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On Rasmussen's s-invariant and the search for exotic definite 4-manifolds

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Resumo

Quando se trata da topologia de variedades, uma das características marcantes que distingue a dimensão 4 das demais é a diferença notável entre variedades topológicas e variedades diferenciáveis. Em [MP21], Manolescu e Piccirillo tentaram construir cópias exóticas de S^4 e $\#n\mathbb{CP}^2$ partindo de pares de nós com cirurgias de Dehn com coeficiente 0 (0-cirurgias) homeomorfas e obstruindo a sua sliceness usando o s-invariant definido em [Ras10]. O objetivo desta tese é fornecer uma visão completa desse programa, começando pelos fundamentos da topologia de variedades de dimensão 4 e da homologia de Khovanov, e culminando no trabalho de Manolescu-Piccirillo. Por fim, seguimos o trabalho de [Nak22] e estudamos esses homeomorfismos de 0-cirurgias, estabelecendo condições para quando se pode usar o s-invariant como obstrução.

Palavras-chave: Variedades exóticas de dimensão 4, Cálculo de Kirby, Homologia de Khovanov, s-invariant, Nós slice, 0-cirurgia

Abstract

When it comes to the topology of manifolds, one of the remarking features which distinguishes dimension 4 from the rest, is the striking difference between topological and smooth 4-manifolds. In [MP21], Manolescu and Piccirillo made an attempt at constructing exotic copies of S^4 and #$n\mathbb{CP}^2$ by taking pairs of knots with homeomorphic 0-surgeries and obstructing their sliceness via the s-invariant defined in [Ras10]. The goal of this thesis is to give a complete overview of this program, starting from the basics of 4-manifold topology and Khovanov homology and leading up to the work of Manolescu-Piccirillo. Finally, we follow the work of [Nak22] and study these 0-surgery homeomorphisms, giving conditions on when one might be able to use the s-invariant as an obstruction.

Keywords: Exotic 4-manifolds, Kirby calculus, Khovanov homology, s-invariant, Slice knots, 0-surgery

Contents

Chapter 1

Introduction

When it comes to the topology of manifolds, dimension 4 sits in a special place. It is high enough to be able to exhibit a lot more complexity than its lower dimensional counterparts, but not so high that we have enough room to simplify things and reduce its complexity. Indeed, compact manifolds of dimension at most 2 are completely classified, and those of dimension 3 can be understood via Thurston's Geometrization conjecture (now known to be a theorem) [Mor04]. On the other hand, it is known to be impossible to classify manifolds of dimension 4 or higher. If we assume the manifolds to be simply connected, then in dimension 5 or higher, these can be understood via the h-cobordism theorem [Sma61] and surgery theory. The key difference between dimension 4 and higher dimensions, is that for the h-cobordism theorem to work we need access to what is known as the Whitney trick. This is a technique which allows us to separate two intersecting 2-disks inside a manifold, but for it to work, we need to make use of the transversality theorem which in this case requires at least 5 dimensions to be able to separate the disks (it is a common saying that 4-manifolds are hardest because $2 + 2 < 5$). Another distinguishing feature of dimension 4 is that it is the lowest dimension where there is a difference between topological and smooth manifolds (up to dimension 3 every manifold admits a unique smooth structure). For example, if $n = 4$, then $\mathbb{R}^n$ admits uncountably many smooth structures, while for any other n there is exactly one smooth structure. In a similar vein we have the smooth Poincaré conjecture, which is probably the most well-known problem in topology.

Conjecture 1.0.1. *If X is a closed, simply connected, smooth n-manifold which is homotopy equivalent to S^n, then X is diffeomorphic to S^n.*

In dimensions lower than 4, the previous classification results imply the smooth Poincaré conjecture, where the last case of Thurston's Geometrization conjecture (which is actually the one which implies the Poincaré conjecture) was recently solved by Perelman [Per02]. In dimensions higher than 4, the topological Poincaré conjecture (if we consider homeomorphisms instead of diffeomorphisms) follows from the h-cobordism theorem and is due to Smale [Sma61], while the smooth case is known to be false in general - the first exotic spheres (spheres which are homeomorphic but not diffeomorphic to the standard sphere) were constructed in dimension 7 (it is also the lowest dimension, other than 4, where the smooth Poincaré conjecture is known to be false) and are due to Milnor [Mil56]. Since in dimension 4 the

Whitney trick fails and there is no result analogous to Thurston's Geometrization conjecture, then even the topological Poincaré conjecture posed an incredibly hard challenge. It was only proven twenty years after the high dimensional case and it is due to Freedman [Fre82]. In fact, Freedman was able to classify simply connected, closed, topological 4-manifolds by their intersection form. This was done by adapting Smale's proof of the h-cobordism theorem to dimension 4, which turned out to be an incredibly difficult process. The last remaining open case is the smooth Poincaré conjecture in dimension 4. Due to the weird exotic phenomena which only appear in dimension 4, it is mostly believed to be false. The difficulty in distinguishing smooth structures on the 4-sphere comes from the fact that most invariants which might be able to do so, vanish on homotopy 4-spheres. This is where the following recently discovered invariant might shed some light.

In [Kho00] Khovanov defined a homology theory which categorifies the Jones polynomial and produces an invariant of links in S^3. This is known to actually be stronger than the Jones polynomial and it is functorial, up to sign, with respect to cobordisms [Jac04]. By exploiting the behaviour of Khovanov homology under cobordisms (actually a deformation due to Lee [Lee05]), Rasmussen was able to produce the s-invariant. This is a knot invariant which gives a lower bound on the slice genus of a knot - which is the lowest genus of any properly embedded surface in the 4-ball having boundary the given knot (which is embedded in the 3-sphere given by the boundary of the 4-ball). In particular if a knot K has s-invariant $s(K) \neq 0$, then K can't be slice. In contrast with other slice obstructions coming from gauge theory, it is not known whether the s-invariant vanishes in homotopy 4-balls, so one could hope to construct an exotic 4-sphere as follows - Take a homotopy 4-sphere S and find a K which is slice in $S - \text{int } D^4$ such that $s(K) \neq 0$. This means that K is not slice in the standard 4-sphere, so S and S^4 can't be diffeomorphic and the sphere S provides a counterexample to the smooth Poincaré conjecture.

This was exactly the plan of Freedman, Gompf, Morrison and Walker (FGMW) in [FGMW10]. They pursued this strategy for a family of homotopy spheres Σ_m, coming from the Cappel-Shaneson construction [CS76]. In particular they found knots in Σ_{-1} and Σ_1 and computed their s-invariant, but after many hours of computation it turned out that both knots had s-invariant 0. In fact, just a few days after, Akbulut showed that all the spheres Σ_m are in fact standard [Akb10]. While this didn't mean that the strategy couldn't work, it unfortunately slowed down the research in this area.

In 2018 some hope was restored to this strategy when Piccirillo proved that the Conway knot was not slice [Pic20]. The Conway knot is an 11 crossing knot which was discovered in 1970 by Conway, and since then the sliceness of the Conway knot has been an open question. Piccirillo proved that the Conway knot is not slice by exploiting a property of the s-invariant that similar invariants coming from -gauge theory didn't share - the s-invariant is not a 0-trace invariant [Pic19] (the 0-trace is a 4-manifold associated with a knot). While the Conway knot has vanishing s-invariant, she used this property to construct another knot K' which shares a 0-trace with the Conway knot, and computed the s-invariant of K' which turned out to be non-zero. This meant that K' is not slice and so neither can the Conway knot be.

This revived some interest in the s-invariant and the approach of FGMW. This strategy was then again attempted by Piccirillo and Manolescu in [MP21], except this time with a slight change. Instead of using known homotopy 4-spheres and having to construct slice knots, they instead found a way to construct homotopy spheres which already come equipped with slice knots. To do so, they suggested that we find pairs of knots K and K' such that K is slice, $s(K') \neq 0$ and such that K and K' have homeomorphic 0-surgeries. This gives us a recipe to construct a homotopy 4-sphere S such that K' is slice in $S - \text{int } D^4$ and since $s(K') \neq 0$ in the standard S^4, then S has to be an exotic sphere. In fact, it turns out that this strategy could be used not only to construct exotic spheres, but also other positive definite 4-manifolds. The s-invariant is defined for knots in the boundary of the 4-ball, but recently in [MMSW23] an adjunction inequality was given which gave a bound for the s-invariant of knots in $\#n\mathbb{CP}^2$. In particular this works when K bounds a null-homologous surface in $\#n\mathbb{CP}^2 - \text{int } D^4$, so with a slight tweak to the strategy, they were able to expand their approach to H-slice knots in $\#n\mathbb{CP}^2$ (a knot is H-slice if the surface it bounds is null-homologous). The core strategy still stands, but now we are trying to find knot pairs K and K' such that K is H-slice in $\#n\mathbb{CP}^2$, $s(K') < 0$ and such that K and K' have homeomorphic 0-surgeries.

To construct knot pairs which could be used to pursue this strategy they defined a type of three component link which can be shown to have associated knots K and K' with homeomorphic 0-surgeries. These are called RBG-links, which are named after the color of the three components - Red, Blue and Green. In particular, they looked at a family of 3375 RBG-links, searching for links with associated knots K and K' such that one has vanishing s-invariant while the other has negative s-invariant. They found 23 pairs of knots such that if the knot of the pair which had s-invariant 0 was shown to be H-slice in $\#n\mathbb{CP}^2$ then we have an exotic $\#n\mathbb{CP}^2$.

Just last year however it was shown by Nakamura [Nak22] that none of the 23 knots are H-slice in any $\#n\mathbb{CP}^2$. Furthermore, he extended the work of Piccirillo and Manolescu to show that the s-invariant cannot obstruct the H-sliceness of any knot that would be obtained from the family of links which Piccirillo and Manolescu used to produce the previous 23 knots. This doesn't seem to completely shatter the strategy proposed in [MP21] however. In fact, Manolescu and Piccirillo used a very restrictive family of RBG-links and as such it is still open whether this strategy fails for all RBG-links. With this in mind, Nakamura gives some conditions on the RBG-links, so that they can avoid the problems that he had run into with the Manolescu-Piccirillo family. This was based on a conjecture from [MMSW23] which states that the s-invariant of a knot K in $\#n\mathbb{CP}^2$ also admits an adjunction inequality when the surface bound by K is non-trivial in homology.

1.1 Objective and thesis outline

The goal of this thesis is to provide an introduction, as self-contained as possible, to the uses of the s-invariant in 4-manifold topology. In particular, we're interested in the possible applications of the s-invariant to the detection of exotic structures in 4-manifolds. At the moment we are limited to $\#n\mathbb{CP}^2$ and S^4, but currently there is still research being done in this direction which might open up new possibilities for the s-invariant. Most of the work presented in the final chapter of this thesis was developed in the last four years, so we hope to have given an up to date exposition of the latest research in this area, which along with the introductory chapters on 4-manifold topology and Khovanov homology, aims to give a fairly detailed quick start guide to research in this topic.

The text is divided into four chapters, which includes the introduction given in Chapter 1.

In Chapter 2 we introduce the necessary topological machinery to be able to reason about 4-manifolds. We start with handlebody theory in its full generality, culminating in a proof of the h-cobordism theorem in high dimensions. Handlebody theory provides a framework to understand manifolds by decomposing them into simpler pieces, called handles. One can then approach the diffeomorphism type of a manifold by studying how different handle decompositions relate to each other. In dimension 4 this takes the form of Kirby calculus, which provides a diagramatic description of 4-manifolds as framed link diagrams. The rest of the chapter is dedicated to studying these diagrams and how they relate to the topology of 3- and 4-manifolds.

Chapter 3 is concerned with Khovanov homology and how it relates to the slice genus of a knot. We start by constructing the cube of resolutions of a link, where to each vertex we associate a graded vector space and to each edge a map, which after some work makes it into a bigraded chain complex. The Khovanov homology is a link invariant defined as the homology of this chain complex. By slighty changing the edge maps, we can recover another bigraded chain complex, whose homology is known as Lee homology and provides another link invariant. The Lee homology of a knot turns out to be surprisingly simple, but the grading on its generators provides us with a particularly useful invariant. Working in this direction, we then define the s-invariant of a knot to be the average among the generators of its Lee homology. We finish the chapter by proving that this gives a lower bound on the slice genus of the knot and we see how we can generalize this to the case of links.

Finally, in Chapter 4 we see how one can use the s-invariant defined in the previous chapter, to obtain results concerning the topology of 4-manifolds. We start by using the s-invariant to construct exotic copies of $\mathbb{R}^4$. After this, we see how the s-invariant can be used to answer a problem, which has been open for for 50 years, concerning the sliceness of the Conway knot. Finally, we study how one can more generally use the s-invariant to detect exotic structures in S^4 and $\#n\mathbb{CP}^2$. To do so, we start by extending the obstruction to sliceness given by the s-invariant to the case of $\#n\mathbb{CP}^2$. We follow by describing a particular type of framed link, known as RBG-link, which we can then use to produce pairs of knots such that one is H-slice in $\#n\mathbb{CP}^2$ (the slice disk is null-homologous) while the other is H-slice in a 4-manifold homotopy equivalent to $\#n\mathbb{CP}^2$. This will give us a family of potential exotic $\#n\mathbb{CP}^2$ which we can obstruct by using the s-invariant. We finish by giving some limitations on these techniques.

Chapter 2

Handlebodies and Topology of 4-manifolds

In this chapter we give an introduction to the theory of 4-manifolds through the lens of handlebody theory. The goal of handlebody theory is to understand smooth manifolds by decomposing them into smaller pieces called handles (we do this in Section 2.1). These behave in a similar fashion to cells in the theory of CW-complexes, except now we have access to the tools of smooth topology. Most notable results in topology (namely the notorious Poincaré conjecture) are linked in some way to handlebody theory as it provides the fundamental tools necessary to work in this area. We can see this directly in Section 2.2, where we prove Smale's h-cobordism theorem ([Sma61]), which in turn is used to prove the high dimensional Poincaré conjecture. Our focus is on 4-manifolds, where handlebody theory shines in the form of Kirby calculus. This is an approach introduced by Kirby in [Kir78], where one represents 4-manifolds by diagrams of framed links (named Kirby diagrams) and we can be assured two diagrams represent diffeomorphic 4-manifolds if we can go from one diagram to another by a series of modifications (namely handle slides and handle cancellations). This comprises most of the work in this chapter and is studied in the remaining sections. In Section 2.6 we make use of the fact that every closed, oriented 3-manifold bounds a 4-manifold, to extend the work of the previous section to be able to study 3-manifolds. We will give an exposition mostly based on [GS99], but some other useful sources are [Sco05] and [Akb16].

2.1 Handles and Handlebodies

Definition 2.1.1. Let X be an n-manifold with boundary. For $0 \leq k \leq n$, an n-dimensional k-handle h is a copy of $D^k \times D^{n-k}$, attached to ∂X along $\partial D^k \times D^{n-k}$ by an embedding $\varphi\colon \partial D^k \times D^{n-k} \to \partial X$.

The manifold $X \cup_\varphi h_k$ obtained by attaching a k-handle to X is in general a manifold with corners. We can however always smooth these corners and get a differential structure on $X \cup_\varphi h_k$ (see Section 2.6 of [Wal16]), making it into a smooth manifold.

For an n-dimensional k-handle we call $D^k \times \{0\}$ the core, $\{0\} \times D^{n-k}$ the cocore, φ the attaching map, $\partial D^k \times D^{n-k}$ (identified with $\varphi(\partial D^k \times D^{n-k})$) the attaching region, $\partial D^k \times \{0\}$ (identified with $\varphi(\partial D^k \times \{0\})$) the attaching sphere and $\{0\} \times \partial D^{n-k}$ the belt sphere. For a 2-dimensional 1-handle these are depicted in Figure 2.1.1.

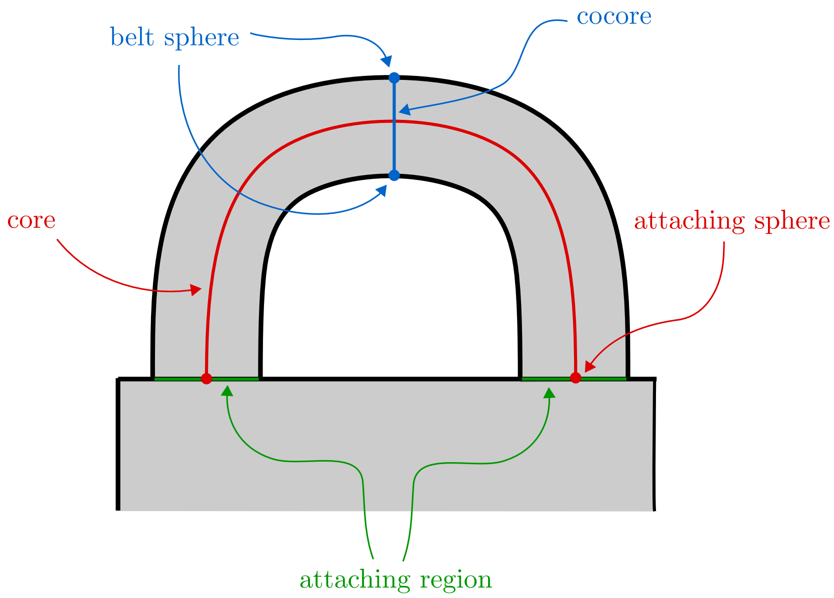

Figure 2.1.1: A 2-dimensional 1-handle.

Note that up to homotopy, attaching a k-handle is the same as attaching a k-cell. If we have a k-handle h, then there is a deformation retract of the handle to its core, which in turn gives us a deformation retract from $X \cup_\varphi h$ to $X \cup_{\varphi|\partial D^k \times \{0\}} D^k \times \{0\}$. A k-handle can then be thought of as a k-cell which we thicken up so that we can smooth the resulting manifold and work in the smooth category. An immediate consequence of this is that the manifold obtained by gluing h to an n-manifold X only depends on the isotopy class of the attaching map. Indeed, if $\varphi, \varphi' \colon \partial D^k \times D^{n-k} \to \partial X$ are two isotopic attaching maps for h, then by using the Isotopy extension theorem (Theorem A.0.1) we can find an ambient isotopy of ∂X which extends the previous isotopy. We then use this to construct a diffeomorphism between $X \cup_\varphi h$ and $X \cup_{\varphi'} h$.

Proposition 2.1.2. *Let h be a k-handle attached to a smooth n-manifold X by $\varphi \colon \partial D^k \times D^{n-k} \to \partial X$. Then the diffeomorphism type of $X \cup_\varphi h$ is determined by the isotopy class of the embedding φ.* □

If we have an embedding $\varphi_0 \colon \partial D^k \times \{0\} \to \partial X$ with trivial normal bundle $\nu\varphi_0(S^{k-1}) \cong S^{k-1} \times \mathbb{R}^{n-k}$, then the Tubular neighborhood theorem tells us that we can extend this to an embedding $\varphi \colon \partial D^k \times D^{n-k} \to \partial X$ by specifying a framing f of the normal bundle, i.e, an identification $f \colon S^{k-1} \times D^{n-k} \to \nu\varphi_0(S^{k-1})$. By the previous proposition, we then have that the diffeomorphism type of a manifold X obtained by attaching a k-handle h is determined, up to isotopy, by an embedding $\varphi_0 \colon S^{k-1} \to \partial X$ with trivial normal bundle, and a choice of framing $f : S^{k-1} \times D^{n-k} \longrightarrow \nu\varphi_0(S^{k-1})$. So if we fix a base embedding φ_0, then the diffeomorphism type of $X \cup_\varphi h$ is completely determined by the isotopy class of the framing f on $\nu\varphi_0(S^{k-1})$. It turns out that the collection of such framings form a torsor.

Proposition 2.1.3. *If X is a smooth n-manifold, then the isotopy classes of framings of an embedding $\varphi\colon S^{k-1} \to \partial X$ with trivial normal bundle (fixing $\varphi(S^{k-1})$) are in bijection with elements of $\pi_{k-1}(O(n-k))$.*

Proof. Fix a framing $f_0\colon S^{k-1} \times D^{n-k} \to \nu\varphi(S^{k-1})$ and let $f\colon S^{k-1} \times D^{n-k} \to \nu\varphi(S^{k-1})$ be any other framing. For every point $p \in S^{k-1}$ we'll then have that the map $f^{-1} \circ f_0|_{\{p\}\times D^{n-k}}\colon D^{n-k} \to D^{n-k}$ corresponds to an element of $GL(n-k)$, and since the map $f^{-1} \circ f_0$ is a diffeomorphism, then the element of $GL(n-k)$ varies smoothly as we change the point $p \in S^{k-1}$. This gives us a correspondence between framings f and maps $S^{k-1} \longrightarrow GL(n-k)$. Furthermore, since $GL(n-k)$ deformation retracts to $O(n-k)$ via Gram-Schmidt orthogonalization, then we get a correspondence between framings and maps $S^{k-1} \longrightarrow O(n-k)$. If we now fix $p \in S^{k-1}$, then there will be a diffeomorphism of the second factor of $D^k \times D^{n-k}$ which will identify p with the identity $I \in O(n-k)$. This leads us to an identification between isotopy classes of framings of φ and elements of $\pi_{k-1}(O(n-k))$. □

Example 2.1.4. A 0-handle has attaching sphere $\partial D^0 \times \{0\} = \emptyset$, so it is attached to a manifold by taking a disjoint union. A 1-handle has attaching sphere $\partial D^1 \times \{0\} = S^0$ which equals two disjoint points. Since in a connected manifold every two points are isotopic (take an isotopy along a path), then if X is connected, there is a unique isotopy class of embeddings $\varphi\colon \partial D^1 \times \{0\} \to \partial X$. Furthermore since $\pi_0(O(n-1)) \cong \mathbb{Z}_2$ for $n \geq 2$, then there are two possible framings on the attaching sphere and thus two manifolds that can be obtained from X by attaching a 1-handle, which are distinguished by their orientability. If $n = 2$, one gives us the trivial D^1-bundle over S^1 while the other gives us the Möbius band (Figure 2.1.2).

Example 2.1.5. For $n \neq 2$, there is a unique framing on $(n-1)$- and n-handles since $\pi_{n-2}(O(1)) = \pi_{n-1}(O(0)) = 1$ (for the case $n = 2$, attaching a $(n-1)$-handle corresponds to attaching a 1-handle, which as we've seen in the previous example, has $\pi_0(O(n-1)) \cong \mathbb{Z}_2$). A n-handle has attaching sphere $\partial D^n \times \{0\}$ and for $n \leq 4$, every diffeomorphism $S^{n-1} \longrightarrow S^{n-1}$ is isotopic to either the identity or a reflection, so there is a unique way to attach a n-handle to an n-manifold with an S^{n-1} boundary component.

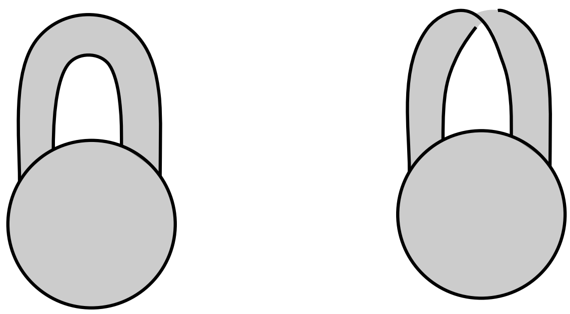

Figure 2.1.2: The two possible 2-manifolds obtained from attaching a 1-handle to a 0-handle.

The 0-handle is the only one that can attach to the empty set, so in this sense we might consider D^4 to be the simplest 4-manifold. The second next most simple 4-manifold is obtained by attaching a k-handle to D^4. In this case we can identify therein a bundle strcture.

Proposition 2.1.6. *Let X be a smooth n-manifold obtained by attaching a k-handle h to a 0-handle. If the attaching sphere of the k-handle bounds an embedded k-disk D in ∂D^n, then X is a D^{n-k}-bundle over a smooth manifold homeomorphic to S^k.*

Proof. Denote by A the attaching sphere of h, and push in the interior of D into the 0-handle D^n, so that $A = \partial D \subset \partial D^n$ and int $D \subset$ int D^n. Notice now that we can think of D^n as $D^k \times D^{n-k}$ constructed from D via a tubular neighborhood $\nu D \cong D^k \times D^{n-k}$, so we get an identification $X \cong (D^k \times D^{n-k}) \cup_{\partial D^k \times D^{n-k}} (D^k \times D^{n-k})$. We will now have that both factors admit a projection onto D^k - the first corresponding to the disk D and the second to the core of the handle h. These glue together along the attaching sphere $A = S^{k-1} \times \{0\}$, so we get an identification of X with a fiber bundle with fiber D^{n-k}. By Theorem A.0.3 any two ways of gluing the disks D^k result in homeomorphic manifolds, so the base must be homeomorphic to S^k and X is a D^{n-k}-bundle over S^k. □

We can generalize this to the case of gluing a k-handle h to the boundary of a generic smooth manifold Y. Indeed, if the attaching sphere bounds a disk $D \subset \partial Y$, then we can similarly push the interior of D into the interior of Y. In a similar fashion we can identify a D^{n-k}-bundle over S^k in Y, except this time it won't be the whole manifold since we don't have the initial identification of νD with Y. What we have instead is a boundary sum between Y and the bundle.

Corollary 2.1.7. *Let Y be a smooth n-manifold and X the manifold obtained by attaching a k-handle h to Y. If the attaching sphere of the k-handle bounds an embedded k-disk D in ∂Y, then X is diffeomorphic to $Y \natural E$, where E is a D^{n-k}-bundle over a smooth manifold homeomorphic to S^k.* □

Fiber bundles constructed by adding a k-handle to a 0-handle are classified by the framing on the attaching sphere of the k-handle, so they are in bijection with elements of $\pi_{k-1}(O(n-k))$. Notice that the proof of Proposition 2.1.6 works in the smooth category up until the last point of gluing the base disks. For $k \leq 3$, every topological manifold is smoothable and furthermore the smooth structure is unique up to diffeomorphism (Theorem A.0.10), so the base S^k will be a standard sphere.

