Anyons and Topological Quantum Computation

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Abstract

The aim of this text is to provide an introduction to the theory of topological quantum computation. We give an introduction to the theory of anyons (two-dimensional quasi-particle excitations that have exotic statistics) and how we can use these to perform fault-tolerant quantum computation. Additionally, we give a complete description of an exactly solvable spin lattice model whose local low-energy excitations of the Hamiltonian behave as anyons. We conclude by indicating how this model can be generalized so as to perform universal quantum computation.

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1 Introduction

1.1 Particle Statistics

In classical physics, identical particles[1] can always be distinguished by their position, in quantum mechanics however this isn't the case. The position of quantum particles is determined by their wavefunction and thus if we take two particles, their wavefunctions might overlap (and do not peak at exactly the same position) making the position, in general, not a good property to distinguish particles. This is where statistical physics comes into play. In a system of many identical particles instead of characterizing each particle by its own wavefunction, we use a single wavefunction to describe the whole system (where its properties need to be compatible with the symmetry imposed by the indistinguishability of particles). In $(3 + 1)$ dimensions this then allows us to classify particles as one of two types: bosons that have a symmetric wavefunction, and fermions that have an anti-symmetric wavefunction. This limitation into one of two types of symmetry comes from the observation that adiabatically exchanging twice two particles is the same as taking one particle around the other. In $(3+1)$ dimensions the process of taking one particle around another is topologically equivalent to the process of not moving either, therefore the wavefunction is left unchanged. This means that by doing a single exchange it limits the factors on our wavefunction to ± 1, corresponding to bosons and fermions respectively. Consider N indistinguishable particles, along with all possible trajectories, from initial positions $R_1, \ldots, R_N$ to final positions $R'_1, \ldots, R'_N$. We then have that these trajectories fall into topological classes corresponding to the elements of the permutation group S_N, where each element specifies how the initial positions are permuted to obtain the final position (fermions and bosons correspond to the only one-dimensional irreducible representations of S_N).

Consider now instead that these particle live in a $(2 + 1)$-dimensional system. In

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[1]Identical particles have all their intrinsic properties (spin, mass, angular momentum, charge ...) equal.

this case the process of taking a particle around another causes a non-trivial winding, and as a result the system doesn't necessarily come back to the same state. If we have a two-particle system and we exchange these in a counter-clockwise manner, its state can change by an arbitrary phase, that is $P|\psi\rangle = e^{i\theta}|\psi\rangle$, and furthermore a double exchange $P^2|\psi\rangle = e^{2i\theta}|\psi\rangle$ only acts trivially when $\theta = 0, \pi$, which corresponds to the cases of bosons and fermions respectively. For any other values of θ, these particles are named anyons (a term coined in [Wil90], as they can take "any" phase). If we take a similar approach to the $(3+1)$ case and consider N identical particles, along with all possible trajectories, from initial positions $R_1, \ldots, R_N$ to final positions $R'_1, \ldots, R'_N$, then the topological classes of trajectories are instead in a one-to-one correspondence with the elements of the braid group $\mathcal{B}_N$.

1.2 Quantum Computation

Over the last few decades there has been an increasing interest in the construction of a quantum computer, that is, a device that makes use of quantum mechanics to perform computation. This interest started with Richard Feynman, as a feasible way to simulate quantum physics [Fey82], but in the last decade of the 20th century this interest has been gradually increasing due to the discovery of new algorithms, such as Shor's integer factorization algorithm [Sho94] and Grover's database search algorithm [Gro96], that could use these quantum mechanical properties to perform some tasks much more efficiently than any classical computer could ever hope to do. The main problem with constructing this quantum computer is the intrinsic vulnerability of quantum particles to errors and decoherence. Any interaction with the environment causes a projection onto a quantum state, effectively disrupting any sort of computation we were hoping to achieve. Furthermore, quantum gates are also increasingly vulnerable (compared to their classical counterparts). One of the approaches proposed to overcome this issue, was introduced by Kitaev and Freedman [Fre+03]. The main idea is to employ the

anyon particles we described in the previous section, to perform quantum computation. Due to their topological nature, these are inherently protected from errors. This type of computer is therefore called a topological quantum computer.

1.3 Overview

In this text we start by giving a brief introduction to Quantum Information Theory and the Braid group, in chapter 2. In chapter 3, we give an in-depth overview of the theory of anyons. The main property of this type of quasi-particles (besides their exotic statistics) is known as fusion which is a non-local way of associating groups of anyons. From the non-locality of this process we then get particles which are more resilient to errors. Still in this chapter we give some examples of specific anyon models and see how one can use these properties to perform quantum computation. The main result of the text is provided in chapters 4 and 5. In the first we give a complete description of an exactly solvable model, whose local quasi-particle excitations are anyons, and see how the properties of the ground state allow us to have a fault-tolerant quantum computer. In the last chapter we aim to generalize this model so as to allow universal quantum computaion.

2 Preliminaries

We will start by introducing some useful concepts in quantum information and braid theory. For further reading about these topics we recommend [NC10] and [KT08] respectively.

2.1 Quantum Information

In classical information, the basic unit is the bit, which takes values in $\{0, 1\}$. When we pass to quantum information, the bit takes the form of the qubit (quantum bit). This is a state in a two-dimensional Hilbert space $\mathcal{H}_2 \cong \mathbb{C}^2$. If we assign an orthonormal basis on this space $\{|0\rangle, |1\rangle\}$, we then have that a qubit is a state of the form

$$|\psi\rangle = a |0\rangle + b |1\rangle, \tag{2.1}$$

where a, $b \in \mathbb{C}$ such that $|a|^2 + |b|^2 = 1$. We can perform a measurement on this qubit that projects it onto the basis $\{|0\rangle, |1\rangle\}$ (known as the computational basis). We then obtain the outcome $|0\rangle$ with probability $|a|^2$ and the outcome $|1\rangle$ with probability $|b|^2$. As one would expect from quantum mechanics, the space on n qubits is a state in $\mathcal{H}_2^{\otimes n}$. If $|i_j\rangle$ denotes a basis for the jth space, a n-qubit state in this tensor product is then

$$|\phi\rangle = \sum_{i_1,i_2,...,i_n=0}^{1} a_{i_1,i_2,...,i_n} |i_1, i_2, ..., i_n\rangle, \qquad \sum_{i_1,i_2,...,i_n=0}^{1} |a_{i_1,i_2,...,i_n}|^2 = 1, \tag{2.2}$$

where $a_{i_1,i_2,...,i_n} \in \mathbb{C}$. We are adopting the convention that $|i_1, i_2, ..., i_n\rangle = |i_1\rangle |i_2\rangle ... |i_n\rangle = |i_1\rangle \otimes |i_2\rangle \otimes ... \otimes |i_n\rangle$. Note that we can also represent information by states in higher-dimensional Hilbert spaces isomorphic to $\mathbb{C}^d$, which we call qudits.

2.1.1 Entanglement

Quantum Entanglement is one of the most curious features in quantum mechanics and one of the main reasons why certain quantum algorithms are much faster than their classical counterparts. To put it simply, an entangled state is one where there are some sort of correlations, not necessarily local, between its constituent states. If we have a n-qubit state $|\psi\rangle$ and states $|\phi_1\rangle$, $|\phi_2\rangle$, ..., $|\phi_m\rangle$, such that

$$|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_m\rangle, \tag{2.3}$$

then $|\psi\rangle$ is said to be separable, otherwise it is said to be entangled. Consider the state $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \in \mathcal{H}_2 \otimes \mathcal{H}_2$ and let $|\phi_1\rangle = a_1 |0\rangle + b_1 |1\rangle$, with $|a_1|^2 + |b_1|^2 = 1$, and $|\phi_2\rangle = a_2 |0\rangle + b_2 |1\rangle$, with $|a_2|^2 + |b_2|^2 = 1$, be two qubits. Assuming that $|\phi^+\rangle$ is separable, then

$$|\phi^+\rangle = |\phi_1\rangle \otimes |\phi_2\rangle = (a_1 |0\rangle + b_1 |1\rangle) \otimes (a_2 |0\rangle + b_2 |1\rangle). \tag{2.4}$$

Therefore there are a_1, b_1, a_2, b_2, such that

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = a_1 a_2 |00\rangle + a_1 b_2 |01\rangle + b_1 a_2 |10\rangle + b_1 b_2 |11\rangle. \tag{2.5}$$

For this to be true we need to ensure that both $|01\rangle$ and $|10\rangle$ are zero by imposing that $a_1 b_2 = b_1 a_2 = 0$, however by doing so we will have that the whole right-hand-side of the equation must be zero. This then guarantees us that $|\phi^+\rangle \neq |\phi_1\rangle \otimes |\phi_2\rangle$ and therefore it is an entangled state. In fact $|\phi^+\rangle$ is one of the Bell states, which represent the simplest examples of entanglement. These are given by

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad\qquad |\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$
$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \qquad\qquad |\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle), \tag{2.6}$$

and exhibit perfect correlation, giving them the name of maximally entangled states [Wei+03].

2.1.2 Quantum Gates

In classical computation, we can perform operations through the action of logic gates on our units of information. A similar approach is taken in quantum computing, but unlike the classical case, where the gates are mostly irreversible, our gates must be reversible so as to preserve the probability interpretation of quantum mechanics. As a result we must then require these quantum gates to be unitary operators that act on our qubit space.

Consider the Pauli operators (in Quantum Information Theory, these are known as X, Y and Z gates)

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.7}$$

Since these are unitary, they are in fact one-qubit gates. The Pauli-X gate represents the classical NOT gate, that flips the state of the (qu)bit. There are also more interesting one-qubit gates, like the Hadamard gate, that does not have a classical counter part

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{2.8}$$

This gate takes a basis state and puts it in a superposition of different basis states, that is $H\left|0\right\rangle = \frac{\left|0\right\rangle + \left|1\right\rangle}{\sqrt{2}}$ and $H\left|1\right\rangle = \frac{\left|0\right\rangle - \left|1\right\rangle}{\sqrt{2}}$.

