Model recap

Parameters

- There are 11 parameters of the model. In BGP equilibrium, all parameters are constant over time.

$$\{\rho, \gamma, \tilde{\gamma}, \tau, s, \alpha, \tilde{\alpha}, \lambda, \delta, \phi, \tilde{\phi}\}$$

Remarks on value function discretization

1. In this model, m denotes the difference between numbers (steps) of innovations that the leader and the follower have had up to now, in a given sector. For instance, if a sector has its state at $m = 5$, then it means that the leader in this sector has had 5 more innovations than its follower, there are 5 productivity steps between them.
2. Importantly, value functions and sector distribution are functions of $m = 0, 1, \ldots, \bar{m}$.
3. Notice that we put a cap $\bar{m} \in \mathbf{N}$ on the maximum number that m can get. That is, we do not allow leaders open up the difference more than $\bar{m}$ steps.
4. Therefore, state space of the model is all integers between 0 and $\bar{m}$, including them.
5. We represent value functions and distribution as vectors/arrays on the computer. For the length/structure of these vectors, you have several options. First note that we have three kinds of value functions: values for leaders, followers, and firms in neck-and-neck sectors. Secondly, leaders and followers operate in sectors with $m \geq 1$, while neck-and-neck firms operate in sectors with the state $m = 0$.

   - I personally prefer to define three different objects for each kind of value functions. Therefore, I represent leaders' and followers' value functions by two separate vectors (arrays) of length $\bar{m}$. Finally, *the* value of neck-and-neck firms is just a single number.
   - Another option is to define two vectors of the same length $\bar{m} + 1$, one for leaders, one for followers. However, in this case, one element of two vectors are the same since that element commonly represents neck-and-neck sectors. I find this complicated.
   - Last option is to have a single vector of length $2\bar{m} + 1$. First $\bar{m}$ elements represent follower values, the element in the middle ($\bar{m} + 1$'th element) represents neck-and-neck value, while the last remaining $\bar{m}$ elements (from $\bar{m} + 2$ to $2\bar{m} + 1$'th element) represent leaders. This structure might be useful in MATLAB. In this structure, the first element corresponds to followers in sector $m = \bar{m}$, the second element corresponds to followers in sector $m = \bar{m} - 1$, and so on, up to the neck-and-neck value at index $\bar{m} + 1$. On the other hand, $\bar{m} + 2$'th element corresponds to leader value in sectors $m = 1$, $\bar{m} + 3$'th element corresponds to leader value in sectors $m = 2$, and so on, up to the last element of the vector.
6. For the distribution function, you have two options, since it is defined over sectors not over the firms as value functions do. In particular,
   - One vector with the length of $\bar{m}$ for the unleveled sectors. And a single number that is calculated every time is for the mass of leveled (neck-and-neck) sectors. Note that mass of

neck-and-neck sectors is defined as 1 minus the sum of masses of all other types of sectors with $m = 1, 2, \ldots, \bar{m}$. This structure is my preference.
  - Another option is to have a single vector of length $\bar{m} + 1$ for all types of sectors altogether. That is, the first element of that vector holds mass of neck-and-neck sectors, while the rest holds that of unleveled sectors from $m = 1$ to $m = \bar{m}$ in an increasing order.
7. The choice of structure depends on your taste and the way you code. I use Julia. Therefore, I do not need to vectorize my code as I'd do in MATLAB. For this reason, I follow the more intuitive way which is first options above.
8. This model is exceptionally simple in that its state space is already discrete. It is a finite set of integers becuase of the step-by-step innovation structure. Hence, you do not need to discretize value function and distribution.
9. In the problem set, you are asked to take $\bar{m} = 100$.

BGP equilibrium equations

A BGP is defined as a collection of

$$\left\{\omega, v_l(m), v_f(m), v_{nn}, x_l(m), x_f(m), x_{nn}, x_e(m), x_{enn}, \mu(m), \mu_{nn}\right\}$$

where

- $0 \leq \omega \leq 1$: Normalized wage rate
- $v_l(m)$: Leader value function, $m = 1, 2, \ldots, \bar{m}$
- $v_f(m)$: Follower value function, $m = 1, 2, \ldots, \bar{m}$
- v_{nn}: Neck-and-neck firm value
- $x_l(m)$: Leader innovation rates, $m = 1, 2, \ldots, \bar{m}$
- $x_f(m)$: Follower innovation rates, $m = 1, 2, \ldots, \bar{m}$
- x_{nn}: Neck-and-neck incumbent innovation rate
- $x_e(m)$: Entrant innovation rates, $m = 1, 2, \ldots, \bar{m}$
- x_{enn}: Innovation rate of entrants at neck-and-neck sector
- $\mu(m)$: Mass of unleveled sectors, $m = 1, 2, \ldots, \bar{m}$
- μ_{nn}: Mass of neck-and-neck sectors

Note: Sometimes, I might have used below $m = 0$ and neck-and-necks respectively. For instance, keep in mind that $x_e(0) = x_{enn}$, or $v_l(0) = v_{nn}$, if you ever encounter them below.