Corollary 2.1.8. *For $n \geq 3$, let X be an oriented, smooth n-manifold obtained by attaching l 1-handles to a single 0-handle. Then X is diffeomorphic to $\natural l S^1 \times D^{n-1}$.*

Proof. First consider the case of gluing a single 1-handle to a 0-handle. Since $\pi_0(O(n-1)) \cong \mathbb{Z}_2$ then there is a unique way to glue the 1-handle so as to make X orientable. From the previous proposition and discussion, we then have that X is diffeomorphic to $S^1 \times D^{n-1}$. To generalize to the case of l 1-handles, we simply identify the 0-handle $D^n = \natural l D^n$ with the boundary sum of l copies of D^n, and to each one of those we add a 1-handle. On each copy of D^n this corresponds to the inicial case, so it follows that X is diffeormophic to $\natural l S^1 \times D^{n-1}$. □

Constructing manifolds by attaching handles to the 0-handle, and then handles on top of these, seems like a natural thing to do, similar to how one constructs a CW-complex by attaching higher dimensional cells to a lower dimensional skeleton. In our situation, we can however be assured that a space constructed in this way will always be a smooth manifold. A manifold obtained exclusively by attaching handles is known as handlebody.

Definition 2.1.9. Let X be a compact n-manifold. A handle decomposition of X is an identification of X with a manifold obtained by attaching handles to $\emptyset$. If X has boundary $\partial X = \partial_+ X \sqcup \overline{\partial_- X}$ then a relative handle decomposition of X is given by identifying X with a manifold obtained from attaching handles to $I \times \partial_- X$. A manifold with a given handle decomposition is called a handlebody.

If one is familiar with the Morse theory then it should be immediate that every compact, smooth manifold X admits a handle decomposition. The idea is that if we have a smooth function $f\colon X \to I$ such that $f^{-1}(0) = \partial_- X$ and $f^{-1}(1) = \partial_- X$, then we can perturb it to a function with no critical points on ∂X and such that every critical point is non-degenerate. Such a function is called Morse and it has the special property that around every critical point c, we have a local chart $(x_1, \dots, x_n)$ such that $f(x_1, \dots, x_n) = f(c) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2$. The integer k is called the index of the critical point and it can be shown that each critical point of index k corresponds to the attachment of a k-handle (see [Mil63], [MSS65] for more details).

Theorem 2.1.10. *Every compact, smooth n-manifold admits a handle decomposition.* □

Example 2.1.11. Every compact, oriented surface of genus g admits a handle decomposition with one 0-handle, one 2-handle and $2g$ 1-handles. Notice the similarities between the number of handles and the Euler characteristic of the surface. We'll shortly see that the homology of a manifold can be read off of its handle decomposition.

Example 2.1.12. Both $\mathbb{RP}^n$ and $\mathbb{CP}^n$ admit handle decompositions with $n+1$ handles - for $\mathbb{RP}^n$ we have a k-handle for $k = 0, \dots, n$, while for $\mathbb{CP}^n$ we have a have a k-handle for every even k (for $k = 0, 2, \dots, 2n$). For an explicit description of a handle decomposition of $\mathbb{CP}^n$ see Example $4.2.4$ of [GS99].

Example 2.1.13. Recall that we can obtain exactly two manifolds by attaching a 1-handle to the 0-handle (Example 2.1.4). In dimension two it is either the annulus or the Möbius band (Figure 2.1.2). If we now attach a 2-handle to the Möbius band then we obtain $\mathbb{RP}^2$, on the other hand if we attach a 2-handle to the annulus we obtain a disk. Notice that in the latter case, we started with a disk (given by the 0-handle) and attached a 1-handle followed by a 2-handle only to recover a disk again. This hints at two things - we might have multiple handle decompositions for the same manifold and there might be a way to move between these handle decompositions by removing some handles (and maybe some other operations).

The previous example highlights that it's possible for a manifold to have multiple handle decompositions. This shouldn't be a surprise since as we've seen before, every handle decomposition is associated to a Morse function, but the Morse functions are by no means unique. In fact, if X is a compact, smooth manifold, then the set of Morse functions is dense in the set of all smooth functions $C^\infty(X, \mathbb{R})$. The remainder of this section is devoted to studying how these different handle decompositions relate to each other.

We start by noticing that we can attach handles in any order.

Proposition 2.1.14. *Let X be a compact, smooth n-manifold. Then X admits a handle decomposition where the handles are attached in order of increasing index.*

Proof. For $k \geq l$, let Y be an n-manifold and $Y \cup h_k \cup h_l$ be the manifold obtained by attaching a k-handle h_k followed by an l-handle h_l to Y. Denote by A the attaching sphere of h_l and by B the belt sphere of h_k. Since the attaching sphere A has dimension $l-1$ and the belt sphere B has dimension $n-k-1$, then $dim(A)+dim(B) = l-1+n-k-1 \leq k-1+n-k-1 = n-2$. Notice now that $dim(\partial X) = n-1$, so using transversality we can perturb A to be disjoint from B. By further taking an open neighborhood of B which contains the remainder of h_k, then by Theorem A.0.2 we can construct an isotopy which separates h_k and h_l. □

From here on we will then assume that all handle decompositions are ordered by increasing index, unless stated otherwise.

Notice that if we have a handle decomposition and we turn it upside down we end up with a different handle decomposition of the same manifold. Indeed, each k-handle can be interpreted as a $(n-k)$-handle by reversing the roles of the core and cocore - any k-handle can be seen as being glued along its belt sphere $\{0\} \times S^{n-k-1}$ to the part "above it", which makes it into a $(n-k)$-handle glued upside down. Such a decomposition is called a dual handle decomposition and in terms of Morse theory amounts to replacing the Morse function f with $1-f$.

Another relevant aspect of handlebodies is that there can be handle decompositions with more handles than strictly necessary - for instance in Example 2.1.13 we saw that attaching the 1- and 2-handles to the 0-handle didn't change the diffeomorphism type of the manifold. This is due to the fact that if two handles of consecutive indices are attached in a specific manner to an n-manifold X, then they will form an n-disk which is glued to the manifold via a boundary sum. Since boundary summing with a disk doesn't change the diffeomorphism type of X, then we recover the original manifold. We thus have that attaching the handle pair doesn't change the manifold, so we can safely remove it. Such a pair is called a cancelling handle pair.

Proposition 2.1.15. *Let Y be a compact, smooth n-manifold and $X = Y \cup h_k \cup h_{k+1}$ be the manifold obtained by attaching a k-handle h_k, followed by a $(k+1)$-handle h_{k+1} to Y. If the attaching sphere A of h_{k+1} intersects the belt sphere B of h_k transversely in exactly one point, then h_k and h_{k+1} form a cancelling handle pair and X is diffeomorphic to Y.*

Proof. Consider the belt sphere of h_k and take a tubular neighborhood $\nu B \subset Y \cup h_k$, which we can identify with $D^k \times \partial D^{n-k}$ since B has trivial normal bundle in $Y \cup h_k$. If we now take p to be the point of intersection between A and B, then A intersects the boundary of h_k along the fiber of νB at p, that is $A \cap \partial h_k = D^k \times \{p\}$. Since A intersects ∂h_k along a disk $D = D^k \times \{p\}$, then A will also intersect ∂Y along another k-disk D' which glues to D along $S = S^{k-1} \times \{p\}$, i.e. along the boundary of the attaching region of h_k. Notice now that S is in the boundary of a tubular neighborhood νA of the attaching sphere A of h_k, so A and S share a tubular neighborhood. As such, A and S can be assumed to be connected by a k-disk D^*, such that if we glue $D' = A \cap \partial Y$ with D^* along S, then we will have a disk in ∂Y which is bounded by A. (see Figure 2.1.3 for the 3-dimensional case). We are now in a position where we can use Corollary 2.1.7 to identify $Y \cup h_k$ with $Y \natural E$ (where E is a D^{n-k}-bundle as in the corollary) and obtain a diffeomorphism between $X = Y \cup h_k \cup h_{k+1}$ and $(Y \natural E) \cup h_{k+1}$. Furthermore since A is completely contained in the boundary of E, then we can safely glue in h_{k+1} before taking the boundary sum (the operation is still well-defined), leading to a diffeomorphism between X and $Y\natural(E \cup h_{k+1})$. Finally notice that A is simply a parallel copy of the 0-section of E, so we can identify the attaching region of h_{k+1} with a tubular neighborhood $\nu A \subset \partial E$. If we decompose the boundary of the fibers of E as $S^{n-k-1} = D_-^{n-k-1} \cup D_+^{n-k-1}$, then we can identify the attaching region of h_{k+1} with D_-^{n-k-1}, leading to a diffeomorphism between $E \cup h_{k+1}$ and D^n. It then follows that X is diffeomorphic to $Y\natural D^n$, but since D^n acts as the identity for the boundary sum, then X is diffeomorphic to Y. $\square$

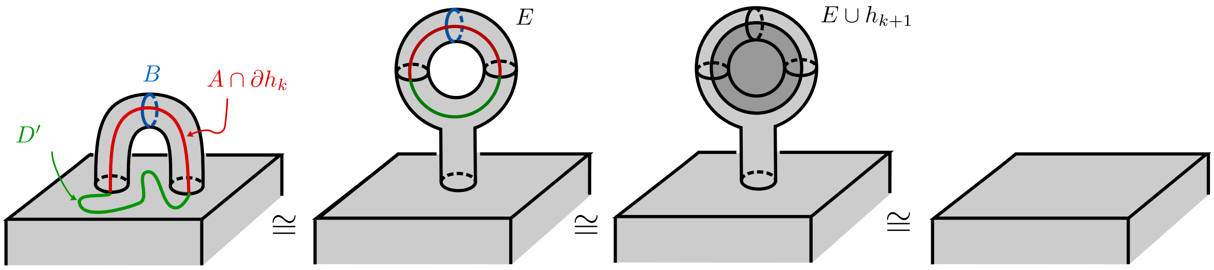

Figure 2.1.3: Cancelling a 3-dimensional $1/2$-handle pair.

Consider now two k-handles h and h' attached to an n-manifold X $(0 < k < n)$. If we isotope the attaching sphere A of h in $\partial(X \cup h')$ through h', then there will be a point where A intersects the belt sphere B' of h' transversely in exactly one point p. Since $T_pA \oplus T_pB'$ has codimension 1 in $T_p\partial(X \cup h')$, then the proof of Proposition 2.1.14 gives us exactly two ways of pushing A off of B'. One way brings us back to our initial state while the other pushes A over h'.

Definition 2.1.16. If X is a smooth n-manifold and h and h' are k-handles attached to X $(0 < k < n)$, then the procedure of isotoping A over h', through B', until A no longer intersects h' (as described in the previous paragraph) is called a handle slide.

Example 2.1.17. We can obtain a handlebody description of the Klein bottle by attaching two 1-handles (with opposite framing) to the 0-handle and capping it off with a 2-handle (Figure 2.1.4). Notice now that we can slide one of the 1-handles over the other to change the attaching map of one the 1-handles. This new handlebody describes $\mathbb{RP}^2\#\mathbb{RP}^2$, so by handle sliding, we have shown that the Klein bottle and $\mathbb{RP}^2\#\mathbb{RP}^2$ are actually diffeomorphic.

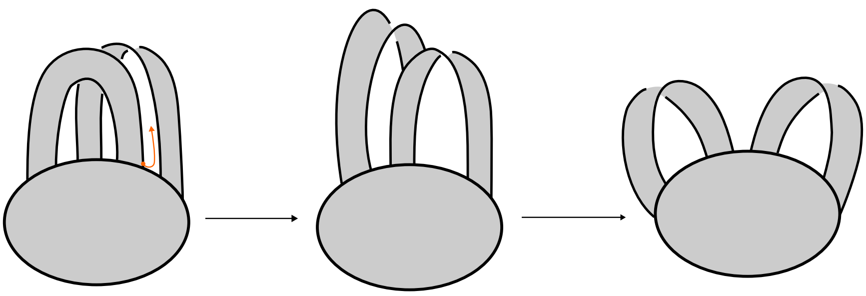

Figure 2.1.4: Handle slide induces a diffeomorphism between the Klein bottle and $\mathbb{RP}^2\#\mathbb{RP}^2$.

Since a handle slide simply amounts to isotoping one handle over another, then it doesn't change the diffeomorphism type of the manifold. Together with the operation of adding or removing cancelling handle pairs, we get a way of changing between different handle decompositions of the same manifold. It turns out that these two moves are all that we need to relate any handle decompositions of the same handlebody.

Theorem 2.1.18 (Cerf - [Cer70]). *If X is a compact, smooth manifold, then any two handle decompositions of X are related by a sequence of creation/removal of cancelling handle pairs or handle slides.* □

The ideia of the proof is that for such a manifold X, we can find two self-indexing Morse functions $f_i \colon X \to [0, 1]$, for $i = 0, 1$, which correspond to the two handle decompositions of X. Since $[0, 1]$ is contractible, then the two functions are homotopic (rel ∂X) and so we can talk about a generic homotopy f_t between them. We will have that at finitely many t, f_t will not be a self-indexing Morse function. These points will correspond to handle creation/cancellations (case when the function is not Morse) or a handle slide (case when the function is not self-indexing).

Proposition 2.1.19. *Let X be a compact, connected, smooth n-manifold. Then X admits a handle decomposition with no 0-handles ($\partial_- X \neq \emptyset$) or with a single 0-handle ($\partial_- X = \emptyset$). Furthermore, X also admits a handle decomposition with no n-handles ($\partial_+ X \neq \emptyset$) or a single n-handle ($\partial_+ X = \emptyset$).*

Proof. For the case where $\partial_- X = \emptyset$, any handle decomposition will necessarily have one 0-handle, since it's the only type of handle that attaches to $\emptyset$. If there is more than a single 0-handle, then two 0-handles must be connected by a 1-handle (all handles of higher index have connected attaching spheres) so as to make the manifold connected. In this situation, the attaching sphere of the 1-handle intersect the belt-sphere of both 0-handles transversely in exactly one point each, thus we have two cancelling $0/1$-handle pairs which share a 1-handle. We can then remove one of the $0/1$-handle pairs and we are left with a 0-handle. Since we can do this for all 0-handles, then after cancelling all handle pairs, we are left with a handle decomposition with a single 0-handle. If $\partial_- X \neq \emptyset$, then the same argument follows, except now we can cancel all 0-handles, since we don't need a 0-handle to attach to $\emptyset$, as we already have $\partial_- X \times I$ fulfilling that role. The second part of the statement, follows by dualizing the handlebody and applying the same argument. □

Before proceeding, it remains to deal with the algebraic topology of a handlebody (as promised in Example 2.1.11). As we've seen before, a k-handle is essentially the same as thickened k-cell, so the algebraic topology of a handle decomposition can be retrieved from the CW-complex formed from the cores of the handles. Consider a handle decomposition of a closed n-manifold X with a single 0-handle. In this case the core of each 1-handle gives a generator of $\pi_1(X)$, while the attaching spheres of the 2-handles gives a relation. Since all other handles of higher dimension have simply connected attaching sphere, then they don't affect the fundamental group, so $\pi_1(X)$ is generated by the 1-handles with relations given by the 2-handles. In a similar fashion, if X is a compact, oriented n-manifold, then we can recover its homology from its handle decomposition. Up to homotopy a k-handle is the same as its core, so we can compute the homology of X by looking at the underlying CW-complex structure, obtained by deformation retracting each handle to its core. We then define the chain groups $C_k(X_k, \partial_- X)$ to be $H_k(X_k, X_{k-1}; \mathbb{Z})$, which is freely generated by the oriented cores of the k-handles. Furthermore we take the differential to be $\partial_k h = \sum_i (B_i \cdot A) h_i$ (which can be shown to coincide with the usual differential in cellular homology), where A denotes the attaching sphere of the k-handle h, B_i the belt sphere of the $(k-1)$-handles h_i and $B_i \cdot A$ their algebraic intersection number - if X is a smooth, oriented manifold and Y_1 and Y_2 are two embedded, oriented, smooth manifolds of complementary dimensions, then due to tranversality they will intersect in a finite number of points. At each point $p \in Y_1 \pitchfork Y_2$, we concatenate positive bases of T_pY_1 and T_pY_2 to get a basis for T_pX. If this basis is positive, then the intersection of Y_1 and Y_2 is said to be positive at p, and negative otherwise. We can then assign $\epsilon(p) = \pm 1$ to each intersection point, and we define the algebraic intersection number (or signed intersection number) $Y_1 \cdot Y_2$ to be the sum of $\epsilon(p)$ along all intersection points p.

It remains to see how the handle moves presented above affect the homology of X. If h is a k-handle which cancels a $(k-1)$-handle h_i, then the attaching sphere A of h intersects the belt sphere B_i of h_i geometrically once, and thus $B_i \cdot A = \pm 1$ depending on the sign of the intersection. Since these form a cancelling handle pair, then we can remove the pair by removing the generators h and h_i from the chain groups. Since under the differential we have that $\partial_{k+1} h = \pm h_i + \sum_{j \neq i} a_j h_j$, then this amounts to having $h = 0$ and $h_i = \mp \sum_{j \neq i} a_j h_j$. It then follows that removing a cancelling handle pair has no effect on homology. Sliding a k-handle h over another k-handle h' has the effect of replacing the generator h by $h \pm h'$, depending on the direction in which we push h. In fact, every basis change of $C_k(X, \partial_- X)$ can be realized by handle slides along with changing the orientation of the core disks of the handles.

2.2 The h-cobordism theorem

The previous section gives us the necessary tools to prove the famous h-cobordism theorem of Smale ([Sma61]). This result earned him a Fields medal and has as a corollary the high dimensional topological Poincaré conjecture, which is probably one of the most well known results in topology, which still to this day motivates research (in particular the smooth four 4-dimensional case).

Definition 2.2.1. Let X and Y be closed, oriented, smooth n-manifolds. We say that X and Y are cobordant ($X \sim Y$) if there is a compact, oriented, smooth $(n+1)$-manifold W with boundary $\partial W = X \sqcup \overline{Y}$. The manifold W is said to be a cobordism between X and Y.

Note that $\sim$ forms an equivalence relation. Indeed, for smooth, closed n-manifolds X, Y and Z we have that $X \sim X$ can always be achieved via the trivial cobordism $X \times I$. Furthermore if $X \sim Y$ via a cobordism W and $Y \sim Z$ via a cobordism W', then the desired cobordism can be constructed by gluing W and W' along Y, via an orientation reversing diffeomorphism. Finally, if $X \sim Y$ via a cobordism W, then by reversing the orientation on W, we get that $Y \sim X$.

As the name implies, an h-cobordism is something like a cobordism, except now we also have some conditions on homotopy (and hence homology).

Definition 2.2.2. Let X and Y be two closed, simply connected, oriented smooth n-manifolds, and W

a cobordism between them. We say that X and Y are h-cobordant, if the inclusions $i_X : X \to W$ and $i_Y : Y \to W$ are homotopy equivalences between X and W, and Y and W respectively.

The h-cobordism theorem states that if we have an h-cobordism W between simply connected, closed, oriented smooth manifolds, then W is actually diffeomorphic to an annulus, i.e. the trivial cobordism. To prove this theorem, we will give a handle decomposition to W and then slide handles until we are in a position where we can cancel every single handle, leaving a trivial cobordism. We start by dealing with the borderline cases.

Proposition 2.2.3. *Let W be a simply-connected, smooth $(n+1)$-manifold with boundary $\partial W = X_0 \cup \overline{X_1}$, such that X_i is connected for $i = 1, 2$. Then W admits a handle decomposition with no $0-, 1-, n-$ or $(n+1)$-handles.*

Proof. As before, if we can remove the 0- and 1-handles, then by dualizing the handlebody we can do the same for the n- and $(n+1)$-handles. For the 0-handles we can proceed as in the proof of Proposition 2.1.19. If we have 0-handles, they must be connected by 1-handles to $X_0 \times I$. In this situation, we can cancel the 0-handles along with the 1-handle pair, until we have no 0-handles. Note that this doesn't necessarily eliminate all 1-handles, as we might have 1-handles attached directly to $X_0 \times I$. To remove these remaining 1-handles we introduce cancelling $2/3$-handle pairs, such that the 2-handles form cancelling $1/2$-handle pairs with the 1-handles that are left (the existence of such pairs is not immediate, for more details see Proposition $9.2.3$ of [GS99]). We now cancel the $1/2$-handle pairs until we remove all 1-handles and the result follows. ☐

The technique used to remove the 1-handles is known as handle trading, since it can be interpreted as "trading" 1-handles for 3-handles. The last ingredient we need in order to prove the h-cobordism theorem, is the Whitney trick. The idea is that after sliding, we will be faced with $(k+1)$-handles which intersect k-handles algebraically once, but not necessarily geometrically (which is the condition required in order to cancel them). The Whitney trick gives us a procedure to remove these extraneous intersections, turning these algebraic intersections into geometric ones.

Lemma 2.2.4 (Whitney trick)**.** *Let W be a connected, simply connected, smooth n-manifold and X_0 and X_1 be smooth, connected submanifolds of dimensions k and l respectively, with $k + l = n \geq 5$. Suppose that X_0 and X_1 have algebraic intersection number ± 1 and that $\pi_1(W - X_0 \cup X_1) = 1$. Then there is a smooth connected k-manifold X_0' such that X_0' is isotopic to X_0 and X_0' intersects X_1 tranversely in exactly one point.*

Proof (Sketch). Assuming that X_0 and X_1 intersect transversely in more than one point, then since $X_0 \cdot X_1 = \pm 1$, there must be points $x, y \in X_0 \cap X_1$ with opposite signs. We can now construct a path α that connects x and y in X_0 and a path β that connects y and x in X_1, such that $\gamma = \alpha\beta$ forms a circle which is isotopic to a loop in $W - X_0 \cup X_1$. Since $\pi_1(W - X_0 \cup X_1) = 1$, then γ bounds an immersed disk D in $W - X_0 \cup X_1$, and furthermore since $n \geq 5$, then by transversality we can perturb this disk to be embedded. We can now isotope X_0 over D to remove these intersection points. Doing this for all pairs of intersection points with opposite signs gives us the manifold X_0', which satisfies the conditions of the statement. □

We are now ready to prove the h-cobordism theorem.

Theorem 2.2.5 (h-cobordism theorem - [Sma61]). *Let X_0 and X_1 be simply connected, closed, oriented, smooth n-manifolds ($n \geq 5$), and W an h-cobordism between them. Then W is diffeomorphic to $X_0 \times I$.*

Proof. The proof follows by induction on the index of the handles - we prove that if W admits a handle decomposition with no l-handles for $l < k$, then it admits a handle decomposition with no l-handles for $l < k+1$. The base case is solved by Proposition 2.2.3. For the induction step, we start by noting that since W is an h-cobordism, then the inclusion $i\colon X_0 \to W$ is a homotopy equivalence, and so by the long exact sequence on homology we have that $H_*(W, X_0) = 0$. Since every chain group $C_k(W, X_0)$ is freely generated by the k-handles, and by hypothesis $C_{k-1}(W, X_0) = 0$, this means that for each k-handle $h \in C_k(W, X_0)$, there must be an element $\alpha = \sum_i a_i h_i$ in $C_{k+1}(W, X_0)$ such that $\partial_{k+1}\alpha = h$. We can now do handle slides on the $(k+1)$-handles to turn α into $\alpha' = h'$, where h' is a $(k+1)$-handle. We now have that $\partial_{k+1}h' = (B \cdot A')h = h$, so the belt sphere B of h intersects the attaching sphere A' of h' algebraically one time. We can now apply the Whitney trick to remove all extra intersections, so that B and A' intersect geometrically in exactly one point, but by Proposition 2.1.15, this means that h and h' form a cancelling handle pair, so we can remove both without changing W. Since we can do this for every handle h generating $C_k(W, X_0)$, then W admits a handle decomposition with no l-handles for $l < k+1$. It then follows by induction that W admits a handle decomposition with no handles, so W is diffeomoprhic to $X_0 \times I$. □

Equipped with the h-cobordism theorem, we can now prove the notorious topological Poincaré conjecture in dimensions greater than 6. Note that Smale actually extended his proof to dimension 5 (where the smooth case also holds), but this goes beyond the scope of this thesis so we simply leave it as a remark. The following statement is equivalent to the Poincaré conjecture, where we used the Whitehead theorem (theorem 4.5 of [Hat00]) and the Hurewicz theorem to convert between homotopy and homology.

Theorem 2.2.6 (High-dimensional Poincaré conjecture - [Sma61]). *Let X be a closed, connected, simply-connected, smooth n-manifold, with $n \geq 6$. If $H_\star(X;\mathbb{Z}) \simeq H_\star(S^n;\mathbb{Z})$, then X is homeomorphic to S^n.*

Proof. Consider X satisfying the conditions of the statement, then for any pair of distinct points we can find neighborhoods D_0^n and D_1^n, such that they're disjoint and each point is in one of the disks. We can now remove these disks so as to obtain a decomposition of $X = D_0^n \cup W \cup D_1^n$, where $W = M - \text{int}(D_0^n \cup D_1^n)$. By construction, W is a smooth cobordism between spheres S_0^{n-1} and S_1^{n-1}, which by the conditions on homology can be shown to be an h-cobordism. We can now apply the h-cobordism theorem to get a diffeomorphism Φ between W and $S_0^{n-1} \times [0,1]$. All that remains is to reglue X back together. Note that the diffeomorphism $\Phi\colon W \to S^{n-1} \times [0,1]$ restricts to homeomorphisms $\phi_i\colon S_i^{n-1} \to S^{n-1}$ on the boundary (for $i = 0, 1$), so by the Alexander trick (Theorem A.0.3) we can extend such homeomorphisms to homeomorphisms of the disks $D_i^n \longrightarrow D^n$. Since these homeomorphisms agree with Φ on the boundary, then by gluing the maps piecewise we get a homeomorphism between $D^n \cup S^{n-1} \times [0,1] \cup D^n$ and S^n, and thus a homeomorphism between X and S^n. $\square$

Note for the proof to work we needed the h-cobordism theorem, which in turn required the Whitney trick. The latter requires at least 5 dimensions to work, since we need to use transversality to separate the disks but $2+2<5$. This means that we can't extend the proof of the h-cobordism theorem (at least not in a straightforward manner) to lower dimensions. The failure of the Whitney trick thus highlights the distinction between low and high dimensions.