Let us now look at two-qubit gates. Since this ia a bigger space, we will have more unitary operators and therefore more gates. An important class of gates is the class of controlled gates, CU. These act by taking one of the qubits as the "controller" and the other as the "target". If the controller is in the state $\left|0\right\rangle$ then the target is left

unaffected, but if it is in state $|1\rangle$ then a unitary operator U is applied to the target. One reason why this is important is that the CNOT gate can be used to entangle two previously unentangled qubits. For example, take the state $\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle$, then

$$\text{CNOT}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |0\rangle\right) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{2.9}$$

which is a maximally entangled state.

2.2 The Braid Group

In two-dimensional particle statistics it turns out that particle world-lines are in one-to-one correspondence with elements of the Braid group, and as such we can not avoid introducing it. The braid group $\mathcal{B}_n$ is the group with generators $\sigma_1, \ldots, \sigma_{n-1}$ that satisfy the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{for } |i - j| \geq 2, \tag{2.10}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \tag{2.11}$$

where the final relation is known as the Yang-Baxter equation. If we take n strands and place them side by side, then these generators can be understood through their action on the strands. The generator σ_i acts by braiding the ith and $(i+1)^{\text{th}}$ strands, as depcicted by fig. 2.1. The action of the strands given by the relations between these generators, eqs. (2.10) and (2.11), are depicted in figs. 2.2 and 2.3 respectively. To interpret this in terms of particles, the braid generator σ_i can be thought of as a counter-clockwise exchange (as viewed from the top) between the ith and $(i+1)^{\text{th}}$ particles.
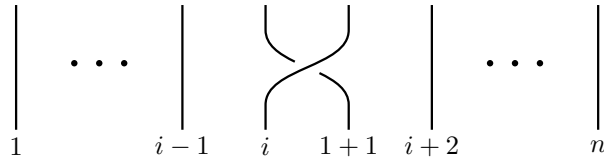
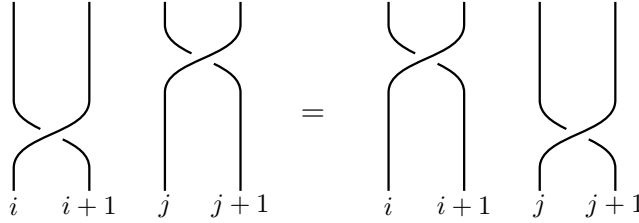Figure 2.1: Action of the braid group generator σ_i on n strands.

Figure 2.2: The first braid relation: the braid group generators commute if they act on strands that are sufficiently far apart.
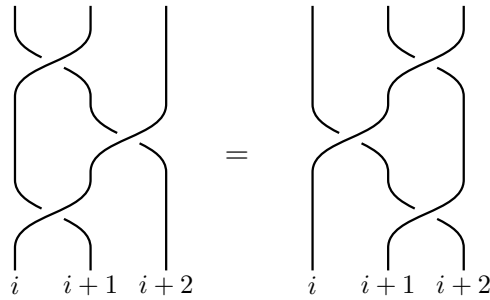

Figure 2.3: The second braid relation: two braids are the same under continuous deformation of the strands. This is known as the Yang-Baxter equation in statistical physics.

The Permutation group S_n is defined by the same generator relations and must also satisfy that $\sigma_i^2 = 1$. While this additional restriction appears minor, it makes an enormous difference. For one, the Permutation group is finite, the number of elements in the group is $|S_n| = n!$, while the Braid group is infinite for $n \geq 2$.

2.2.1 Braid Group Representations

Let us now classify unitary one-dimensional representations of B_n. These are functions $\rho : \mathcal{B}_n \to \mathbb{C}$ such that to each generator σ_j we assign a complex number $e^{i\theta_j}$, that is

$$\rho(\sigma_j) = e^{i\theta_j}, \tag{2.12}$$

for $\theta \in [0, 2\pi[$. If we wish to represent the braid group we must then satisfy the relations between its generators. Eq. (2.10) is trivially satisfied. In the case of eq. (2.11) we thus have

$$e^{i\theta_j} e^{i\theta_{j+1}} e^{i\theta_j} = e^{i\theta_{j+1}} e^{i\theta_j} e^{i\theta_{j+1}}, \tag{2.13}$$

which enforces $\rho(\sigma_j) = \rho(\sigma_{j+1})$, for all j. Therefore θ is the only parameter characterizing a particular representation

$$\rho_\theta : \sigma_j \to e^{i\theta}, \tag{2.14}$$

for all j, with $\theta \in [0, 2\pi[$. One should bear in mind that this representation is not faithful since we cannot distinguish between braid group elements $\sigma_j \sigma_{j+1}$ and $\sigma_{j+1} \sigma_j$. Due to the commutativity of the representation we therefore can only represent abelian anyons. Higher dimensional representations of the braid group are much richer and some allow for non-abelian anyons. We also have that $\sigma_i^2 \neq 1$ meaning that this is not a representation of the Permutation group S_n, but we can make it into one by taking $\rho(\sigma_i) = 1$ or $\rho(\sigma_i) = -1$. In terms of particle statistics, this represents in fact bosonic and fermionic representations respectively.

3 Anyon Theory

As we've seen before, anyons are particles that live in two-dimensional space and may exhibit exotic statistics. In this chapter we will introduce the theory of anyons and the fundamental tools to analyze and classify specific anyon models, as well as how to use these to perform quantum computation. For alternative/complementary approaches to the subject we recommend [Pac12], [Sim16], [Pre], [Bon07], [Wan10] and [Nay+08].

3.1 Fusion

An anyon model has a finite set of labels $(a, b, c, ...)$, called particles types, topological or anyonic charges, anyon types and many other names; as well as a set of labels ($\overline{a}, \overline{b}, \overline{c}, ...$) which represent antiparticles of the corresponding anyon under the fusion process, which we will introduce shortly. There must also be a unique trivial particle (1) which represents the vacuum. These charges must satisfy the equation

$$a \times b = \sum_c N_{ab}^c c \tag{3.1}$$

where the fusion multiplicities N_{ab}^c are non-negative integers which indicate the number of different ways the particles a and b can be combined to produce c. These obey the relations

$$N_{ab}^c = N_{ba}^c = N_{b\overline{c}}^{\overline{a}} = N_{\overline{a}b}^{\overline{c}} \tag{3.2}$$

$$\sum_e N_{ab}^e N_{ec}^d = \sum_f N_{af}^d N_{bc}^f. \tag{3.3}$$

We will justify these later when we define the Hilbert space associated with this process. A theory is said to be non-abelian if there are a and b such that
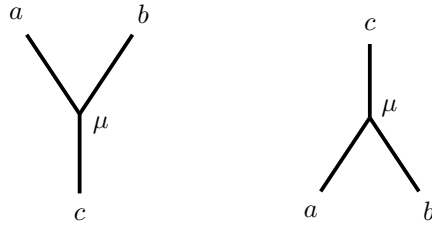
$$\sum_c N_{ab}^c > 1 \tag{3.4}$$

Figure 3.1: If we take time as going downwards then on the left we have the fusion of a and b into c and on the right we have the splitting of c into a and b.

and a charge a is said to be abelian if $\sum_c N_{ab}^c = 1$ for every b. This process of combining particles to form others is known as fusion and eq. (3.1) is known as the fusion rules (each model has its own). Fusion can be thought of as putting two anyons in a box and evaluating the total charge of the box. As such the anyons don't need to interact and can be kept far apart from each other. The process inverse to fusion is called splitting and can be interpreted by reversing time in a fusion process. To analyze this it is useful to employ diagrams known as fusion trees (see fig. 3.1). We will not construct a full diagrammatic calculus, but doing so might be a good idea. For further reading on this see [Bon07]. From our description of fusion it is only natural that this be commutative. For each particle a there is also one associated anti-particle $\bar{a}$ such that $a \times \bar{a} = 1 + \sum_{c \neq 1} N_{a\bar{a}}^c c$ (here we considered the vacuum to be multiplicity-free but this need not be the case). The vacuum also acts trivially on any particle, that is $a \times 1 = a$.

