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Problem Set 1

## Problem [redacted]2

COLLABORATION: On problem 2, I collaborated with no one, and received help from Alex Willisson and Robert Sloan, and referred to wolframalpha.com, wikipedia.org, and mathisfun.com.

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using template on pages 35-36 of text

Proof. By contradiction assume that $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ is false. Then, some nonnegative integers serve as counterexamples to it. Let's collect them in a set:

$$C ::= \left\{ n \in \mathbb{N} \;\middle|\; \sum_{k=0}^{n} k^2 \neq \frac{n(n+1)(2n+1)}{6} \right\}. \qquad \not\exists$$

Assuming there are counterexamples, C is a nonempty set of nonnegative integers. So, by the ~~well ordering principle~~ Well-Ordering Principle, C has a minimum element, which we'll call c. Among the nonnegative integers, c is the smallest counterexample.

Since c is the smallest counterexample we know that $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ is false for $n=c$ but true for all nonnegative integers $n<c$. But it is true for $n=0$, so $c>0$. This means $c-1$ is a nonnegative integer, and since it is less than c the equation must be true for it. That is,

$$1^2+2^2+3^2+\ldots+(c-1)^2 = \frac{(c-1)c(2c-1)}{6} = \frac{c(c-1)(2c-1)}{6}$$

But then, adding $c^2$ to both sides, we get

$$1^2+2^2+3^2+\ldots+(c-1)^2+c^2 = \frac{c(c-1)(2c-1)}{6} + c^2$$

$$\frac{c(c-1)(2c-1)}{6} + \frac{6c^2}{6} = \frac{(c^2-c)(2c-1)+6c^2}{6} = \frac{2c^3-2c^2-c^2+c+6c^2}{6}$$

$$= \frac{2c^3+3c^2+c}{6} = \frac{c(c+1)(2c+1)}{6}$$

which means $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ does hold for c.

This is a contradiction, so C must be empty, so there are no counterexamples and $\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ is true. $\square$