2.3 4-dimensional handlebodies and Kirby diagrams

Now that we're done with high dimensional manifolds, we can finally focus on the 4-dimensional case. The goal of this section and in fact most of the remainder of the chapter, is to provide a diagrammatic description of 4-manifolds (constructed as handlebodies) which is now known as Kirby Calculus (earning its name from the mathematician who first developed these results in [Kir78]). This will allow us to interpret 4-dimensional handlebodies as framed link diagrams in $\mathbb{R}^3$ (Kirby diagrams) and perform handle slides and handle cancellations by moving and removing knots from these diagrams. This will provide a very powerful tool to help deal with the complexity of 4-manifolds.

We focus on the case of a connected, compact, smooth 4-manifold X. Start with the 0-handles, which can be taken to be a single D^4 since our manifold is connected. We identify the boundary of the 0-handle $S^3 = \mathbb{R}^3 \cup \{\infty\}$ so that we can see the attaching region of the other handles in $\mathbb{R}^3$. The attaching region of a 1-handle is given by $S^0 \times D^3$ and since $\pi_0(O(3)) \cong \mathbb{Z}_2$, then there is a unique way of gluing them so as to get an orientable manifold. This can be seen in $\mathbb{R}^3$ as a pair of balls B^3 where on the boundary they are identified by the map $(x,y,z) \mapsto (x,-y,z)$ as in Figure 2.3.1.

Figure 2.3.1: Attaching a 1-handle to the 0-handle.

If we want to attach n 1-handles to the 0-handle then, by Corollary 2.1.8 the union of the 0-handle with the 1-handles X_1 is diffeomorphic to $\natural n S^1 \times D^3$ which we can see as $2n$ 3-balls, glued in pairs with the boundary identifications as before. The 2-handles are now attached to ∂X_1 by an embedding $\varphi\colon \partial D^2 \times D^2 \to \partial X_1$ which is determined by the image of $S^1 \times \{0\}$ along with a framing $f \in \pi_1(O(2)) \cong \mathbb{Z}$. Unlike the previous case, we now have a choice of framing (which we will get back to shortly). Note that since the attaching spheres of the 2-handles are circles, then in $\mathbb{R}^3$ they can be knotted and linked between each other, furthermore since we are attaching to X_1, then these circles can also go over the 1-handles (Figure 2.3.2).

It remains to deal with 3- and 4-handles. Luckily the following theorem guarantees that for a closed manifold, these are uniquely attached.

Theorem 2.3.1. *Let X be a closed, connected, oriented, smooth 4-manifold. Then X is completely determined by the union of the 0-, 1- and 2-handles X_2.*

Proof. Since X is connected, then it admits a handle decomposition with a single 4-handle, and since X_1 is diffeomorphic to $\natural n S^1 \times D^3$ (n is the number of 1-handles), then by dualizing X_1, the union of 3- and 4-handles will be diffeomorphic to $\natural m S^1 \times D^3$, with m the number of 3-handles. So for X to be closed, we must have that $\partial(X_2) \cong \partial(\{m \text{ 3-handles}\} \cup \{\text{4-handle}\}) = \partial(\natural m S^1 \times D^3) = \# m S^1 \times S^2$, but by Theorem A.0.4 any self-diffeomorphism of $\# m S^1 \times S^2$ extends to a diffeomorphism of $\natural m S^1 \times D^3$, so any two ways of gluing the 3- and 4-handles result in diffeomorphic manifolds. $\square$

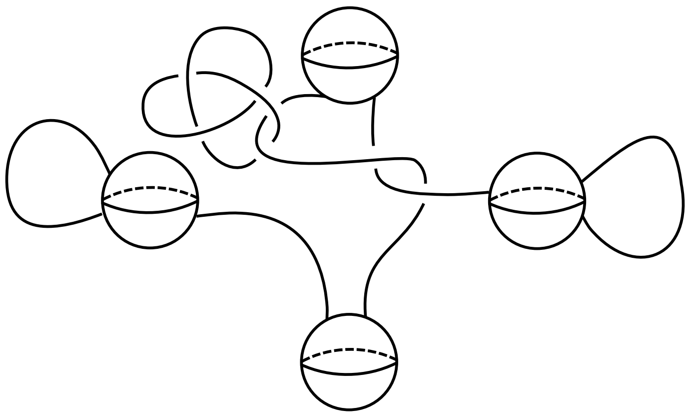

Figure 2.3.2: Kirby diagram mod framing of the 2-handles.

Remark 2.3.2. If X is not closed, but is simply connected with connected boundary, then X is determined by X_2 and the number of 3-handles (see Proposition 5 of [Tra82]).

Before proceeding, we are left to deal with the framings on the 2-handles. Notice that while we know that framings are in bijection with elements of $\pi_1(O(2)) \cong \mathbb{Z}$, we don't know what the correspondence is since the framings form a torsor and not a group. We must thus define a 0-framing to specify the bijection. With that in mind, consider a knot K embedded in an oriented, compact 3-manifold M. We can determine a framing on K by taking a non-vanishing transverse vector field v of K. Then the orientation of K, coming from an embedding $S^1 \hookrightarrow M$, along with the orientation of M, extend v to an oriented basis of the normal bundle of K. This means that we can specify the attachment of a 2-handle by a knot K and such a vector field v (which we usually call itself the framing). An equivalent way to do this is to construct a parallel knot K' by pushing K in the direction of v, more specifically if we have a 2-handle attachment given by $\varphi\colon S^1 \times D^2 \to M$, then $K = \operatorname{im}(\varphi|_{S^1\times\{0\}})$ and $K' = \operatorname{im}(\varphi|_{S^1\times\{p\}})$ for some $p \in D^2 - \{0\}$ (Figure 2.3.3).

Once we choose a framing, we get all the other framings by adding twists as in Figure 2.3.4. In particular if we take $M = S^3$ and give it the standard orientation, then a positive framing n corresponds to n right handed twists while a negative n corresponds to n left-handed twists. Now that we know how

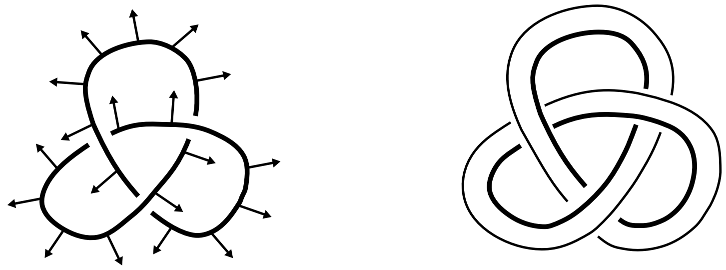

Figure 2.3.3: Framing specified by a vector field and by a parallel knot.

to determine all the other framings from a specific framing, it remains to find out how to specify the 0-framing.

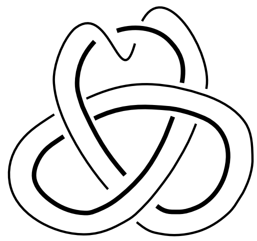

Figure 2.3.4: Another framing of the same knot.

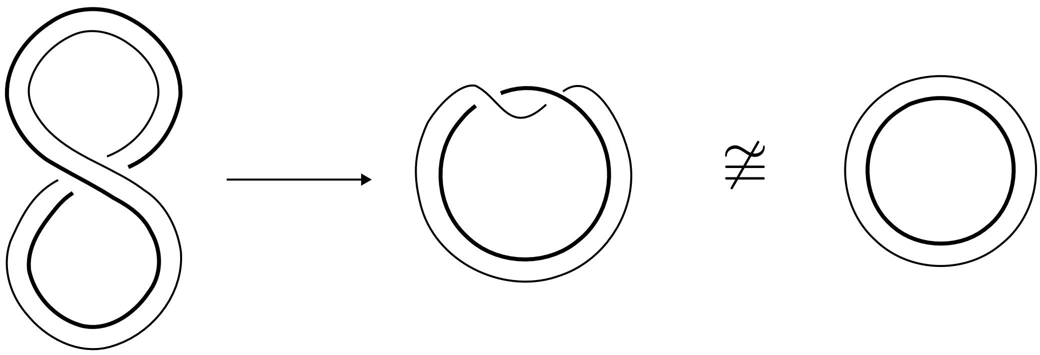

Figure 2.3.5: The blackboard framing is not invariant under isotopies of the knot.

For now we'll focus on 2-handles that are attached directly to S^3, i.e that don't go over 1-handles, since there is some care we have to take in this situation. At first glance, we could take a knot K and choose a vector field that lies in $\mathbb{R}^2$ but Figure 2.3.5 shows that this choice is not invariant under isotopy. This is called the blackboard framing and it equals the writhe of the knot $w(K)$, so it is not invariant under Reidemeister I moves. This is a convenient choice since it is easy to compute, but we wish to define an isotopy invariant 0-framing. To do this we make use of the linking number.

Let K_1, K_2 be knots in S^3 and $\nu K_1 \cong S^1 \times D^2$ a tubular neighborhood of K_1. Since $H_i(S^3;\mathbb{Z}) = 0$ for $i \neq 0, 3$ then the Mayer-Vietoris sequence of the pair $(S^3 - \text{int}\ \nu K_1, \nu K_1)$

$$\longrightarrow H_2(S^3;\mathbb{Z}) \longrightarrow H_1(S^1 \times S^1;\mathbb{Z}) \longrightarrow H_1(S^3 - \text{int}\ \nu K_1;\mathbb{Z}) \oplus H_1(\nu K_1;\mathbb{Z}) \longrightarrow H_1(S^3;\mathbb{Z}) \longrightarrow$$

gives us an isomorphism $H_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z} \cong H_1(S^3 - \text{int}\ \nu K_1) \oplus H_1(\nu K_1)$. Since $H_1(\nu K_1) \cong \mathbb{Z}$ is generated by a longitude of the boundary torus, then $H_1(S^3 - K_1) \cong H_1(S^3 - \text{int}\nu K_1) \cong \mathbb{Z}$ is generated by a meridian μ of K_1 i.e. a circle μ in $S^3 - K_1$ that bounds a disk D in νK_1 intersecting K_1 transversely exactly one time. We fix the orientation of μ using the right-hand rule, that is such that K_1 and D have positive intersection. We are now ready to define the linking number.

Definition 2.3.3. Let K_1, K_2 be knots in S^3 and μ a generator of $H_1(S^3 - K_1)$ as before. Then the linking number $lk(K_1, K_2)$ is defined as the unique $n \in \mathbb{Z}$ such that $[K_2] = n[\mu]$ for $[K_2] \in H_1(S^1 - K_1)$.

This works for any nullhomologous knot in an oriented, compact 3-manifold, but we are mainly interested in knots in the boundary S^3 of a 0-handle. Note that if any of the components K_1, K_2 or S^3 changes orientation, then the linking number changes sign, i.e. $lk(-K_1, K_2) = lk(K_1, -K_2) = -lk(K_1, K_2)$ and $lk(K_1, K_2)_{\overline{S^3}} = -lk(K_1, K_2)_{S^3}$.

Proposition 2.3.4. *Let K_1, K_2 be knots in S^3. Then $lk(K_1, K_2) = K_1 \cdot F_2$, where F_2 is a Seifert surface of K_2 (a compact, oriented surface in S^3 such that $\partial F_2 = K_2$).*

Proof. Assume that F_2 intersects K_1 in a single point. If we now take a band-sum of K_2 with the meridian μ of K_1, then this new knot $K_2' = K_2 \# - \mu$ ($+\mu$ if the intersection is negative) admits a Seifert surface F_2' that no longer intersects K_1 (Figure 2.3.6). Without loss of generality, if we now assume that $n = K_1 \cdot F_2$ is positive and do the same procedure for the remaining intersections, then $K_2 \# - n\mu$ is the boundary of a Seifert surface that no longer intersects K_1, so $[K_2 - n\mu] = 0 \in H_1(S^3 - K_1)$. It follows that $[K_2] = n[\mu]$, so $lk(K_1, K_2) = n = K_1 \cdot F_2$. □

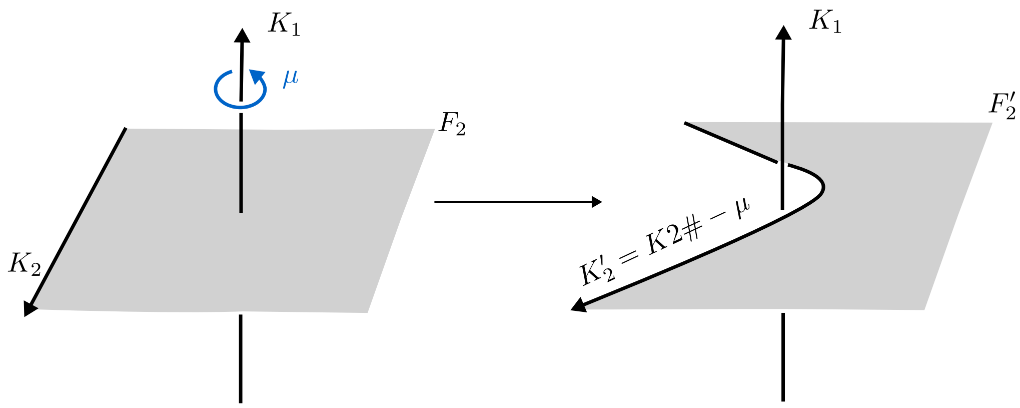

Figure 2.3.6: Removing an intersection by taking a band-sum.

Note that for any nullhomologous knot in an oriented 3-manifold, we can find a Seifert surface via the Seifert algorithm (see Section 5.A of [Rol03]), so the previous characterization is always well-defined. If the knots are represented by diagrams instead, the following propostion gives us a simple way to compute their linking number.

Proposition 2.3.5. *Let K_1 and K_2 be knots in S^3, given by link diagrams. Then $lk(K_1, K_2)$ equals the signed number of times that K_2 crosses underneath K_1.*

Proof. It follows from the same idea as the previous proof. Indeed, if K_2 crosses underneath K_1 n signed times, then by taking n band-sums of K_2 with the meridian μ of K_1, we can remove every undercrossing leading to a null-homologous knot $[K_2 - n\mu] \in H_1(S^3 - K_1)$. It then follows that $[K_2] = n[\mu]$ and thus $lk(K_1, K_2) = n$. □

Symmetry of the linking number quickly follows from this proposition - if we have diagrams for K_1 and K_2 and we look at the them from the other side of $\mathbb{R}^2$, then any undercrossing of K_2 will become an undercrossing of K_1, so from the previous proposition we have that $lk(K_1, K_2) = lk(K_2, K_1)$.

Proposition 2.3.6. *The linking number is an isotopy invariant.*

Proof. We just need to check that the Reidemeister moves don't change the linking number. Indeed, since the first Reidemeister move only affects one of the knots, then the linking number doesn't change, and furthermore since the third Reidemeister move simply moves the position of two crossings, then it also doesn't affect the linking number. Finally, the second Reidemeister move adds a positive and a negative crossing, so by the previous proposition, the linking number stays the same. □

Now that we have an isotopy invariant, we are ready to define the framing coefficient.

Definition 2.3.7. Let (K, v) be a framed knot in the boundary of a 0-handle. Then the framing coefficient is defined as $lk(K, K')$, where K' is obtained from K by pushing in the direction of v (as we've seen before), and such that K' has the same orientation as K.

So now we have a way to associate integers to the framings of knots in S^3 and any twist will change this framing by ± 1 depending on if it's a right-handed or left-handed twist. To define the 0-framing, we just have to find a vector field that produces a knot K' such that $lk(K, K') = 0$. From Proposition 2.3.4 this amounts to having $K' \cdot F = 0$, where F is a Seifert surface for K, thus the 0-framing is usually also called the Seifert framing, which can be obtained by pushing K' in the direction of any Seifert surface of K. Another useful and easier to compute choice is the blackboard framing, which we discussed earlier.

Example 2.3.8. For a manifold X, if we have a 2-handle going over a 1-handle, then the attaching sphere won't, in general, be a nullhomologous knot in $\partial(\{0\text{-handle}\} \cup \{1\text{-handles}\})$, so we can't find a Seifert surface and get a 0-framing. If we specify a framing by a parallel copy of the knot, then by doing an isotopy of the 2-handle, we may change the framing by an even number (Figure 2.3.7). To get around this problem, we'll have to introduce a new notation for 1-handles, which we will discuss in Section 2.7.

Example 2.3.9. As we've seen before, $\natural n S^1 \times D^3$ is constructed by gluing n 1-handles to the 0-handle, which gives us a diagram consisting of $2n$ 3-balls glued in pairs. The n-sphere is obtained by gluing an n-handle to the 0-handle, so since no 1-handle or 2-handles are present, we get an empty diagram.

Example 2.3.10. Consider the manifold X obtained by attaching a 2-handle along an n-framed unknot to the 0-handle. Since in a Kirby diagram a 2-handle appears as a knot along with its framing, then a diagram for X will be particularly simple (Figure 2.3.8). Note that by Proposition 2.1.6 X will be a D^2-bundle over S^2 - since K is the unknot, then we can easily find a Seifert disk D in S^3 and by pushing it into the 0-handle and gluing the core of the 2-handle along K we get a sphere S realizing the 0-section of the bundle and generating the homology of X. We can now construct a parallel copy S' of S to compute

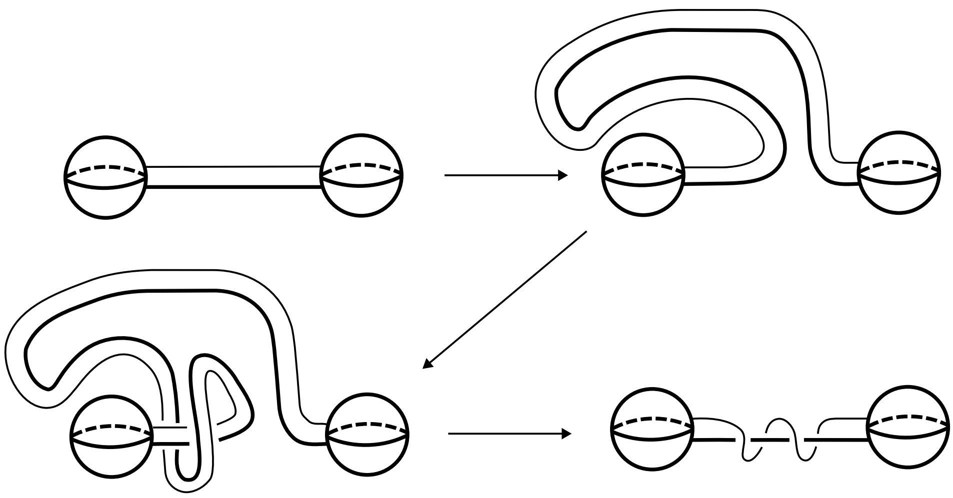

Figure 2.3.7: Isotopy of a 2-handle going over a 1-handle which changes its framing.

the self-intersection of S as follows. We take a parallel copy of the core $D^2 \times \{p\}$ which will intersect the 0-handle along a parallel copy of the knot $K' = S^1 \times \{p\}$ and glue in a Seifert disk D' for K'. Now push K', along with D', into the 0-handle and extend it by a collar of the boundary so that $K' \times I$ is an annulus in $S^3 \times I$ (and D' will be glued at the end of it) and such that S will intersect $K' \times \{t\}$ at some t (see Figure 2.3.9). Since K has framing n then by Proposition 2.3.4, S will intersect K' in exactly $|n|$ points, either positively or negatively depending on the sign of n. We then conclude that the self-intersection of the 0-section is given by the framing n and so X is a D^2-bundle over S^2 with Euler number n. In particular, if $n = 0$, then X is simply $S^2 \times D^2$. If $n = 1$ (or $n = -1$), then by gluing a 4-handle to X we get $\mathbb{CP}^2$ (or $\overline{\mathbb{CP}^2}$), where the sphere S realizes the $\mathbb{CP}^1$ (or $\overline{\mathbb{CP}^1}$) sitting inside X.

Manifolds obtained by attaching a single 2-handle, along an n-framed knot, to the 0-handle (as in the previous exemple) will be a recurring theme in this thesis, so we give them a name.

Definition 2.3.11. A 4-manifold obtained from the 4-ball by attaching an n-framed 2-handle along a knot K is called an n-framed knot trace and denoted by $X_n(K)$.

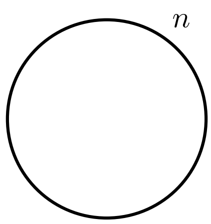

Figure 2.3.8: D^2-bundle over S^2 with Euler number n.

Example 2.3.12. We now describe a Kirby diagram for $S^2 \times S^2$. Notice that a product of a k-handle and an l-handle is a $(k + l)$-handle, so to get a handle decomposition of $S^2 \times S^2$, we give each S^2 a handle decomposition with a single 0-handle D_- and a 2-handle D_+ and take their product. Then $S^2 \times S^2 = (D_- \times D_-) \cup (D_- \times D_+) \cup (D_+ \times D_-) \cup (D_+ \times D_+)$, where we can identify $(D_- \times D_-)$ with a 0-handle, $(D_+ \times D_+)$ with a 4-handle, and $(D_- \times D_+)$ and $(D_+ \times D_-)$ with a 2-handle each. Now note

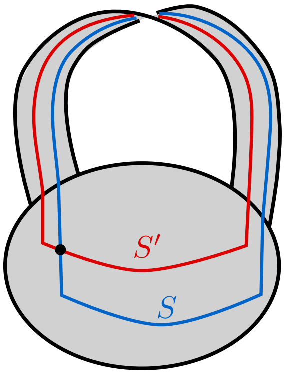

Figure 2.3.9: Computing the self-intersection of the 0-section by pushing along a framing.

that $(D_- \times D_-) \cup (D_- \times D_+) = D_- \times S^2 \cong D^2 \times S^2$ and $(D_- \times D_-) \cup (D + \times D_-) = S^2 \times D_- \cong S^2 \times D^2$, so we have two 0-framed unknots, and since the 4-handle $(D_+ \times D_+)$ is uniquely attached, we just need to find how these are knotted. Note that the attaching sphere of the 2-handles are the circles $S^1 \times \{0\}$ and $\{0\} \times S^1$ in $\partial(D_- \times D_-) = S^3$, and after an isotopy we see that $S^1 \times \{0\}$ is a meridian of $\{0\} \times S^1$. On the boundary, this corresponds to the usual decomposition of S^3 as two solid tori and so a Kirby diagram for $S^2 \times S^2$ is given by a 0-framed Hopf link (Figure 2.3.10).

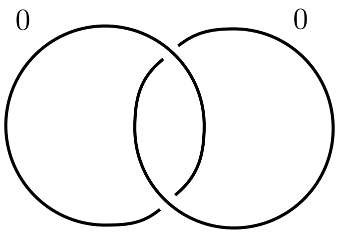

Figure 2.3.10: $S^2 \times S^2$.

Example 2.3.13. For connected manifolds, the boundary sum operation doesn't depend on the choice of disks, so if we have two 4-dimensional handlebodies, we can construct their boundary sum by summing along disks D^4. A Kirby diagram for a boundary sum is then simply a disjoint union of the diagrams for each handlebody. In a similar fashion, if we have closed manifolds, then the connected sum amounts to removing 4-handles and gluing along the remaining boundary. But since 4-handles don't influence the diagram, then a Kirby diagram for a connected sum is a disjoint union of the diagrams of each handlebody.

Example 2.3.14. Let X be a compact, oriented, smooth manifold with boundary. We define the double of X as $DX = X \cup_{id_{\partial X}} \overline{X}$. If X is a handlebody without 3- or 4-handles, then we can get a handle description for DX by gluing X to its dual decomposition. Since DX is always closed, then 3- and 4-handles, obtained from the 1- and 0-handles respectively, will be uniquely attached. For each 2-handle h we obtain a dual 2-handle h' with the roles of the core and cocore reversed, such that we attach h' to ∂X along the belt sphere of h. Now note that the core of h' shares a boundary sphere with the cocore of h, so they glue together to a trivial D^2-bundle over S^2. Thus the dual 2-handles are attached to the belt-sphere of the original 2-handles along 0-framed unkots. In terms of Kirby diagrams, this amounts to adding 0-framed meridians to each 2-handle (Figure 2.3.11).

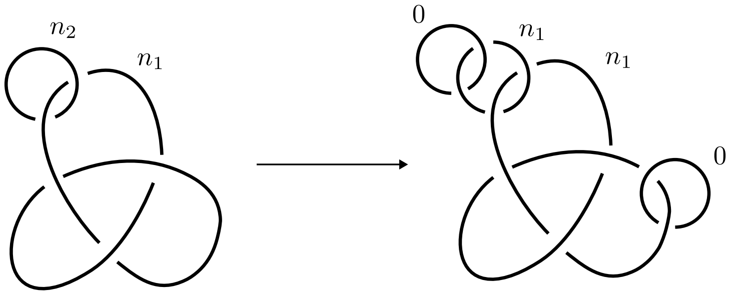

Figure 2.3.11: Doubling a manifold.

2.4 The Intersection form

We now turn our attention to perhaps the most important invariant of any closed 4-manifold - their intersection form.

Definition 2.4.1. Let X be a closed, oriented 4-manifold. Then we define the intersection form Q_X as the bilinear form

$$Q_X : H^2(X;\mathbb{Z}) \times H^2(X;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

$$Q_X(a,b) = (a \cup b)([X])$$

where $[X] \in H_4(X;\mathbb{Z})$ is the fundamental class.

By Poincaré duality we have that $H^2(X) \cong H_2(X)$, so Q_X is also well-defined on $H_2(X;\mathbb{Z}) \times H_2(X;\mathbb{Z})$. Since Q_X is a map to $\mathbb{Z}$ (which has no torsion) then the intersection form must vanish on any torsion elements, hence Q_X descends to a map on $H_2(X;\mathbb{Z})/Tor(H_2(X);\mathbb{Z}) \times H_2(X;\mathbb{Z})/Tor(H_2(X);\mathbb{Z})$. We can then represent Q_X by a matrix, and since the determinant is independent of the choice of basis, we denote it by det Q_X.

Note that the intersection form doesn't make use of the smooth structure of the manifold, so it is also well-defined for topological manifolds. It turns out however, that when we have a smooth manifold X, then $H_2(X)$ will be generated by closed, embedded surfaces in X and the intersection form will gain a more geometric interpretation.