Let us now define the Fusion Hilbert space V (Hilbert space associated with a fusion process), whose states represent different fusion outcomes and as such they are orthogonal. To anyons a and b with fusion rules given by eq. (3.1) we assign a state $|a, b; c, \mu\rangle$ to each possible fusion outcome, where $\mu = 1, ..., N_{ab}^c$. If $N_{ab}^c = 0$ then this does not represent a valid fusion and therefore $V_{ab}^c = \varnothing$. If $N_{ab}^c \neq 0$ then fusion multiplicity uniquely defines the dimension of the Hilbert space

$$\dim(V_{ab}^c) = N_{ab}^c \tag{3.5}$$

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Let us now consider three anyons with charge a, b, c that fuse to d. The result of the fusion should always be d and as such it shouldn't matter in which order we fuse them. Thus we can decompose the space V_{abc}^d into a tensor product of two anyon fusion spaces, in two isomorphic ways

$$V_{abc}^d \cong \bigoplus_i V_{ab}^e \otimes V_{ec}^d \cong \bigoplus_f V_{af}^d \otimes V_{bc}^f. \tag{3.6}$$

Since these two different ways of fusing anyons span the same space, we are able to switch between these two different bases using a set of isomorphisms called F-moves that act as

$$|a, b; e, \alpha\rangle \, |e, c; d, \beta\rangle = \sum_{f,\mu,\nu} [F_{abc}^d]_{(e,\alpha,\beta)(f,\mu,\nu)} \, |b, c; f, \mu\rangle \, |a, f; d, \nu\rangle \tag{3.7}$$

and are unitary for anyon models. We can best think of these by looking at their action on fusion trees

$$\tag{3.8}$$

These ensure for us that fusion is an associative process. If we increase the size of our space, by adding additional anyons, we must impose that the F-moves satisfy a specific constraint known as the Pentagon equation and given by

$$\sum_\delta [F_{fcd}^e]_{(g,\beta,\gamma)(l,\delta,\nu)} [F_{abl}^e]_{(f,\alpha,\delta)(k,\lambda,\mu)}$$

$$= \sum_{h,\sigma,\psi,\rho} [F_{abc}^g]_{(f,\alpha,\beta)(h,\sigma,\psi)} [F_{ahd}^e]_{(g,\sigma,\gamma)(k,\lambda,\rho)} [F_{bcd}^k]_{(h,\psi,\rho)(l,\mu,\nu)}. \tag{3.9}$$

This simply states that one should end up with the same basis decomposition if we take two distinct series of F-moves. We can best think of this constraint by looking at it in its diagrammatic form in fig. 3.2, which is where we get the name. This means that
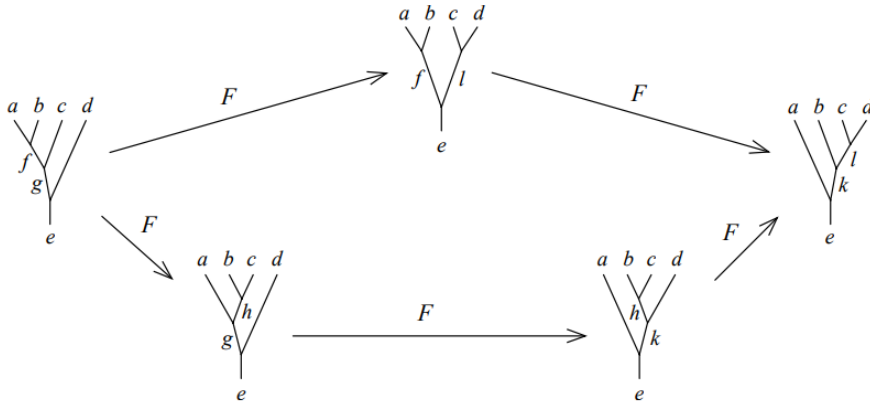
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Figure 3.2: Pentagon equation: starting from an initial fusion basis, two different series of F-moves should lead us to the same final fusion basis. Image taken from [Bon07].

starting from one fusion basis we should be able to take either the upper path or the lower path (the diagram commutes) and so by taking two different series of F-moves we should end up in the same final fusion basis. We must also impose an additional physical condition that fusion with the vacuum commutes with the F-moves. This imposes that fusion with the vacuum mustn't change the state, that is, $[F^d_{abc}] = 1$ if either a, b or c is 1 (assuming that particular F-move is a valid one).

Let us now increase the size of our fusion space by taking a family of anyons $\{a_i\}_{i=0}^{n-1}$ with the same charge that fuse to a final anyon a_n. As we've seen before it doesn't matter how we fuse them since we can always apply F-moves to move between fusion bases; thus we will fuse them in numerical order (standard basis), with inbetween outcomes e_1, e_2, ... e_{n-3}. This particular decomposition is then given by

$$V^{a_n}_{a_1,a_2,...,a_{n-1}} = \bigoplus_{e_1,e_2,...,e_{n-3}} V^{e_1}_{a_1 a_2} \otimes V^{e_2}_{e_1 a_3} \otimes \cdots \otimes V^{a_n}_{e_{n-3} a_{n-1}}. \tag{3.10}$$

It then follows from this and equation (3.5) that the dimension of this larger space is

given by

$$\dim\left(V^{a_n}_{a_1,a_2,\ldots,a_{n-1}}\right) = \dim\left(\bigoplus_{e_1,e_2,\ldots,e_{n-3}} V^{e_1}_{a_1 a_2} \otimes V^{e_2}_{e_1 a_3} \otimes \cdots \otimes V^{a_n}_{e_{n-3} a_{n-1}}\right)$$

$$= \sum_{e_1,e_2,\ldots,e_{n-3}} N^{e_1}_{a_1 a_2} N^{e_2}_{e_1 a_3} N^{c_3}_{e_2 a_4} \cdots N^{a_n}_{e_{n-3} a_{n-1}}. \tag{3.11}$$

A more intuitive way to look at this is through a quantity d_i known as the quantum dimension of an anyon i. It refers to the dimension of the Hilbert space associated with a particular anyonic charge and can be defined from the fusion rules, eq. (3.1), as

$$d_a d_b = \sum_c d_c. \tag{3.12}$$

It follows from the unitarity of anyon models that $d_i \geq 1$, with equality when i is an abelian anyon. If we have a space with n anyons, all with charge i (as in the previous case), then the quantum dimension characterizes how fast the dimension of the Hilbert space grows when we add an additional anyon. We therefore obtain the relation

$$\dim\left(V^{a_n}_{a_1,a_2,\ldots,a_{n-1}}\right) \sim d_a^n. \tag{3.13}$$

This obviously can't be an equality as the dimension of the space needs be an integer while d_a^n doesn't. From this we can now get the total dimension of an anyon model, which is defined as

$$\mathcal{D} = \sqrt{\sum_i d_i^2}. \tag{3.14}$$

3.2 Braiding

The main property of anyons is their non-trivial statistical behaviour. We have that fusion is commutative hence there is an isomorphism R, between fusion spaces

$$R : V_{ab}^c \longrightarrow V_{ba}^c \tag{3.15}$$

called an R-move. These are responsible for the exchange of two particles and they are defined by their action on the basis states as

$$R_{ab}^c \, |a, b; c, \mu\rangle = \sum_\nu [R_{ab}^c]_{\mu\nu} \, |b, a; c, \nu\rangle \, . \tag{3.16}$$

In this case it is particularly helpful to look at this in a diagrammatic form. As such the action of these R-moves on the fusion tree is given by

$$\tag{3.17}$$

As with the F-moves, these are also unitary. They must also satisfy an equation that ensures that braiding is consistent with fusion. This is known as the Hexagon equation, due to its diagrammatic form in fig. 3.3. This equation is given by

$$\sum_{\lambda,\gamma} [R_{ca}^e]_{\alpha\lambda} [F_{acb}^d]_{(e,\lambda,\beta)(g,\mu,\gamma)} [R_{cb}^g]_{\gamma\nu}$$

$$= \sum_{f,\sigma,\delta,\psi} [F_{cab}^d]_{(e,\alpha,\beta)(f,\sigma,\delta)} [R_{cf}^d]_{\sigma\psi} [F_{abc}^d]_{(f,\delta,\psi)(g,\mu,\nu)}. \tag{3.18}$$
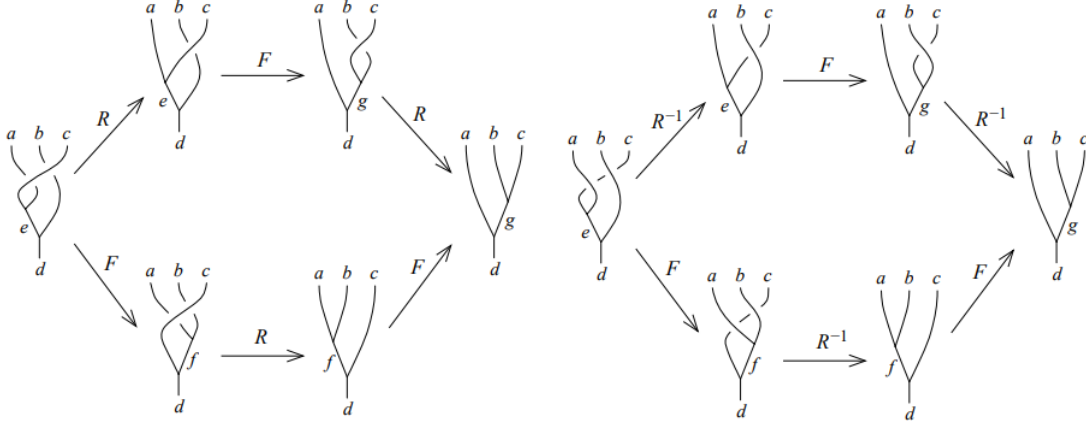
Figure 3.3: Hexagon equation: Enforces that fusion is compatible with braiding, as in, sequences of F-moves and R-moves from the same starting configuration to the same final configuration give the same results (the diagram commutes). Image taken from [Bon07].

As one should be able to do the same when braiding them clockwise we thus end up with

$$\sum_{\lambda,\gamma}[(R^e_{ac})^{-1}]_{\alpha\lambda}[F^d_{acb}]_{(e,\lambda,\beta)(g,\mu,\gamma)}[(R^g_{bc})^{-1}]_{\gamma\nu}$$

$$= \sum_{f,\sigma,\delta,\psi}[F^d_{cab}]_{(e,\alpha,\beta)(f,\sigma,\delta)}[(R^d_{fc})^{-1}]_{\sigma\psi}[F^d_{abc}]_{(f,\delta,\psi)(g,\mu,\nu)}. \tag{3.19}$$

This imposes that braiding commutes with fusion and implies the Yang-Baxter equation introduced in the context of braid theory in eq. (2.11) (Chapter 2). As we've stated before, and as is the case for the F-moves, this equation can be best seen by looking at its diagrammatic form, where the condition is enforced by making the diagram commute. From the triviality of fusion with the vacuum and the Hexagon equation we then have that $R^a_{a1} = R^b_{1b} = 1$. The F-moves and R-moves completely specify an anyon model and from the Mac Lane coherence theorem [Mac98], the Pentagon and Hexagon equations are sufficient to ensure consistency.