1. **Euler equation**

$$g = r - \rho$$

where

$$g \equiv \frac{d \ln Y_t}{dt}$$

Note that BGP growth rate is constant over time.

1. **Incumbents:** *Normalized* HJBs and innovation rates

   - **Leaders** $m = 1, 2, \ldots, \bar{m}$**:** Value functions v_l and innovation rates $x_l(m)$

$$\rho v_l(m) = (1-\tau)\left(1-\frac{1}{\lambda^m}\right) + \max_x \left\{-(1-s)\,\alpha\frac{x^\gamma}{\gamma}\omega + x\left[v_l(m+1)-v_l(m)\right]\right\}$$
$$+\left(\phi x_f(m) + \delta + \tilde{\phi} x_e(m)\right)\left[v_{nn} - v_l(m)\right]$$
$$+\left((1-\phi)\,x_f(m) + \left(1-\tilde{\phi}\right)x_e(m)\right)\left[v_l(m-1)-v_l(m)\right]$$

$$x_l(m) = \left(\frac{v_l(m+1)-v_l(m)}{(1-s)\alpha\omega}\right)^{\frac{1}{\gamma-1}}$$

- **Followers $m = 1, 2, \ldots, \bar{m}$:** Value functions $v_f(m)$ and innovation rates $x_f(m)$

$$\rho v_f(m) = \max_x \left\{-(1-s)\,\alpha\frac{x^\gamma}{\gamma}\omega + x\left[\phi v_{nn} + (1-\phi)\,v_f(m-1) - v_f(m)\right]\right\}$$
$$+\delta\left[v_{nn} - v_f(m)\right]$$
$$+x_l(m)\left[v_f(m+1) - v_f(m)\right]$$
$$+x_e(m)\left[0 - v_f(m)\right]$$

$$x_f(m) = \left(\frac{\phi v_{nn} + (1-\phi)\,v_f(m-1) - v_f(m)}{(1-s)\,\alpha\omega}\right)^{\frac{1}{\gamma-1}}$$

- **Neck-and-neck $m = 0$:** Value v_{nn} and innovation rate x_{nn}

$$\rho v_{nn} = \max_x \left\{-(1-s)\,\alpha\frac{x^\gamma}{\gamma}\omega + x\left[v_l(1) - v_{nn}\right]\right\}$$
$$+x_{nn}\left[v_f(1) - v_{nn}\right]$$
$$+x_{enn}\left[\frac{1}{2}v_f(1) - v_{nn}\right]$$

$$x_{nn} = \left(\frac{v_l(1) - v_{nn}}{(1-s)\,\alpha\omega}\right)^{\frac{1}{\gamma-1}}$$

2. **Entrants:** Innovation rates

- **Unleveled $m = 1, 2, \ldots, \bar{m}$:** Entry problem and entrant innovation rate $x_e(m)$

$$\max_m \left\{-\tilde{\alpha}\frac{x^{\tilde{\gamma}}}{\tilde{\gamma}}\omega + x\left[\tilde{\phi} v_{nn} + \left(1-\tilde{\phi}\right)v_f(m-1) - 0\right]\right\}$$

$$x_e(m) = \left(\frac{\tilde{\phi} v_{nn} + \left(1-\tilde{\phi}\right)v_f(m-1)}{\tilde{\alpha}\omega}\right)^{\frac{1}{\tilde{\gamma}-1}}$$

- **Neck-and-neck $m = 0$:** Entry problem and entrant innovation rate in neck-and-neck sectors x_{enn}

$$\max_m \left\{-\tilde{\alpha}\frac{x^{\tilde{\gamma}}}{\tilde{\gamma}}\omega + x\left[v_l(1) - 0\right]\right\}$$

$$x_{enn} = \left(\frac{v_l(1)}{\tilde{\alpha}\omega}\right)^{\frac{1}{\tilde{\gamma}-1}}$$

3. **Kolmogorov Forward equations:** How sector distribution $\mu(m)$ and μ_{nn} evolves over time

- **For** $m = 2, \ldots, \bar{m} - 1$**:**

$$0 = \mu(m-1)x_l(m-1) + \mu(m+1)\left[(1-\phi)\,x_f(m+1) + \left(1-\tilde{\phi}\right)x_e(m+1)\right] - \mu(m)\left[x_l(m) + x_f(m) + \delta + x_e(m)\right]$$

- **For** $m = \bar{m}$**:**

$$0 = \mu(\bar{m}-1)x_l(\bar{m}-1) - \mu(\bar{m})\left[x_f(\bar{m}) + \delta + x_e(\bar{m})\right]$$