Theorem 2.4.2. *Let X be an oriented, compact, smooth 4-manifold. Then every element of $H_2(X;\mathbb{Z})$ can be represented by a closed, oriented, embedded surface.*

Proof. First assume that there are no 1-handles. Then we know that $H_2(X)$ is generated by the 2-handles and we can find embedded representatives as follows - each 2-handle has an oriented core disk $D = D^2 \times \{0\}$ attached to S^3 along a knot K, and this knot admits a Seifert surface F in $S^3 = \partial D^4$. By gluing along K, we then have that $D \cup_K \overline{F}$ will be an embedded surface representing an element of $H_2(X)$.

Assume now that we have 1-handles. If the 2-handles are attached directly to the boundary of the 0-handle, then by the previous case we are done; otherwise, consider a cycle $z \in H_2(X)$ represented by a collection of oriented core disks of 2-handles $D = \bigcup_k D^2_k \times \{0\}$ with boundary the attaching spheres of the 2-handles, given by an oriented link L in $\partial(\{0\text{-handle}\} \cup \{1\text{-handles}\})$. From the boundary formula we have that $\partial z = \sum_i (B_i \cdot L)h_i$ equals 0 if the belt spheres of the 1-handles intersect L an even number of times with opposite sign, i.e. L has an even number of strands going over each 1-handle, where half has one orientation and the other half the opposite. We can now form an oriented surface $\hat{F}$ by adding bands to the core disks along strands with opposite orientation and the new link L' can now be slid off the 1-handle (Figure 2.4.1). Since L' will now sit in $\partial D^4 = S^3$, then we can find a Seifert surface F for L', and by gluing along L', we then have that $\hat{F} \cup \overline{F}$ will be an embedded generator of $H_2(X)$. Note that we can also have 3-handles, but since we can attach handles in any order, then we can do this procedure before attaching the 3-handles. As a result, the 3-handles will only mod out certain cycles, so the conclusion still remains. □

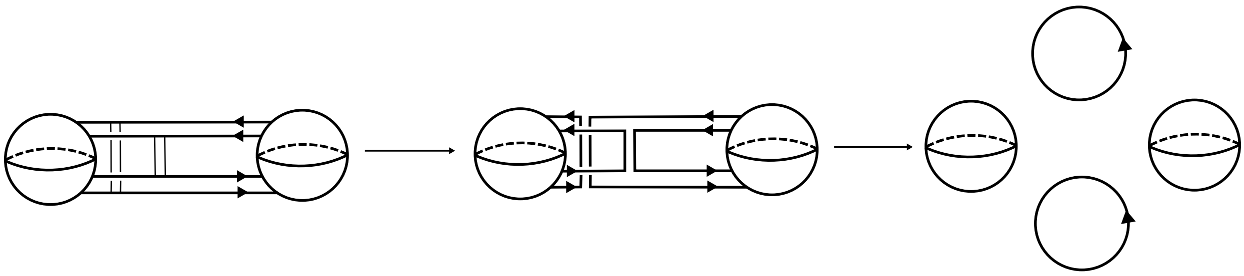

Figure 2.4.1: We can slide the 2-handles off the 1-handles by band-summing the attaching spheres.

For a closed, oriented, smooth 4-manifold X, denote by $PD(a) \in H_2(X)$ the Poincaré dual of an element $a \in H^2(X)$. If we have two cohomology classes $a, b \in H^2(X)$, then by the previous theorem we have surfaces Σ_α and Σ_β representing their Poincaré duals $\alpha = PD(a)$ and $\beta = PD(b)$. We can make Σ_α and Σ_β intersect transversely inside X and using de Rham cohomology it can be shown that $Q_X(a, b) = \Sigma_\alpha \cdot \Sigma_\beta$.

Proposition 2.4.3 (Proposition 1.2.5 of [GS99]). *Let X be a closed, oriented, smooth 4-manifold. Then for $a, b \in H^2(X; \mathbb{Z})$ and $\alpha, \beta \in H_2(X; \mathbb{Z})$ given by the Poincaré duals as before, the intersection form $Q_X(a, b)$ will equal the number of points in $\Sigma_\alpha \cap \Sigma_\beta$ counted with sign.* □

Note that the for each intersection, the sign doesn't depend on the order of the intersection and only on the orientations of the surfaces and X, so if we reverse the orientation of X we get that $Q_{\overline{X}} = -Q_X$. The intersection form also behaves nicely with respect to the connected sum. Indeed if we have two 4-manifolds X_1 and X_2, then taking a connected sum amounts to removing a 4-handle from each and gluing along the remaining boundaries. Note however that the second homology is only affected by 1-, 2- and 3-handles, so it follows that $H_2(X_1 \# X_2; \mathbb{Z}) \cong H_2(X_1; \mathbb{Z}) \oplus H_2(X_2; \mathbb{Z})$ and thus $Q_{X_1 \# X_2} \cong Q_{X_1} \oplus Q_{X_2}$.

Consider a closed, oriented, smooth 4-manifold which can be described by a 2-handlebody. Since we have no 1- or 3-handles, then $H_2(X; \mathbb{Z}) \cong Z^m$, where m is the number of the 2-handles. Theorem 2.4.2

gives us a way to construct a canonical basis for $H_2(X)$ - for each 2-handle we have a knot K_i in S^3 (representing its attaching sphere) which admits a Seifert surface F_i. Now push F_i into the interior of D^4 and glue the core $D^2 \times \{0\}$ along K_i. This will give us an oriented, closed surface $\hat{F}_i$ for each 2-handle and the classes $\alpha_i = [\hat{F}_i]$ represented by the surfaces, will give us a canonical basis for $H_2(X)$.

Proposition 2.4.4. *Let X be a closed, oriented, smooth 4-manifold X which admits a handle decomposition without 1- or 3-handles. Then the matrix of Q_X with respect to the basis α_1, ..., α_m given before, is given by the linking matrix of the link L describing the 2-handlebody.*

Proof. For $i \neq j$ take two surfaces $\hat{F}_i$ and $\hat{F}_j$ representing the classes α_i and α_j. Then using the technique of Example 2.3.10, we push $\hat{F}_j$ deeper into D^4, such that F_i intersects K_j in a collar $S^3 \times I$ of the boundary (F_j starts after the collar, and before we have an annulus $K_j \times I$). Then by Proposition 2.3.4 we have that $Q_X(\alpha_i, \alpha_j) = \hat{F}_i \cdot \hat{F}_j = F_i \cdot K_j = lk(K_i, K_j)$ and by symmetry that $Q_X(\alpha_j, \alpha_i) = lk(K_j, K_i)$. If $i = j$ then we proceed exactly as in Example 2.3.10 - take a parallel copy $\hat{F}_i'$ starting from a parallel copy of the core $D^2 \times \{pt\}$ which will intersect S^3 along a knot K_i' which is a parallel copy of K_i given by the framing. Now replicate the previous scenario to get $Q_X(\alpha_i, \alpha_i) = \hat{F}_i \cdot \hat{F}_i' = \hat{F}_i \cdot K_i' = lk(K_i, K_i')$, which is by definition the framing coefficient of K_i. Combining both cases, we get the linking matrix of L. □

Example 2.4.5. By the previous proposition, D^2-bundles over S^2 with Euler number n, have intersection form (n). In particular $Q_{\mathbb{CP}^2} = (1)$ and $Q_{\overline{\mathbb{CP}^2}} = (-1)$, so since their intersection forms don't agree, then there doesn't exist a orientation reversing self-diffeomorphism of $\mathbb{CP}^2$.

Example 2.4.6. By the previous proposition, the trivial S^2-bundle over S^2 has intersection form $Q_{S^2 \times S^2} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Notice that the handlebody described in Figure 2.4.2 has the same framings and linking number as $S^2 \times S^2$, so it has the same intersection form, but its boundary is a non-trivial homology sphere so it can't be $S^2 \times S^2$. Thus the intersection form doesn't uniquely determine the manifold.

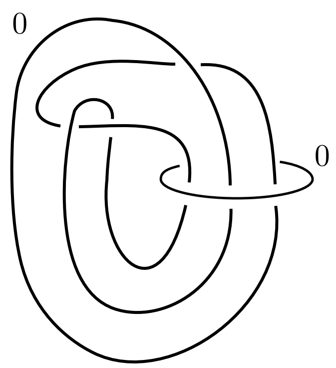

Figure 2.4.2: Manifold with the same intersection form as $S^2 \times S^2$.

Definition 2.4.7. Let Q_X be the intersection form of a closed, oriented, smooth 4-manifold X. Then

(a) the rank of Q_X is the rank of $H^2(X;\mathbb{Z})/Tor(H^2(X;\mathbb{Z}))$, i.e. $rk(Q_X) = b_2(X)$ the second Betti number of X.

(b) if we extend and diagonalize Q_X over $\mathbb{R}$, the signature of Q_X is given by $\sigma(Q_X) = b_2^+(X) - b_2^-(X)$, where $b_2^\pm$ are the number of positive or negative eigenvalues of Q_X.

(c) Q_X is positive (negative) definite if $Q_X(\alpha, \alpha) > 0$ ($Q_X(\alpha, \alpha) < 0$) for all non-zero $\alpha \in H_2(X; \mathbb{Z})$; and it is indefinite otherwise.

(d) Q_X is even if $Q_X(\alpha, \alpha)$ is even for all non-zero $\alpha \in H_2(X; \mathbb{Z})$, and it's odd otherwise.

(e) Q_X is unimodular if det $Q_X = \pm 1$.

Note that for a closed 4-manifold X, since the intersection form descends to a map on homology mod torsion, then Q_X is always unimodular (see Exercise 1.2.10 in [Fre82]).

The ultimate goal of any invariant is to be able to distinguish objects as best as possible and we strive for the same with the intersection form. We first try to classify all 4-manifolds, but the following theorems show that this can't be done for arbitrary fundamental group.

Theorem 2.4.8. *(Markov) For every finitely presented group G there is a smooth, closed, oriented 4-manifold X with $\pi_1(X) \cong G$.* □

Proof. Given a finite presentation of G, we can construct an oriented, smooth 4-manifold Y by attaching to D^4 a 1-handle for each generator and a 2-handle realizing each relation, so that $\pi_1(Y) \cong G$. To make Y closed we take its double DY. We attach a $(n-k)$-handle for each k-handle, but since 3- and 4- handles don't affect the fundamental group, we just need to worry about the 2-handles. These are attached along 0-framed meridians of the attaching spheres of the 2-handles in Y, so they don't go over any 1-handles, and thus they don't contribute to the fundamental group. Thus $DY = X$ is a manifold with fundamental group G. □

Theorem 2.4.9. *([Rab58]) There doesn't exist an algorithm which determines whether a finite group presention is trivial.*

We thus focus on the case of a simply connected, closed 4-manifold X. Since $\pi_1(X) = 0$, then by Hurewicz's theorem we have that $H_1(X; \mathbb{Z}) = 0$ and thus by the universal coefficient theorem we have that $H^1(X; \mathbb{Z}) \cong Hom(H_1(X), \mathbb{Z}) \oplus Ext(H_0(X), \mathbb{Z}) = 0$. By Poincaré duality, it then follows that $H_3(X) \cong H^3(X) = 0$. For the second homology, using the universal coefficient theorem we have that $H^2(X; \mathbb{Z}) \cong Hom(H_2(X), \mathbb{Z}) \oplus Ext(H_1(X), \mathbb{Z}) \cong Hom(H_2(X), \mathbb{Z})$ and by Poincaré duality we conclude that $H_2(X; \mathbb{Z}) \cong Hom(H_2(X), \mathbb{Z})$, so we have no torsion. The intersection form then contains all the homological information of X. Much work has been done in trying to classify simply connected 4-manifolds by their intersection form, and the first major result comes in the form of the following theorem.

Theorem 2.4.10 (Whitehead - Theorem $1.2.25$ of [GS99])**.** *Let X_1 and X_2 be simply connected, closed, topological 4-manifolds. Then X_1 and X_2 are homotopy equivalent if and only if $Q_{X_1} \cong Q_{X_2}$.* □

It turns out that this result can be improved for topological manifolds. The proof is due to Freedman and it earned him a Fields medal.

Theorem 2.4.11 ([Fre82]). *For every unimodular, symmetric, bilinear form Q there exists a simply connected, closed, topological 4-manifold X such that $Q_X \cong Q$. If Q is even, then this manifold is unique up to homeomorphism. If Q is odd, then there are exactly two different homeomorphism types of manifolds with the given intersection form and at most one of these homeomorphism types carries a smooth structure.* □

Thus for a simply connected, closed, smooth 4-manifold its homeomorphism type is completely determined by its intersection form. Furthermore, since S^4 is such a manifold, then the topological 4-dimensional Poincaré conjecture follows.

Corollary 2.4.12. *Let X be a topological 4-manifold homotopy equivalent to S^4. Then X is homeomorphic to S^4.* □

The next step is to wonder if we can once again improve upon this, to get some idea on how many differential structures a manifold carries and whether we can distinguish these using the intersection form. The following result was established by Donaldson, and once again, earned him a Fields medal.

Theorem 2.4.13. *([Don83]) Let X be a simply connected, closed, smooth 4-manifold. If Q_X is positive (or negative) definite, then Q_X is diagonalizable over $\mathbb{Z}$. That is Q_X is equivalent to $n(1)$ (or $n(-1)$).* □

2.5 Kirby Calculus

The goal of this section is to see how handle slides and handle cancellations are described in the context of Kirby diagrams. As we saw with Theorem 2.1.18, any two handlebodies that are related by these moves (and isotopies) are diffeomorphic, so in the realm of 4-manifolds, if we can see how these moves affect Kirby diagrams, then we can develop a diagramatic way to construct diffeomorphisms between 4-manifolds.We start by looking at handle slides. If we have two k-handles h_1 and h_2 attached to a manifold X, then a handle slide amounts to taking the attaching sphere of h_1 and isotoping it in $\partial(X \cup h_2)$, through a disk $D^k \times \{0\} \subset \partial(D^k \times D^{h-k}) = \partial h_2$, and getting back to ∂X.

For the case of 1-handle slides in a Kirby diagram, we simply take one of the attaching 3-balls of the 1-handle h_1 and slide it over the other 1-handle by pushing it through one of the 3-balls of h_2 and coming out through the other 3-ball. This is depicted in Figure 2.5.1.

When 2-handles go over the 1-handles, we need to keep track of their framings when doing 1-handle slides, since this amounts to isotoping the 1-handle, which as we've seen in Example 2.3.8 might cause problems, so we postpone the rest of this discussion to Section 2.7. We now focus on 2-handle slides. To illustraste the idea, we start with the 3-dimensional case. Looking at Figure 2.5.2, we note that sliding a 2-handle h_1 over another 2-handle h_2 amounts to, at the level of the core, sliding the core disk of h_1 over a parallel copy of the core disk of h_2. Since the attaching spheres are the boundaries of the cores, then when we slide the core of h_1, this is the same as taking a band-sum of the attaching sphere A_1 with a parallel copy A_2' of the attaching sphere A_2 of h_2. Since $\pi_2(O(1)) \cong 1$, then we don't have to worry about framings, and so the only choice we have is if the band sum agrees with the orientation of the attaching spheres or not.

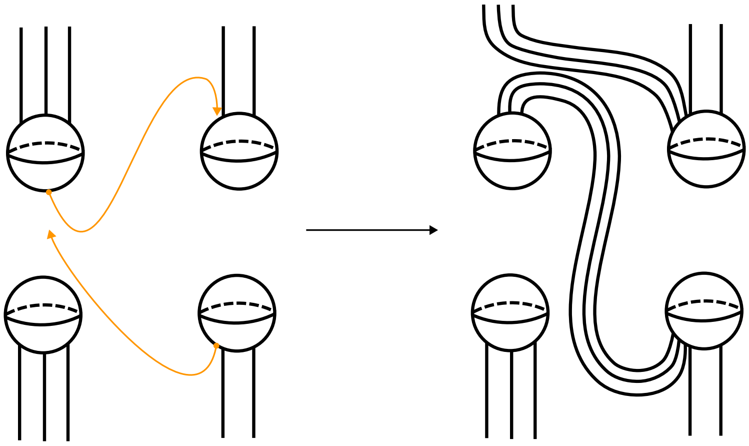

Figure 2.5.1: 1-handle slide in a Kirby diagram.

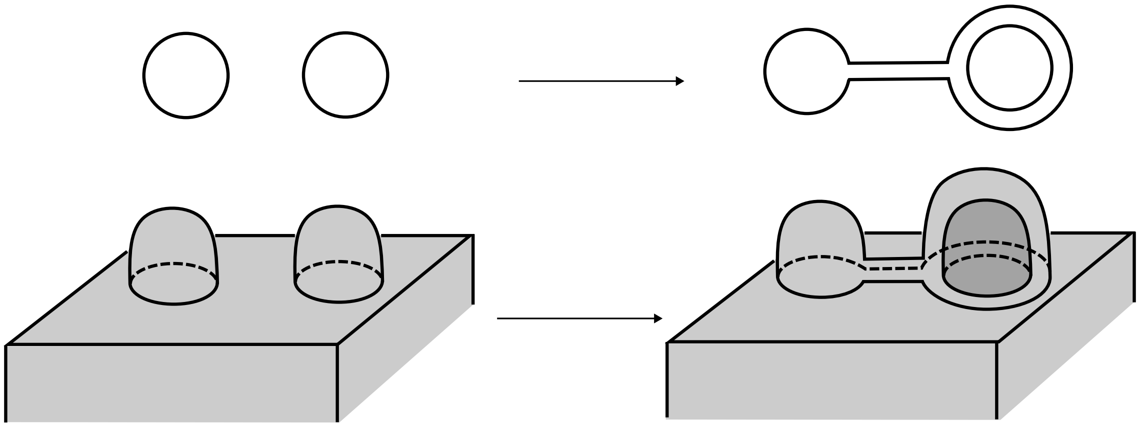

Figure 2.5.2: 3-dimensional 2-handle slide.

The 4-dimensional case follows in a similar fashion. Indeed, if the attaching spheres of the 2-handles are given by framed knots K_1 and K_2 then sliding h_1 over h_2 in a Kirby diagram, amounts to taking a band-sum of K_1 with a parallel copy K_2' determined by the framing on K_2. Note that the handle slide may give different results depending on which band we choose. If K_1 and K_2 are oriented and the band-sum preserves this orientation, then we call it a handle addition; otherwise we call it a handle subtraction (Figure 2.5.3).

It remains to deal with the framing of h_1. A simple way to do this is by using the double-strand notation (Figure 2.5.4) - we specify the framing of a 2-handle by making a parallel copy of K and adding a box labelled n, which corresponds to n twists on the strands going inside the box. To do a handle slide, we just need to slide both strands along two parallel copies of the core disk of h_2, which amounts to taking a band-sum for each strand.

If the 2-handles are attached directly to the 0-handle, then a more convenient way to determine the framing is by using framing coefficients. Indeed for a 2-handlebody represented by an oriented, framed link L, then by Proposition 2.4.4 we have a basis α_1, ..., α_n for $H_2(X;\mathbb{Z})$ for which the intersection form of X is given by the linking matrix of L. If we now slide h_i over h_j, then at the level of $H_2(X;\mathbb{Z})$, this amounts to replacing α_i by $\alpha_i' = \alpha_i \pm \alpha_j$, with the sign depending on whether we are perfoming a handle

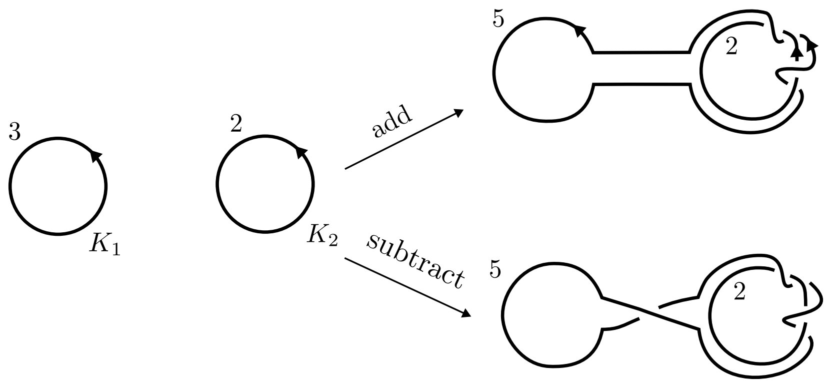

Figure 2.5.3: 2-handle slide on a Kirby diagram.

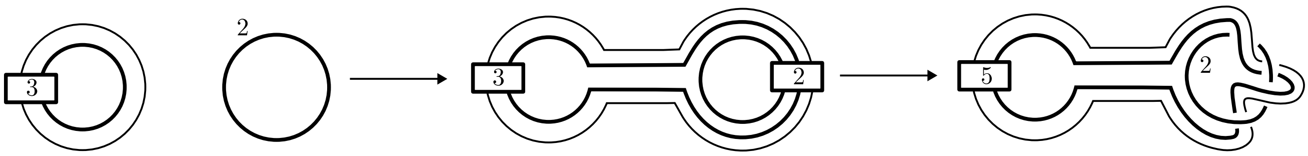

Figure 2.5.4: 2-handle slide on a Kirby diagram using double-strand notation.

addition or subtraction (Figure 2.5.5).

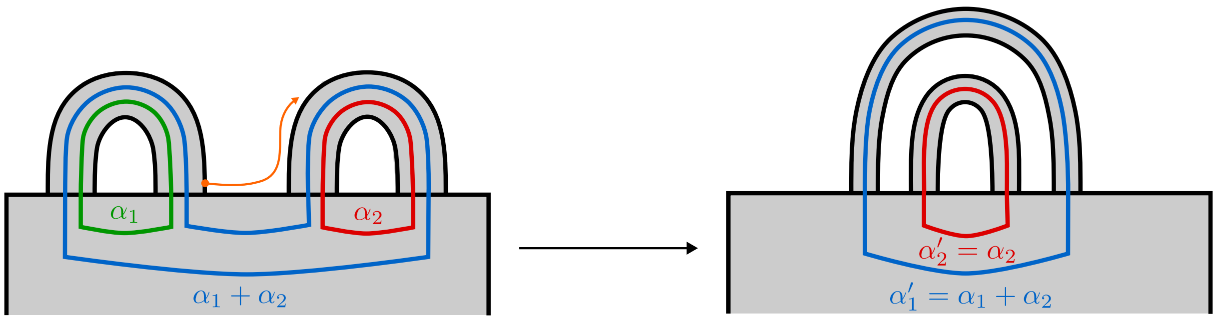

Figure 2.5.5: Effect of a 2-handle slide on a basis of $H_2(X;\mathbb{Z})$.

If we denote by n_k the framing coefficient of K_k, then we can compute the new framing coefficient

$$\begin{aligned}\alpha_i'^2 &= (\alpha_i \pm \alpha_j)^2 = \alpha_i^2 + \alpha_j^2 \pm 2\alpha_i\alpha_j \\ &= n_i + n_j \pm 2lk(K_i, K_j)\end{aligned}$$

Note that changing the orientation of either K_i or K_j results in changing both the linking number and the sign before it, so the framing doesn't depend on the orientation of L. Sometimes it is useful to slide many parallel strands of an attaching sphere of a single 2-handle h_1 over another 2-handle h_2, so instead of sliding each strand individually, we can slide them all at once using the same argument as before - for each strand, we slide over a parallel copy $D^2 \times \{p_i\}$ of the core of h_2, so if we want to slide l strands, we must take l band-sums with parallel copies of K_2 (the attaching sphere of h_2), given by the framing. If we now take the previous basis for $H_2(X;\mathbb{Z})$ and $\alpha_1, \alpha_2 \in H_2(X;\mathbb{Z})$ are classes representing h_1 and h_2 respectively, then what we are doing is replacing α_1 by $\alpha_1' = \alpha_1 + \hat{l}\alpha_2$, where $\hat{l}$ is the number of strands counted with sign. The framing is then given by

$$\begin{aligned}\alpha_1'^2 &= (\alpha_1 + \hat{l}\alpha_2)^2 = \alpha_1^2 + \hat{l}^2\alpha_2^2 + 2\hat{l}\alpha_i\alpha_j \\ &= n_1 + \hat{l}^2 n_2 + 2\hat{l} \cdot lk(K_1, K_2)\end{aligned}$$

where n_1 and n_2 are the framings on K_1 and K_2 respectively.

Example 2.5.1. Every S^2-bundle over S^2 can be realized as the double of a D^2-bundle over S^2. By handle-sliding we can show that there are exactly two different S^2-bundles over S^2. This is depicted in Figure 2.5.6 - from a D^2-bundle over S^2 with Euler number n, we obtain an S^2-bundle over S^2 by adding a 0-framed meridian. By sliding the n-framed unknot over the 0-framed one we change its framing by 2, but the knots still form a Hopf link. We can then conclude that there are two distinct S^2-bundles over S^2 - the one given by the trivial bundle $S^2 \times S^2$ as in Example 2.3.12 (representing the class with even n) and the twisted bundle $S^2 \tilde{\times} S^2$ given by a Hopf link with a 0-framed unknot and a 1-framed unknot (representing the class with odd n).

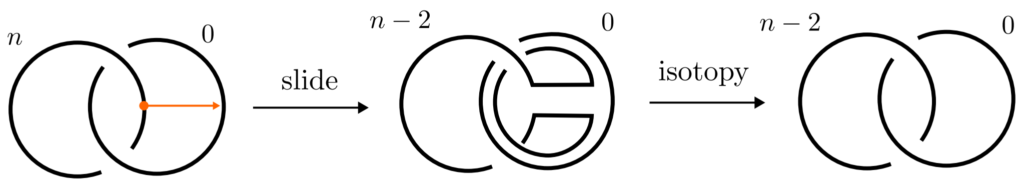

Figure 2.5.6: S^2-bundles over S^2 are determined by the parity of their framing.

Example 2.5.2. The twisted sphere bundle $S^2 \tilde{\times} S^2$ is diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ (Figure 2.5.7).

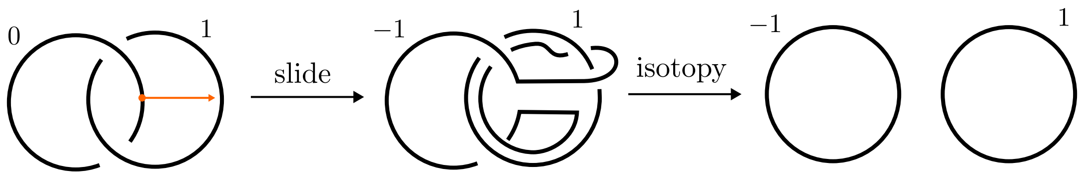

Figure 2.5.7: Diffeomorphism between $S^2 \tilde{\times} S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.