Let us now see how braiding affects anyons without a direct fusion channel. Consider the fusion of three anyons with charge a, b and c into a fourth d in the basis

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$|a, c; e, \alpha\rangle |e, b; d, \beta\rangle$. This basis seems rather arbitrary but it represents the fusion process when anyons b and c are braided (see fig. 3.4). In this basis we can't directly braid them, thus to circumvent this we must first apply F-moves so that they have a direct fusion channel. Now we can apply R-moves, effectively braiding them and if we wish we can return to the original fusion basis through the action of inverse F-moves. This can be better seen by the changes in fusion trees as depicted by fig. 3.4.
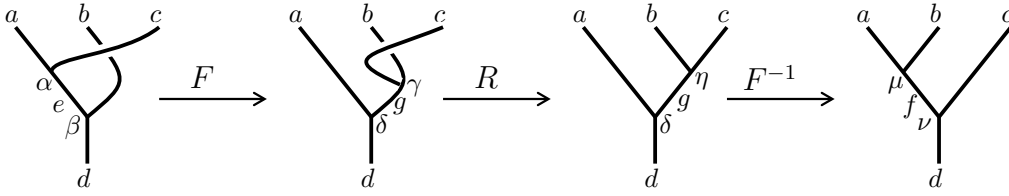
Figure 3.4: Description of a B-move.

These are called the B-moves. Since the F-moves and the R-moves are unitary so are these. It is easy to see in fact that these are representations of the Braid group $\mathcal{B}_n$ and furthermore they are irreducible. As for the previous moves these are indexed by $(6 + 4)$ indices and are explicitly given by

$$[B_{abc}^d]_{(e,\alpha,\beta)(f,\mu,\nu)} = \sum_{g,\gamma,\delta,\eta} [F_{acb}^d]_{(e,\alpha,\beta)(g,\gamma,\delta)} [R_{cb}^g]_{\gamma,\eta} [(F_{abc}^d)^{-1}]_{(g,\eta,\delta)(f,\mu,\nu)}, \tag{3.20}$$

or from fig. 3.4, we can define it in its diagrammatic form as

$$= \sum_{f,\mu,\nu} [B_{abc}^d]_{(e,\alpha,\beta)(f,\mu,\nu)} \tag{3.21}$$

3.3   Twisting

In an anyon model, to each particle a there is an associated quantity known as the topological spin or twist factor, θ_a. This is associated with a 2π rotation of an anyonic

charge a around its own axis. Similarly to the previous cases, this is best seen by looking at this process in its diagrammatic form

$$\theta_a \bigg| = \bigg| \theta_a \bigg| = \bigg| , \tag{3.22}$$

and

$$\theta_a^* \bigg| = \theta_a^* \bigg| = \bigg| . \tag{3.23}$$

When applicable, this is related to the (ordinary angular momentum) spin or CFT conformal scaling dimension h_a, by

$$\theta_a = e^{i2\pi h_a}. \tag{3.24}$$

One can best visualize this if we treat our anyonic world-lines as world-ribbons instead. By doing so, if we pull a twisted ribbon straight we end up with topologically different surfaces. This represents in fact the twist of a particle around its own axis, or the exchange of two identical particles. In a more general context this is known as the spin-statistics theorem. If we take h_a to be an integer, then the topological spin is therefore 1 and the particle is a boson. On the other hand if we take h_a to be a half-(odd)integer, then the topological spin equals -1 and the particle is a fermion. Every other value represents a different species of anyons.

Let us now see how twisting relates to braiding. Consider a fusion process of two anyons a, b into a third c. If we apply a double braiding to this fusion process we thus end up with a phase factor that depends on the topological spin of each anyon as depicted by fig. 3.5. This process of double braiding is known as monodromy and is
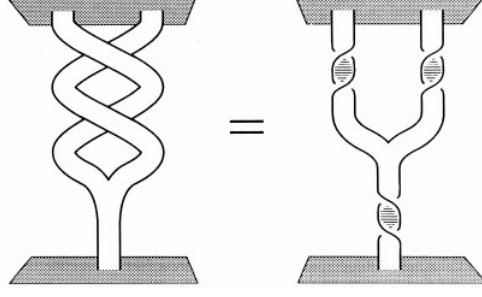
Figure 3.5: Double braiding is equivalent to twisting each particles world-ribbon. Image taken from [BS09].

governed by the monodromy equation

$$\sum_\lambda [R^c_{ab}]_{\mu\lambda}[R^c_{ba}]_{\lambda\nu} = \frac{\theta_c}{\theta_a\theta_b}\delta_{\mu,\nu} = e^{2\pi i(h_c-h_b-h_a)}\delta_{\mu,\nu}. \tag{3.25}$$

where δ represents the usual Kronecker delta.

3.4 Examples

We will now present some examples of anyon theories. All these are multiplicity-free, therefore we can drop the Greek indices from the F-moves (3.9), R-moves (3.19) and B-moves (3.20). These are then given by

$$[F^e_{fcd}]_{gl}[F^e_{abl}]_{fk} = \sum_h [F^g_{abc}]_{fh}[F^e_{ahd}]_{gk}[F^k_{bcd}]_{hl}, \tag{3.26}$$

$$[(R^e_{ac})^{-1}][F^d_{acb}]_{eg}[(R^g_{bc})^{-1}] = \sum_f [F^d_{cab}]_{ef}[(R^d_{fc})^{-1}][F^d_{abc}]_{fg}, \tag{3.27}$$

$$[R^e_{ca}][F^d_{acb}]_{eg}[R^g_{cb}] = \sum_f [F^d_{cab}]_{ef}[R^d_{cf}][F^d_{abc}]_{fg}. \tag{3.28}$$

The B-moves follow naturally from the others and as such we we not will state or derive them explicitly.

3.4.1 Fibonacci

This theory has two types of particles τ and 1, where the latter is the vacuum, and is non-abelian since $\tau \times \tau = \tau + 1$. Its name comes from the quantum dimension of τ particles which is the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ and hence the fusion space grows according to the Fibonacci sequence. From the three examples below, this is the only one which can be used for universal quantum computation, as it has a dense braid group representation [FLW02]. For further insight and derivations of the model see [Tre+08].

<div align="center">Data</div>

$1 \times 1 = 1, \quad 1 \times \tau = \tau, \quad \tau \times \tau = \tau + 1$	
$F^{\tau}_{\tau\tau\tau} = \begin{bmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{bmatrix}$	$R_{\tau\tau} = \begin{bmatrix} \text{-}e^{i4\pi/5} & 0 \\ 0 & e^{i3\pi/5} \end{bmatrix}$
$d_1 = 1, \ d_\tau = \phi, \ \mathcal{D} = \sqrt{1 + 2\phi}$	$\theta_1 = 1, \ \theta_\tau = e^{i\pi\frac{4}{5}}$

3.4.2 Ising

This is another non-abelian theory known as the Ising model. It is of particular interest since it arises from a particular solvable model introduced by Kitaev in [Kit06]. Even though the model is non-abelian and there are more particle species, this theory does not allow for universal quantum computation. For further insight and derivations of the model see [Kit06].

<div align="center">Data</div>

$1 \times a = a, \quad \psi \times \psi = 1, \quad \sigma \times \psi = \sigma + 1, \quad \sigma \times \sigma = 1 + \psi$	
$F^{\sigma}_{\sigma\sigma\sigma} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$	$R_{\sigma\sigma} = \begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i3\pi/8} \end{bmatrix}$
$[F^{\psi}_{\sigma\psi\sigma}]_{\sigma\sigma} = [F^{\psi}_{\psi\sigma\psi}]_{\sigma\sigma} = -1$	$R^{\sigma}_{\sigma\psi} = R^{\sigma}_{\psi\sigma} = e^{-i\pi/2}, \quad R^{1}_{\psi\psi} = -1$
$d_1 = d_\psi = 1, \ d_\sigma = \sqrt{2}, \ \mathcal{D} = 2$	$\theta_1 = 1, \ \theta_\psi = -1, \ \theta_\sigma = e^{i\pi/8}$

3.4.3 $\mathcal{D}(\mathbb{Z}_2)$

This theory arises from a spin system embedded in a torus and is the main result of this text, which we will explore in depth in the next section. This model is obviously abelian since the dimension of all particles is one and therefore does not allow for universal quantum computation.

