Long Paths in Polynomial Divisor Graphs

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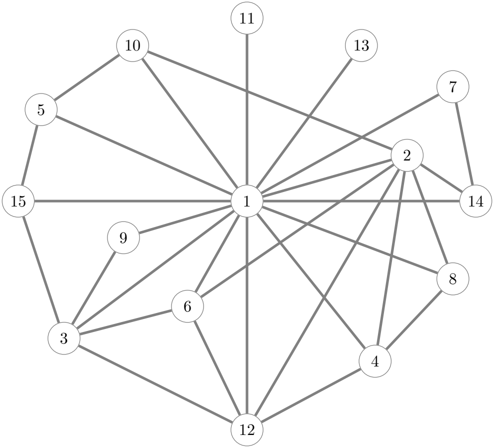
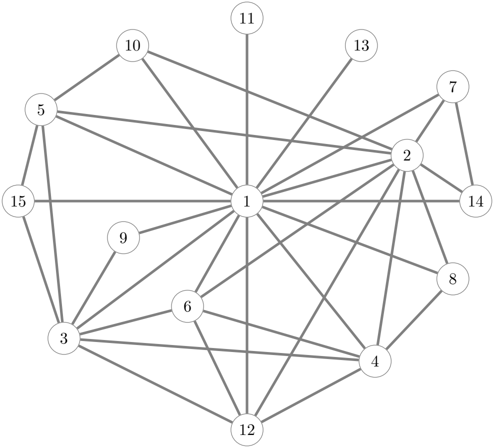
Divisor Graphs

A divisor graph, $D(n)$ contains vertices $\{1, 2, ..., n\}$ and an edge between two vertices, u and v, if $u|v$ or $v|u$.



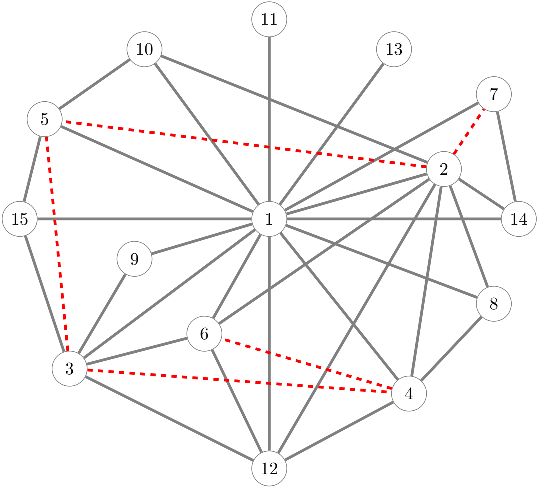
LCM Graphs

An LCM graph, $L(n)$ contains vertices $\{1, 2, \ldots, n\}$ and an edge between two vertices, u and v, if $[u, v] \leq n$.



LCM Graphs

Note: $D(n)$ is a subgraph of $L(n)$ because if $u|v$, $[u, v] = v \leq n$.



Previous Results

Let $f(n), g(n)$ denote the length of the longest path in $D(n), L(n)$ respectively. Note that $f(n) \le g(n)$. $f(n)$ has been previously studied by Pollington, Pomerance, Tenenbaum, and Saias. In particular Saias shows that

Theorem

For sufficiently large n there exist constants c_1, c_2 s.t

$$c_1 \frac{n}{\log n} \le f(n) \le g(n) \le c_2 \frac{n}{\log n}.$$

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We are interested in the analogous question for polynomials over a finite field (also called function fields).

Some Notation

- Let $\mathbb{F}_q$ be a finite field of order q. For example $\mathbb{Z}_2 = \{0, 1\}$ with addition and multiplication mod 2 is the finite field of order 2.

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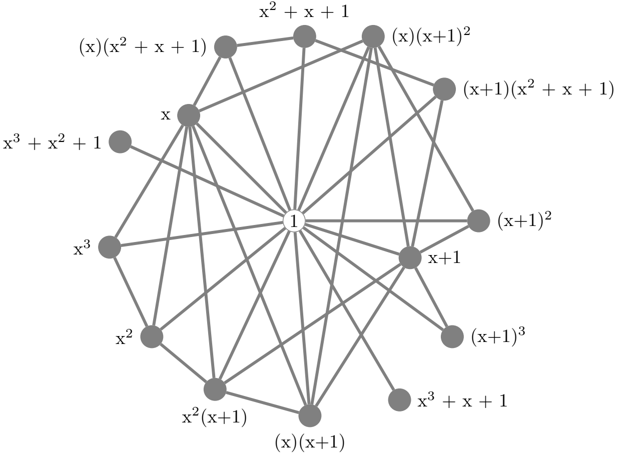
$$\mathcal{M}_q^n = \left\{\text{monic } F \in \mathbb{F}_q[x] : \deg F = n\right\}$$

- We denote the polynomials with degree at most n by

$$\mathcal{M}_q^{\leq n} = \left\{\text{monic } F \in \mathbb{F}_q[x] : \deg F \leq n\right\}$$

Polynomial Divisor Graphs

The Polynomial Divisor Graph $D_q(n)$ has vertices $\mathcal{M}_q^{\leq n}$.
Below is the case $D_2(3)$ with vertices $\mathcal{M}_2^{\leq 3}$.



