Jonathan Parlett

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- A simplified encoding scheme from polynomials with complex coefficients to vectors with complex coefficients
- The actual encoding scheme from polynomials with integer coefficients to vectors of complex coefficients.

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- This fact will become important when we are discussing the mapping of polynomials with integer coefficients to complex vectors

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- How do they achieve this map from complex vectors to polynomials of integer coefficients

- To start we will first understand the simpler map from complex vectors to polynomials with complex coefficients

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- The forward map may be easier to consider

$$\sigma : \frac{\mathbb{C}[X]}{X^N + 1} \to \mathbb{C}^N$$

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- Its clear from this equation that given one we can compute the other and since this is a square matrix there is one and only one solution
- so it should be intuitive that this transformation defines an isomorphism between $\mathbb{C}^4$ and $\frac{\mathbb{C}[X]}{X^4+1}$

Simplified encoding scheme

- At this point we have defined our simplified map and its inverse as essentially the equation we showed in the previous slide

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- The CKKS map/algorithm $\sigma^{-1} : \mathbb{C}^N \to \frac{\mathbb{Z}[X]}{X^N+1}$ adds further structure in order to place restrictions on this map to ensure that we encode our complex vectors as polynomials with integer coefficients only

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EVALUATING POLYNOMIALS AT COMPLEX COEFFICIENTS

- To understand something about the structure of our map well need the fact that evaluating polynomials with real coefficients at complex conjugates produces complex conjugates
- This should be straight forward if we understand the simpler statement that powers of complex conjugates are still complex conjugates
- This is simple to show using Euler's formula so we will state it here for clarity

- For some $z \in \mathbb{C}$ we have that $z = re^{ix}$ for some $r, x \in \mathbb{R}$

Evaluating polynomials at complex coefficients

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$$(\overline{z})^n = (re^{-ix})^n \tag{1}$$
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$$(\overline{z})^n = (re^{nix})^{-1} \tag{3}$$
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$$\tag{6}$$

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- Then if we constructed a general polynomial with real coefficients we could use this fact to make our conclusion

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- Does this hold for polynomials with complex coefficients?
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- Now that we have shown this we can always say for any complex z and polynomial with real coefficients $P(x)$, $\overline{P(z)} = P(\overline{z})$
- Now lets return to CKKS and our map $\sigma : \frac{\mathbb{C}[X]}{X^N+1} \to \mathbb{C}^N$

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- So more formally our goal is to define the image of R under our transformation σ denoted $\sigma(R)$
- To map some $P(x) \in R$ to $\mathbb{C}^N$ we evaluate at the roots of our cyclotomic

Integer Polynomials map to vectors of complex conjugates

- We eventually want to figure out how to map our complex vectors to polynomials with integer coefficients only
- So a good place to start would be looking at what integer coefficient polynomials map to under our current transformation σ^{-1}
- Lets first define $R = \frac{\mathbb{Z}[X]}{X^N+1}$ to be the space of polynomials with integer coefficients
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- Recall that the roots of our cyclotomic are complex conjugates of each other.

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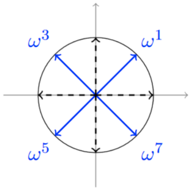
- 

$\Phi_8(X) = 1 + X^4$

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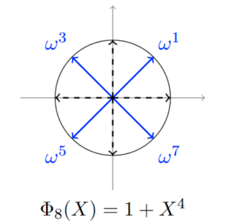
- 

- We can see that $\omega^1 = \overline{\omega^7}$ and $\omega^3 = \overline{\omega^5}$

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- So we have that any polynomial with real coefficients maps to a vector of complex conjugates
- Based on how we chose to order the conjugates it specifically maps to a complex vector $\vec{z} \in \mathbb{C}^N$ s.t $z_i = \overline{z_{N-i}}$

- Now consider the set of all complex vectors who's i-th coordinates are the complex conjugate of their $N - i$-th

$$\mathbb{H} = \{z \in \mathbb{C}^N | z_i = \overline{z_{N-i}}\} \subset \mathbb{C}^N$$

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- Now stepping back we can see that we have reached our goal of defining the image of R. Namely it is a subset of $\mathbb{H}$

$$\sigma(R) \subset \mathbb{H} = \{z \in \mathbb{C}^N | z_i = \overline{z_{N-i}}\}$$

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- The inverse map $\pi(z) \in \mathbb{C}^{N/2}$ simply cuts the vector in half discarding the 2nd half of conjugates

- Now by composing these maps we can get from any complex vector to a polynomial with real coefficients

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- This is however where I get pretty lost but I will present the theory covered in the paper regardless

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- From my understanding the goal is essentially to compute the closest lattice vector to our given input vector thereby transforming our input into a vector that maps to a polynomial in R

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- This operation/map is denoted $\lfloor \cdot \rceil_{\sigma(R)}$
- Then we now have a way to get $\mathbb{C}^{N/2} \to R$ via a composition of maps

$$\sigma^{-1} \circ \lfloor \pi^{-1}(\mathbf{z}) \rceil_{\sigma(R)} : \mathbb{C}^{N/2} \to R$$

- They also note that the error resulting from rounding may destroy significant figures of the message so they recommend multiplying by a scaling factor Δ before rounding to preserve precision

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- This changes our current map to

$$\sigma^{-1} \circ \lfloor \Delta \cdot \pi^{-1}(\mathbf{z}) \rceil_{\sigma(R)} : \mathbb{C}^{N/2} \rightarrow \sigma(R)$$

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- The decoding procedure is just the inverse of the encoding procedure. We simply apply the inverse maps in reverse order to recover the encoded plaintext

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\mathbb{C}^{\phi(M)/2} & \xrightarrow{\pi^{-1}} & \mathbb{H} & \xrightarrow{\lfloor\cdot\rceil_{\sigma(\mathcal{R})}} & \sigma(\mathcal{R}) & \xrightarrow{\sigma^{-1}} & \mathcal{R} \\
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- Thank you for your time!