<div align="center">

Data

$1 \times a = a, \quad e \times e = 1, \quad m \times m = 1, \quad e \times m = \epsilon, \quad \epsilon \times \epsilon = 1$	
$F_{abc}^{a \times b \times c} = 1,$ for any $a, b, c \in \{1, e, m, \epsilon\}$	$R_{ee}^1 = R_{mm}^1 = R_{em}^\epsilon = R_{\epsilon m}^e = R_{e\epsilon}^m = 1,$ $R_{\epsilon\epsilon}^1 = R_{me}^\epsilon = R_{m\epsilon}^e = R_{\epsilon\epsilon}^m = -1$
$d_1 = d_e = d_m = d_\epsilon = 1, \mathcal{D} = 2$	$\theta_1 = \theta_e = \theta_m = 1, \; \theta_\epsilon = -1$

</div>

3.5   Topological Quantum Computation

In the most general sense, a quantum computation is performed by preparing some qubits (in the form of states), evolving them through the action of unitary gates, and reading the output through measurements on these qubits. Our aim is now to employ non-abelian anyons to perform fault-tolerant quantum computation in a similar fashion to that described above. We start by creating pairs of anyons from the vacuum (preparing the fusion states). As we've seen before, braiding is unitary and hence we can simulate quantum gates by braiding anyons. Finally we simply read the output by measuring the total anyonic charge of the system, that is, by fusing the anyons (see fig. 3.6). Note that from (3.13) and (3.14) we have that the dimension of the fusion space need not be an integer and therefore might not admit a tensor product structure. Nevertheless we can still encode quantum information on a subspace that does so. The advantage of this scheme over others is the fault-tolerance that accompanies it. The gates are more robust since the braidings depend only on the topology and not on the specific details of paths spanned by the anyons. The qubits are also more resilient to
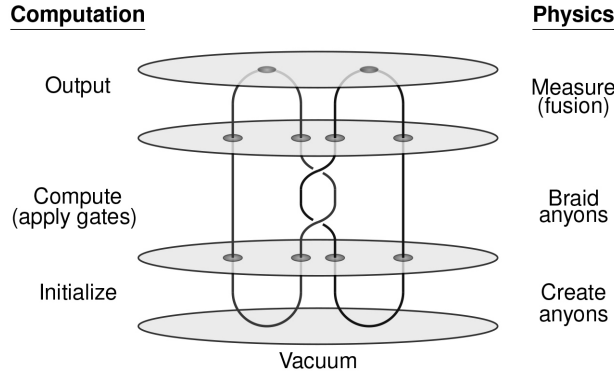
Figure 3.6: Scheme for performing computation using anyons. Image credits go to Eric C. Rowell.

the environment since we can keep them far apart from each other, making any environmental error acting in the form of local perturbations of the Hamiltonian, not cause any error on the logical qubits (fusion state).

Let us now provide an example of a computation in the Fibonnaci model, introduced in section 3.4.1. We'll use a two-dimensional space (one qubit space), corresponding to the fusion of 4 τ anyons into the vacuum, with states $|0\rangle = |\tau, \tau; 1\rangle |\tau, \tau; 1\rangle$ and $|1\rangle = |\tau, \tau; \tau\rangle |\tau, \tau; \tau\rangle$. We start by creating two pairs of τ aynons from the vacuum, meaning we are in the state $|0\rangle$. We can now act with a gate on the qubit, by braiding the two middle anyons, that is,

$$R_{23} |0\rangle = B_{23} |0\rangle = \phi^{-1}(-e^{i4\pi/5})\phi^{-1} |0\rangle + \phi^{-1/2}e^{i3\pi/5}\phi^{-1/2} |1\rangle \neq |0\rangle . \qquad (3.29)$$

Since the initial state changed from $|0\rangle$ to a superposition, we have therefore performed a computation on this qubit.

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4 The Toric Code

In this chapter we will describe a model known as the toric code. It was first introduced by Kitaev [Kit03] as a simple realization of topological order, i.e., a system that can be described at low temperature and long wavelength by a topological quantum field theory. In the context of this text, this simply means that it can represent an anyon model, and therefore a quantum computer (while not directly stated, the error-correction properties that we will see later in this chapter, are related to the existence of these anyons). For alternative/complementary approaches to the subject see [Kit03], [Pac12], [Sim16] and [Wan10].

4.1   The Space

Consider a $k \times k$ lattice embedded in a torus (this can be done by imposing periodic boundary conditions). To each edge we attach a spin (or qubit), that is, a two-dimensional Hilbert space $\mathcal{H}_i$ with orthonormal basis $\{|0\rangle, |1\rangle\}$ representing spin up and spin down respectively. The Hilbert space of our model, $\mathcal{H}$, will be a tensor product of these local two-dimensional Hilbert spaces. Since there are exactly $N = 2k^2$ edges (spins) the dimension of $\mathcal{H}$ will then be 2^N. For each vertex v and each plaquette p we define the following operators

$$A(v) = \prod_{i \in star(v)} \sigma_i^z, \qquad B(p) = \prod_{j \in \partial p} \sigma_j^x, \tag{4.1}$$

where the index $i \in star(v)$ runs over all four edges around a vertex v, the index $j \in \partial p$ runs over all four edges on the boundary of a plaquette p, and σ^x and σ^z are the usual Pauli operators specified in eq. (2.7) in chapter 2.
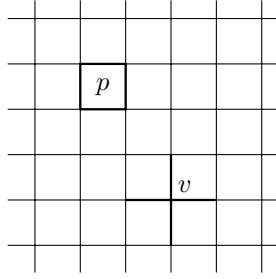
Figure 4.1: Square lattice with periodic boundary conditions.

4.2 The Algebra of Operators

Let us now see the commutation relations between these operators. The Pauli operators obey the commutation relation established by the commutator $[\sigma^a, \sigma^b] = 2i\epsilon_{abc}\sigma^c$, where ϵ is the Levi-Civita symbol and $a, b, c \in \{x, y, z\}$. This then allows us to establish the following relations

$$[\sigma^x, \sigma^x] = 0, \tag{4.2}$$

$$[\sigma^z, \sigma^z] = 0, \tag{4.3}$$

$$\sigma^x \sigma^z = -\sigma^z \sigma^x. \tag{4.4}$$

Furthermore, it is also simple to verify that all Pauli operators square to the identity, that is $(\sigma^a)^2 = I$, for $a \in \{x, y, z\}$. From this we can conclude that the $A(v)$ $(B(p))$ operators also square to the identity and they also commute, since these are simply products of Pauli operators with the same label, thus leading us to the relations

$$[A(v_1), A(v_2)] = 0, \quad A(v)^2 = I \tag{4.5}$$

$$[B(p_1), B(p_2)] = 0, \quad B(p)^2 = I \tag{4.6}$$

for any vertices v, v_1, v_2 and plaquettes p, p_1, p_2. Let us now see how the $A(v)$ and $B(p)$ operators interact with each other. If we take the star of a vertex v and the boundary

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of a plaquette p, then these either share no edges and the operators trivially commute, or they share exactly two edges (the case where v is a vertex of the plaquette). From eq. (4.4) we then have

$$
\begin{aligned}
A(v)B(p) = \prod_{i\in star(v)}\sigma_i^z \prod_{j\in\partial p}\sigma_j^x &= \sigma_1^z\sigma_2^z\sigma_3^z\sigma_4^z\sigma_3^x\sigma_4^x\sigma_5^x\sigma_6^x \\
&= \sigma_1^z\sigma_2^z(-1)\sigma_3^x\sigma_3^z(-1)\sigma_4^x\sigma_4^z\sigma_5^x\sigma_6^x \\
&= (-1)^2\sigma_3^x\sigma_4^x\sigma_5^x\sigma_6^x\sigma_1^z\sigma_2^z\sigma_3^z\sigma_4^z \\
&= B(p)A(v),
\end{aligned}
\tag{4.7}
$$

where in the third and fourth equalities we used the fact that sigma operators with the same label commute and sigma operators with different labels anti-commute. This then ensures that vertex and plaquette operators always commute. We thus end up with the commutator

$$
[A(v), B(p)] = 0,
\tag{4.8}
$$

for any v, p. One can now see what happens when we act with these operators on every vertex and plaquette simultaneously. If we apply $A(v)$ $(B(p))$ operators on two adjacent vertices (plaquettes) then they will act on different $\mathcal{H}_i$ spaces, except for the one edge they share. From equations (4.2) and (4.3) we know that this shared edge will be left unchanged. If we now apply these operators to every vertex and plaquette then we will leave the whole space unchanged (due to the periodic boundary conditions imposed on the lattice), $i.e$,

$$
\prod_v A(v) = I, \qquad \prod_p B(p) = I.
\tag{4.9}
$$

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Let us now evaluate the eigenvalues of the $A(v)$ and $B(p)$ operators. If we take $|\psi_v\rangle$ and $|\psi_p\rangle$ to be eigenvectors of $A(v)$ and $B(p)$ respectively, we then have

$$A(v)\,|\psi_v\rangle = a\,|\psi_v\rangle\,, \quad B(p)\,|\psi_p\rangle = b\,|\psi_p\rangle\,. \tag{4.10}$$

From the commutation relations established before we can now find the eigenvalues of these operators:

$$\begin{aligned} A(v)^2\,|\psi_v\rangle =&I\,|\psi_v\rangle = a^2\,|\psi_v\rangle\,, \\ B(p)^2\,|\psi_p\rangle =&I\,|\psi_p\rangle = b^2\,|\psi_p\rangle\,, \end{aligned} \tag{4.11}$$

thus telling us that their eigenvalues are ± 1.