Result

Let $f_q(n), g_q(n)$ to be the longest path in the polynomial divisor graph $D_q(n)$, and the polynomial LCM graph $L_q(n)$ respectively. We will show that

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We don't yet have a complete proof for the upper bound.

Polynomials (Why do we care?)

Polynomials over a finite field have similar properties to the integers.

Integers	Polynomials
$n \in \mathbb{N}$	monic $F \in \mathbb{F}_q[x]$
$\log n$	$\deg F$
$\lvert n \rvert = n$	$\lvert\lvert f \rvert\rvert = q^{\deg F}$
Primes, p	Irreducible polynomials, P
$\pi(n) = \#\{p \leq n : p \text{ prime}\}$	$\pi_q(n) = \#\left\{ \begin{array}{c} P \in \mathcal{M}_q^n : \\ P \text{ irreducible} \end{array} \right\}$
Unique factorization into primes	Unique factorization into irreducibles

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- $\Phi(F)$ has connections to many important questions in number theory.

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- For example, over $\mathbb{Z}_2$, x^4 does while $x^2 + x + 1$ does not.

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- *Answer:* Exactly when $\Phi(F) - \deg F \leq 1$.

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- Then the difference in degree of consecutive divisors, $\deg D_{i+1} - \deg D_i$, is at most m exactly when $\Phi(F) - \deg F \leq m$.
- This difference is also called the divisor gap of F.

Divisor Gaps

- Andreas Weingartner provides estimates for

$$\#\left\{F \in \mathcal{M}_q^n : \Phi(F) \leq n + m\right\}.$$

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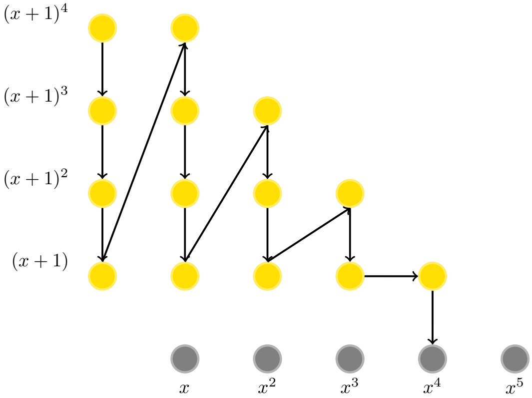
- We will see later that the size of our longest path is limited by this set.
- Now we will show the main ideas of the proofs starting with the lower bound.

Dot Diagram

The following base case represents $\Gamma(6, x+1)$. The path is constructed by connecting multiples of irreducible polynomials that are adjacent in our ordering.



Polynomial Path

Our polynomial path is

$$\Gamma(d,P) = \begin{cases} 1 \to \Gamma_0 \to \Gamma_1 \to \dots & \\ \to \Gamma_b \to^* \Gamma(d,P^\dagger) & P \le P_d \text{ and } b = a^\dagger \\ 1 \to \Gamma_0 \to \Gamma_1 \to \cdots \to \Gamma_b & \\ \to PP^{\dagger a^\dagger} \to^* \Gamma(d,P^\dagger) & P \le P_d \text{ and } b \neq a^\dagger \\ \Gamma(d,P_d) & P > P_d \end{cases}$$

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- Base cases are $\Gamma(d,x)$ and $\Gamma(d,x+1)$.
- Assume the conditions hold for any irreducibles $< P$.

Lower Bound Proof: Conditions

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3. $\Gamma(d, P)$ consists of all polynomials F such that $P^+(F) \leq P$ and $\Phi(F) \leq d - 1$.

Lower Bound Proof: Counting $\Gamma(d, P)$

- Since our path $\Gamma(n, P)$ contains all polynomials $F \in \mathcal{M}_q^{\leq n}$ such that $\Phi(F) \leq n-1$ we have that

$$\Gamma(n, P) \supseteq \left\{ F \in \mathcal{M}_q^{n-2} : \Phi(F) \leq n-1 \right\}.$$

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- Using the results of Weingartner discussed previously we can show that

$$|\Gamma(n, P)| \geq \# \left\{ F \in \mathcal{M}_q^{n-2} | \Phi(F) \leq n - 1 \right\}$$
$$\geq q^{n-2} \cdot \frac{c}{n-1} \geq c' \frac{q^n}{n}.$$

Where $c' = \frac{c}{q^2}$

Upper-Bound: $\mathcal{A}(n,m), \mathcal{B}(n,m)$

- $\mathcal{A}(n,m) = \left\{F \in \mathcal{M}_q^{\leq n} : \Phi(F) \leq n+m\right\}$. This set is closely related to the set of polynomials with divisor gap at most m.

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- So we can count them using Weingartner! $|\mathcal{A}(n,m)| \le c\frac{q^n}{n}$ for some constant c.