Definition 2.5.3. Let X be a smooth 4-manifold. Then the operation of taking a connected sum of X with $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$ is called blowing up X. The reverse operation is called blowing down.

In terms of Kirby diagrams, taking a connected sum with $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$ amounts to simply adding a ± 1-framed unknot to the diagram, so blowing up is particularly simple in this setting. This description is not especially useful though. A more convenient way of describing blow-ups in this context, is to take a bunch of strands corresponding to parts of attaching spheres, and adding a full ± 1 twist in the bundle, along with a ± 1-framed unknot linking it. Figure 2.5.8 gives a diagram of this description, along with a diffeomorphism relating it to the first one.

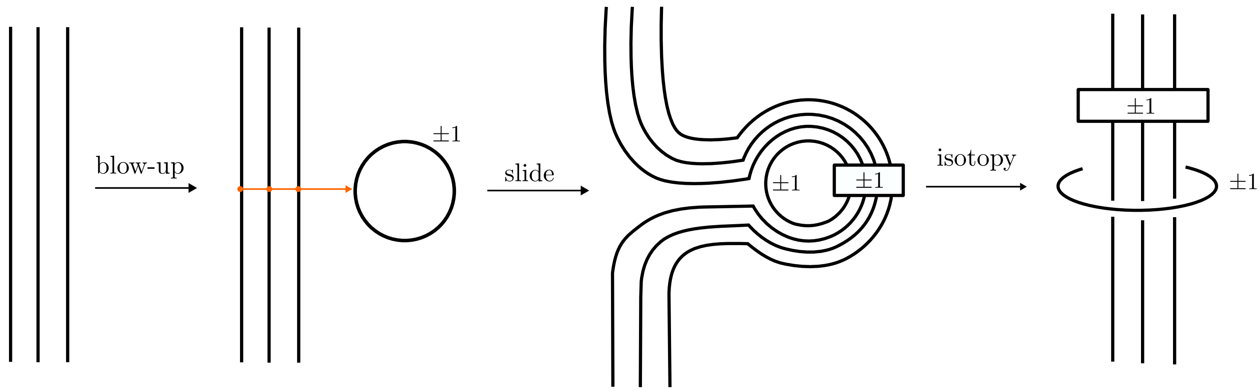

Figure 2.5.8: Blowing up a Kirby diagram.

It remains to deal with the framings on the strands. If the strands belong to different attaching spheres, then we isotope each one individually and since the unknot starts unlinked, then the framing on each strand changes by ±1 depending on whether we do a positive or negative blow-up. If the strands all belong to the same attaching sphere, then we can slide them all simultaneously and since no strands link the unknot, then the framing changes by $\pm k^2$, where k is the signed number of strands.

We can reverse this process to obtain a blow-down. Indeed, if we have a ±1-framed unknot K linking a bunch of strands, then we can separate it by applying a ∓1 twist to these strands, and then finish the blow-down by completely removing the unknot (representing a $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$) from the diagram. This is depicted in Figure 2.5.9. If each strand K_i belongs to a different attaching sphere, then their framing changes by $\pm1 \mp 2lk(K_i, K) = \mp1$ since $lk(K_i, K) = 1$. If all the strands belong to the same attaching sphere, then the framing changes by $\pm k_i^2 + 2k_i lk(K_i, K)$ but $lk(K_i, K) = \mp k_i$, since each strand contributes with ∓1 to the linking number, so the framing changes by $\mp lk(K_i, K)^2$.

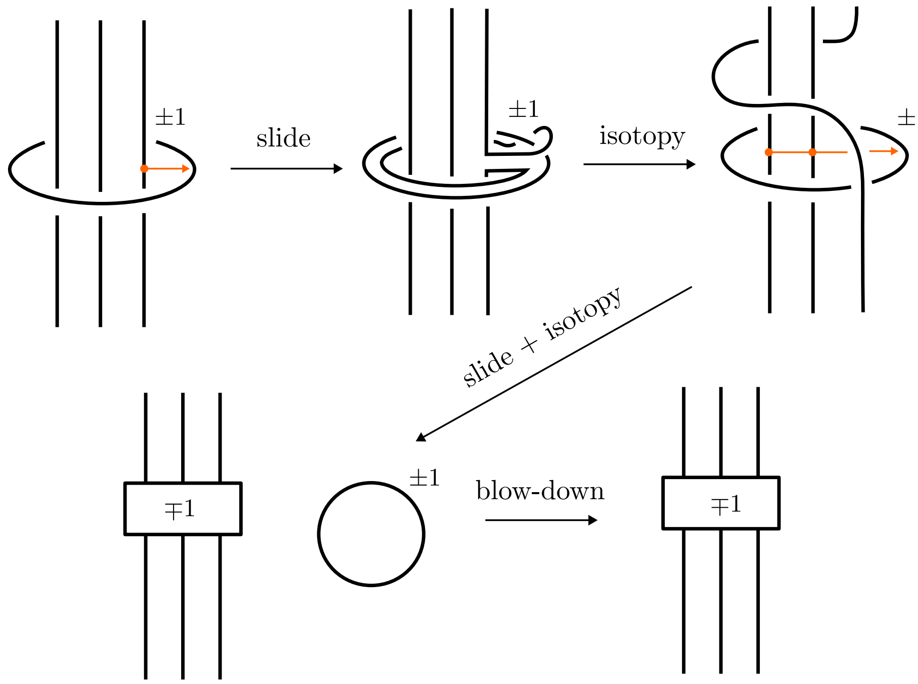

Figure 2.5.9: Blowing down a Kirby diagram.

Note that since $\mathbb{CP}^2$ is closed, then blowing up or down any 4-manifold has no effect on the boundary. So if we are only interested in the 3-manifold bounding our 4-manifold, then this operation is particularly helpful since it doesn't change the diffeomorphism type of the boundary. As we will see in Section 2.6, it turns out that any closed, oriented, connected 3-manifold M can be realized as the boundary of a 4-manifold and as a consequence M can be described by a framed link. In this setting, if we have two framed links L and L' representing diagrams for two diffeomorphic 3-manifolds, it turns out that L can be transformed into L' by blowing up and down.

Example 2.5.4. We can reverse a crossing in a knot by blowing up (Figure 2.5.10). There is no change to the framing since the blow-up was done in a section where there was a strand going up and another going down. Note that this can also be done for the opposite crossing and orientations. Another useful application of blow-ups is in undoing clasps as in Figure 2.5.11.

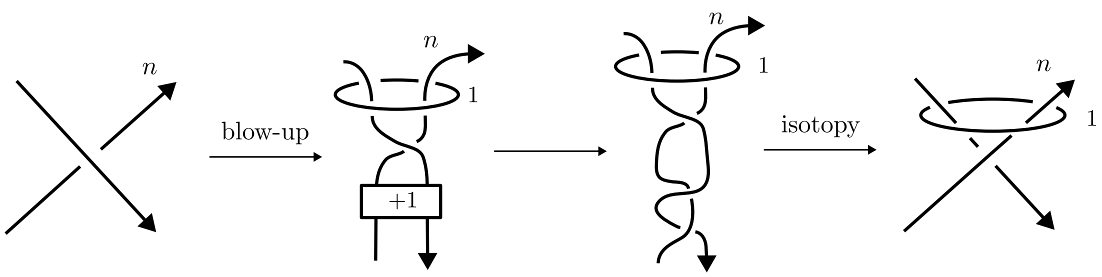

Figure 2.5.10: Blowing up to reverse a crossing.

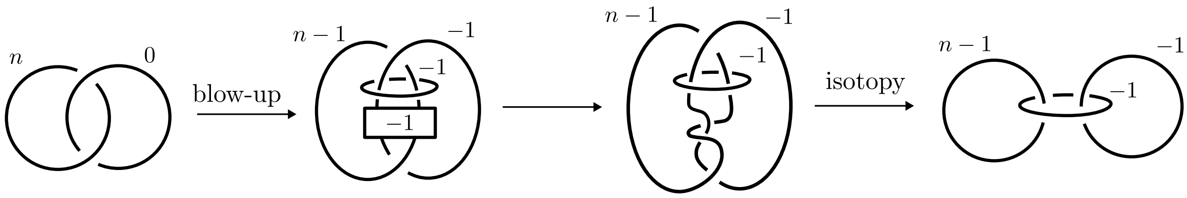

Figure 2.5.11: Blowing up to undo a clasp.

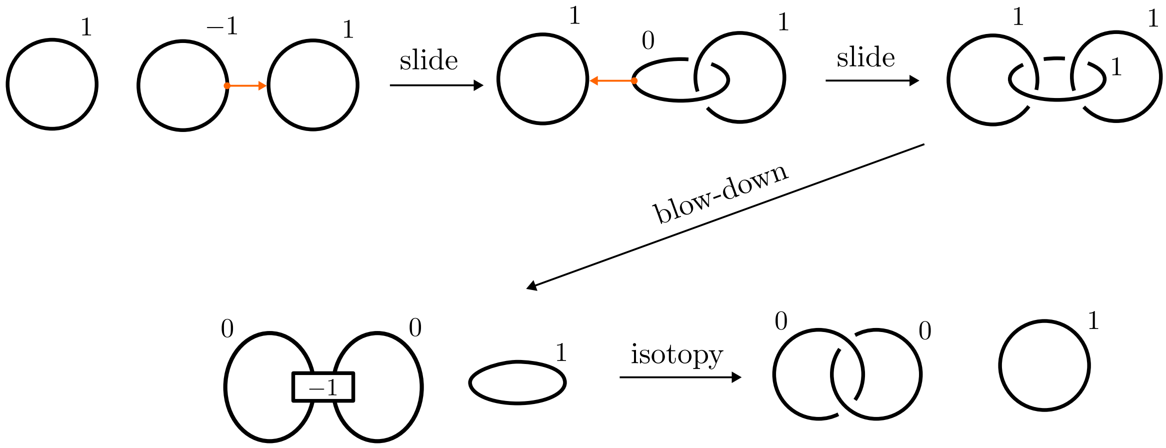

Figure 2.5.12: Diffeomorphism between $\overline{\mathbb{CP}^2}\#2\mathbb{CP}^2$ and $S^2 \times S^2\#\mathbb{CP}^2$.

Example 2.5.5. By reversing the previous example, we can prove that $\overline{\mathbb{CP}^2}\#2\mathbb{CP}^2 \cong S^2 \times S^2\#\mathbb{CP}^2$ (Figure 2.5.12). We can use this, along with Example 2.5.2, to prove that the two distinct sphere bundles become diffeomorphic after taking a connected sum with $\mathbb{CP}^2$. Indeed, $S^2 \times S^2\#\mathbb{CP}^2 \cong \overline{\mathbb{CP}^2}\#2\mathbb{CP}^2 \cong S^2 \tilde{\times} S^2\#\mathbb{CP}^2$.

We now turn our attention to handle cancellations. If we have a $(k-1)$-handle and a k-handle, then by Proposition 2.1.15 we can cancel them if the attaching sphere of the k-handle intersects the belt sphere of the $(k-1)$-handle in exactly one point. Since we are working with handlebodies with a single 0-handle, we have no way to attach a cancelling 1-handle (since the attaching sphere is disconnected), so we don't need to worry when $k = 1$. The case $k = 2$ corresponds to a cancelling $1/2$-handle pair and in terms of Kirby diagrams, is depicted in Figure 2.5.13.

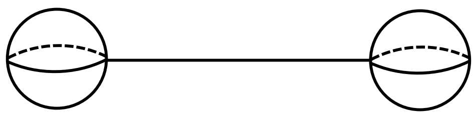

Figure 2.5.13: Model for a cancelling $1/2$-handle pair.

Note that any cancelling $1/2$-handle pair can be made to look like Figure 2.5.13. Indeed, we have three ways in which a cancelling $1/2$-handle pair can differ from the figure - either the 2-handle is knotted, or the 2-handle runs over more than one 1-handles or there are other 2-handles running over the 1-handle. In the first situation we can unknot the 2-handle by simply isotoping one of the attaching balls of the 1-handle through the knotted part of the diagram of the 2-handle (think about pushing the ball through the 2-handle). Since the framing doesn't matter when cancelling, then we are done. The second situation can be solved by sliding the cancelling 1-handle over the other 1-handles (Figure 2.5.14). The last case can be resolved by sliding the 2-handles over the cancelling 2-handle, so that they don't go over the 1-handle (Figure 2.5.15).

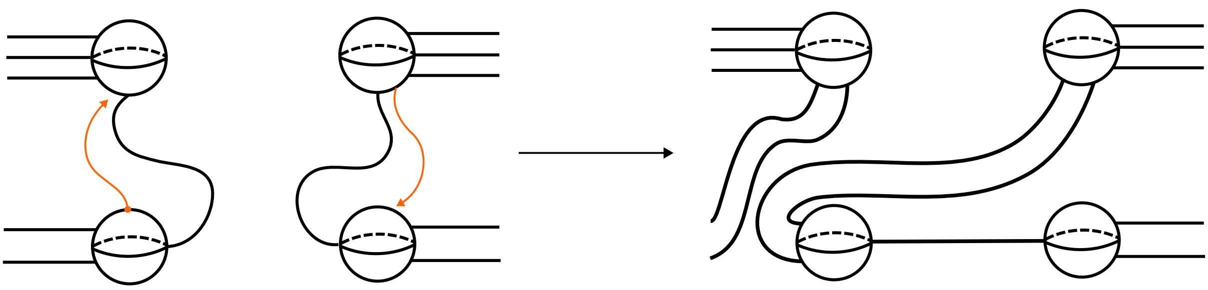

Figure 2.5.14: Sliding 2-handle off of the other 1-handles.

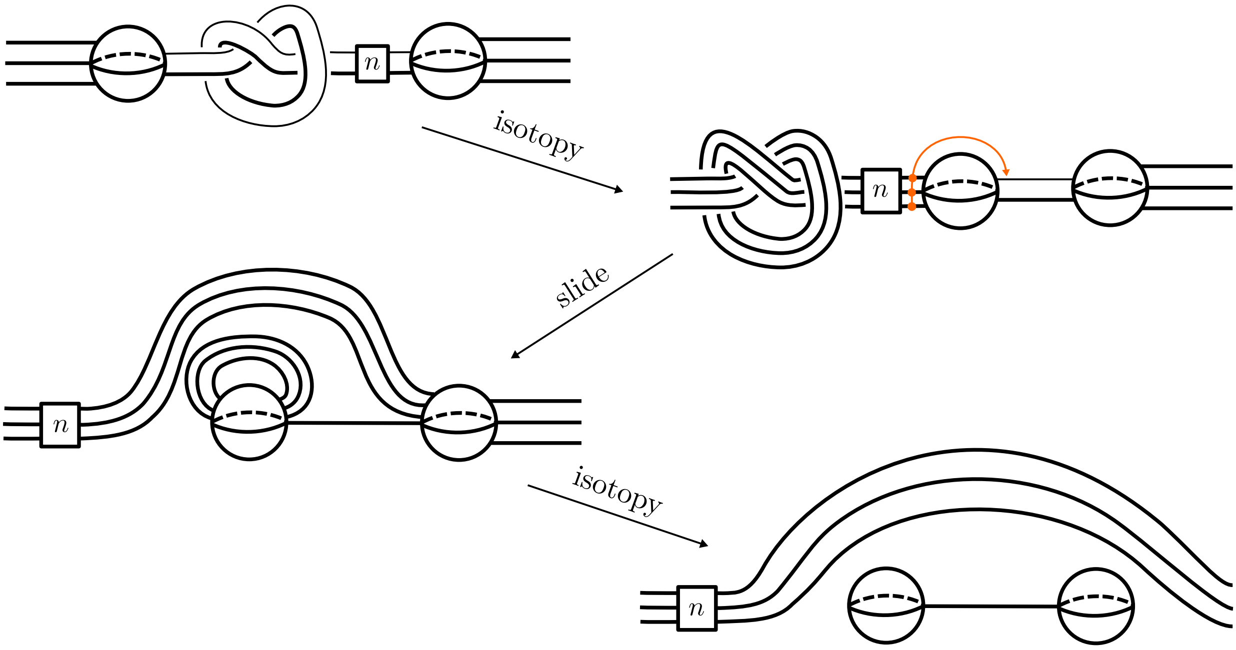

Figure 2.5.15: Sliding extra 2-handles off of the 1-handle.

Consider now the case of a cancelling $2/3$-handle pair. As with the previous case, if our handlebody has a cancelling $2/3$-handle pair it turns out we can do handle slides and isotopies so that we end up in the situation depicted by Figure 2.5.16, where we can remove both handles from the diagram. In this situation, the attaching sphere of the 3-handle is a $S^2 \times \{0\}$ glued to the boundary of $D^4 \cup 2$-handle along an $S^2 \times \{p\} \subset S^2 \times S^1 = \partial(S^2 \times D^2)$. In the diagram, the belt sphere of the 2-handle will be a meridian $\{p\} \times \partial D^2$ of the unknot, so it will intersect the attaching sphere of the 3-handle geometrically once, as required.

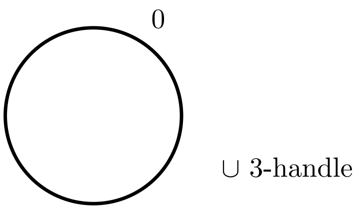

Figure 2.5.16: Model for a cancelling $2/3$-handle pair.

We now delve deeper into the previous remark. Suppose that X is a closed, oriented 4-manifold with a cancelling $2/3$-handle pair. Without loss of generality, we can assume that the attaching sphere A_3 of the 3-handle h_3 intersects the 2-handle h_2 along a disk $D^2 \times \{p\} \subset D^2 \times \partial D^2 = \partial h_2$. If we denote by Y the manifold obtained from the union of all handles of index lower than 3 except h_2, that is $Y = D^4 \cup \{1\text{-handles}\} \cup \{2\text{-handles} - h_2\}$, then $D = A_3 - h_2$ will be an embedded disk in ∂Y (compare with Proposition 2.1.15). Note that D might go over other 1- and 2-handles, but we can always find a tubular neighborhood $\nu\partial D$ of its boundary that contains the attaching sphere K of h_2. We can then think of D as being connected to h, and by handle-sliding K we can drag D to ∂D^4, where it will be a spanning disk for K (more details in Section 5.1 of [GS99]). Now K will be a 0-framed unknot in S^3, which looks like Figure 2.5.16 and we can cancel it with the 3-handle.

On the other hand, if we have a diagram representing X with a 2-handle h attached along an isolated 0-framed unknot, then we can always cancel it with a 3-handle. Indeed since X is closed and oriented then $\partial(Y \cup h) \cong \#mS^1 \times S^2$ and h represents one of these $S^1 \times S^2$. By Theorem A.0.5 the summands are uniquely determined, so it follows that $\partial Y \cong \#(m-1)S^1 \times S^2$. We can then cancel h with a 3-handle and then add the other 3- and 4-handles, since we always have the correct boundary.

2.6 Dehn Surgery

In this section we will discuss a general technique used to modify manifolds which goes by the name of surgery. There are many different types of surgery but the main idea is that we remove a piece of the manifold and glue in another piece by a homeomorphism of the remaining boundary. There is a deep relation between surgery and handle attachment which will provide us with useful techniques for working with both 4-manifolds and their boundaries.

Definition 2.6.1. Let M be a compact, smooth n-manifold and $\varphi\colon S^k \times D^{n-k} \to M$ be an embedding into M. The procedure of removing $\varphi(S^k \times \text{ int } D^{n-k})$ and gluing in $D^{k+1} \times S^{n-k-1}$ along $S^k \times S^{n-k-1}$ is called a surgery on φ.

As before in the case of handle attachments, the result of a surgery (up to diffeomorphism) is completely determined by the isotopy class of the embedding $\varphi_0\colon S^k \times \{0\} \to M$ along with a framing f on the normal bundle of $\varphi_0(S^k)$. In fact, there are many similarities between handle attachment and surgery - if M is an n-manifold with boundary, then attaching a $(k+1)$-handle to M is the same as doing a k-surgery (surgery along a k-sphere) on ∂M. Attaching a $(k+1)$-handle h to M alters the boundary as $\partial(M \cup h) = (S^k \times \text{ int } D^{n-k-1}) \cup_{S^k \times S^{n-k-2}} D^{k+1} \times S^{n-k-2}$, which coincides with doing a k-surgery on ∂M (we have an extra -1 factor since we are doing surgery on ∂M which is an $(n-1)$-manifold), assuming that the embedded sphere S^k extends to the same isotopy class of framed embeddings in both cases. We can make use of this to relate any surgeries by a cobordism. Indeed if we take $W = M \times I$ and attach a $(k+1)$-handle to $M \times \{1\}$, then W represents a cobordism between $\overline{M}$ and the manifold obtained from M via a k-surgery. Note that a k-surgery can be reversed to obtain M again - we can find an embedded $D^{k+1} \times S^{n-k-1}$ on the manifold obtained from a k-surgery, which corresponds to a tubular neighborhood of the belt sphere of the corresponding $(k+1)$-handle attachment, and by doing surgery on this S^{n-k-1} we re-obtain M. What we are really doing is dualizing the handlebody $M \times I \cup_{\varphi \times \{1\}} h$ (which can be thought of as attaching a $(n-k-1)$-handle to the other end of the cobordism) so we again obtain a cobordism reversing the surgery.

We are mostly interested in surgery on 3-manifolds since these correspond to the boundary of 4-manifolds constructed by attaching handles. Of particular interest to us, is a generalization of surgery on circles, which goes by the name of Dehn surgery or (rational) surgery, when this generates no confusion.

Definition 2.6.2. Let K be a knot embedded in a closed, oriented 3-manifold, along with a closed tubular neighborhood $\nu K \cong S^1 \times D^2$. The procedure of removing int νK and gluing in $D^2 \times S^1$ by a homeomorphism of the boundary $S^1 \times S^1$ is called a Dehn surgery on K.

The group of self-homeomorphisms of $S^1 \times S^1$ correponds to $GL(2,\mathbb{Z})$ (see Secion 2.D of [Rol03]) but it turns out that we only need to keep track of a meridian of the gluing solid torus. Indeed, notice that $S^1 \times D^2$ has a handle decomposition given by a 0-handle and a 1-handle, so by dualizing the handlebody we get a decomposition given by a 2-handle and a 3-handle glued to $S^1 \times S^1$. Now, if we take this $S^1 \times S^1$ to be the boundary component of $M - \nu K$, then the attaching sphere of the 2-handle determines the gluing, since the 3-handle is uniquely attached. The attaching sphere is determined by its homology class in $\alpha \in H_1(\partial\nu K;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Note that $H_1(\partial\nu K;\mathbb{Z})$ is generated by a meridian and a longitude of the torus, so as before we get a meridian μ by the right-hand rule and we can find a longitude λ by giving a parallel copy of K (if $M = S^3$ then we can use the 0-framing). Note that since we are working over $\mathbb{Z}$, then the matrix $A = \left(\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right)$ corresponding to a self-homeomorphism of the boundary torus, must have $ps - qr = \pm 1$. If (μ, λ) is a basis for $H_1(\partial\nu K;\mathbb{Z})$, then the homology class of the attaching sphere must be of the form $\alpha = p\mu + q\lambda$ with p and q coprime (up to sign). We can solve this sign issue by noting that changing the orientation of K or α changes both the signs of μ and λ (hence p and q) but doesn't change the surgery, so we can take the matrices to have determinant 1. Since we are mainly interested in working in the boundary of the 0-handle, then for $M = S^3$, a knot K along with integers p and q completely determine the surgery and we call $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$ the surgery coefficient (where $\frac{p}{0} = \infty$ corresponds to the trivial surgery since we are simply regluing the solid torus in the same way). We use $S^3_{\frac{p}{q}}(K)$ to denote the Dehn surgery on K (in S^3) with coefficient $\frac{p}{q}$. This notion extends to links by doing the surgeries one at a time for each component, but now we need to specify a fraction $\frac{p_i}{q_i}$ for each link component L_i. Note that attaching a 2-handle to D^4 along a knot K leads to an integral Dehn surgery ($q = \pm 1$) on the boundary, where now the α is given by a parallel copy of K determined by the framing and thus the surgery coefficient equals the framing coefficient.

Example 2.6.3. Note that S^3 admits a decomposition as a union of two solid tori glued along their boundaries. The gluing sends the meridian of one of the solid torus to the longitude of the other, so if we do a Dehn surgery with coefficient 0 on an unknot U, then what we're doing is removing one of the solid torus and regluing it back, but this time sending a meridian to a meridian. This corresponds to gluing in $D^2 \times S^1$ along the boundary component $S^1 \times \{p\} \subset \partial D^2 \times S^1$ so we conclude that $S^3_0(U) = S^2 \times S^1$. We can check that this is in fact correct, since attaching a 2-handle to D^4 along a 0-framed unknot has boundary $\partial(S^2 \times D^2) = S^2 \times S^1$.

Example 2.6.4. We define the Lens space $L(p,q)$ as the Dehn surgery on the unknot with coefficient $-\frac{p}{q}$. In particular $L(p,1)$ is the boundary of a D^2-bundle over S^2 with Euler number n (Example 2.3.10), $L(0,1) = S^2 \times S^1$ is our previous example, and $L(\pm 1, 1) = S^3$ which is the boundary of $\mathbb{CP}^2 -$ int D^4 and $\overline{\mathbb{CP}^2} -$ int D^4 respectively (D^2-bundles over S^2 with Euler number ± 1).

As we've discussed before, attaching a 2-handle to D^4 leads to an integral surgery on the boundary. So if we wish to realize a 3-manifold by surgery, we might look at the boundary of a 4-manifold. For closed, oriented 3-manifolds, this can always be done.

