1.0 Differential Equations

Differential equation: an equation containing one or more derivatives.
Order of differential equation: the order of its highest derivative.

Eulars method
$y_1 = y_0 + f(x_0, y_0) * \Delta x$
$y_2 = y_1 + f(x_1, y_1) * \Delta x$

This process repeats for n approximations and can be described by the formula
$y_{n+1} = y_n + \frac{dy}{dx}\Delta x$

Exponential Growth and Decay

growth $= \frac{dp}{dt} = kp$
decay $= \frac{dp}{dt} = -kp$
where t is time and p is population.

Integrated Formulas
growth $= y = y_0 * e^{kt}$
decay $= y = y_0 * e^{-kt}$

For decay problems if given $t_{1/2}$ then set $y = y_0 * \frac{1}{2}$ and $t = t_{1/2}$ to solve for k

Population growth $= \frac{dy}{dt} = k(1 - \frac{y}{L})y$
Integrated Population growth $= y = \frac{y_0 * L}{y_0 + (L - y_0)e^{-kt}}$

Newton Law of Cooling

$$T(t) = T_s + (T_0 - T_s)e^{-kt}$$

The temp at time t is equal to temp of surroundings + the initial temp - temp of surroundings time e to the $-kt$ power.

Integrated Formula

$$\frac{dT}{dt} = k(T_s - T), k > 0$$

1.1 Integrating factors

a first order differential equation is said to linear if it is expresible in the form $\frac{dy}{dx} + p(x)y = q(x)$ if this is the case we can use the method of integrating factors to solve this problem.

$$\mu = e^{\int p(x)dx}$$
$$\frac{d}{dx}(\mu y) = \mu q(x)$$

This gives us the result

$$y = \frac{1}{\mu} * \int \mu * q(x)dx$$

by rearranging a given equation into the linear format and calculating μ we can solve for y.

1.2 Increasing and Decreasing (Monotone) Sequences

Theorem 9.2.3-4
if sequence a_n is eventually increasing then either a or b is true.
(a): There is a constant M, called an upper bound for the sequence, such that $a_n \leq M$ for all n, in which case the sequence converges to a limit L satisfying $L \leq M$ (b): No upper bound exists, in which case, $\lim_{n\to\infty} a_n = +\infty$.

Likewise for decreasing sequences.
(a): There is a constant M, called a lower bound for the sequence, such that $a_n \geq M$ for all n, in which case the sequence converges to a limit L satisfying $L \geq M$. (b): No lower bound exists, in which case, $\lim_{n\to\infty} a_n = -\infty$.

Monotone testing table
$a_{n+1} - a_n > 0$ $a_{n+1}/a_n > 1$ Strictly increasing
$a_{n+1} - a_n < 0$ $a_{n+1}/a_n < 1$ Strictly decreasing
$a_{n+1} - a_n \geq 0$ $a_{n+1}/a_n \geq 1$ increasing
$a_{n+1} - a_n \leq 0$ $a_{n+1}/a_n \leq 1$ decreasing

1.3 Convergence Tests

u_k is defined as the "general" term of a series. For example in the harmonic series. $\sum_{k=1}^{\infty} \frac{1}{k}$, $u_k = \frac{1}{k}$.

Theorem 1
if $u_k \neq 0$ as k approaches infinity then the series diverges. Else the series may diverge or converge. $\lim_{k\to\infty} \neq 0$.

Theorem 2
if $\sum u_k$ and $\sum v_k$ are convergent series then thier sum and difference are also convergent series. $\sum u_k - v_k$ and $\sum u_k + v_k$.

Thier sums and differences are also related. $\sum u_k - v_k = \sum u_k - \sum v_k$

$\sum u_k + v_k = \sum u_k + \sum v_k$

The integral Test

if u_k is a decreasing function on $[a, +\infty]$ then. $\sum_{k=1}^{\infty} u_k$ and $\int_a^{\infty} u_k$ both converge or diverge.

