HW #10

1 For Y_1, Y_2 $\quad \underline{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ $\quad \underline{U} = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$

as $Y_1 = \sqrt{-2\ln U_1}\,\sin(2\pi U_2)$, $Y_2 = \sqrt{-2\ln U_2}\,\cos(2\pi U_2)$

For each $\underline{U}$ there is only one $\underline{Y}$, meaning one-to-one mapping

$\therefore f_{\underline{Y}}\, d\underline{Y} = f_{\underline{U}}\, d\underline{U}$ $\quad \therefore f_{\underline{Y}} = f_{\underline{U}} \cdot |\det J_{\underline{Y},\underline{U}}|^{-1}$

$\therefore f_{\underline{Y}}(Y_1, Y_2) = f_{\underline{U}}(U_1, U_2) \cdot |\det J_{\underline{Y},\underline{U}}|^{-1}$

$$J_{\underline{Y},\underline{U}} = \begin{pmatrix} \frac{1}{U_1}(-2\ln U_1)^{-\frac{1}{2}}\sin(2\pi U_2) & 2\pi(-2\ln U_1)^{\frac{1}{2}}\cos(2\pi U_2) \\ \frac{1}{U_1}(-2\ln U_1)^{-\frac{1}{2}}\cos(2\pi U_2) & -2\pi(-2\ln U_1)^{\frac{1}{2}}\sin(2\pi U_2) \end{pmatrix}$$

$\therefore$ And for $0 \le U_1 \le 1$, $0 \le U_2 \le 1$ Y_1 and Y_2 are matched to whole $\mathbb{R}\times\mathbb{R}$ space. (as $\ln U_1$ is in $(-\infty, 0]$, and $\sin(2\pi U_2)$ and $\cos(2\pi U_2)$ rotates whole radian of 2π)

$\therefore f_{\underline{Y}}(Y_1, Y_2) = 1 \cdot \frac{U_1}{2\pi} = \frac{U_1}{2\pi}$

and $-2\ln U_1 = Y_1^2 + Y_2^2$ $\quad \therefore U = \exp\left(-0.5\left(Y_1^2 + Y_2^2\right)\right)$

$= \exp\left(-\frac{1}{2}\underline{Y}^T \cdot \underline{Y}\right)$

$\therefore f_{\underline{Y}}(Y_1, Y_2) = \frac{1}{2\pi} \cdot \exp\left(-\frac{1}{2}\underline{Y}^T \cdot \underline{Y}\right)$

$\therefore \underline{Y} \sim N(\underline{0}, \underline{I})$ $\quad$ ($\Sigma = I$, $\sqrt{\det \Sigma} = 1$ and bivariate Normal distribution's PDF is $\frac{1}{2\pi\sqrt{\det\Sigma}}\exp\left(-\frac{1}{2}\underline{Y}^T\Sigma^{-1}\underline{Y}\right)$)

2. Sample statistics 와 Theoritical Marginal CDF - Sample CDF plot 첨부했습니다.

As we can see in graph. Sample statistics and CDF Converge to the that of theoritical caes. Wanted to save graphs as matlab figure, but it occured error, so graphs are jpg file.

3. Speed of Importance sampling at $\Sigma = I$ was the fastest. And Speed of Monte Carlo and Importance sampling by $\Sigma = 2^2 I$ was almost same. (Used sample size 10^5)

1) Previous case

P_f by Monte Carlo : 0.04313 C.O.V : 0.014895

P_f by Importance Sampling $\Sigma = I$: 0.043393 C.O.V : 0.0051435

P_f by Importance Sampling $\Sigma = 2^2 I$: 0.043331 C.O.V : 0.013282

2) zero-Correlation.

P_f Monte Carlo : 0.07246 C.O.V : 0.011314

P_f Imp Sample $\Sigma = I$: 0.074386 C.O.V : 0.0048348.

P_f Imp Sample $\Sigma = 2^2 I$: 0.074512 C.O.V : 0.01192.

(P_f, C.O.V plot 첨부했습니다. + Code도 첨부했습니다. P_3 실행시 결과 추출)

As shown, Importance Sampling by $\Sigma = 2^2 I$ give slow Convergence than $\Sigma = I$. Cause. It make samples in more wide areas than $\Sigma = I$. Therefore It gives slow Convergence as Monte Carlo simulation.