Theorem 2.6.5 (Rokhlin - [Rok51])**.** *Any closed, oriented 3-manifold M bounds a compact, oriented 4-manifold X.* □

Theorem 2.6.6 (Lickorish-Wallace)**.** *Any closed, oriented, connected 3-manifold M is realized by integral surgery on a link L in S^3.*

Proof. By the previous theorem, we can write M as the boundary of a compact, oriented 4-manifold X, which admits a handle decomposition. The union of the 0- and 1-handles is diffeomorphic to $\natural n S^1 \times D^3$ (for n 1-handles), and notice that $\partial(\natural n S^1 \times D^3) = \# n S^1 \times S^2 = \partial(\natural n S^2 \times D^2)$ which corresponds to the boundary of n 0-framed 2-handles attached along unlinked unknots to D^4. We can then replace every 1-handle (by doing surgery on the circles generating the homology of $\natural n S^1 \times D^3$) by a 2-handle without changing the boundary. By dualizing the handlebody, every 3-handle will correspond to a 1-handle and we can do the same procedure. All that remains are 2-handles, so we are done. □

Note that the previous theorem also implies Theorem 2.6.5, since any such 4-manifold can be constructed by attaching 2-handles, along a framed link realizing its boundary, to D^4. For a proof of the previous theorem not relying on Rokhlin's theorem see Section 9.I of [Rol03].

By Theorem 2.6.6 every closed, oriented 3-manifold can be described by a diagram of framed links (called surgery diagram), so as we did with Kirby diagrams, we would like to find a way to relate 3-manifolds by looking at their surgery diagrams. Since every closed, oriented 3-manifold can be realized as the boundary of some compact, oriented 4-manifold, then any diffeomorphism of the underlying 4-manifold (via handle slides, handle cancellation or isotopy) will also restrict to a diffeomorphism on the boundary, so Theorem 2.1.18 gives us some moves that preserve the diffeomorphism type of the boundary. We may also have moves that change the underlying 4-manifold without changing the boundary, for instance blowing up and blowing down. It turns out that these form a complete set of moves (along with isotopies) for surgery diagrams.

Theorem 2.6.7. *(Kirby, Fenn-Rourke - [Kir78]) Let L and L' be two framed links in S^3 describing integral surgery diagrams for diffeomorphic 3-manifolds. Then L can be transformed into L' by blowing up and down (and isotopy) and any orientation-preserving diffeomorphism can be realized in this manner.* □

Since every compact, oriented 3-manifold can be realized via integral surgery, then by the previous theorem blowing up and down is all that we need to relate such manifolds, but sometimes it's convenient to also allow rational coefficients. If we have a surgery diagram with an unknot K linking a bunch of strands K_i, then we can add n twists to the strands going through K as in Figure 2.6.1 - such a move is called a Rolfsen twist. If $r = \frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$ is the surgery coefficient of K, then the Rolfsen twist will change it to $r' = \frac{p}{q+np}$, and for each strand K_i, the surgery coefficient changes from r_i to $r_i' = r_i + n \cdot lk(K_i, K)^2$. A proof can be found in Section 9.H of [Rol03].

Example 2.6.8. If we have an unknot, with coefficient ± 1, linking a bunch of strands, then by doing a ∓ 1 Rolfsen twist we can remove the unknot (Figure 2.6.2). Note that the linking number of the strands changes in the same way as a blow-up, so a blow-up is just a special case of a Rolfsen twist.

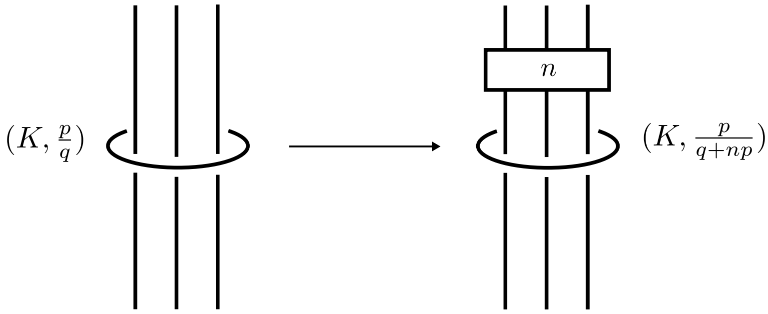

Figure 2.6.1: Rolfsen twist.

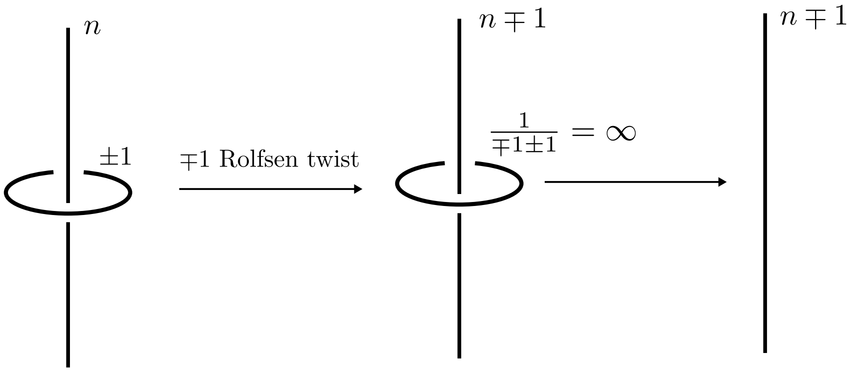

Figure 2.6.2: Blowing-up by Rolfsen twisting.

From the Rolfsen twist, we can derive another move called a slam-dunk. If we have a surgery diagram where a knot K, with rational coefficient $r = \frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$, is a meridian of another knot K_1 with integer coefficient $n \in \mathbb{Z}$, then we can remove K and change the surgery coefficient of K_1 to $n' = n - \frac{1}{r}$. This is depicted in Figure 2.6.3. Although it can be derived from Rolfsen twists, we will prove it directly.

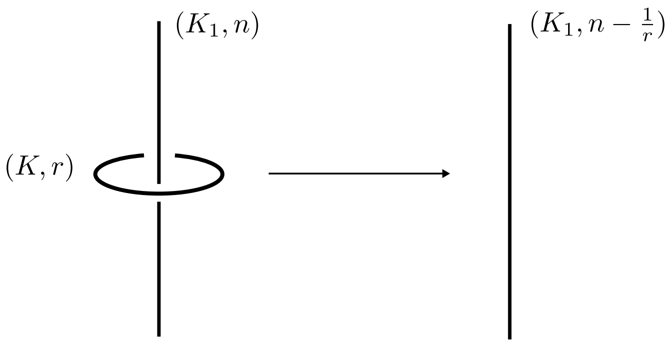

Figure 2.6.3: Slam-dunk.

Proposition 2.6.9. *Let L and L' be links with rational coefficients representing 3-manifolds M and M' respectively. If L' is obtained from L by a slam-dunk then M is diffeomoprhic to M'.*

Proof. If L' is obtained from L by a slam-dunk, then there is a knot K with coefficient $r = \frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$ isotopic to a meridian of a knot K_1 with coefficient $n \in \mathbb{Z}$. We start by doing the surgery on K_1. We remove a tubular neighborhood νK_1 from S^3 and glue in $T = S^1 \times D^2$ by the homeomorphism $\varphi \colon \partial T \to \partial \nu K_1$

given by

$$\varphi = \begin{pmatrix} n & -1 \\ 1 & 0 \end{pmatrix} \qquad\qquad \varphi^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & n \end{pmatrix}$$

Since K is a meridian of K_1, then we can isotope K to the boundary of $S^3 - \text{int}\,\nu K_1$ so that under the inverse homeomorphism φ^{-1} we have

$$\varphi^{-1}(K) = \begin{pmatrix} 0 & 1 \\ -1 & n \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

which means that $\varphi^{-1}(K)$ is isotopic to a longitude of T which doesn't knot (intersects a disk $D^2 \times \{pt\}$ in T exactly once). If we now identify K with $\varphi^{-1}(K)$ and take a tubular neighborhood $\nu K \cong S^1 \times D^2$ in T, then we get a diffeomorphism $\nu K \cong T$. It now remains to do the surgery on K, but by the previous observation, this amounts to removing $T \subset S^3_n(K_1)$ and regluing it again by a different homeomorphism of the boundary. It then follows that doing the surgery on K will be the same as doing another surgery on K_1, so we can remove K from the diagram. Since the gluing map for $S^3_r(K)$ is given by $\left(\begin{smallmatrix} p & p' \\ q & q' \end{smallmatrix}\right)$, then to get the new surgery coefficient (given by the slam-dunk) we compose the maps and see how it maps a meridian μ of T

$$\psi(\mu) = \begin{pmatrix} n & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} p & p' \\ q & q' \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} np-q & np'-q' \\ p & p' \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} np-q \\ p \end{pmatrix}$$

thus the surgery coefficient is given by $\frac{np-q}{p} = n - \frac{1}{r}$ and we get the manifold $S^3_{n-\frac{1}{r}} = (K_1)$ as expected.

$\square$

Example 2.6.10. Consider a surgery diagram given by a Hopf link with coefficients $-n$ and $-m$. By slam-dunking as in Figure 2.6.4, we show that this corresponds to the lens space $L(nm-1, m)$. In particular, if $m = 0$ we get $L(-1, 0) = S^3$, so if we double any D^2-bundle over S^2 by adding a 0-framed meridian to an n-framed unknot, then by slam-dunking we can verify that its boundary is indeed S^3.

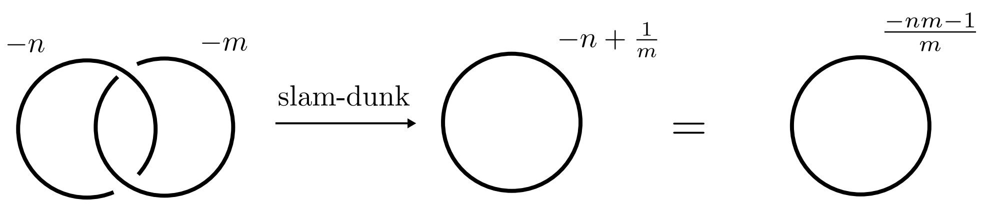

Figure 2.6.4: Slam-dunk on the Hopf link gives $L(nm-1, m)$.

As we've seen, Rolfsen twists contain both blow-ups and slam-dunks, so in particular these form a complete set of moves for integral surgery diagrams. It turns out, that this is also true in the rational case.

Theorem 2.6.11 (Rolfsen - Proposition $5.3.10$ of [GS99]). *Let L and L' be links with rational coefficients in S^3 describing rational surgery diagrams for diffeomorphic 3-manifolds. Then L can be transformed into L' by Rolfsen twists (along with isotopies and inserting and deleting components with coeficient ∞) and any orientation-preserving diffeomorphism can be realized in this manner.*

Proof. (Sketch) If we start with a rational surgery diagram for a link $\tilde{L}$, then by applying the reverse slam-dunk operation we obtain a surgery diagram where $\tilde{L}$ will now have integer coefficient along with a meridian with rational coefficient. By successively repeating this process on the meridians obtained by reverse slam-dunking, we will arrive at an integer surgery diagram equivalent to that of $\tilde{L}$. By now applying this idea to L and L' we will get two integer surgery diagrams, to which we can apply Theorem 2.6.7 to get the desired result. □

Before finishing this section, it remains to consider the homology of a 3-manifold obtained by rational surgery on a link.

Theorem 2.6.12. *Let M be a 3-manifold obtained by rational surgery on a link $L \subset S^3$ composed of knots K_i with coefficient $\frac{p_i}{q_i}$ (for $i = 1$, ..., n). Then $H_1(M;\mathbb{Z})$ is generated by the meridians μ_1, ..., μ_n of the components, with a complete set of relations given by $p_i\mu_i + q_i\sum_{j\neq i} lk(K_i, K_j)\mu_j = 0$ (for $i = 1$, ..., n).*

Proof. Recall from the discussion on the linking number that for a link $L \subset S^3$, the homology of its exterior $H_1(S^3 - L;\mathbb{Z})$ is freely generated by the meridians μ_i of its components. Also, from earlier in this section, we noted that the Dehn surgery on a link, corresponds to attaching 2- and 3-handles to $S^3 -$ int νL, where νL is a tubular neighborhood for L in S^3, so $H_1(M;\mathbb{Z})$ is generated by $H_1(S^3 - L;\mathbb{Z})$ along with the relations given by the 2-handle attachements. In particular, since each K_i has coefficient $\frac{p_i}{q_i}$ and we can define a 0-framed longitude λ_i, then $H_1(M)$ is generated by the meridians μ_i with relations given by $p_i\mu_i + q_i\lambda_i = 0$. Notice however that the 0-framed longitude of each K_i is defined via a Seifert surface F_i, and thus in $S^3 -$ int νL each component K_j ($j \neq i$) will intersect F_i, so we must remove these intersection as in the proof of Proposition 2.3.4. Since each intersection with K_j is removed by adding $\pm\mu_j$ according to the orientation (thus the linking number), then we get the relations $\lambda_i = \sum_{j\neq i} lk(K_i, K_j)\mu_j$ (for $i = 1$,..., n). Replacing λ_i in the previous relations give us the desired result. □

Corollary 2.6.13. *If M is a 3-manifold obtained from a $\frac{p}{q}$-surgery on a knot $K \subset S^3$ then $H_1(M;\mathbb{Z}) \cong \mathbb{Z}_p$ (where $\mathbb{Z}_0$ corresponds to $\mathbb{Z}$).*

Proof. This follows immediately from the previous theorem, since for a manifold M obtained by surgery on a knot K with coefficient $\frac{p}{q}$, we have that $H_1(M)$ is generated by the meridian μ with relation $p\mu = 0$, which corresponds to $\mathbb{Z}_p$. □

The previous theorem also allows us to determine the homology of an integral surgery on the boundary of a 4-dimensional 2-handlebody.

Corollary 2.6.14. *Let X be a 4-manifold which admits a handle decomposition with no 1- or 3-handles. Then any matrix for the intersection form of X will also be a presentation matrix for $H_1(\partial X;\mathbb{Z})$. In particular, $H_1(\partial X;\mathbb{Z})$ is finite if and only if det $Q_X \neq 0$, in which case we have $|H_1(\partial X;\mathbb{Z})| = |$det $Q_X|$*

Proof. Since 2-handles are integrally framed, then the surgery coefficients on the boundary ∂X are also integral, so we can recover the homology of the boundary from Corollary 2.6.13, by setting the q_i to 1. Now by Proposition 2.4.4, the matrix for Q_X with respect to the standard basis (obtained from the cores of the 2-handles) is the linking matrix, which in turn gives a presentaion of $H_1(\partial X)$. Any change of basis in $H_2(X)$ comes from handle moves, and since these have the same effect on the boundary, then they give rise to a change of presentation of $H_1(\partial X)$. $\square$

2.7 Alternative description of 1-handles

As we've seen before in Example 2.3.8, there are problems that arise with the framings of 2-handles that run over 1-handles. The goal of this section is to introduce a new notation, and way to look at 1-handles, that fixes this issue. As a bonus, this new description of 1-handles, will also allow us to extend the discussion of surgery on a link in S^3 (given as the boundary of a 2-handlebody) to the case of a handlebody containing both 1- and 2-handles.

Recall that if we have a 1-handle attached to a manifold X, then we can always find a 2-handle that will cancel it and give us back X. This remark is the main idea behind our new description for 1-handles - we can think of adding a 1-handle as removing the 2-handle that would cancel it. Note that if we have an unknot K in the boundary S^3 of the 0-handle, then we can find a Seifert disk D, push its interior into D^4 and remove a tubular neighborhood νD. We can then construct a diffeomorphism $D^4 - \text{int}\,\nu D \cong S^1 \times D^2 \times I \cong S^1 \times D^3$ which identifies $D^4 - \text{int}\,\nu D$ with $D^4 \cup \{1\text{-handle}\}$. So removing a tubular neighborhood of an unknotted disk embedded in D^4, with boundary in S^3, is the same as attaching a 1-handle to D^4 (This also works for a compact manifold X, by working in a collar $\partial X \times I$ of ∂X). Since the cocore $D^2 \times \{0\}$ of a 2-handle will always be an unknotted disk, then adding a 1-handle is equivalent to removing a tubular neighborhood of the cocore of a cancelling 2-handle.

To draw a diagram for the 1-handle, we start by identifying a 2-handle inside D^4 and find the cocore disk D. The boundary circle of D will be an unknot, which we isotope to ∂D^4, so that D is identified as before (Figure 2.7.1 depicts the 3-dimensional case). This boundary circle will now be visible in our diagram and to identify the 1-handle obtained by removing a tubular neighborhood of D, we add a dot to the circle as in Figure 2.7.2.

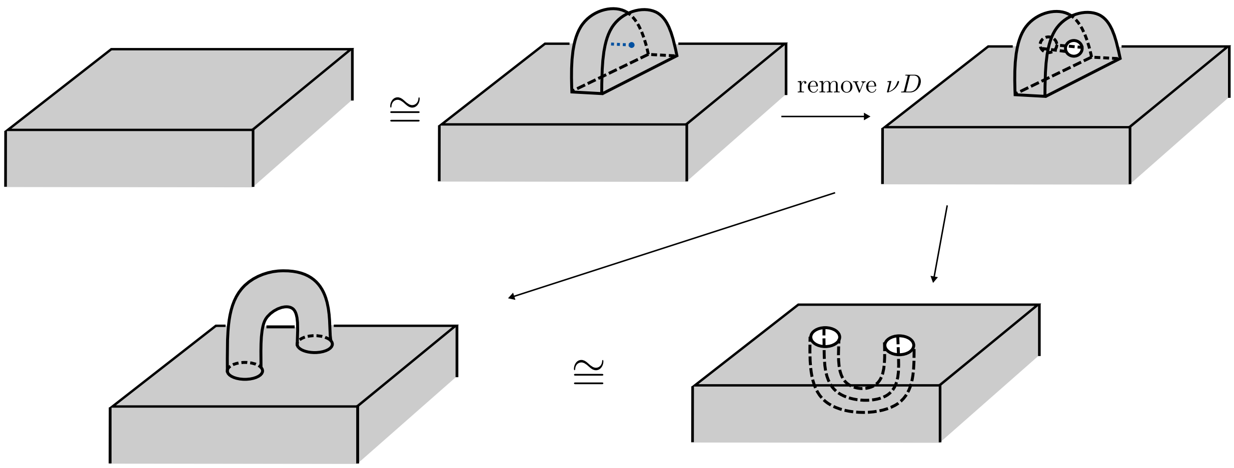

Figure 2.7.1: Removing the cocore of a 2-handle to obtain a 1-handle.

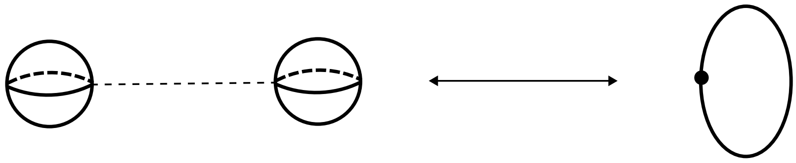

Figure 2.7.2: Dotted 1-handle notation.

Note that a cancelling $1/2$-handle pair can always be represented as in Figure 2.5.13, and since the cocore of the 2-handle is depicted by a disk intersecting the attaching sphere exactly once, then its boundary will be the circle depicting our 1-handle and we can convert between notations, by pushing the balls together, as in Figure 2.7.3. Any 2-handle going over the 1-handle will now go through the dotted circle, and since every 2-handle will now be a knot in S^3, then we can properly define its framing coefficient. The problem with framings we had before was due to the fact that our old notation was hiding a slide of a 2-handle going under a 1-handle. In Example 2.3.8, the framing changed because the attaching sphere of the 2-handle crossed the region between the attaching balls of the 1-handle, i.e. in our new notation, the attaching sphere of the 2-handle went through the dotted circle (Figure 2.7.4). While this situation depicts a 2-handle sliding under a 1-handle, we refer to it as a 2-handle sliding over a 1-handle, since it looks exactly like a 2-handle sliding over a 0-framed 2-handle. In fact, note that the handlebody obtained by attaching a 0-framed 2-handle to D^4 is diffeomorphic to $S^2 \times D^2$ and if we do surgery on the $S^2 \times \{0\} \subset S^2 \times D^2$ obtained from pushing a Seifert disk of the attaching sphere into D^4 and gluing the core of the 2-handle, then we get $S^1 \times D^3$, which corresponds to a handlebody with a 1-handle attached to D^4. Since the surgery happens in the interior of the handlebody, then when we slide the 2-handle (with attaching sphere K_1) over a 1-handle, the framing should change in the same way as sliding a 2-handle over a 0-framed unknot K_2, that is the framing changes by $\pm 2lk(K_1, K_2)$.

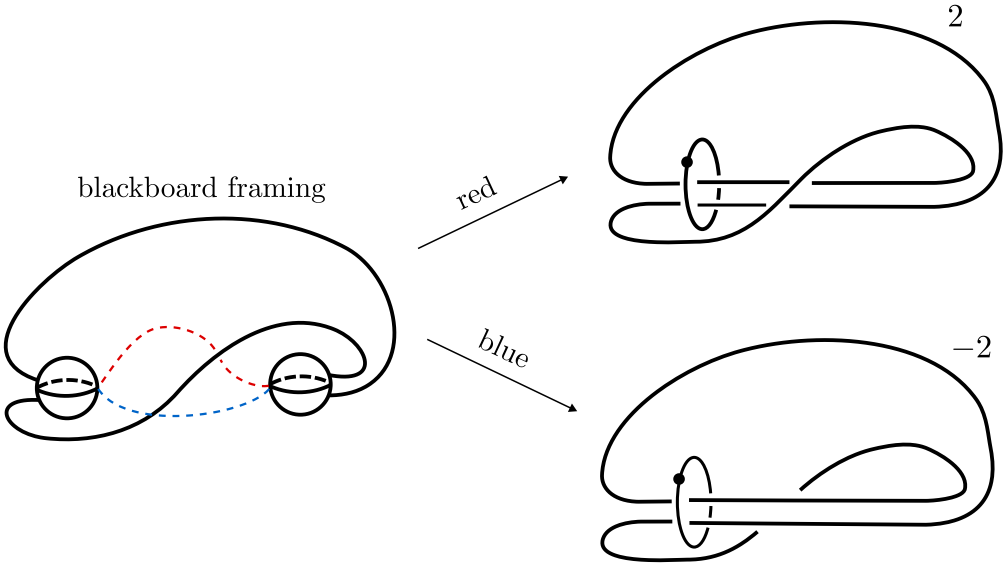

Figure 2.7.3: Switching between 1-handle notations.

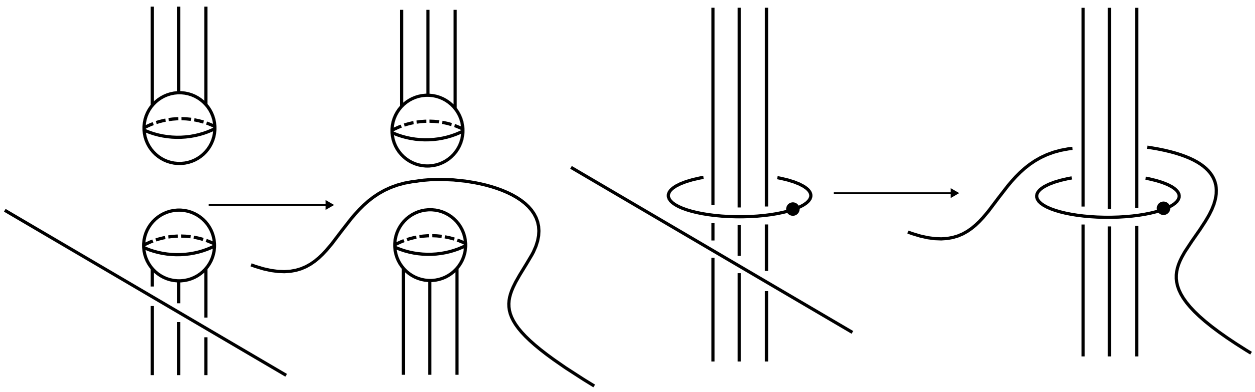

Figure 2.7.4: Sliding a 2-handle under a 1-handle.

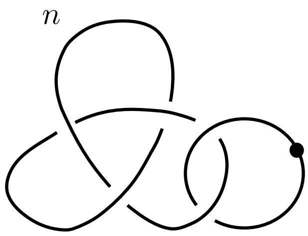

Figure 2.7.5: Cancelling $1/2$-handle pair.

If we want to draw 1-handles in our old notation, we then have to specify an arc between the two balls which tells us how to move the balls together as in Figure 2.7.3. In this sense, different arcs may lead to different framings, such that any 2-handle that crosses the arc will get a framing change induced by sliding such a 2-handle over the dotted 1-handle in our new notation.

We now see how the other handle moves change in this new notation. Cancelling a $2/3$-handle pair remains the same, since they don't interact with the 1-handles. We can cancel a $1/2$-handle pair if the attaching circle of the 2-handle goes through the dotted circle exactly once (Figure 2.7.5). Note that as before, we have to slide any other 2-handles going through the 1-handle. Sliding 2-handles remains the same, except now we are allowed to slide a 2-handle over a 1-handle, as seen before. Sliding a 1-handle over another 1-handle is similar to sliding 0-framed 2-handles, but we have to be careful not to choose a band that will knot the dotted circles, since these must remains unknotted and disjoint (Figure 2.7.6).

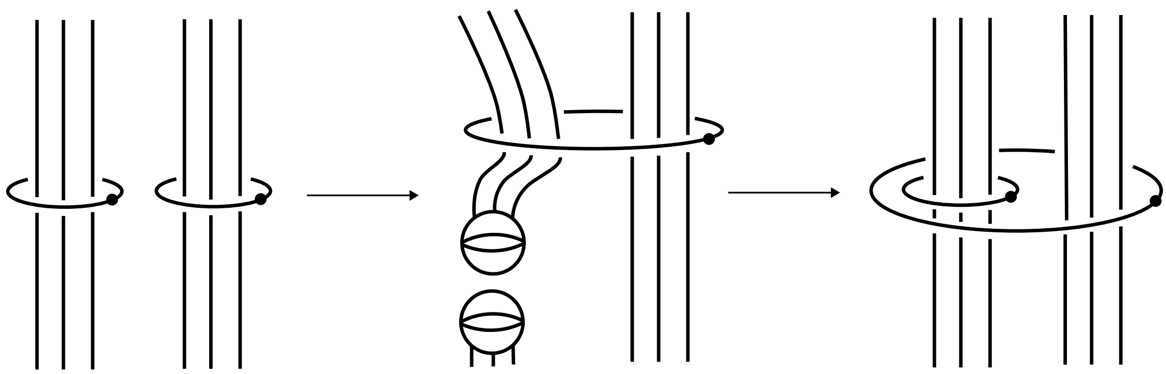

Figure 2.7.6: Sliding 1-handle over another 1-handle.

