Introduction to Measurement Error and Misclassification

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Introduction

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  - Discrepancies between the 'true' and the observed value
  - The result of poorly defined construct and/or an imperfect measurement process
- Examples in the Social Sciences
  - Elusive constructs, loosely defined:
    e.g. happiness, ethnicity, political decentralisation
  - Subjectively elicited data:
    e.g. survey data, affected by memory failures (*when was the last time you went to a pub?*), social desirability (*for how long have you been unemployed?*), or content analysis affected by inter-rater unreliability
  - Administrative/official data used as proxies:
    e.g. using earnings to measure poverty, or measuring violent crime from police records
- Question: Can you think of examples form your research area?

Introduction

- Why does it matter?
  - We cannot describe reality accurately

    e.g. What is the true prevalence of covid? What is the true extent of property crime? Has it increased compared to last year? Is it higher in Leeds than in Bradford?
  - But also, our causal inferences will be biased

    e.g. Does education affect violent crime? To what extent a guilty plea lowers the probability of receiving a custodial sentence?

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    e.g. Does education affect violent crime? To what extent a guilty plea lowers the probability of receiving a custodial sentence?
- There are ways to anticipate the prevalence and impact of measurement error
  - And to some extent adjust for it
  - But to do so we first need to define these errors formally using measurement error models

Defining Measurement Error Formally

- The classical measurement error model (random errors)
  - $\overbrace{X^*}^{observed} = \overbrace{X}^{true\ value} + \overbrace{U}^{noise}$
  - with the errors taken to be randomly distributed, $U \sim N(0, \sigma_U)$

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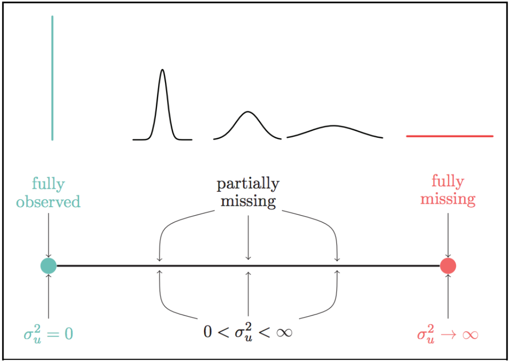
- Only the variance is affected
  - $\sigma^2_{X^*} = \sigma^2_X + \sigma^2_U$; but the mean is unaffected since $E(U) = 0$
  - Taking repeated observations we can estimate the prevalence of classical measurement error
  - The reliability ratio: $\rho_{X^*} = \dfrac{\sigma^2_X}{\sigma^2_X + \sigma^2_U} = \dfrac{\text{true variability}}{\text{observed variability}}$

Conceptualising Random Errors



Source: Blackwell et al. (2017)

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  - Question: Can you think of any examples?
    E.g. crime recorded by the police, self-reported unemployment spells
- Repeated observations won't pick up systematic errors
  - We need a *gold standard* (at least for a subgroup of our sample)
  - E.g. Unemployment register, victimisation surveys

Multiplicative Errors

- What if the error is proportional to the true value of the quantity being measured?
  - E.g. memory failures;

    How many alcohol units do you drink per week?

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    How many alcohol units do you drink per week?

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    How many sex partners have you had in your life?
- These can be better specified using a multiplicative rather than an additive model
  - I.e., as $X^* = X \cdot U$, rather than $X^* = X + U$,

    with $E(U) = 1$ if random, and $E(U) \neq 1$ if systematic

Misclassification

- Up to now we have taken examples from continuous variables
- What if the variable affected by measurement error is discrete?
  - E.g. self-reported work status, ethnicity determined through subjects' names
- We have a misclassification problem
  - which, for a binary variable, can be specified through a 2x2 misclassification matrix

$$\begin{cases} P(X^* = 1|X = 1) = \theta_{1|1}; & \text{Sensitivity} \\ P(X^* = 0|X = 0) = \theta_{0|0}; & \text{Specificity} \end{cases}$$

$$\begin{cases} P(X^* = 1|X = 0) = \theta_{1|0}; & \text{Probability false positive} \\ P(X^* = 0|X = 1) = \theta_{0|1}; & \text{Probability false negative} \end{cases}$$

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  - Could be seen in hierarchical or longitudinal data; e.g. interviewer effects
- <u>Differential errors</u> (errors correlated with the residuals form the outcome model)
  - E.g. in studying the effect of employment status on self-esteem we should consider that the probability of misreporting employment status is correlated with self-esteem

Multiple Error Mechanisms

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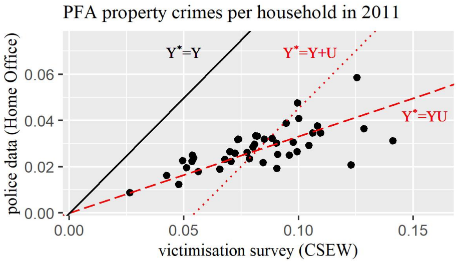
- This is what we found in the Recounting Crime project
  - Measurement error in police recorded crime rates
- We defined this measurement error as:
  - *systematic*, since not all crime is reported to the police
  - *random*, subject to variability across areas, as a result of the different recording practices across police forces
  - *multiplicative*, errors seem proportional to the true extent of crime in the area

