Mathematical Odds and Ends

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Contents

1 Trigonometric Identities and Relationships

1.1 Euler's Formula

Euler's formula is given by

$$e^{ix} = \cos x + i \sin x \tag{1}$$

where e is the base of the natural logarithm and $i = \sqrt{-1}$. Many of the standard trigonometric identities can be quickly derived from this one formula.

1.2 Additive Identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \tag{2}$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \tag{3}$$

1.3 Law of Sines and Cosines

Given the triangle shown in Figure 1, the *law of sines* states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{ constant } . \tag{4}$$

The *law of cosines* states that

$$c^2 = a^2 + b^2 - 2ab \cos C \tag{5}$$

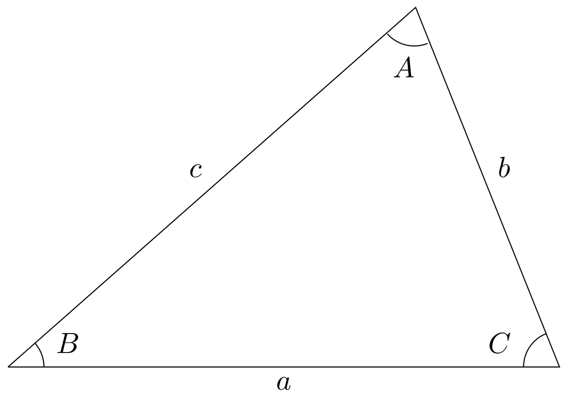

Figure 1: A triangle.

2 Special Functions

2.1 Delta Functions

The Dirac delta function is a generalized function defined by

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a) \tag{6}$$

for some function $f(x)$, and

$$\delta(x-a) = \begin{cases} 0, & x \neq a, \\ \text{undefined}, & x = a \end{cases} . \tag{7}$$

The fact that the Delta function is a *generalized* function means that it is only really defined with respect to integration. Pragmatically this presents no difficulty if it used in probability density functions or other functions that must be integrated to yield physically meaningful results.

The discrete analog of the Dirac delta function is the Kronecker delta, defined

$$\delta_{i,j} = \begin{cases} 0 & i \neq j, \\ 1 & i = j. \end{cases} \tag{8}$$

2.2 Legendre Polynomials

The Legendre polynomials are a set of polynomials, $\{P_n\}_{n=0}^{\infty}$. For a given n, the polynomial $P_n(x)$ is a polynomial of degree n. For example, the first few Legendre polynomials are

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right), \\ P_3(x) &= \frac{1}{2}\left(5x^3 - 3x\right), \\ P_4(x) &= \frac{1}{8}\left(34x^4 - 30x^2 + 3\right). \end{aligned}$$

The Legendre polynomials are orthogonal over the interval $(-1, 1)$ and satisfy the following orthogonality relationship:

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{n,m}, \tag{9}$$

where $\delta_{n,m}$ is the Kronecker delta.

The polynomials also satisfy the following two recurrence relations:

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0, \tag{10}$$

$$\left(1 - x^2\right) \frac{dP_n}{dx} = -nxP_n(x) + nP_{n-1}(x). \tag{11}$$

The Legendre polynomials form a complete set over the interval $[-1, 1]$. This means that a bounded, piecewise-continuous function $f(x)$ may be written as linear combination of the Legendre polynomials. Namely,

$$f(x) = \sum_{n=0}^{\infty} f_n P_n(x), \quad x \in [-1, 1] \tag{12}$$

where

$$f_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx. \tag{13}$$

3 The Direction Vector

The unit direction vector can be expressed in terms of a polar angle, θ, and a azimuthal angle, φ, as shown in Figure 2. In a Cartesian coordinates the unit direction vector can be written

$$\hat{\Omega} = \Omega_x \mathbf{i} + \Omega_y \mathbf{j} + \Omega_z \mathbf{k} \tag{14}$$

where

$$\Omega_x = \sin\theta \cos\varphi, \tag{15a}$$

$$\Omega_y = \sin\theta \sin\varphi, \tag{15b}$$

$$\Omega_z = \cos\theta. \tag{15c}$$

The differential solid angle can be written

$$d\hat{\Omega} = \sin\theta d\varphi d\theta. \tag{16}$$

The direction vector is said to subtend 4π steradians, as

$$\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi = 4\pi. \tag{17}$$

For this reason, integration over all solid angles is commonly denoted $\int_{4\pi} d\hat{\Omega}$. It is common to make the change of variable $\theta \to \mu = \cos\theta$. In this case

$$\int_{4\pi} d\hat{\Omega} = \int_{-1}^{1} d\mu \int_0^{2\pi} d\varphi. \tag{18}$$

It is common for directions to be described probabilistically, given some probability density $p\left(\hat{\Omega}\right)$. For example, if a direction is selected isotropically, then we have

$$p\left(\hat{\Omega}\right) = \frac{1}{4\pi}. \tag{19}$$

In one-dimensional problems the isotropic distribution can be reduced to a function of θ alone. This is done by integrating over the circles of radius $\sin\theta$ formed by tracing out φ over 2π angles for each θ:

$$p(\theta) = \int_0^{2\pi} p\left(\hat{\Omega}\right) \sin\theta d\varphi = \frac{1}{2}\sin\theta. \tag{20}$$

This distribution, for example, describes isotropic neutron emission in a CM scattering event.

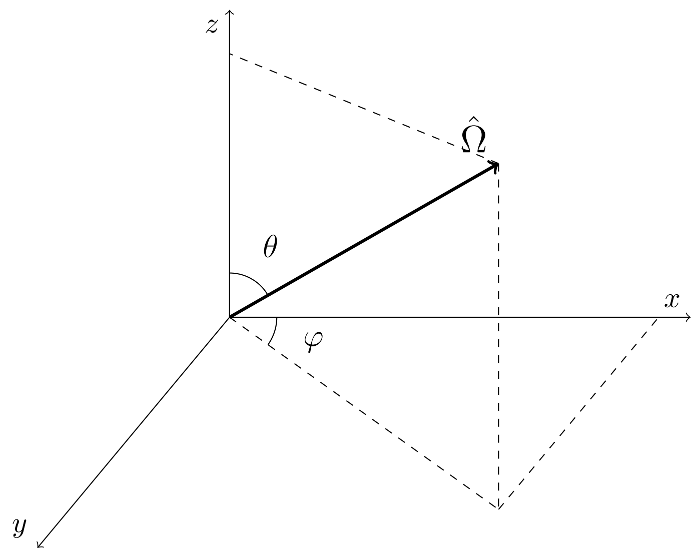

Figure 2: Description of the unit direction vector.