4.3 Path Operators

We will now introduce two operators which we will frequently use throughout the rest of this chapter. These are strings of Pauli operators along particular paths of local Hilbert spaces and as such are defined as:

$$X(\gamma) = \prod_{j\in\gamma} \sigma_j^x \qquad Z(\gamma^*) = \prod_{i\in\gamma*} \sigma_i^z, \tag{4.12}$$

where γ is a path along the lattice and γ^* is a cut on the lattice, that is a path along the dual lattice (the Z operator will act on the $\mathcal{H}_i$ spaces of edges that intersect this curve). Consider γ (γ^*) to be an open path on the lattice (dual lattice). If we act with a $X(\gamma)$ ($Z(\gamma*)$) operator on the space, then it will act on edges belonging to any star(v) (∂p) along the curve twice, except at both ends of the path where it acts exactly once. As we've seen before, Pauli operators square to the identity and so $X(\gamma)$ ($Z(\gamma*)$) will act trivially everywhere except at the ends of the curve, where it will change the eigenvalue of the corresponding $A(v)$ ($B(p)$) operator (due to the anti-commutation
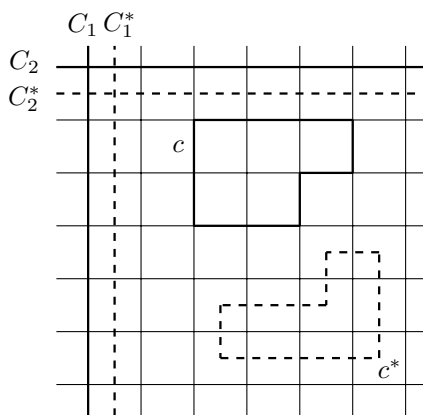
Figure 4.2: Various curves on the lattice. Closed contractible curves are the product of $A(v)$ or $B(p)$ operators.

between Pauli operators with different label). This will be the mechanism used to create and move excitations in the next section. If we take c to be a closed curve on the lattice [1] instead, then $X(c)$ is simply a product of $B(p)$ operators on the enclosed plaquettes. The product of these $B(p)$ operators will act with σ^x operators on all spaces on the curve and all spaces enclosed by the curve and since Pauli operators square to the identity, the spaces inside the curve will then be left unchanged. We thus end up with the relation

$$X(c) = \prod_{p \ inside \ c} B(p). \tag{4.13}$$

The same will happen for Z operators on closed curves. If we take a closed curve c^* on the dual lattice then this is simply a product of $A(v)$ operators on all vertices enclosed by the curve and thus we end up with the relation

$$Z(c^*) = \prod_{v \ inside \ c^*} A(v). \tag{4.14}$$

A better way to understand this is by looking at figure 4.2. Until now we've only considered contractible curves, that is, curves that can be deformed into a point, but

---

[1]Since our lattice is on a torus there can be non-contractible closed curves. Unless stated otherwise all curves are contractible.

since our lattice is on a torus there can also exist non-contractible curves that can not be written as a product of $A(v)$ and $B(p)$. These are curves that wind all the way around the torus and will be important when we introduce the Hamiltonian of the model. As such these shall receive names C_1, C_2, C_1^* and C_2^* where the indices correspond to the two possible orientations on the torus.

4.4   The Hamiltonian

Let us now introduce the Hamiltonian for the model as a sum of $A(v)$ and $B(p)$ operators over all vertices and plaquettes of the lattice,

$$H = -\sum_v A(v) - \sum_p B(p). \tag{4.15}$$

We've seen before that the $A(v)$ and $B(p)$ operators commute, hence they also commute with the Hamiltonian making all these operators simultaneously diagonalizable and therefore the model exactly solvable. The next step is to find the ground state of the system. This is the state of lowest energy, hence it will be such that the value of the Hamiltonian is minimal. Since we have a minus sign in front of each of the terms in eq. (4.16), we need to maximize the eigenvalues of every $A(v)$ and $B(p)$ operator. It then follows that the ground state is given by

$$GS = \{|\xi\rangle : A(v)|\xi\rangle = |\xi\rangle,\ B(p)|\xi\rangle = |\xi\rangle \text{ for all } v,\ p\}. \tag{4.16}$$

We will now find an explicit form for the ground state. From eqs. (4.5) and (4.6) we know that both $A(v)$ and $B(p)$ operators square to the identity, therefore we will also have that $A(v)(I + A(v)) = A(v) + I$ and $B(p)(I + B(p)) = B(p) + I$. We now claim

that an explicit form of the ground state is

$$|\psi_0\rangle = \frac{1}{\sqrt{2}} \prod_p (I + B(p)) \bigotimes_{\mathcal{H}_i} |0\rangle_i \,, \qquad (4.17)$$

where the product runs over all plaquettes, p, and the tensor product runs over all edges (local Hilbert spaces $\mathcal{H}_i$) on the lattice, thus denoting a state in $\mathcal{H}$. $1/\sqrt{2}$ is simply a normalization factor. Let us verify that this is indeed a ground state. We will start by acting with a $A(v)$ operator on an arbitrary vertex v, that is

$$\begin{aligned} A(v)|\psi_0\rangle &= A(v)\frac{1}{\sqrt{2}} \prod_p (I + B(p)) \bigotimes_{\mathcal{H}_i} |0\rangle_i \\ &= \frac{1}{\sqrt{2}} \prod_p (I + B(p)) A(v) \bigotimes_{\mathcal{H}_i} |0\rangle_i \\ &= \frac{1}{\sqrt{2}} \prod_p (I + B(p)) \bigotimes_{\mathcal{H}_i} |0\rangle_i \\ &= |\psi_0\rangle \,, \qquad (4.18) \end{aligned}$$

where in the second equality we used the commutativity between the $A(v)$ and $B(p)$ operators and in the third we used the fact that $\sigma^z |0\rangle = |0\rangle$. If we now act with a $B(p)$ operator, then from the identity $B(p)(I + B(p)) = B(p) + I$ we obtain

$$\begin{aligned} B(p')|\psi_0\rangle &= B(p')\frac{1}{\sqrt{2}} \prod_p (I + B(p)) \bigotimes_{\mathcal{H}_i} |0\rangle_i \\ &= \frac{1}{\sqrt{2}} \prod_p (I + B(p)) \bigotimes_{\mathcal{H}_i} |0\rangle_i \\ &= |\psi_0\rangle \,. \qquad (4.19) \end{aligned}$$

This confirms that $|\psi_0\rangle$ is indeed a ground state for our model.

Let us now see the effect of the path operators, given by eq. (4.12), on the ground state. We need not bother with open curves since these take us out of the ground state by changing the eigenvalues of the $A(v)$ or $B(p)$ operators at their ends. If

we instead take a closed curve c, our path operator will act as $X(c)\,|\psi_0\rangle$ and from eqs. (4.13) and (4.14) we obtain $\prod_p B(p)\,|\psi_0\rangle = |\psi_0\rangle$, thus ensuring us that X type operators along closed curves leave the ground state unchanged. We can make the same argument for $Z(c^*)$ operators. We simply take $Z(c^*)\,|\psi_0\rangle = \prod_v A(v)\,|\psi_0\rangle = |\psi_0\rangle$. Consider now a non-contractible loop. Since it is a loop, two Pauli operators will act on each star (or plaquette boundary) along the curve and so there will be no changes in eigenvalues, meaning $A(v)$, $B(p)$ commute with $X(C)$ and $Z(C*)$, effectively keeping us in the ground state but since the loop isn't contractible, it isn't a product of vertex or plaquette operators. Let us then see the action of $X(C_1)$ on the ground state

$$
\begin{aligned}
X(C_1)\,|\psi_0\rangle &= X(C_1)\frac{1}{\sqrt{2}}\prod_p (I + B(p)) \bigotimes_{\mathcal{H}_i} |0\rangle_i \\
&= \frac{1}{\sqrt{2}}\prod_p (I + B(p)) X(C_1) \bigotimes_{\mathcal{H}_i} |0\rangle_i \\
&= \frac{1}{\sqrt{2}}\prod_p (I + B(p)) \bigotimes_{j\in C_1} |1\rangle_j \bigotimes_{i\notin C_1} |0\rangle_i \\
&= |\psi_1\rangle\,,
\end{aligned}
\tag{4.20}
$$

so we end up with a different state. We've pointed out before that this must indeed be a ground state and we will formally verify this now. Let us start by acting with a vertex operator on this state:

$$
A(v)\,|\psi_1\rangle = A(v)\frac{1}{\sqrt{2}}\prod_p (I + B(p)) \bigotimes_{j\in C_1} |1\rangle_j \bigotimes_{i\notin C_1} |0\rangle_i
\tag{4.21}
$$

$$
= \frac{1}{\sqrt{2}}\prod_p (I + B(p)) \bigotimes_{j\in C_1} |1\rangle_j \bigotimes_{i\notin C_1} |0\rangle_i\,.
\tag{4.22}
$$

Now if $A(v)$ acts on a vertex that is not on the curve C_1 then it is the same case as $|\psi_0\rangle$ and the action is trivial. If it acts on a vertex on the curve then it should act trivially as well since there are exactly two edges of the star v on the curve. Since $B(p)$ acts trivially on $|\psi_1\rangle$, due to the relation $B(p)(I + B(p)) = B(p) + I$, we thus confirm

that $|\psi_1\rangle$ is in fact a ground state different from the original $|\psi_0\rangle$. If we take another non-contractible loop around the other direction of the torus, $X(C_2)$, then we end up in yet a different ground state

$$X(C_2)|\psi_0\rangle = |\psi_2\rangle. \tag{4.23}$$

We can also act with operators X around both non-contractible loops of the torus at the same time, leading us to a ground state different from any of the previous ones:

$$X(C_1)X(C_2)|\psi_0\rangle = |\psi_3\rangle. \tag{4.24}$$

We will not prove these claims but with similar arguments to the first one, it should be fairly straightforward to do so.

By embedding the lattice in a torus we thus end up with a four-fold degenerate ground state and can change between these by taking non-contractible loops around the torus. In fact this ground state degeneracy depends only on the topology of the surface. If we increase the number of non-contractible loops by embedding the lattice in a surface with higher genus g, we end up with a 4^g-fold ground state degeneracy.

4.5   Anyons

Let us now classify low energy excitations of the Hamiltonian (4.15). The ground state is such that $A(v)|\xi\rangle = |\xi\rangle$ and $B(p)|\xi\rangle = |\xi\rangle$ for all v and p, therefore a particle excitation will occur when one of these constraints is violated. From the relations $\prod_v A(v) = 1$ and $\prod_p B(p) = 1$ in (4.9), we aren't allowed to create a single particle excitation, thus these must always come in pairs.

