If one row of a A is multiplied by k to produce B then det $B = k * detA$

Theorem 4: Any square matrix is invertible iff $detA \neq 0$

Theorem 5: If A is an n x n matrix then det $A^T = detA$

Theorem 6: If A and B are n x n matrices, then det $AB = (detA)(detB)$

Theorem 7: Cramers Rule Let A be an invertible n x n matrix. For any b in $\mathbb{R}^n$, the unique solution x of $Ax = b$ has entires given by
$x_i = \frac{det(A_i)(b)}{detA}, i = 1, 2, \cdots, n$

Theorem 8: A formula for Inverse of a matrix let A be an invertible n x n matrix. Then

$A^{-1} = \frac{1}{detA} * adjA$

Determinates as Area If A is a 2 x 2 matrix, the area of the parallelogram determined by the columns of A is —$detA$—. If A is a 3x3 matrix, the volume of the parallelpiped determined by the columns of A is —$detA$—.

Theorem 10: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transform determined by the 2x2 matrix A. If S is a parallelogram in $\mathbb{R}^2, then$

$areaT(S) = |detA| * areaofS$

If T is determined by a 3x3 matrix A and if S is a parallelpiped in $\mathbb{R}^3$, then
$volumeT(S) = |detA| * volumeS$

3.2 Eigenvalues and Eigenvectors

An eigenvector of an n x n matrix A is a non-zero vector x such that $Ax = \lambda x$ for some scaler λ. A scaler λ is called an eigenvalue of A if there is a nontrivial solution x of $Ax = \lambda x$; such an x is called an eigenvector corresponding to λ.

An eigenvector must be non-zero but an eigenvalue may be zero.

λ is an eigenvalue of matrix A iff $(A - \lambda I)x = 0$ has a non-trivial solution.
$NUll(A - \lambda I)$ is called the eigen space of A.

Theorem 1: The Eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem 2: If $v_1, \cdots, v_r$ are eigenvectors that correspond to distint eigenvalues $\lambda_1, \cdots .\lambda_r$ of an n x n matrix A, then the set $v_1, \cdots, v_1$ is linearly.
Note: 0 is an eigen value of A iff A is not invertible.

The Characteristic Equation

Finding all the eigenvalues of a matrix can be defined as finding all cases where the matrix $(A - \lambda I)$ has a zero determinant.

IVT Extended:
Let A be an n x n matrix. Then A is invertible iff:
s. The number 0 is not an eigenvalue of A.
t. The deteminant of A is not zero.

$\det(A - \lambda I) = 0$ is called the characteristic equation. λ is an eigenvalue of a n x n matrix A iff λ satifies the characteristic equation.
This implies that the solution set of characteristic equation is the roots of the polynomial formed from the product of the diagonal of a triangular matrix formed from $(A - \lambda I)$.
The multiplicity of a eigenvalue is its multiplicity as a root of the characteristic equation (the number of times it occurs as a factor).

Similarity
If A and B are n x n matrices, then A is similar to B if there is an invertible matrix P such that $P^{-1}AP = B$ equivalently $A = PBP^{-1}$.
Writing Q for P^{-1}, we have $Q^{-1}BQ = A$.
So B is also similar to A and thus A and B are similar.

Theorem 4: If n x n matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues.

Theorem 5: Diagonalization Theorem
An n x n matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In fact $A = PDP^{-1}$, with D a diagonal matrix iff the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that

correspond, respectively, to the eigenvectors in P.

Theorem 6: An n x n matrix with n distinct eigenvalues is diagonalizable.