Multiview Clustering

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Abstract

Index Terms

I. MODEL

$$\sum_{v=1}^{V}\left\|X^{(v)}-V^{(v)}U^{\mathrm{T}}\right\|_{2,1}+\alpha\sum_{v=1}^{V}(\beta_v)^r\operatorname{tr}(U^{\mathrm{T}}L^{(v)}U) \quad s.t. U^{\mathrm{T}}U=I, U\geqslant 0 \tag{1}$$

II. OPTIMIZATION

The objective function in above is not convex in four variables but is convex if we update the $2V+1$ variables alteratively. Thus, we use Alternating Direction Method of Multiplier(ADMM) to optimize the objective function. by introducing four auxiliary variables $E^{(v)} = X^{(v)} - V^{(v)}U^{\mathrm{T}}$ and $Z_1 = U$. The objective function can be rewritten into the following equivalent problem:

$$\arg\min_{U,V^{(v)},E_v,Z_1}\sum_{v=1}^{V}\left\|E^{(v)}\right\|_{2,1}+\alpha\sum_{v=1}^{V}(\beta_v)^r\operatorname{tr}(Z_1^{\mathrm{T}}L^{(v)}U) \quad s.t. E^{(v)}=X^{(v)}-V^{(v)}U^{\mathrm{T}}, Z_1=U, U^{\mathrm{T}}U=I, Z_1\geqslant 0 \tag{2}$$

which can be solved by the following ADMM problem

$$\begin{aligned}\min_{U,V^{(v)},E^{(v)},Z_1,\lambda^{(v)},\mu} \sum_{v=1}^{V} \left\|E^{(v)}\right\|_{2,1} + \alpha \sum_{v=1}^{V} (\beta_v)^r \operatorname{tr}(Z_1^{\mathrm{T}} L^{(v)} U) \\ + \sum_{v=1}^{V} < \lambda^{(v)}, X^{(v)} - V^{(v)} U^{\mathrm{T}} - E^{(v)} > + < \lambda_1, Z_1 - U > \\ + \frac{\mu}{2} (\|Z_1 - U\|_F^2 + \sum_{v=1}^{V} \left\|X^{(v)} - V^{(v)} U^{\mathrm{T}} - E^{(v)}\right\|_F^2) \\ s.t. U^{\mathrm{T}} U = I, Z_1 \geqslant 0\end{aligned} \tag{3}$$

Update $E^{(v)}$: To update $E^{(v)}$, we fixed other variables except $E^{(v)}$ and remove terms that are irrelevant to $E^{(v)}$. The the objective function becomes

$$\min_{E^{(v)}} \frac{1}{2} \left\|E^{(v)} - (X^{(v)} - V^{(v)} U^{\mathrm{T}} + \frac{1}{\mu} \lambda^{(v)})\right\|_F^2 + \frac{1}{\mu} \left\|E^{(v)}\right\|_{2,1} \tag{4}$$

Let$B = X^{(v)} - V^{(v)} U^{\mathrm{T}} + \frac{1}{\mu} \lambda^{(v)}$, then $E^{(v)}$ can be update as

$$e_{vi} = \begin{cases} (1 - \frac{1}{\mu \|b_i\|}) b_i, & if \ \|b_i\| \geq \frac{1}{\mu} \\ 0, & otherwise \end{cases} \tag{5}$$

Update $V^{(v)}$:

To update $V^{(v)}$, we fix other variables except $V^{(v)}$, we obtain the following objective function:

$$\min_{V^{(v)}} \frac{\mu}{2} \left\|X^{(v)} - V^{(v)} U^{\mathrm{T}} - E^{(v)} + \frac{1}{\mu} \lambda^{(v)}\right\|_F^2 \tag{6}$$

Considering that $U^{\mathrm{T}} U = I$, we can rewrite the above objective function:

$$\min_{V^{(v)}} \frac{1}{2} \left\|V^{(v)} - (X^{(v)} - E^{(v)} + \frac{1}{\mu} \lambda^{(v)}) U\right\|_F^2 \tag{7}$$

then $V^{(v)} = (X^{(v)} - E^{(v)} + \frac{1}{\mu} \lambda^{(v)}) U$.