Finally we can see that this notation offers a clear advantage if we wish to discuss the 3-manifold bounding our 4-manifold X. If we assume there are no 3-handles, then we can simply surger out the 1-handles (replace dotted unknots by 0-framed unknots representing 2-handles) without changing the boundary. Now we have a handlebody with no 1- and 3-handles and thus we can apply Corollary 2.6.14 to get a presentation of $H_1(\partial X;\mathbb{Z})$.

Chapter 3

Khovanov homology and the s-invariant

In this chapter we give an introduction to Khovanov homology along with one of its variants, in the form of Lee homology, with the goal of describing the s-invariant which gives a lower bound on the slice genus of a knot. In Section 3.1 we describe the Khovanov homology of a link. This is an invariant initially described in [Kho00], which categorifies the Jones polynomial. It is obtained by applying a Topological quantum field theory (TQFT) to the cube of resolutions (cube formed by smoothing every crossing) of a link diagram, yielding a bigraded chain complex whose homology is a link invariant. Just a few years later, Lee introduced in [Lee05] another homology theory defined in a similar fashion to Khovanov homology, except this time she applied a different TQFT to the cube of resolutions (Section 3.2). This produced yet another link invariant associated to a bigraded chain complex, which was surprisingly simple when computed on knots - there were only two generators on homology. While it appears that this doesn't have much value, it turns out that the q-grading (non-homological) grading of these generators are invariants of the knot, whose average gives a lower bound on its slice genus. This was the work of Rasmussen in [Ras10], where he defined the aforementioned s-invariant which is the main obstruction tool we will use in this thesis. This is described in Section 3.3. We will give an exposition mostly based on [BN02] along with the previously mentioned articles.

3.1 Cube of resolutions and Khovanov Homology

Let $L \subset S^3$ be an oriented link and D a diagram for L with $n = n_+ + n_-$ crossings, where n_+ denotes the number of positive crossings and n_- the number of negative crossings. Then despite of the orientation on L, there are exactly two different ways to resolve a crossing - the 0- and 1-resolutions as depicted in Figure 3.1.1 (these are sometimes also called smoothings).

There are two choices of resolution at each crossing, so in total there are a 2^n possible resolutions of D, each corresponding to an element $v \in \{0,1\}^n$, i.e. a tuple where the i-th entry has value of either 0 or 1 depending on which type of resolution was done at the i-th crossing (for $i = 1$, ..., n). To each

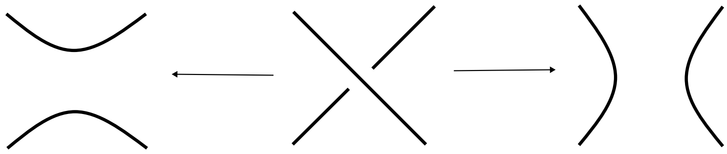

Figure 3.1.1: The two possible resolutions of a crossing.

element v there is an associated a diagram D_v obtained from D by resolving each crossing according to v. Note that D_v will have no crossings, so it will be a collection of simple closed curves in the plane, that is, a closed 1-manifold. We can form an n-dimensional cube by placing a diagram D_v at each vertex v - this is called the cube of resolutions. We depict the cube of resolutions for the trefoil in Figure 3.1.2.

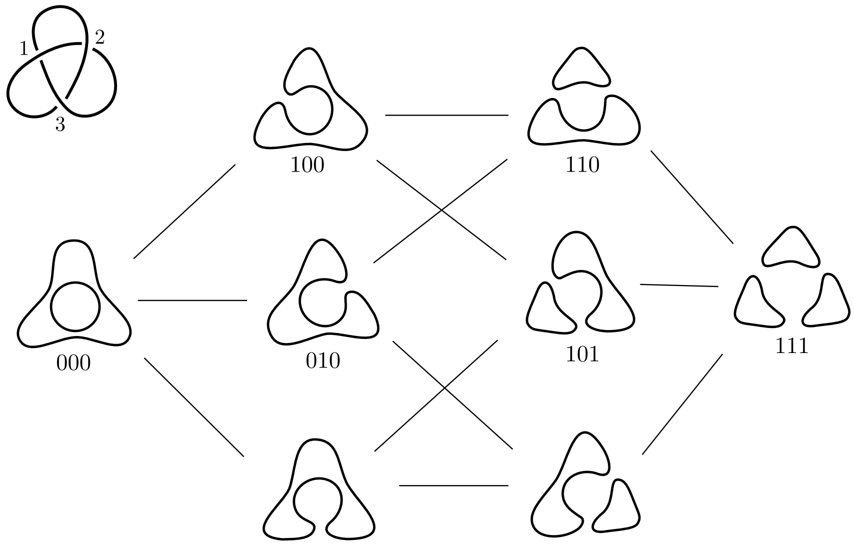

Figure 3.1.2: Cube of resolutions of the Trefoil.

If two resolutions $v_i(0)$ and $v_i(1)$ of D differ only in the i-th entry (the i-th crossing has the 0-resolution for $v_i(0)$ and the 1-resolution for $v_i(1)$) then there will be an edge in the cube of resolutions which connects the vertices $D_{v_i(0)}$ and $D_{v_i(1)}$. We denote this edge by $e_i\colon v_i(0) \to v_i(1)$, which will enduce a map between the diagrams $\Sigma_{e_i}\colon D_{v_i(0)} \to D_{v_i(1)}$. As we've seen before, each diagram D_v is a closed 1-manifold, so the induced map Σ_{e_i} describes a cobordism between 1-manifolds, which is the product cobordism except in a neighborhood of the i-th crossing, where it will be a saddle between the 0- and 1-resolutions.

The cube of resolutions is given by a collection of closed 1-manifolds and cobordisms between them, so the goal is now to apply a TQFT $\mathcal{A}\colon Cob_{1+1} \to Mod_R$ to it in order to obtain a chain complex whose homology will be a link invariant. We start with the vertices - take $R = \mathbb{Q}$ (a $\mathbb{Q}$-module is simply a vector space over $\mathbb{Q}$) and $V = \langle v_-, v_+ \rangle$ to be a 2-dimensional graded vector space generated by v_- and v_+ with gradings $deg(v_+) = 1$ and $deg(v_-) = -1$. Now to each vertex D_v we associate the vector space $\mathcal{A}(D_v) = V^{\otimes k_v}$, where k_v is the number of circles in D_v. Note that we can extend the grading on V to the product $V^{\otimes k_v}$ by $deg(v_1 \otimes v_2 \otimes \ldots \otimes v_{k_v}) = deg(v_1) + deg(v_2) + \ldots + deg(v_{v_k})$. Now that every vertex of the cube has an associated vector space, we define the chain vector spaces as $C^k(D) = \bigoplus_{|v|=k} \mathcal{A}(D_v)$, where $|v|$ is the number of 1s in the coordinates of v (if we look at Figure 3.1.2, what we are really doing is summing along vertical lines). We define the Khovanov complex by applying a homological shift $C^k_{Kh}(D) = C^k[-n_-]$, so that the homological grading is given by $gr(v) = |v| - n_-$.

Now that we have defined the objects, we focus on the differentials. Note that every edge $\Sigma_{e_i} : D_{v_i(0)} \to D_{v_i(1)}$ connects diagrams which differ in their number of circles by 1 - either two circles merge to one or one circle splits into two. For the first situation, $\mathcal{A}(\Sigma_{e_i})$ will correspond to a map $m\colon V \otimes V \to V$ where the vector spaces in the product correspond to the two circles that merge and the image corresponds to the obtained circle. The second situtation follows in a similar fashion, instead now we have a map $\Delta\colon V \to V \otimes V$ where the vector space in the domain corresponds to the circle that splits and the image to the two circles obtained. These maps are explicitly given by

$$\begin{aligned} &m(v_+ \otimes v_+) = v_+ && \Delta(v_+) = v_+ \otimes v_- + v_- \otimes v_+ \\ &m(v_+ \otimes v_-) = m(v_- \otimes v_+) = v_- && \Delta(v_-) = v_- \otimes v_- \\ &m(v_- \otimes v_-) = 0 \end{aligned} \tag{3.1.1}$$

Note that these maps act only on the circles involved in the merge/split, but we can extend this to a product map by acting with the identity on the remaining circles.

While not necessary to construct the complex, if we wish to define a TQFT, then we also need to define the map $\iota\colon \mathbb{Q} \to V$ associated with the appearence of a circle in the diagram, i.e. the attachment of a 0-handle, and the map $\epsilon\colon V \to \mathbb{Q}$ associated with the removal of a circle from the diagram, i.e. the attachment of a 2-handle. These are explicitly given by

$$\begin{aligned} &\epsilon(v_-) = 1 && \iota(1) = v_+ \\ &\epsilon(v_+) = 0 \end{aligned} \tag{3.1.2}$$

Now that we're done with the maps, we get back to the task of defining the differentials. This will come in the form of a sum of m and Δ maps, explicitely given by

$$d^k_{Kh} = \sum_{i=1}^{k} (-1)^{sgn(e_i)} \mathcal{A}(\Sigma_{e_i})$$

where e_i denotes the edge which corresponds to changing the i-th crossing from the 0-resolution to the 1-resolution. The term $(-1)^{sgn(e_i)}$ is a way to distribute signs over each face of the cube so that if the faces commute, then they will anticommute after factoring in the sign term, so that $d^2_{Kh} = 0$ and we get a legitimate differential. A way to do this is for instance to take $sgn(e_i)$ to be the sum of the entries of $v_i(0)$ with index lower than i mod 2 (for example if e_2 maps $(1,0,1)$ to $(1,1,1)$ then $sgn(e_2) = 1$, so $(-1)^{sgn(e_2)} = -1$). With this identification, it just remains to check that the faces actually commute without signs.

Proposition 3.1.1. *The map d_{Kh} is a differential for C_{Kh}, that is $d^2_{Kh} = 0$.*

Proof Sketch. We just need to check that the faces commute. Every face of the cube is given by four vertices $v_{(i,j)}(0,0), v_{(i,j)}(1,0), v_{(i,j)}(0,1), v_{(i,j)}(1,1)$, so we need to check that the two ways of going between $v_{(i,j)}(0,0)$ and $v_{(i,j)}(1,1)$ lead to the same result after applying the TQFT $\mathcal{A}$. We need to check 8

different cases. If one believes our claim that $\mathcal{A}$ is indeed a TQFT, then we can do this by working directly in Cob_{1+1}. If so, we just need to check that both paths around the face lead to isotopic cobordisms. Otherwise, we can check every case by working directly with the maps m and Δ. □

Note that if $v \in V^{\otimes k}$ is a homogeneous element, then $deg(\mathcal{A}(\Sigma_e(v))) = deg(v) - 1$ since both m and Δ decrease the degree by 1. We use this fact to define a new grading q on $C_{Kh}(D)$ which makes d into a q-degree preserving map. If v is as before, then we define $q(v) = deg(v) + gr(v) + n_+ - n_-$. Both n_+ and n_- don't change under d, furthermore d decreases deg by 1, as we've just seen, and increases the homological degree gr by 1, so it follows that $q(d(v)) = q(v)$ and thus it preserves the q-grading. This splits the chain modules $C_{Kh}(D)$ into a direct sum of complexes, one for each q-degree.

Definition 3.1.2. Let $L \subset S^3$ be an oriented link and D a diagram for L. Then the homology of the chain complex $(C_{Kh}(D), d_{Kh})$ is called the Khovanov homology of L and is denoted by $Kh(L)$.

Recall that to construct the Khovanox complex of a link L, we first had to choose a diagram for L, from which we constructed the cube of resolutions. Notice however that in the previous definition we made no use of this information when defining the homology. This is because Khovanov homology is in fact a link invariant. This is by no means a trivial result and the proof involves some serious work, so for the sake of brevity we won't include it here.

Theorem 3.1.3 (Theorem 1 of [Kho00]). *If $L \subset S^3$ is an oriented link, then $Kh(L)$ is an invariant of L.* □

Besides the mentioned reference, this theorem was also proven in [BN02] and [BN05]. The last article is particularly interesting - instead of constructing the Khovanov complex directly as we did, the author instead constructs a pre-additive category whose objects are chain complexes formed from the cube of resolutions (in a pure topological way, without referencing the maps m and Δ) and the morphisms are chain maps. To prove invariance in such a setting, he showed that the chain complexes associated with the diagrams before and after the Reidemeister moves, are chain homotopic. Now the Khovanov homology is simply obtained by taking a TQFT out of this category, and we don't need to worry about invariance, since it was already proven without having to refer to any particular TQFT. This gives rise to a universal theory, where we can take different TQFTs and produce different link invariants (in fact, his construction also works for tangles).

Before proceeding, recall that the Kauffman bracket $\langle D \rangle \in \mathbb{Z}[q^{-1}, q^{+1}]$ of a diagram D is defined recursively as

$$\langle D \rangle = \langle D_- \rangle - q\langle D_+ \rangle$$
$$\langle k \text{ circles in the plane } \rangle = (q + q^{-1})^k$$

where D_- and D_+ correspond to the diagram D with a specific crossing having either the 0-resolution or the 1-resolution respectively. This is not a link invariant, as it isn't invariant under Reidemeister I moves. We can however use it to construct the unnormalized Jones polynomial $\hat{J}(D) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle$ which is a link invariant. Notice the similarities between this definition and the construction of the Khovanov complex (in our definition of q-grading, we have the same $n_+ - 2n_-$ shift and we also have a

n_- shift in the homological grading). After unwinding both definition more carefully, one can reach the following conclusion.

Theorem 3.1.4. *If $L \subset S^3$ is an oriented link, then the graded Euler characteristic of $C_{Kh}(L)$ equals the Jones polynomial, i.e.*

$$\chi(C_{Kh}(L)) = \sum_{i,j}(-1)^i q^j dim(Kh^{i,j}(L)) = J(L) \qquad \square$$

Note that this means that Khovanov homology is at least as powerful as the Jones polynomial, since we can recover the Jones polynomial from its Euler characteristic. In [BN02] Bar-natan computed the Khovanov homology of all prime knots with up to 11 crossings and found 18 pairs of knots with the same Jones polynomial, such that their Khovanov homologies differ. This means that the Khovanov homology is indeed a stronger invariant than the Jones polynomial. A perhaps more meaningful result is due to Kronheimer and Mrowka, who proved that Khovanov homology detects the unknot ([KM11]), a result which is unknown for the Jones polynomial.

Example 3.1.5 (Khovanov homology of the trefoil)**.** Consider the Khovanov complex of the trefoil T in Figure 3.1.3, obtained from Figure 3.1.2 by taking the TQFT $\mathcal{A}$. We'll skip the tedious details of computing the homology of the complex and claim that the Khovanov homology of the trefoil is generated by

$$\begin{aligned}
&Kh^0(T) = \langle v_- \otimes v_-, v_+ \otimes v_- - v_- \otimes v_+ \rangle \\
&Kh^1(T) = 0 \\
&Kh^2(T) = \langle (v_- \otimes v_+, v_- \otimes v_+, v_- \otimes v_+) \rangle \\
&Kh^3(T) = \langle v_+ \otimes v_+ \otimes v_+ \rangle
\end{aligned}$$

Since all three crossings of T are positive, then for an element $v \in C_{Kh}(T)$, we'll have that $q(v) = deg(v) + |v| + 3$. This leads to the following decomposition into q-gradings

$$\begin{aligned}
&Kh^0(T) = Kh^{0,1}(T) \oplus Kh^{0,3}(T) \\
&Kh^1(T) = 0 \\
&Kh^2(T) = Kh^{2,5}(T) \\
&Kh^3(T) = Kh^{3,9}(T)
\end{aligned}$$

We use this to compute the Euler characteristic of the chain complex, which gives us $\chi(C_{Kh})(T) = q^1 + q^3 + q^5 - q^9 = (q + q^{-1})(q^2 + q^6 - q^8)$, and thus agrees with the unnormalized Jones polynomial of the trefoil.

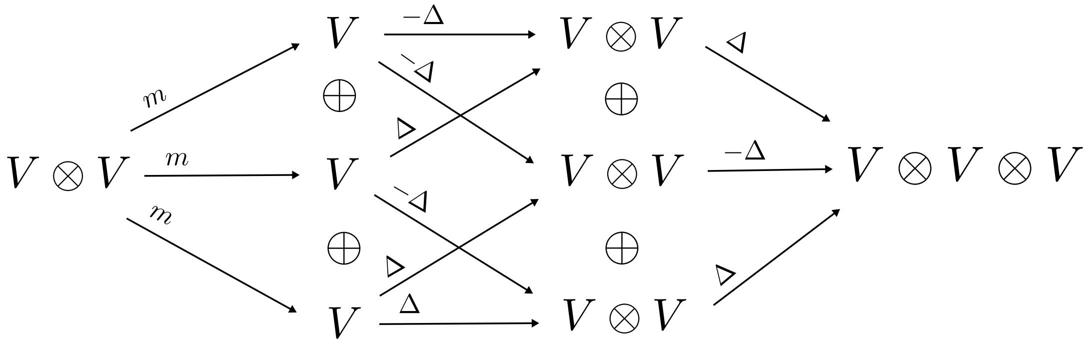

Figure 3.1.3: Khovanov complex of the trefoil.

3.2 Lee homology

In the previous section we introduced the Khovanov homology of a link L, which is obtained by applying the TQFT $\mathcal{A}$ to the cube of resolutions of a diagram of L. This yielded a link invariant which recovers the Jones polynomial. But what if instead of applying $\mathcal{A}$ to the cube of resolutions we considered another TQFT? This was the idea of Lee - she constructed another TQFT $\mathcal{A}'$ which in a similar fashion to Khovanov homology, produces a homology theory which gives rise to a link invariant. This section follows mostly the original work of Lee in [Lee05].

Instead of using the TQFT $\mathcal{A}$, we now consider another TQFT $\mathcal{A}'$. The underlying $\mathbb{Q}$-vector spaces will be the same but now we'll have a deformation of the maps associated with the cobordisms (the maps $\iota\colon \mathbb{Q} \to V$ and $\epsilon\colon V \to \mathbb{Q}$ associated with 0- and 2-handle attachments remain the same). These are now $m'\colon V \otimes V \to V$ and $\Delta'\colon V \to V \otimes V$ and are explicitly given by

$$\begin{aligned} m'(v_+ \otimes v_+) &= m'(v_- \otimes v_-) = v_+ & \Delta'(v_+) &= v_+ \otimes v_- + v_- \otimes v_+ \\ m'(v_+ \otimes v_-) &= m'(v_- \otimes v_+) = v_- & \Delta'(v_-) &= v_- \otimes v_- + v_+ \otimes v_+ \end{aligned} \tag{3.2.1}$$

Comparing the maps of $\mathcal{A}$ with those of $\mathcal{A}'$ we can see they decomponse as sums $m = m' + \delta_m$ and $\Delta = \Delta' + \delta_\Delta$ where $\delta_m\colon V \otimes V \to V$ and $\delta_\Delta\colon V \to V \otimes V$ are given by

$$\begin{aligned} \delta_m(v_+ \otimes v_+) &= \delta_m(v_+ \otimes v_-) = \delta_m(v_- \otimes v_+) = 0 & \delta_\Delta(v_+) &= 0 \\ \delta_m(v_- \otimes v_-) &= v_+ & \delta_\Delta(v_-) &= v_+ \otimes v_+ \end{aligned} \tag{3.2.2}$$

In the same way as we did with Khovanov homology, we can now construct a chain complex by using the $\mathbb{Q}$-vector spaces previously defined but now we use the differential $d_{Lee} = d_{Kh} + \delta$ where δ is obtained as before by replacing the cobordism maps by the ones defined in Equation (3.2.2). Note that $d_{Lee}^2 = (d_{Kh}+\delta)^2 = d_{Kh}+d_{Kh}\delta+\delta d_{Kh}+\delta^2$ and since $d_{Kh}^2 = 0$ and $\delta^2 = 0$, then for d_{Lee} to be a differential we just need to have $d_{Kh}\delta = -\delta d_{Kh}$. This follows by checking all cases as in Proposition 3.1.1.

Definition 3.2.1. Let $L \subset S^3$ be an oriented link and D a diagram for L. Then the homology of the chain complex $(C_{Lee}(D) = C_{Kh}(D), d_{Lee})$ is called the Lee homology of L and is denoted by $Lee(L)$.

Note that as before, the definition of the Lee complex requires a specific diagram of the link, but as before, we didn't use this information when defining the homology. As it should be expected by now, it turns out that as in the case of Khovanov homology, this is an invariant of the link, so we don't require a particular diagram when specifying the homology. This can be easily proven by applying Lee's TQFT to Bar-natan's category (Section 9.2 of [BN05]).

Recall that the differential d_{Kh} is degree preserving, but we can't say the same for d_{Lee}. Since $\delta_m(v_-\otimes v_-) = v_+$, $\delta_\Delta(v_-) = v_+\otimes v_+$ and δ is 0 otherwise, then δ increases the degree by 4, so d_{Lee} can't be degree preserving. In fact, d_{Lee} is not even graded since for instance $\Delta'(v_-) = v_-\otimes v_- + v_+\otimes v_+$ is not homogenous. If $v \in C_{Lee}$ is a sum of monomials $v = v_1 + v_2 + \ldots + v_k$, then we can assign a q-degree to v by setting $deg(v) = min\{deg(v_i) \mid i = 1,\ldots,k\}$ which in turn means that $q(v) = min\{q(v_i) \mid i = 1,\ldots,k\}$. This means that if v a homogeneous element of C_{Lee}, then $q(d_{Lee}(v)) \geq q(v)$, so while d_{Lee} doesn't preserve the q-grading, it induces a filtration $\mathcal{F}C_{Lee}(D)$ on $C_{Lee}(D)$ given by

$$C_{Lee}(D) = \mathcal{F}^nC_{Lee}(D) \supset \mathcal{F}^{n+1}C_{Lee}(D) \supset \cdots \supset \mathcal{F}^{m-1}C_{Lee}(D) \supset \mathcal{F}^mC_{Lee}(D) = \{0\}$$

where $\mathcal{F}^kC_{Lee}(D) = \{v \in C_{Lee}(D) \mid q(v) \geq k\}$, for $k = n, ..., m$. This gives rise to a filtered complex, which in turn gives us a spectral sequence.

Theorem 3.2.2. *Let $L \subset S^3$ be a link. Then there is a spectral sequence $\{E^r, d^r\}$ with E^1 page $Kh(L)$ and E^∞ page $Lee(L)$. Furthermore, for $j \geq 1$ all E^j pages are invariants of L.*

Proof. By the previous paragraph we have a spectral sequence $\{E^r, d^r\}$ with $E^\infty = Lee(L)$. The E^1 page will be the homology of $(E_0, d_0) = (C_{Kh}(D), d_{Kh})$, so $E^1 \cong Kh(L)$. The remaining claim is proven in Section 6 of [Ras10]. □

Before proceeding we introduce a new basis $\{a, b\}$ for V by defining $a = v_- + v_+$ and $b = v_- - v_+$. These new elements are mapped by m' and Δ' as follows

$$\begin{aligned} m'(a\otimes a) &= a & \Delta'(a) &= a\otimes a \\ m'(a\otimes b) &= m(b\otimes a) = 0 & \Delta'(b) &= b\otimes b \\ m'(b\otimes b) &= -2b \end{aligned} \tag{3.2.3}$$

Under this basis, we can construct a collection of elements of $C_{Lee}(D)$ in the following manner. Let L be an oriented link and D a diagram for L with orientation o. Resolve each crossing of D in the only way compatible with the orientation of the strands involved, i.e. such that after resolving all the crossings, each circle in the diagram D_o will have an orientation induced by the original orientation o. To each circle $C \in D_o$ we now assign an element $\tau(C) \in \mathbb{Z}_2$ as follows

- If C is oriented counter clockwise, then $\tau(C)$ equals the number of circles separating C from infinity mod 2.
- If C is oriented clockwise, then $\tau(C)$ equals the number of circles separating C from infinity plus 1 mod 2.

This gives us a way of associating elements of V to each circle by defining $\mathfrak{s}_C = a$ if $\tau(C) = 0$ and $\mathfrak{s}_C = b$ if $\tau(C) = 1$. This will in turn give us an element $\mathfrak{s}_o = \bigotimes_{C \in D_o} \mathfrak{s}_C$ of $C_{Lee}(L)$ depending on the original orientation of D. Since L has n components, then there are 2^n possible orientations and thus 2^n different elements $\mathfrak{s}_o$ in $C_{Lee}(L)$.

It turns out that the collection $\{\mathfrak{s}_o\}$ will form a complete set of generators for $Lee(L)$, but before proving that we need the following lemma.

Lemma 3.2.3. *Let $L \subset S^3$ be a link with diagram D, such that D_o is an oriented diagram for the oriented resolution of D as before. Assume D_o is labeled by $\mathfrak{s}_o$ and consider a region containing two strands of D_o as in Figure 3.2.1. Then, in that region either the orientation of both strands is the same and the labels are different, or the orientation is different and the labels are the same.*

Proof. There are three possible scenarios - either the strands belong to the same circle, or they belong to different circles such that one is included in the other, or they belong to different circles such that neither is included in the other. Note that the only way for two strands to have the same orientation in a region is to introduce a Reidemeister I move, since D_o is a collection of circles in the plane with no crossings, then the first case is immediately satisfied. With this in mind, the remaining cases also easily follow - if a circle is contained in the other and both circles have the same orientation, then their labels will differ since there is an extra circle separating one of them. If their orientation is different, then they'll have the same label because the clockwise circle will change label (when compared to the previous case). If none of the circles are inside the other, then if they have the same orientation in the region, they must have opposite orientations globally and thus opposite labels. If they have different orientation in the region, then the opposite follows and they must have the same label. $\square$

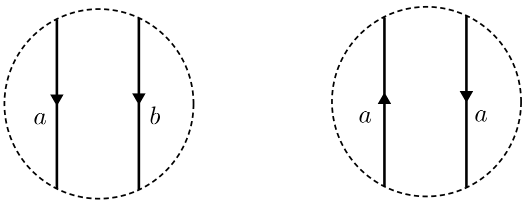

Figure 3.2.1: The two possible situations in a region of D_O.