- **For** $m = 1$**:**

$$0 = \mu_{nn}\left[2x_{nn} + x_{enn}\right] + \mu(2)\left[(1-\phi)\,x_f(2) + \left(1-\tilde{\phi}\right)x_e(2)\right] - \mu(1)\left[x_l(1) + x_f(1) + \delta + x_e(1)\right]$$

- **For** $m = 0$**:**

$$0 = \mu(1)\left[(1-\phi)\,x_f(1) + \left(1-\tilde{\phi}\right)x_e(1)\right] + \sum_{k=1}^{\bar{m}} \mu(k)\left[\phi x_f(k) + \delta + \tilde{\phi} x_e(k)\right] - \mu_{nn}\left[2x_{nn} + x_{enn}\right]$$

4. **Labor market clearing**

$$1 = \sum_{m=0}^{\bar{m}} \mu(m)\left[l_l(m) + h_l(m) + h_f(m) + h_e(m)\right]$$

where $h(\cdot)$ and $l(\cdot)$ denote R&D and production labor demands, respectively. In particular

$$h_i(m) = \alpha\frac{x_i(m)^\gamma}{\gamma}, \quad i = l, f$$

$$h_e(m) = \tilde{\alpha}\frac{x_e(m)^{\tilde{\gamma}}}{\tilde{\gamma}}$$

$$l_l(m) = \frac{1}{\lambda^m}\omega^{-1}$$

1. **Aggregate output and BGP growth rate:**

$$\ln Y_t = \int_0^1 q_t(j)dj - \ln\omega - \ln\lambda\sum_{m=0}^{\bar{m}} m\mu(m)$$

$$g \equiv \frac{d\ln Y_t}{dt} = \ln\lambda\left[\mu(0)\left(2x_{nn} + x_{enn}\right) + \sum_{m=1}^{\bar{m}}\mu(m)x_l(m)\right]$$

Numerical algorithm

Our goal is to find numerical solutions of the following variables altogether.

$$\left\{\omega, v_l(m), v_f(m), v_{nn}, x_l(m), x_f(m), x_{nn}, x_e(m), x_{enn}, \mu(m), \mu_{nn}\right\}$$

In what follows, I outline a numerical algorithm for **my** structure of vector specifications, such as separate vectors for leaders, followers, and neck-and-neck firms, and a vector of distribution for unleveled sectors, and remaining mass of neck-and-neck sectors. Bold characters such as $\mathbf{v}, \mathbf{x}, \mathbf{M}$ denote vector representations of value functions, innovation rates, and distribution function, respectively.

Steps

1. [OUTER LOOP] Guess a wage $0 < \omega < 1$.
2. [BACKWARD ITERATIONS] Find associated value functions and innovation rates: We have to discretize HJBs. Let me denote all right hand side of HJBs by a variable A. Then HJB with time subscripts is of the following general form

$$\rho v_t(m) - \dot{v}_t(m) = A_t$$
$$\dot{v}_t(m) = \rho v_t(m) - A_t$$

   Note that in BGP $\dot{v}_t(m) = 0$, but for the sake of iteration, we will keep the time subscript, and discretize the value function over the time dimension. We will assume implicit method, i.e. all variables except time derivative will be indexed by future values $t + dt$. Using time subscripts, we will have the following equation for value function iteration

$$\frac{v_{t+dt}(m) - v_t(m)}{dt} = \rho v_{t+dt}(m) - A_{t+dt}$$
$$v_t(m) = v_{t+dt}(m) - \left[\rho v_{t+dt}(m) - A_{t+dt}\right] dt$$

   Note that the expression A_{t+dt} might depend on value function at indexes $m, m+1$ or $m-1$. That is why we need a full guess for the entire value function at the beginning.

   Given a guess for the function v_{t+dt}, we can now solve for the current value v_t by iterating above expression (That is why it is called backward iteration). Note that this procedure is the same for leader, follower and neck-and-neck value functions.

   A. [INNER LOOP 1] Given ω, guess value function vectors $\mathbf{v}^l_{t+dt}, \mathbf{v}^f_{t+dt}, \mathbf{v}^{nn}_{t+dt}$. Zero vectors as a guess work fine. These can be considered as values in $t + dt$.
   B. [INNER LOOP 1] Given guesses, solve for innovation rates in $t + dt$ from first order conditions, i.e. vectors $\mathbf{x}^l_{t+dt}, \mathbf{x}^f_{t+dt}, \mathbf{x}^{nn}_{t+dt}, \mathbf{x}^e_{t+dt}, \mathbf{x}^{enn}_{t+dt}$.
   C. [INNER LOOP 1] Given values and innovation rates, solve for current value functions in t from discretized HJBs. That is, find new vectors $\mathbf{v}^l_t, \mathbf{v}^f_t, \mathbf{v}^{nn}_t$ for current values.
   D. [INNER LOOP 1] Check whether value functions (vectors) at t and $t + dt$ are close enough with respect to some certain tolerance. In other words, check