Multiplicative Errors: Crime Rates



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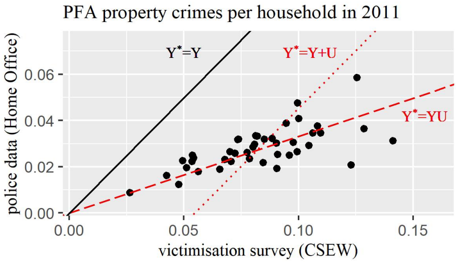

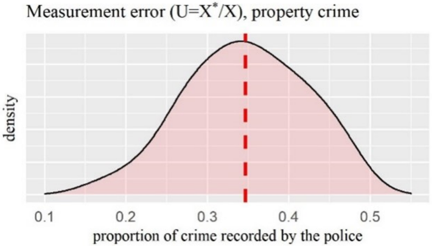

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- But how do (non-differential) errors affect estimates from multivariate (regression) models?
- Assuming only one variable is prone to measurement error, its impact will depend on:
  1. the outcome model (whether linear, Poisson, etc.)
  2. the measurement error model (additive, random, etc.)
  3. where is the affected variable introduced in the model (as a response or an explanatory variable)

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- Question: Will β be biased in any of those scenarios?

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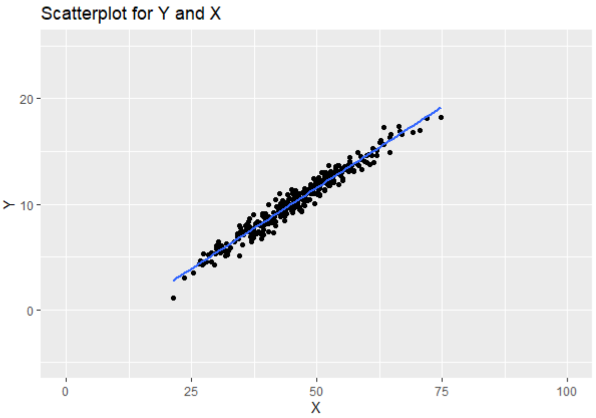

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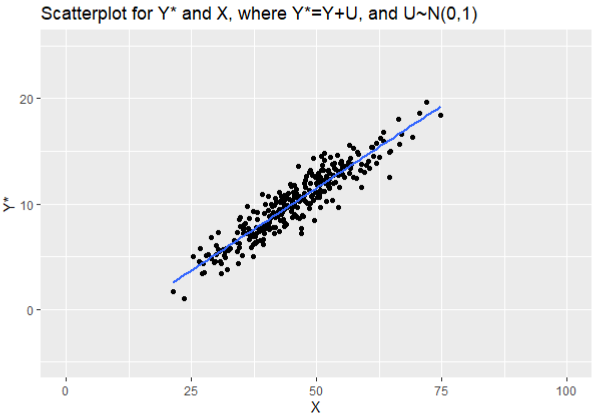

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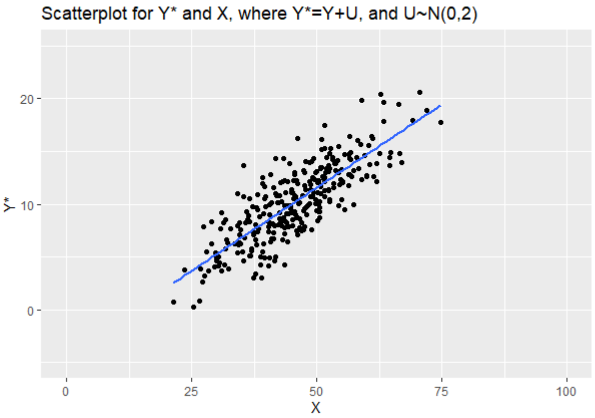

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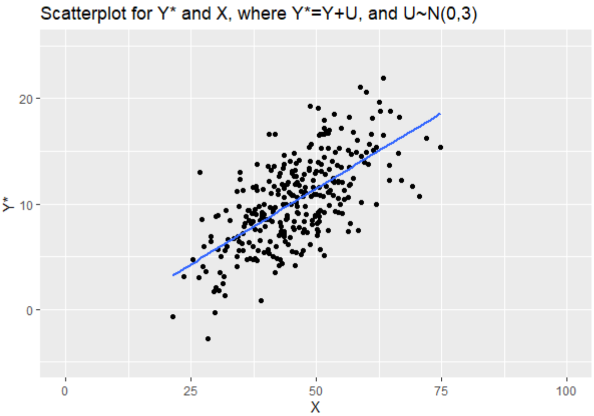