Update Z_1:

$$\min_{Z_1 \geq 0} \frac{\mu}{2} \|Z_1 - U\|_F^2 + < \lambda_1, Z_1 - U > + \alpha \sum_{v=1}^{V} (\beta_v)^r \operatorname{tr}(Z_1^{\mathrm{T}} L^{(v)} U) \tag{8}$$

we obtain

$$\min_{Z_1 \geq 0} \|Z_1 - K\|_F^2 \tag{9}$$

where $K = (U - \frac{1}{\mu}\lambda_1 - \frac{\alpha}{\mu}\sum_{v=1}^{V}(\beta_v)^r L^{(v)}U)$. The above object function can be further decomposed to element-wise optimization problem as

$$\min_{Z_{1ij} \geq 0} \left\| Z_{1ij} - K_{ij} \right\|_F^2 \tag{10}$$

Therefor, the optimal solution of above problems is

$$Z_{1ij} = \max(K_{ij}, 0) \tag{11}$$

Update U:

$$\begin{aligned} &\min_{U^{\mathrm{T}}U=I} < \lambda_1, Z_1 - U > + \sum_{v=1}^{V} < \lambda^{(v)}, X^{(v)} - V^{(v)}U^{\mathrm{T}} - E^{(v)} > \\ &+ \frac{\mu}{2}(\|Z_1 - U\|_F^2 + \sum_{(v=1)}^{V} \left\| X^{(v)} - V^{(v)}U^{\mathrm{T}} - E^{(v)} \right\|_F^2) + \alpha \sum_{v=1}^{V} (\beta_v)^r \operatorname{tr}(Z_1^{\mathrm{T}} L^{(v)} U) \end{aligned} \tag{12}$$

removing the irrelevant terms, we arrive at

$$\min_{U^{\mathrm{T}}U=I} \frac{\mu}{2} \|U\|_F^2 - \mu < H, U > \tag{13}$$

where

$$H = \frac{1}{\mu}\lambda_1 + Z_1 - \frac{\alpha}{\mu}\sum_{v=1}^{V} L Z_1 + \sum_{v=1}^{V}(X^{(v)} - E^{(v)} + \frac{1}{\mu} * \lambda^{(v)})^{\mathrm{T}} V^{(v)} \tag{14}$$

Thus, we further obtain

$$\min_{U^{\mathrm{T}}U=I} \|U - H\|_F^2 \tag{15}$$

Denote

$$L(U, \Lambda) = \|U - H\|_F^2 + \Lambda(UU^{\mathrm{T}} - I) \tag{16}$$

we then obtain

$$U = N_u Q_u^{\mathrm{T}} \tag{17}$$

where N_u and Q_u are the left and right singular vectors of the economic singular value decomposition of H. ***PS: How to obtain the N_u and Q_u. The details refer to the pages 38 and 39 of my doctoral dissertation***

Update β_v**:** Denote $p^{(v)} = \mathrm{tr}(Z_1^{\mathrm{T}} L^{(v)} U)$

$$\beta_v = (rp^{(v)})^{\frac{1}{1-r}} / \sum_{v=1} V (rp^{(v)})^{\frac{1}{1-r}} \tag{18}$$

Update $\lambda^{(v)}$**,**λ_1**, and** μ**:**

$$\lambda^{(v)} = \lambda^{(v)} + \mu(X^{(v)} - V^{(v)} U^{\mathrm{T}} - E^{(v)}) \tag{19}$$

$$\lambda_1 = \lambda_1 + \mu(Z_1 - U) \tag{20}$$

$$\mu = \rho\mu \tag{21}$$