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- A simplified encoding scheme from polynomials with complex coefficients to vectors with complex coefficients
- The actual encoding scheme from polynomials with integer coefficients to vectors of complex coefficients.

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- This fact will become important when we are discussing the mapping of polynomials with integer coefficients to complex vectors

Simplified encoding scheme

- The input messages in the CKKS scheme are vectors of complex numbers $z \in \mathbb{C}^N$ where N is called the degree modulus for reasons that will become obvious

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- How do they achieve this map from complex vectors to polynomials of integer coefficients

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- The forward map may be easier to consider

$$\sigma : \frac{\mathbb{C}[X]}{X^N + 1} \to \mathbb{C}^N$$

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- Its clear from this equation that given one we can compute the other and since this is a square matrix there is one and only one solution

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- Here we can see we have the coefficient vector **a** that uniquely determines the polynomial and the output vector **z** $\in \mathbb{C}^N$.
- Its clear from this equation that given one we can compute the other and since this is a square matrix there is one and only one solution
- so it should be intuitive that this transformation defines an isomorphism between $\mathbb{C}^4$ and $\frac{\mathbb{C}[X]}{X^4+1}$

- At this point we have defined our simplified map and its inverse as essentially the equation we showed in the previous slide

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$$\sigma^{-1} : \mathbb{C}^N \rightarrow \frac{\mathbb{C}[X]}{X^N + 1}$$
- The CKKS map/algorithm $\sigma^{-1} : \mathbb{C}^N \rightarrow \frac{\mathbb{Z}[X]}{X^N+1}$ adds further structure in order to place restrictions on this map to ensure that we encode our complex vectors as polynomials with integer coefficients only

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- This should be straight forward if we understand the simpler statement that powers of complex conjugates are still complex conjugates
- This is simple to show using Euler's formula so we will state it here for clarity

- For some $z \in \mathbb{C}$ we have that $z = re^{ix}$ for some $r, x \in \mathbb{R}$

Evaluating polynomials at complex coefficients

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$$(\overline{z})^n = (re^{-ix})^n \tag{1}$$
$$(\overline{z})^n = (re^{-nix}) \tag{2}$$
$$(\overline{z})^n = (re^{nix})^{-1} \tag{3}$$
$$(\overline{z})^n = (z^n)^{-1} \tag{4}$$
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$$\tag{6}$$

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- Then if we constructed a general polynomial with real coefficients we could use this fact to make our conclusion

Evaluating polynomials at complex coefficients:Exampe

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- $P(2 + i) = 4 + 4i$
- So we can see that they are conjugates

- Does this hold for polynomials with complex coefficients?

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- Does this hold for polynomials with complex coefficients?
- No to see why just consider that multiplying 2 complex conjugates by another complex number does not necessarily produce complex conjugates
- Now that we have shown this we can always say for any complex z and polynomial with real coefficients $P(x)$, $\overline{P(z)} = P(\overline{z})$
- Now lets return to CKKS and our map $\sigma : \frac{\mathbb{C}[X]}{X^N+1} \to \mathbb{C}^N$

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- Lets first define $R = \frac{\mathbb{Z}[X]}{X^N+1}$ to be the space of polynomials with integer coefficients

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- Recall that the roots of our cyclotomic are complex conjugates of each other.

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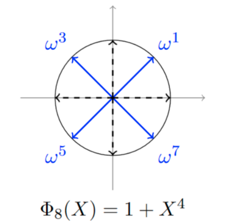
- 

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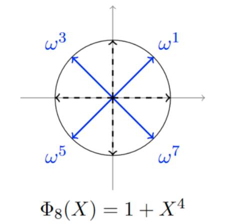
- 

- We can see that $\omega^1 = \overline{\omega^7}$ and $\omega^3 = \overline{\omega^5}$

Integer Polynomials map to vectors of complex conjugates

- Take some polynomial $P(x)$ with real coefficients and use it to transform the vector $\vec{z} = \langle \omega^1, \omega^3, \omega^5, \omega^7 \rangle$, we can see this satisfies the property $z_i = \overline{z_{4-i}}$

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- So we have that any polynomial with real coefficients maps to a vector of complex conjugates
- Based on how we chose to order the conjugates it specifically maps to a complex vector $\vec{z} \in \mathbb{C}^N$ s.t $z_i = \overline{z_{N-i}}$

The image of R

- Now consider the set of all complex vectors who's i-th coordinates are the complex conjugate of their $N - i$-th

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- Now stepping back we can see that we have reached our goal of defining the image of R. Namely it is a subset of $\mathbb{H}$

$$\sigma(R) \subset \mathbb{H} = \{z \in \mathbb{C}^N | z_i = \overline{z_{N-i}}\}$$

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- The inverse map $\pi(z) \in \mathbb{C}^{N/2}$ simply cuts the vector in half discarding the 2nd half of conjugates

- Now by composing these maps we can get from any complex vector to a polynomial with real coefficients

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- This is however where I get pretty lost but I will present the theory covered in the paper regardless

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- Now R has a $\mathbb{Z}$-basis $\{1, X, \ldots, X^{N-1}\}$

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- This means we have a set of basis vectors for $\sigma(R)$, that constitute a lattice
- From my understanding the goal is essentially to compute the closest lattice vector to our given input vector thereby transforming our input into a vector that maps to a polynomial in R

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- This operation/map is denoted $\lfloor \cdot \rceil_{\sigma(R)}$
- Then we now have a way to get $\mathbb{C}^{N/2} \to \sigma(R)$ via a composition of maps

$$\sigma^{-1} \circ \lfloor \pi^{-1}(\mathbf{z}) \rceil_{\sigma(R)} : \mathbb{C}^{N/2} \to \sigma(R)$$

Scaling to preserve precision

- They also note that the error resulting from rounding may destroy significant figures of the message so they recommend multiplying by a scaling factor Δ before rounding to preserve precision

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- This changes our current map to

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- The decoding procedure is just the inverse of the encoding procedure. We simply apply the inverse maps in reverse order to recover the encoded plaintext

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- Thank you for your time!