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Using the properties of the covariance and the variance we can see how the former is not affected by random noise, but the latter is:

$$\sigma_{X^*,Y} = \sigma_{X+U,Y} = \sigma_{X,Y} + \sigma_{U,Y} = \sigma_{X,Y}$$
$$\sigma^2_{X^*} = \sigma^2_{X+U} = \sigma^2_X + \sigma^2_U + \sigma_{X,U} = \sigma^2_X + \sigma^2_U$$

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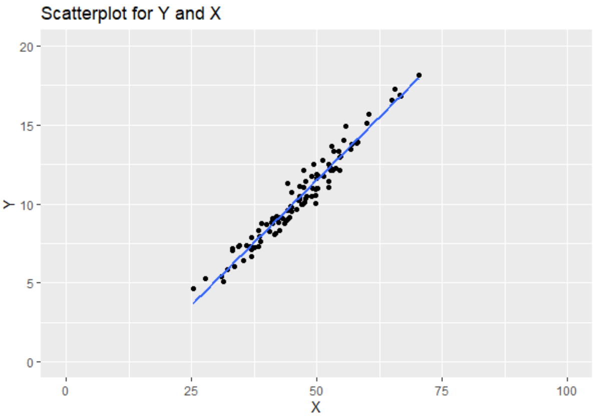
Effect of Random Measurement Error



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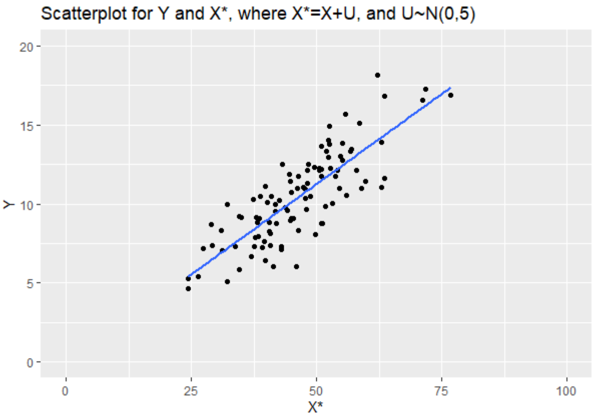

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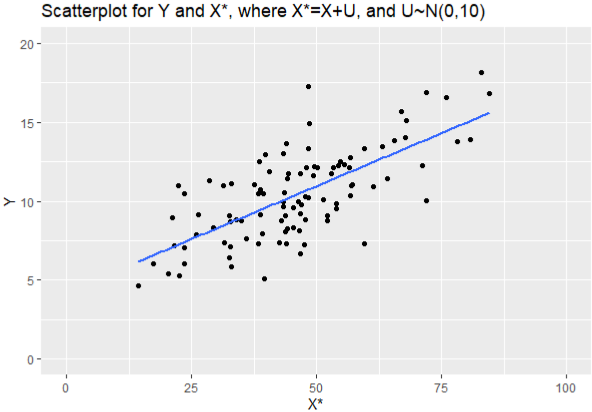

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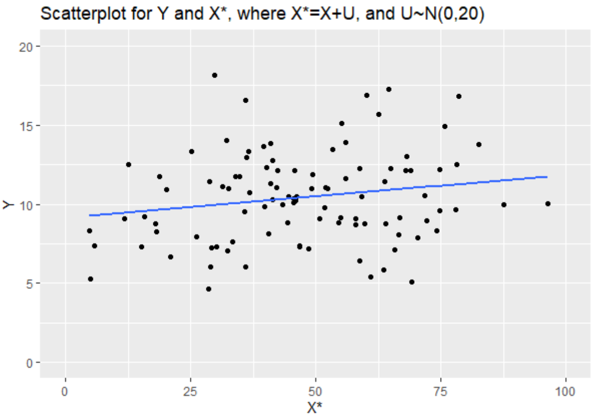

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    $Y \cdot U = \alpha + \beta X + \epsilon$

    $$Y = \frac{\alpha + \beta X + \epsilon}{U}$$
  - All regression coefficients are biased

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    Assume multiple linear regression, with one error-prone variable

    $Y = \alpha + \beta_1 X^* + \beta_2 Z + \epsilon$

    Both slopes will be biased; we will estimate $\widehat{\beta}_1^*$ and $\widehat{\beta}_2^*$ instead

    $$\widehat{\beta}_1^* = \widehat{\beta}_1 \left(\frac{\sigma^2_{X|Z}}{\sigma^2_{X|Z} + \sigma^2_U} \right) = \widehat{\beta}_1 \rho'_{X^*}$$

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    $$\widehat{\beta}_2^* = \widehat{\beta}_2 + \widehat{\beta}_1 (1 - \rho'_{X^*}) \Gamma_Z$$
  - Much harder to assess if we consider more complex models: non-linear, two-stage processes, systems of equations, etc.
  - For most 'real-world' applications we won't be able to trace out the measurement error induced biased algebraically
  - And we have not even consider how measurements of uncertainty are also affected

Impact of Measurement Error

- In the words of Nugent et al. (2000: 60):
  - *"Measurement error is, to borrow a metaphor, a gremlin hiding in the details of our research that can contaminate the entire set of estimated regression parameters"*