P-series

A p-series is a series of the form $\sum \frac{1}{k^p}$. The harmonic series is one such series where $p = 1$.

If $p > 1$ the series converges, otherwise it diverges.
Comparison Test

given $\sum a_k$ and $\sum b_k$ all a_k and b_k positive. If all $b_k \geq a_k$ if $\sum b_k$ converges so does $\sum a_k$. Conversely if $\sum a_k$ diverges so does $\sum b_k$.

in other words in the larger sum converges so does the smallerand if the smaller sum diverges so does the larger.

Limit Comparison Test

given $p = \lim_{k\to\infty} \frac{a_k}{b_k}$ and $p > 0$ and finite.
Then both series converge or diverge.

Ratio Test

given $p = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$.
$p < 1$ Series converges
$p > 1$ or $p = +\infty$ Series diverges
$p = 1$ Nothing is certain

Root Test

given $p = \lim_{k\to\infty} (a_k)^{\frac{1}{k}}$.
$p < 1$ Series converges
$p > 1$ or $p = +\infty$ Series diverges
$p = 1$ Nothing is certain

1.4 Alternating Series

An alternating series converges is these two conditions are satisfied.

$$a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_k \geq \ldots$$
$$\lim_{k\to+\infty} a_k = 0$$

if the above conditions are satisfied then the sum of ther series S is between $s_n \leq S \leq s_{n+1}$ or $s_{n+1} \leq S \leq s_n$ Which ever partial sum is greater.

if S is approximated as s_n then absolute error $|S - s_n| \leq a_{n+1}$ Also the sign of the error is the same as the sign of a_{n+1}

to find n to be accuarate to two decimals it suffices to find n such that $0.005 \leq a_{n+1}$ or the nth plus one term. in general solving the inequality $|error| \leq a_{n+1}$ will give you target n or target $|error|$ depending on the problem.

Absolute Convergence

a series $\sum u_k$ is said to diverge or converge absolutely if the absolute values of the series diverge or converge.

If a series converges absolutely then it converges. If a series converges, but diverges absoluteley then it is said to be conditionally divergent.

Ratio test for absolute convergence

The ratio test can be applied to a series for absolute divergence or convergence.

$p = \lim_{k \to +\infty} \frac{|a_{n+1}|}{|a_n|}$

$p < 1$ Series converges absolutely therefor converges.

$p > 1$ or $p = +\infty$ then series diverges.

$p = 1$ inconclusive.

1.5 Taylor and Maclaurin polynomials

A function can be approximated by a polynomial at a point by finding a polynomial which has similar values around that point. To have similar values at a point the derivatives must match. To accomplish this you must approximate constants times an nth degree polynomial. The constants can be described as the nth deriviative of the func in questions.

$c_0 = f(x)$
$c_1 = f'(x)$
$c_2 = \frac{f''(x)}{2!}$
$c_3 = \frac{f'''(x)}{3!}$
$c_n = \frac{f^n(x)}{n!}$

A taylor polynomial is described as follows.
$P_n(x) = c_0 + c_1(x - x_0) + c_2\frac{(x-x_0)^2}{2!} + c_3\frac{(x-x_0)^3}{3!} + \cdots + c_n\frac{(x-x_0)^n}{n!}$
Recall that $c_n = f^n(x_0)$
A Maclaurin polynomial is basically a Taylor poly nomial at $x_0 = 0$. In which case all $(x - x)_0$ can be replaced with simply x

The nth remainder functions is equal to the error of the taylor polynomial and it is denoted by the difference between the function and its nth taylor polynomial.

$R_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^k(x_0)}{k!}(x - x_0)^k$
This can also be denoted by $f(x) = P_n(x) + R_n(x)$

Remainder estimation theorem
if a function can be differentiated on an interval containing (x_0) n+1 times and M is an upper bound on that interval and $|f^{n+1}(x) \leq M$ in said interval then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}$$

A more useful definition is

$$R_n(x) = \frac{f^{(n+1)}(x) * |x - x_0|^{n+1}}{(n+1)!}$$