Take the action of a path operator $X(\gamma)$ on a single edge. From the anti-commutation of Pauli operators with different label we thus have $A(v)X(\gamma) = -X(\gamma)A(v)$ on the two vertices that share that edge. This puts us in violation of the ground state constraints,
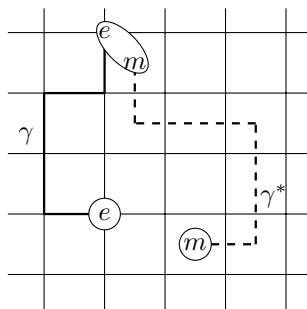
Figure 4.3: Creating two electric charges and two magnetic vortices and moving one of each together to obtain a fermion.

as we have the $A(v)$ operators on both these vertices in the -1 eigenvalue. We will call these particles "electric charges" (e). From the path operator properties established in section 4.3 we can then create and move these particles by acting with X operators on the space. Note that the state $|\psi(\gamma)\rangle = X(\gamma)|\xi\rangle$ depends only on the homotopy class[2] of the path γ while $X(\gamma)$ depends on the path itself. Let us now take the action of a path operator $Z(\gamma^*)$ on a single edge. We have that Z anti-commutes with $B(p)$ on the two plaquettes that share that edge, hence we now have two $B(p)$ operators in the eigenvalue -1, meaning we are no longer in the ground state. These particles live on the plaquettes (or the dual lattice) and are called "magnetic vortices" (m). We can create and move these particles by acting with Z operators on the space.

As we've seen before, if we take $X(c)$ and $Z(c^*)$ where c and c^* are closed curves then their action is trivial on the ground state. One can interpret this as creating two particles, taking one around a path and bringing it back to the first, annihilating the two. This process can be understood as the fusion established in section 3.1, and so we have just named two fusion rules, $e \times e = 1$ and $m \times m = 1$. We can also consider a new particle by taking together (fusing) an e and a m particle. These two will certainly not annihilate since they live on dual spaces. This new particle is denoted by ϵ which we will simply call a fermion, due to its statistical behaviour (we will show this in the

---

[2]Two paths connecting the same endpoints are said to be in the same homotopy class if one can be continuously deformed into the other.

next section). We can see the fusion of ϵ particles as simply fusing their components thus giving rise to the fusion rules $\epsilon \times e = m$, $\epsilon \times m = e$ and $\epsilon \times \epsilon = 1$. The fusion rules for our model are then

$$\boxed{e \times e = 1, \ m \times m = 1, \ e \times m = \epsilon, \ \epsilon \times e = m, \ \epsilon \times m = e, \ \epsilon \times \epsilon = 1} \tag{4.25}$$

Note that moving an anyon to an unoccupied vertex or plaquette is the same as fusing with the vacuum, thus ensuring that the fusion rule $a \times 1 = a$, for $a \in \{e, m, \epsilon\}$, is satisfied.

4.6 Braiding and Statistics

Now that we've figured out all the possible particle excitations of the Hamiltonian, let us evaluate their statistical behaviour, that is, what happens to the overall state when we exchange two particles. We will start by exchanging two electric charges. We can do this by creating two pairs of e particles and exchanging one of each pair, as depicted by fig. 4.4. To simplify things we start by acting with X operators that set up the particles for the exchange, that is, $X(\gamma_{s_1})X(\gamma_{s_2}) \left| \xi \right\rangle = \left| \psi(\gamma_{s_1}\gamma_{s_2}) \right\rangle$. We can now exchange them by acting with X operators on the paths γ_1 and γ_2. Since these form a closed curve and there are no magnetic fluxes inside it ($B(p)$ operators on those plaquettes have $+1$ eigenvalue) then $X(\gamma_1)X(\gamma_2) \left| \psi \right\rangle = \prod_p B(p) \left| \psi(\gamma_{s_1}\gamma_{s_2}) \right\rangle = \left| \psi(\gamma_{s_1}\gamma_{s_2}) \right\rangle$, where p are the plaquettes enclosed by the curve. We can use a similar process process for the exchange of two magnetic particles:

$$\begin{aligned} Z(\gamma_1^*)Z(\gamma_2^*)Z(\gamma_{s_1}^*)Z(\gamma_{s_2}^*) \left| \xi \right\rangle &= Z(\gamma_1^*)Z(\gamma_2^*) \left| \psi(\gamma_{s_1}^*\gamma_{s_2}^*) \right\rangle \\ &= \prod_{v \ inside \ c^*} A(v) \left| \psi(\gamma_{s_1}^*\gamma_{s_2}^*) \right\rangle \\ &= \left| \psi(\gamma_{s_1}^*\gamma_{s_2}^*) \right\rangle. \end{aligned} \tag{4.26}$$
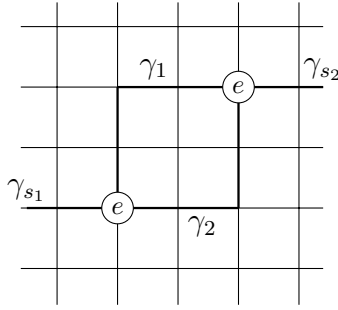
Figure 4.4: Exchange of two electrical charges.

The state of $\mathcal{H}$ remains unchanged after an exchange of both e and m particles, thus pointing us towards the idea that these particles are in fact bosons. Note that since we only took two paths to exchange the particles we have conceptually annihilated the particles and created two new ones along the way, but this isn't a problem because X and Z operators of the same kind commute and so we can always decompose the paths into smaller ones and add a "waiting time" on one of the particles (we will come back to this later in the section), thus not allowing them to fuse. Let us now see the statistics between e and m particles. Since they live on dual spaces we can not directly exchange them, but we can take one around the other, which is equivalent to exchanging them twice. We start by creating a pair of e and a pair of m particles and setting them up for the exchange (see fig. 4.5), that is

$$|\Psi_{initial}\rangle = Z(\gamma^*)X(\gamma)\,|\xi\rangle. \tag{4.27}$$

We now take the e particle around m via an $X(c)$ operator on the closed curve c (remember that the action of a X or Z operator along a closed curve is trivial on the ground state). Since $X(c)$ and $Z(\gamma^*)$ anti-commute, and their action shares exactly one
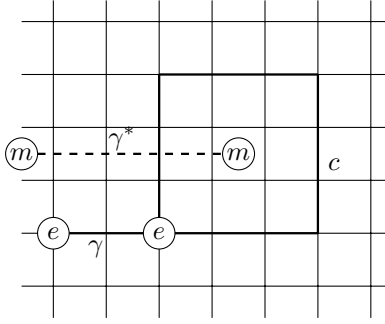
Figure 4.5: Exchange between electric charges and magnetic fluxes reveals the fermionic nature of ϵ.

edge then

$$
\begin{aligned}
|\Psi_{final}\rangle &= X(c) |\Psi_{initial}\rangle \\
&= X(c)Z(\gamma^*)X(\gamma) |\xi\rangle \\
&= - Z(\gamma^*)X(\gamma)X(c) |\xi\rangle \\
&= - |\Psi_{initial}\rangle .
\end{aligned}
\tag{4.28}
$$

The exchange then leads to a minus sign on the initial state, thus revealing the exotic relative statistics of these particles. We can then conclude that these are in fact anyons and not bosons as initially thought. Furthermore from the spin-statistics theorem and eq. (3.25) this exchange also confirms the fermionic nature of the ϵ particle. We also witnessed a phase change without the particles interacting, thus leading us to interpret this as an Aharanov-Bohm effect [AB59]. Note that taking a magnetic flux around an electric charge would lead us to the same result. We can also show this to be true through the direct exchange of two fermions. Consider two ϵ particles prepared for the exchange, as depicted by fig. 4.6. We will exchange them without splitting up the particles, therefore we can only act with X and Z operators on paths with size one. Note that mathematically it doesn't matter if you keep the particles together or not, but we will do so as it makes more sense conceptually. We will also introduce the
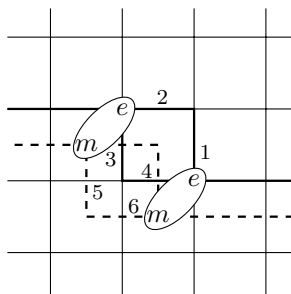
Figure 4.6: Exchange of two fermions.

"waiting time" we mentioned in the previous approach. We thus have

$$
\begin{aligned}
\left|\psi_{final}\right\rangle &= Z(\gamma_3^*)X(\gamma_2)X(\gamma_4)Z(\gamma_6^*)Z(\gamma_5^*)X(\gamma_3)X(\gamma_1)Z(\gamma_4^*)\left|\psi_{initial}\right\rangle \\
&= -X(\gamma_1)X(\gamma_2)X(\gamma_3)X(\gamma_4)Z(\gamma_3^*)Z(\gamma_4^*)Z(\gamma_5^*)Z(\gamma_6^*)\left|\psi_{initial}\right\rangle \\
&= -B(\gamma_1\gamma_2\gamma_3\gamma_4)A(\gamma_3^*\gamma_4^*\gamma_5^*\gamma_6^*)\left|\psi_{initial}\right\rangle \\
&= -\left|\psi_{initial}\right\rangle.
\end{aligned} \tag{4.29}
$$

Since there is exactly one edge where both X and Z operators act, in the second equality we simply reordered these operators and we get a minus sign from the anti-commutation of operators with different label on edge 3. Now that these are in order they are in fact the product of Pauli operators with the same label on a star and a boundary of a plaquette respectively, hence in the third equality we simply used this fact to introduce the $A(v)$ and $B(p)$ operators, thus leading us to the expected statistical phase. Note that the first two operators move the rightmost particle while the next four move the leftmost particle to the initial position of the other particle. This is the "waiting time" we mentioned. We used it to prevent the particles from fusing so as to not cause problems conceptually.