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- $\mathcal{B}(n,m) = \left\{ \substack{F \in \mathcal{M}_q^{\leq n} : F \notin \mathcal{A}(n,m) \\ \text{but any proper divisor } G \text{ of } F \text{ in } \mathcal{A}(n,m)} \right\}$.

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- There are two important facts that characterize $\mathcal{B}(n,m)$.

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- If $B_1 \neq B_2 \in \mathcal{B}(n,m)$, then $\deg[B_1, B_2] > n+m$.
- The LCMs of distinct B are large.

Upper-bound: Sketch

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- First we remove all vertices in $\mathcal{A}(n)$.

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- Let $\mathcal{S}(\mathcal{P})$ be the set of all subpaths.
- $$|\mathcal{P} \setminus \mathcal{A}(n)| = \sum_{\mathcal{S} \in \mathcal{S}(\mathcal{P})} |\mathcal{S}|$$

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- For each $\mathcal{S} = \left(F_i, F_{i+1}, \ldots, F_j\right)$, since $F_k \notin \mathcal{A}(n)$, there is $B_k | F_k$.

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- So for each $\mathcal{S}$, there is a $B(\mathcal{S}) \in \mathcal{B}(n)$ that divides everything in the subpath.
- This means we can break up our sum according to the B-values of the paths!

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- $|\mathcal{P} \setminus \mathcal{A}(n)| = \sum_{\mathcal{S} \in \mathcal{S}(\mathcal{P})} |S| = \sum_{B \in \mathcal{B}(n)} \sum_{\substack{\mathcal{S} \in \mathcal{S}(\mathcal{P}) \\ B(\mathcal{S}) = B}} |\mathcal{S}|$

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- We can bound the inner sum $\sum_{\substack{\mathcal{S} \in \mathcal{S}(\mathcal{P}) \\ B(\mathcal{S}) = B}} |\mathcal{S}|$ in two different ways.
- Since every vertex in these subpaths is divisible by B,

$$\sum_{\substack{\mathcal{S} \in \mathcal{S}(\mathcal{P}) \\ B(\mathcal{S}) = B}} |\mathcal{S}| \leq \#\left(\text{multiples of } B\right) \leq q^{n - \deg B}$$

Upper-bound: Sketch

- The 2nd way requires an observation. Each of these $\mathcal{S}$ implies a path in the LCM graph $L_q(n - \deg B)$

$$\mathcal{S} = \left(F_i, F_{i+1}, \ldots, F_j\right) = \left(BG_i, BG_{i+1}, \ldots, BG_j\right)$$

So $\left(G_i, G_{i+1}, \ldots, G_j\right)$ is a path in $L_q(n - \deg B)$.

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- So really we are summing over a collection of paths in $L_q(n - \deg B)$!
- If we know how many paths we have, does that help? Let $K(B) = \left|\left\{\mathcal{S} \in \mathcal{S}(\mathcal{P}) : B(\mathcal{S}) = B\right\}\right|$.

Upper-bound: Sketch

$$\sum_{\substack{\mathcal{S}\in\mathcal{S}(\mathcal{P})\\ B(\mathcal{S})=B}} |\mathcal{S}| \leq \#\begin{pmatrix}\text{vertices we can cover in } L_q(n-\deg B)\\ \text{with } K(B) \text{ paths}\end{pmatrix}$$

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- We can break up our sum strategically so that each of our bounds is as small as possible.

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$$\sum_{\substack{B\in\mathcal{B}(n)\\ K(B),\deg B \text{ are small}}} \sum_{\substack{\mathcal{S}\in\mathcal{S}(\mathcal{P})\\ B(\mathcal{S})=B}} |\mathcal{S}| + \sum_{\substack{B\in\mathcal{B}(n)\\ \text{one of } K(B),\deg B \text{ is big}}} \sum_{\substack{\mathcal{S}\in\mathcal{S}(\mathcal{P})\\ B(\mathcal{S})=B}} |\mathcal{S}|$$

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$$\sum_{B\in\mathcal{B}(n)} \sum_{\substack{\mathcal{S}\in\mathcal{S}(\mathcal{P})\\ B(\mathcal{S})=B \\ K(B),\deg B \text{ are small}}} |\mathcal{S}| + \sum_{B\in\mathcal{B}(n)} \sum_{\substack{\mathcal{S}\in\mathcal{S}(\mathcal{P})\\ B(\mathcal{S})=B \\ \text{one of } K(B),\deg B \text{ is big}}} |\mathcal{S}|$$

- When we don't have too many subpaths, we can use our first bound.
- In the other case, we use our multiples bound.

Upper-bound: Sketch

- Thus we obtain sums of the form

$$\sum_{\substack{A \in \mathcal{A}(n) \\ \text{+other constraints}}} h(n, A)$$

where h depends on n and A only.

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$$\sum_{\substack{A \in \mathcal{A}(n) \\ +\text{other constraints}}} h(n, A)$$

  where h depends on n and A only.
- We are close to bounding these sums using the results of Weingartner to count the A's given our extra constraints.

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