$$\max_{\text{vector elements}} |\mathbf{v}_t - \mathbf{v}_{t+dt}| < 1e - 8$$

   where $\mathbf{v}$ stacks all value vectors together. This is the standard approach in numerical analysis.
   E. [INNER LOOP 1] If above inequality is satisfied, then it means you solved value functions for your initial guess of wage rate ω, break the **inner loop 1**, and go to next step inner loop 2. If above inequality is not satisfied, then update your vectors $\mathbf{v}^l_{t+dt}, \mathbf{v}^f_{t+dt}, \mathbf{v}^{nn}_{t+dt}$ with $\mathbf{v}^l_t, \mathbf{v}^f_t, \mathbf{v}^{nn}_t$, and proceed to step 2.2.

3. [FORWARD ITERATIONS] Find associated distribution: Given ω, values $\mathbf{v}^l, \mathbf{v}^f, \mathbf{v}^{nn}$, and corresponding innovations $\mathbf{x}^l, \mathbf{x}^f, \mathbf{x}^{nn}, \mathbf{x}^e, \mathbf{x}^{enn}$, we will find distribution represented by vector $\mathbf{M}$ in a separate inner loop 2. Similar to value functions, we should discretize Kolmogorov forward equations (KF). Generic form of KF is as follows

$$\dot{\mu}_t(m) = A_t$$

for some generic RHS variable A_t. For discretization, we will use explicit method this time. That is,

$$\frac{\mu_{t+dt} - \mu_t(m)}{dt} = A_t$$
$$\mu_{t+dt}(m) = \mu_t + A_t dt$$

Similar as before, RHS A_t might depend on distribution values at indexes $m+1, m-1$ along with m, so we need a full guess for the entire distribution vector $\mathbf{M}_t$.

Given a guess for current distribution $\mathbf{M}_t$, we can now solve for distributions in the future $\mathbf{M}_{t+dt}$ by iterating above expression (That is why it is called forward iteration).

A. [INNER LOOP 2] Given ω, innovation rates $\mathbf{x}^l, \mathbf{x}^f, \mathbf{x}^{nn}, \mathbf{x}^e, \mathbf{x}^{enn}$, guess current distribution vector $\mathbf{M}_t$. Uniform distribution works fine. Note that mass of neck-and-neck sectors is 1 minus the sum of masses of all other unleveled sectors, i.e. $1 - sum(\mathbf{M})$.
B. [INNER LOOP 2] Given guesses, solve for $\mathbf{M}_{t+dt}$.
C. [INNER LOOP 2] Check whether distribution vectors $\mathbf{M}_t$ and $\mathbf{M}_{t+dt}$ are close enough with respect to the same tolerance 1e-8. this step is similar to 2.4.
D. [INNER LOOP 2] If difference is lower than the tolerance, break the **inner loop 2**. and go to next step. If not, update $\mathbf{M}_t$ with $\mathbf{M}_{t+dt}$, and proceed to step 3.2.

4. [IMPLIED WAGE] Given innovations $\mathbf{x}^l, \mathbf{x}^f, \mathbf{x}^{nn}, \mathbf{x}^e, \mathbf{x}^{enn}$, and distribution $\mathbf{M}$, solve for the implied wage rate that clears the labor market. Denote it by ω'. If $|\omega' - \omega| < 1e-4$, then break the outer loop. That is, you have completed the outer loop and found a BGP with the following associated solutions

$$\left\{\omega', \mathbf{v}^l, \mathbf{v}^f, \mathbf{v}^{nn}, \mathbf{x}^l, \mathbf{x}^f, \mathbf{x}^{nn}, \mathbf{x}^e, \mathbf{x}^{enn}, \mathbf{M}, 1 - sum(\mathbf{M})\right\}$$

from previous steps. If wages have not converged yet, update the initial wage by the following weighted average

$$\omega = 0.25\omega' + 0.75\omega$$

and proceed to step 1, i.e. the beginning to the **outer loop**.

Final hints for the problem set

Above procedure is a workflow to find the BGP for a single set of parameters. In the problem set, you are asked to solve for BGPs across different values of δ by keeping other parameters the same. That is, for each different value of δ, you need to solve for the BGP again, and calculate the corresponding annual growth rate from the above growth formula. Note that, the growth rate above, after multiplied by the length of time periods dt, is the growth rate from one subperiod to another. You need to annualize it.

Hint: If a quantity grows with the rate gdt from one period to the next, what is the compounded growth rate over 50 periods? Recall that $dt = 1/50$, i.e. a year consists of 50 subperiods.