4.7 Encoding Information

Now that we've fully described the model, let us see how we can use it to encode information that is naturally resilient to errors and perform fault-tolerant computation on it. To each ground state we assign a state of a n-qubit. Since we're working on a torus we have a 4-fold degenerate ground state and therefore can encode 2 qubits of information. The idea is that any errors are in the form of excitation of the ground state and so we can correct these by simply moving the excitations back together and fusing them, thus returning to the ground state. Note that if we have a string of errors, we don't know which path the errors took, only the endpoints. Consider a protocol where we correct errors by bringing the excitations back together through the shortest path possible. We can then correct any error that doesn't go over halfway around the torus, in which case we correct the error by moving the excitation through a non-contractible loop therefore causing an error on the logical qubits (information we are trying to protect). This protocol can therefore detect $k-1$ errors and correct $\left\lfloor \dfrac{k-1}{2} \right\rfloor$ errors. If we take a large enough torus we can significantly decrease the chances of errors on the logical qubits. We can also perform quantum computation on these qubits. By intentionally creating two excitations and taking one around the torus before fusing them back together we can then realize some quantum gates. Unfortunately, the only gates we can conceive are the Pauli X and Z gates, which aren't enough for universal quantum computation. Nonetheless the toric code is still a great way to store quantum information, in the form of quantum memories, as you don't need to perform computation on these.

5 Quantum Double Models

We've seen that the toric code provides a great way of storing quantum information; due to its topological nature it's a system that is fault-tolerant for sufficiently large lattices. As our aim is to perform universal quantum computation, we now intend to generalize this model, so as to allow non-abelian anyons.

5.1   The space

Consider a generic lattice (which need not be square) embedded in a two-dimensional orientable surface, and a finite group G. To each edge of the lattice we attach a local Hilbert space of dimension $|G|$, such that $\{|g\rangle : g \in G\}$ is an orthonormal basis (we can consider this as a $|G|$-dimensional spin system). We choose an orientation on each edge such that changing the direction is equivalent to a change of basis $|z\rangle \mapsto |z^{-1}\rangle$ (see fig. 5.1). Let us now define 4 types of linear operators that act on the space as follows

$$
\begin{aligned}
L_+^g \, |z\rangle \ &= \ |gz\rangle\,, & T_+^h \, |z\rangle \ &= \ \delta_{h,z} \, |z\rangle\,, \\
L_-^g \, |z\rangle \ &= \ |zg^{-1}\rangle\,, & T_-^h \, |z\rangle \ &= \ \delta_{h^{-1},z} \, |z\rangle\,,
\end{aligned}
\tag{5.1}
$$

where $g, h \in G$ and δ represents the usual Kronecker delta.

5.2   The Algebra and More Operators

We start by figuring out the relations between the operators defined in 5.1. It should be clear that the T operators commute, since a product of these imposes that all indexed group elements should be equal. On the other hand the L operators only commute if the group is abelian. Consider now the action of two operators with a different label on a state $|z\rangle$. We have that $L_+^g T_+^h \, |z\rangle = L_+^g \delta_{h,z} \, |z\rangle$ which equals 0 if $h = z$ and $|gz\rangle$ otherwise, hence we end up with the relation $L_+^g T_+^h = T_+^{gh} L_+^g$. Similarly we can find
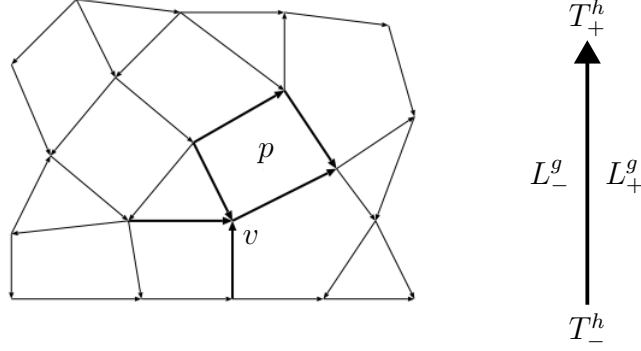
Figure 5.1: Generic lattice on the left and rules for operators $L^g_\pm$ and $T^h_\pm$ on the right. Image taken from [Kit03].

the relations between the remaining operators,

$$L^g_+ T^h_+ = T^{gh}_+ L^g_+, \qquad L^g_+ T^h_- = T^{hg^{-1}}_+ L^g_+,$$
$$L^g_- T^h_+ = T^{hg^{-1}}_+ L^g_-, \qquad L^g_- T^h_- = T^{gh}_- L^g_-. \qquad (5.2)$$

Let us now introduce vertex and plaquette operators similarly to those defined in (4.1). Take i to be an edge and v one of its endpoints. We now define the operators $L^g(i, p) = L^g_\pm(i)$ and $T^h(j, v) = T^h_\pm(j)$ in terms of the orientation of the edge on which they are acting (see fig. 5.1)

$$L^g(i, p) = \begin{cases} L^g_+(i), \text{ if } p \text{ is the adjacent plaquette on the right of the edge } j, \\ L^g_-(i), \text{ if } p \text{ is the adjacent plaquette on the left of the edge } j, \end{cases} \qquad (5.3)$$

and similarly for projection operators T:

$$T^h(j, v) = \begin{cases} T^h_+(j), \text{ if } v \text{ is the endpoint of the arrow on edge } i, \\ T^h_-(j), \text{ if } v \text{ is the origin of the arrow on edge } i \;. \end{cases} \qquad (5.4)$$

We are now ready to introduce the vertex and plaquette operators. These are

$$A^h(v) = \sum_{h_1 \ldots h_n = h} \prod_{k=1}^{n} T^{h_k}(j_k, v), \qquad B^g(p) = \prod_{i \in \partial p} L^g(i, p), \qquad (5.5)$$

where $j_1, \ldots, j_n$ are the edges of the star of v listed in a counterclockwise manner, hence the vertex operator acts as the identity if $h_1 \ldots h_n = h$ and zero otherwise. Let us now identify the relations between these operators. Consider two $A^k(v)$ operators acting on different vertices. They either share no edges and they obviously commute or they share exactly one edge and output zero if the group elements indexing the T operators are different and act as the identity if they're equal and so they commute. A similar argument can be used for $B^k(p)$ operators. Consider now two vertex operators acting on the same vertex. Since these act as the identity if $g_1 \ldots g_n = g$ and $h_1 \ldots h_n = h$ and they output zero otherwise, then we have $A^g(v)A^h(v) = \delta_{g,h} A^h(v)$. If two $B^k(p)$ operators act on the same plaquette p then it is trivial to verify the relation $B^g(p)B^h(p) = B^{gh}(p)$. Consider now operators with a different label. If their action shares no edges then they obviously commute. If they do share edges then from the relations 5.2 we get $B^h(p)A^g(v) = A^{ghg^{-1}}B^g(p)$. Since $(B^g(p))^* = B^{g^{-1}}(p)$ we can then summarize the relations

$$A^g(v)A^h(v) = \delta_{g,h} A^h(v), \qquad B^g(p)B^h(p) = B^{gh}(p),$$
$$B^h(p)A^g(v) = A^{ghg^{-1}}B^g(p), \qquad (B^g(p))^* = B^{g^{-1}}(p). \qquad (5.6)$$

The name quantum doubles come from these relations since they are the same as those determining the quantum double $\mathcal{D}(G)$ of the Hopf Algebra $\mathbb{C}[G]$ [Kas95].

5.3 The Hamiltonian

Let us now generalize the previous operators (5.5) so as to make them commutative. We then define $A(v) := A^h(v)$ and $B(p)$ to be the average of all the plaquette operators

on a specific plaquette p, thus obtaining

$$A(v) = A^h(v), \qquad B(p) = \frac{1}{|G|} \sum_{g \in G} B^g(p). \tag{5.7}$$

From a lattice gauge theory [Oec05] point of view, the vertex operator projects out states with electric charge h through the vertex while the plaquette operator projects out states that are not gauge invariant at the plaquette (it identifies the absense of a magnetic flux at the plaquette). We thus have $[A(v), B(p)] = 0$ and therefore are ready to introduce the Hamiltonian in a similar manner to (4.15), that is

$$H = -\sum_v A(v) - \sum_p B(p). \tag{5.8}$$

Since all $A(v)$ and $B(p)$ commute they also commute with the Hamiltonian and therefore it is exactly solvable. As was the case for the toric code we will then have that the ground state is limited by the same constraints:

$$\{|\xi\rangle : A(v) |\xi\rangle = |\xi\rangle,\ B(p) |\xi\rangle = |\xi\rangle \text{ for all } v,\ p\}. \tag{5.9}$$

5.4   The Toric Code as a Quantum Double

We can now verify that the toric code introduced in chapter 4 is in fact a quantum double model with group $\mathbb{Z}_2$. Since in this group each element is its own inverse we don't need to bother with defining an orientation on the edges. Then

$$A(v) = \sum_{h_1 \ldots h_4} \prod_{k=1}^{4} \delta_{h_k, v} = \begin{cases} 1, \text{ if } h_1 \ldots h_4 = 1, \\ 0, \text{ otherwise.} \end{cases} \tag{5.10}$$

While this is not the same as the $A(v)$ operator defined in (4.1), we can define a projector $A'(v) = \frac{1}{2}(1 - A(v))$ with eigenvalues 0 and 1 that coincides with the vertex

operator defined for a quantum double model in eq. (5.5). For the plaquette operator, since $\mathbb{Z}_2$ is abelian, then from eq. (5.7) we have

$$B(p) = \frac{1}{2} \sum_{g \in \{0,1\}} \prod_{i \in \partial(p)} L_+^g(i), \tag{5.11}$$

which is equivalent to the projector, defined in a similar fashion to the vertex projector, $B'(p) = \frac{1}{2}(1 - B(p))$. We have thus confirmed that the toric code is indeed the quantum double model $\mathcal{D}(\mathbb{Z}_2)$.

We will not go deeper into this model as it goes beyond the scope of this text. For further reading we recommend [Kit03], [BM08], [Naa15] and [Luo+11].

Acknowledgements

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