Theorem 3.2.4. *If $L \subset S^3$ is an oriented link with n components then $Lee(L)$ has rank 2^n.*

Proof. We start by proving that the collection $\{\mathfrak{s}_o\}$ forms an orthonormal set for $Lee(L)$. Each $\mathfrak{s}_o$ is obtained from the oriented resolution of every crossing, so if two strands in D_o share a crossing in D, then they will have the same orientation in a region around the crossing and so by the previous lemma they must have different labels in s_o. Note that the oriented resolution of a crossing amounts to doing the 0-resolution if the crossing is positive and the 1-resolution if the crossing is negative, and since positive crossings translate into the splitting of two circles, then the differential d_{Lee} will be a sum of merging m' maps. Since any two circles coming from a crossing will have different labels and $m'(a \otimes b) = 0$, then $d_{Lee}(\mathfrak{s}_o) = 0$ and thus every $\mathfrak{s}_o$ is a cycle in $Lee(L)$. We now check that they are all orthonormal in $Lee(L)$. Note that the basis $\{a, b\}$ on V induces a basis on $V^{\otimes k}$ formed by all products on a and b. In turn this basis will give rise to an inner product $\langle \cdot, \cdot \rangle$ which will make it into an orthonormal basis for $C_{Lee}(L)$. Under this inner product we will have an adjoint differential d^*_{Lee} (and adjoint chain complex with the same chain modules) such that $\langle d_{Lee}v, w \rangle = \langle v, d^*_{Lee}w \rangle$, for $v, w \in C_{Lee}(L)$. To see that the inner product preserves the orthogonality of the elements $\mathfrak{s}_o \in Lee(L)$ on homology, we just need to assure that $[\mathfrak{s}_o] \neq 0$. This condition is the same as $\mathfrak{s}_o$ not being the image of any element $x \in C_{Lee}$ under the differential. Since it can be proven that $d^*_{Lee}(\mathfrak{s}_o) = 0$ and thus $\mathfrak{s}_o \in ker(d_{Lee}) \cap ker(d^*_{Lee})$, then it follows that if $d_{Lee}(x) = \mathfrak{s}_o$, we would have that

$$\langle \mathfrak{s}_o, \mathfrak{s}_o \rangle = \langle \mathfrak{s}_o, d_{Lee}(x) \rangle = \langle d^*_{Lee}(\mathfrak{s}_o), \mathfrak{s}_o \rangle = 0$$

and thus that $\mathfrak{s}_o = 0$, but this can't happen since $\mathfrak{s}_o \neq 0$ by hypothesis, so we must have that $[\mathfrak{s}_o] \neq 0$. It then follows that every element $\mathfrak{s}_o \in Lee(L)$ is a generator of $Lee(L)$ and thus $dim(Lee(L)) \geq 2^n$. To prove the theorem, we just need to check that there are no more generators, that is $dim(Lee(L)) \leq 2^n$. This is done by induction on the number of crossings of L - for a proof see Theorem 4.2 of [Lee05]. $\square$

3.3 Rasmussen's s-invariant and the slice genus

If K is a knot, then by the previous theorem, its Lee homology will be surprising simple since it only has rank 2. While $Lee(K)$ itself isn't very exciting, we can obtain a lot of information by looking at the q-grading of its 2 generators. This turns out to be a knot invariant which gives a lower bound on the slice genus of K. In this section, we mostly follow the work of Rasmussen in [Ras10].

Recall the filtration $\mathcal{F}C_{Lee}(D)$ on the Lee complex induced by the q-grading, and extend it to homology by defining a class $[x] \in Lee(L)$ to be in $\mathcal{F}^k Lee(D)$ if and only if it has a representative which is an element of $\mathcal{F}^k C_{Lee}(D)$. If $[x] \in Lee(D)$ then we define the q-grading on $[x]$ as the maximum q-grading along its representatives. That is, if we denote the q-grading on $Lee(L)$ by s, then

$$s([x]) = max\{q(v) \mid [v] = x\}$$

For a knot K, its Lee homology is particularly simple and will be generated by two elements. This leads us to the following definitions.

Definition 3.3.1. If $K \subset S^3$ is a knot, then we define

$$s_{min}(K) = min\{s(x) \mid x \in Lee(K), x \neq 0\}$$
$$s_{max}(K) = max\{s(x) \mid x \in Lee(K), x \neq 0\}$$

It turns out that both of these values are invariants of the knot. This was proven in Proposition 3.2 of [Ras10] by verifying that the chain maps induced by the Reidemeister moves will preserve the q-grading and thus will respect the induced filtration on homology. A big part of this section will be devoted to proving that $s_{max}(K) = s_{min}(K) + 2$ which leads us to the definition

Definition 3.3.2. Let $K \subset S^3$ be a knot. We call

$$s(K) = s_{min}(K) + 1 = s_{max}(K) - 1$$

the s-invariant of K.

Before starting our journey to ensure that the s-invariant is well-defined, we need the following lemma.

Lemma 3.3.3 (Lemma 1.7 of [Lew]). *If $L \subset S^3$ is a link with n components, then every element in $C_{Lee}(L)$ will have q-grading congruent to n mod 2.* □

Lemma 3.3.4. *Let $L \subset S^3$ be a link with n components. Then there is a direct sum decomposition*

$$C_{Lee}(L) \cong C^o_{Lee}(L) \oplus C^e_{Lee}(L)$$

where $C^o_{Lee}(L)$ is generated by the elements with q-grading congruent to $2+n$ mod 4 and $C^e_{Lee}(L)$ is generated by the elements with q-grading congruent to n mod 4. Furthermore if o is an orientation on L, then $\mathfrak{s}_o + \mathfrak{s}_{\overline{o}}$ is contained in one summand and $\mathfrak{s}_o - \mathfrak{s}_{\overline{o}}$ is contained in the other.

Proof. By the previous lemma an element of $C_{Lee}(L)$ will have q-grading congruent to n mod 2. As we've seen before the differential d_{Kh} preserves the q-grading while δ increases it by 4, so $d_{Lee} = d_{Kh} + \delta$ increases the q-grading mod 4. Combining these two things, we get that every element will either have q-grading congruent to $2+n$ mod 4 or to n mod 4, so we get a decomposition $C_{Lee}(L) \cong C^o_{Lee}(L) \oplus C^e_{Lee}(L)$.

For the second claim, consider the map $\Psi\colon C^o_{Lee}(L) \oplus C^e_{Lee}(L) \to C^o_{Lee}(L) \oplus C^e_{Lee}(L)$ which acts on $C^e_{Lee}(L)$ with the identity and on $C^o_{Lee}(L)$ with multiplication by -1. Define now the map $\psi\colon V \to V$ such that $\psi(v_-) = v_-$ and $\psi(v_+) = -v_+$, which in turn means that $\psi(a) = b$ and $\psi(b) = a$. This will induce a map on the product $\psi^{\otimes k}\colon V^{\otimes k} \to V^{\otimes k}$ which will act on a standard generator of $V^{\otimes k}$ as the identity if there is an even number of v_+ and with multiplication by -1 if there is an odd number of v_+. If we apply a degree change, so that $deg(v_-) = 0$ and $deg(v_+) = 2$, then this means that $\psi^{\otimes k}$ will act as the identity on elements with q-grading congruent to 2 mod 4 and with multiplication by -1 on elements with q-grading congruent to 0 mod 4. By recovering the original grading, this simply means that $\Psi = \pm\psi^{\otimes k}$. For a fixed orientation o, the generator $\mathfrak{s}_{\overline{o}}$ is obtained by switching every label of $\mathfrak{s}_o$, so under the previous map we will have that $\Psi(\mathfrak{s}_o) = \pm\psi^{\otimes k}(\mathfrak{s}_o) = \pm\mathfrak{s}_{\overline{o}}$. Note now that since Ψ acts as the identity on the even part and as multiplication by -1 on the odd part, then $\mathfrak{s}_o + \Psi(\mathfrak{s}_o) = \mathfrak{s}_o \pm \mathfrak{s}_{\overline{o}}$ will have no odd part while $\mathfrak{s}_o - \Psi(\mathfrak{s}_o) = \mathfrak{s}_o \mp \mathfrak{s}_{\overline{o}}$ will have no even part. It follows that $\mathfrak{s}_o + \mathfrak{s}_{\overline{o}}$ is contained in one summand and $\mathfrak{s}_o - \mathfrak{s}_{\overline{o}}$ is contained in the other. □

So since $C_{Lee}(L)$ decomposes as a direct sum, then so does the spectral sequence induced by the q-grading. In particular, if K is a knot then $Lee(K) \cong \mathbb{Q} \oplus \mathbb{Q}$ only has two generators, and since by the previous remark $Lee(K)$ also decomposes as a direct sum on two generators, then by the previous lemma they must be have different q-gradings.

Corollary 3.3.5. *If $K \subset S^3$ is a knot, then $s_{max}(K) \geq s_{min}(K) + 2$.* □

Corollary 3.3.6. *If $K \subset S^3$ is a knot, then $s(\mathfrak{s}_o) = s(\mathfrak{s}_{\overline{o}}) = s_{min}(K)$,* *Proof.* Assume by contradiction that there is another element $[x] \in Lee(K)$ such that $s(x) \leq s(\mathfrak{s}_o)$ (assuming $s(x) \leq s(\mathfrak{s}_{\overline{o}})$ leads to the same result). Since K is a knot then $Lee(K)$ only has two generators $\{\mathfrak{s}_o, \mathfrak{s}_{\overline{o}}\}$, so by the previous lemma we can write x as linear combination of $\mathfrak{s}_o + \mathfrak{s}_{\overline{o}}$ and $\mathfrak{s}_o - \mathfrak{s}_{\overline{o}}$ (one representing the odd part and the other the even part of $C_{Lee}(K)$). It then follows that

$$s(x) = s((\mathfrak{s}_o + \mathfrak{s}_{\overline{o}}) + (\mathfrak{s}_o - \mathfrak{s}_{\overline{o}})) = s(2\mathfrak{s}_o) = \mathfrak{s}_o$$

which leads to a contradictions since we assumed that $s(x) \leq s(\mathfrak{s}_o)$. The result then follows. □

It remains to prove that $\mathfrak{s}_{max}(K) \leq s_{min}(K) + 2$. To do so we make use of the following lemma.

Lemma 3.3.7. *Let $K_1, K_2 \subset S^3$ be oriented knots. Then there is a short exact sequence on Lee homology*

$$0 \longrightarrow Lee(K_1 \# K_2) \xrightarrow{p_*} Lee(K_1) \otimes Lee(K_2) \xrightarrow{\partial_*} Lee(K_1 \# \overline{K_2}) \longrightarrow 0$$

where the maps p_* and ∂_* are filtered of q-degree -1 and $\overline{K_2}$ denotes K_2 with the reverse orientation.

Proof. Consider the diagram for the connected sum of K_1 and K_2 depicted in Figure 3.3.1. In the diagram for $K_1 \# K_2$ there is a distinguished crossing which we can resolve to obtain diagrams for $K_1 \sqcup K_2$ (0-resolution) and $K_1 \# \overline{K_2}$ (1-resolution), where $\overline{K_2}$ denotes K_2 with the reverse orientation. To each diagram we can associate its Lee chain complex, and since the diagrams are related by a merge, we can construct the Lee chain complex for $K_1 \# K_2$ by taking the cone over such a map, that is $C_{Lee}(K_1 \# K_2) = C(C_{Lee}(K_1 \sqcup K_2) \longrightarrow C_{Lee}(K_1 \# \overline{K_2}))$. After adjusting the q-gradings, we then get a short exact sequence

$$0 \longrightarrow C_{Lee}(K_1 \# \overline{K_2})[-1]\{2\} \xrightarrow{i} C_{Lee}(K_1 \# K_2) \xrightarrow{p} C_{Lee}(K_1 \sqcup K_2)\{1\} \longrightarrow 0$$

where the $C\{l\}$ denotes a q-degree shift of the complex C by l. It then follows that we can derive the following long exact sequence on homology

$$\longrightarrow Lee^{i-1}(K_1 \# \overline{K_2})\{2\} \xrightarrow{i_*} Lee^i(K_1 \# K_2) \xrightarrow{p_*} Lee^i(K_1 \sqcup K_2)\{1\} \xrightarrow{\partial} Lee^i(K_1 \# \overline{K_2})\{2\} \longrightarrow$$

with i_*, p_* and ∂ being filtration perserving. We will have that that $dim(Lee(K_1 \# K_2)) = dim(Lee(K_1 \# \overline{K_2})) + dim(Lee(K_1 \sqcup K_2)) - 2rk(\partial)$ and since both $K_1 \# K_2$ and $K_1 \# \overline{K_2}$ are knots, then their Lee homologies have dimension 2. Furthermore $K_1 \sqcup K_2$ corresponds to two unlinked knots, so $dim(K_1 \sqcup K_2) = 4$. We must then have that $rank(\partial) = 2$, but since by exactness we have $im(\partial) = ker(i_*)$, then i_* must coincide with the zero map. We then conclude that the long exact sequence splits into short exact sequences for each homological degree, leading to the following short exact sequence

$$0 \longrightarrow Lee(K_1 \# K_2) \xrightarrow{p_*} Lee(K_1 \sqcup K_2) \xrightarrow{\partial} Lee(K_1 \# \overline{K_2}) \longrightarrow 0$$

Since we removed the q-degree shifts, then p_* and ∂ are filtered of degree -1. □

Proposition 3.3.8. *Let $K \subset S^3$ be a knot, then $s_{max}(K) = s_{min}(K) + 2$.*

Proof. Since by Corollary 3.3.5 $s_{max}(K) \geq s_{min}(K)+2$, we only need to prove that $\mathfrak{s}_{max}(K) \leq s_{min}(K)+2$. To do so, consider the short exact sequence from the previous lemma, with $K_1 = K$ and $K_2 = U$

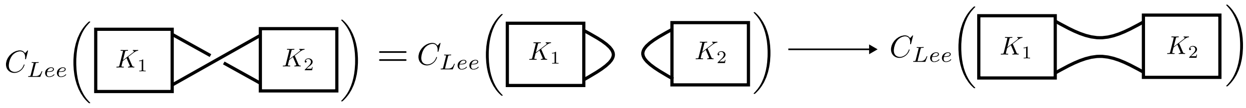

Figure 3.3.1: Cone of the Lee complex associated with a connected sum of two knots.

the unknot, meaning that $Lee(U)$ is generated by $\{a, b\}$. Denote by $\mathfrak{s}_a$ and $\mathfrak{s}_b$ the generators for K depending on if the circle resulting from the resolution of the crossing in Figure 3.3.1 has label a or b. By Lemma 3.3.4, either $s(\mathfrak{s}_a + \mathfrak{s}_b)$ or $s(\mathfrak{s}_a - \mathfrak{s}_b)$ will equal $s_{max}(K)$, so without loss of generality, assume that $s_{max} = s(\mathfrak{s}_a - \mathfrak{s}_b)$ (if $s_{max} = s(\mathfrak{s}_a + \mathfrak{s}_b)$ we simply change the orientation of K to get the other equality). Note now that the map ∂_* coming from the short exact sequence corresponds to a merge of the two circles involved in the crossing, and since $m'(a \otimes b) = 0$ and $m'(a \otimes a) = 2a$ then we have that $\partial_*([\mathfrak{s}_a - \mathfrak{s}_b] \otimes [a]) = [\mathfrak{s}_a]$. Since ∂_* is filtered of degree -1, then $s([\mathfrak{s}_a - \mathfrak{s}_b] \otimes [a]) \leq s([\mathfrak{s}_a]) + 1$. By Corollary 3.3.6 we have that $s([\mathfrak{s}_a]) = s_{min}(K)$ and $s(a) = s(b) = s_{min}(U) = -1$, so it follows that $s_{max}(K) - 1 \leq s_{min}(K) + 1$, which in turn means that $s_{max}(K) \leq s_{min}(K) + 2$. □

We can then conclude that the s-invariant is well-defined and thus Definition 3.3.2 is justified. We now look at some useful properties of the s-invariant.

Proposition 3.3.9 (Proposition 3.10 of [Ras10]). *Let $K \subset S^3$ be a knot and $-K$ its mirror image. Then*

$$s_{max}(-K) = -s_{min}(K)$$
$$s_{min}(-K) = -s_{max}(K)$$
$$s(-K) = -s(K)$$

□

The idea of the proof is to construct an isomorphism between the complex $C_{Lee}(-K)$ and the dual complex of $C_{Lee}(K)$. The dual $C_{Lee}(K)^*$ will admit a dual filtration $\mathcal{F}^{-i}C_{Lee}(K)^* = \{x \in C_{Lee}(K)^* \mid \langle x, y \rangle = 0, y \in C_{Lee}(K)\}$, which will identify the q-grading of an element in $C_{Lee}(K)$ with the inverse in $C_{Lee}(K)^*$. Since the complexes are isomorphic, then this will induce an isomorphism of the spectral sequences induced by the filtration and the result follows.

Note that in the proof of Proposition 3.3.8, we used the short exact sequence from Lemma 3.3.7 to get an inequality between the s-invariant of a connected sum and its components. By virtue of the last proposition, we can do the same for the mirrors and get the opposite result. The equality then follows.

Proposition 3.3.10 (Proposition 3.11 of [Ras10]). *Let $K_1, K_2 \subset S^3$ be knots. Then $s(K_1 \# K_2) = s(K_1) + s(K_2)$.* □

As the main goal of this section (and really this chapter) is to use the s-invariant to obtain a bound on the slice genus, we recall its definition.

Definition 3.3.11. If $L \subset S^3$ is a link, then the (smooth) slice genus of L, denoted by $g_4(L)$, is the minimum genus taken among oriented, properly embedded, smooth surfaces $\Sigma \subset D^4$ such that $\partial\Sigma = L$. In particular, a knot is said to be slice if it bounds a properly embedded, smooth disk $D \subset D^4$ such that $\partial D = K$.

The natural way to study embedded surfaces is through link cobordisms, so if one wishes to relate the s-invariant of a knot with its slice genus, we must study how its Lee homology behaves under cobordisms. We usually work with links sitting inside S^3, so by a cobordism Σ between links L_0 and L_1 we refer to a compact, oriented, properly embedded surface in $S^3 \times I$ such that $\Sigma \cap S^3 \times \{0\} = L_0$ and $\Sigma \cap S^3 \times \{1\} = L_1$. Note that we can think of Σ as sitting inside $\mathbb{R}^3 \times I$ such that it misses the points at infinity of $S^3 = \mathbb{R}^3 \cup \{\infty\}$. If o_Σ is an orientation for Σ, then we say that the orientation o_0 and o_1 of $L_0, L_1 \subset \Sigma$ are compatible with o_Σ if o_1 is the usual orientation induced on L_1 as the boundary of Σ and o_0 is the opposite of the usual orientation induced on L_0 as the boundary of Σ. This is equivalent to saying that the orientation o_0 on L_0 is compatible with the orientation o_1 on L_1, if there is an orientation on a cobordism between them such that they are both compatible with it. If Σ is such a cobordism, then our goal is to define a map $\phi'_\Sigma \colon Lee(L_0) \to Lee(L_1)$ such that L_0 and L_1 both have orientations compatible with that of Σ.

Consider a cobordism between two links and define a vertical Morse function on $S^3 \times I$. Then this defines a sequence of "frames" $F_i = S^3 \times \{p_i\}$ such that $\Sigma \cap F_i$ will either be a link, or a link with tranverse double points which correspond to critical values of the Morse function. As we've seen in Section 2.1, these correspond to the attachment of 0-,1- or 2-handles to Σ, so if p_i is a critical point of the Morse function, then the link in $F_{i-\epsilon}$ will differ from the link in $F_{i+\epsilon}$ as in Figure 3.3.2. If there are no critical points of the Morse function between two frames F_i and F_{i+1} and if $L_i = F_i \cap \Sigma$, then the cobordism corresponding to $\Sigma \cap [F_i, F_{i+1}]$ will be an annulus $L_i \times [i, i+1]$ and the link L_{i+1} will only differ from L_i by isotopies. Since these are generated by the Reidemeister moves, then between any two frames, we'll have that the corresponding links will either differ by a Reidemeister move or by a handle attachment. This means that we can split the cobordism Σ into elementary cobordisms such that each one corresponds to exactly one of the previous moves. If Σ is a cobordism between L_0 and L_1, we would like the map $\phi'_\Sigma \colon Lee(L_0) \to Lee(L_1)$ to be functorial in the sense that we can decompose it into a composition of maps $\phi'_{\Sigma \cap [F_i, F_{i+1}]} \colon Lee(L_i) \to Lee(L_{i+1})$ associated with elementary cobordisms.

Suppose that Σ is an elementary cobordism between L_0 and L_1 induced by a Reidemeister move. As we've pointed out before, if L_0 and L_1 are related by the i-th Reidemeister move, then there is a chain map $\rho'_i \colon C_{Lee}(L_0) \to C_{Lee}(L_1)$ which induces an isomorphism on homology. In this case, we then define ϕ'_Σ to either be ρ'_{i*} or its inverse. While we didn't point this out before, note that that these maps are actually filtered of degree 0 (see Proposition 3.2 of [Ras10]).

Suppose now that Σ is instead an elementary cobordism associated to a handle attachment. If we consider a diagram D_0 for L_0 and a resolutions $D_{0,v}$ of D_0, then the cobordism Σ_v will induce a resolution $D_{1,v}$ for L_1. The resolution $D_{1,v}$ is identical to that of $D_{0,v}$ except now there will either be a new circle appearing (0-handle attachment), there is a crossing in D_0 which is resolved with the opposite resolution (1-handle attachment), or there is a circle that vanishes (2-handle attachment). In every case, we see that an element $x \in \mathcal{A}'(D_{0,v})$ will be mapped to an element $\mathcal{A}'(\Sigma_v)(x)$. Thus at the chain level, we define $\phi_\Sigma \colon C_{Lee}(L_0) \to C_{Lee}(L_1)$ to be the map given by $\phi(x) = \mathcal{A}'(\Sigma_v)(x)$ and we define ϕ'_Σ to be the map induced on homoly by ϕ_Σ. Note that the maps associated to a 1-handle attachment are m' and Δ', which are filtered of degree -1 (since we have a map between two complexes, we pick up a degree shift - see Lemma 3.3.7). The 0- and 2-handle attachments are associated with ι and ϵ, so they are clearly filtered

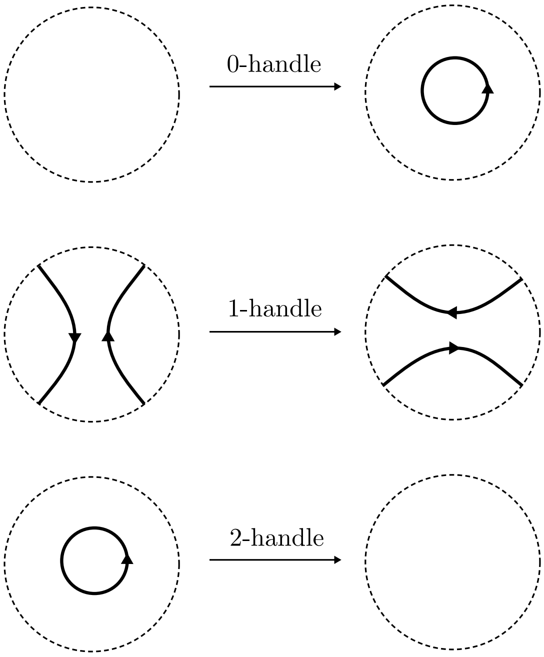

Figure 3.3.2: Change in a link diagram induced by a handle attachment.

of degree 1.

If we wish to relate the s-invariant of two links connected by a cobordism, then we need to see what happens to the canonical generators $\mathfrak{s}_o$ of Lee homology under the map ϕ'_Σ.

Proposition 3.3.12. *Let $L_0, L_1 \subset S^3$ be links and $\Sigma \subset S^3 \times I$ a cobordism between them with no closed components. If o is an orientation on L_0, then*

$$\phi'_\Sigma([\mathfrak{s}_o]) = \sum_{o_I} a_I [\mathfrak{s}_{o_I}]$$

where o_I corresponds to the orientations of L_1 compatible with o and such that each a_I is non-zero.

Proof. We split Σ into elementary cobordisms and the proof follows by induction on the number of these. The base case is a single elementary cobordism, so we need to check that the equation holds for Reidemeister moves and handle attachments. The case of Reidemeister moves follows from Proposition 2.3 of [Ras10], so we focus on the handle attachments.

We start with 0-handles. In this case, a generator $\mathfrak{s}_o \in Lee(L)$ will be mapped under $\phi_\Sigma = \iota$, and thus we have $\iota(\mathfrak{s}_o) = \mathfrak{s}_o \otimes v_+ = \mathfrak{s}_o \otimes \frac{1}{2}(a-b)$. The labels on the second factor of the product refer to the labels on the new circle, so any orientation is compatible with o. Since a and b are the generators of $Lee(U)$, then we have that $\phi'_\Sigma([\mathfrak{s}_o]) = \frac{1}{2}[\mathfrak{s}_o \otimes a] - \frac{1}{2}[\mathfrak{s}_o \otimes b]$, so the equation holds.

The case of 2-handles is especially simple. The induced map will be $\phi_\Sigma = \epsilon$ and since we are simply removing a circle, then the matter of orientations will become trivial. We have that $\epsilon(a) = \epsilon(b) = 1$, so $\mathfrak{s}_o = \mathfrak{s}_{o_1}$ and the equation holds.

The 1-handle case is the hardest of the three since we have different maps and we have to be careful with the orientations. Start with the case where one component of L_0 splits into two components of L_1 via a cobordism Σ. In this situation, there is a unique orientation o_1 on L_1, compatible with the orientation o

on L_0, so by Figure 3.2.1, the two strands involved in the handle attachment must have opposite direction and have the same label. Note that outside of the circles involved in the 1-handle move, every other factor of $\mathfrak{s}_o$ agrees with $\mathfrak{s}_{o_1}$, so the map ϕ_Σ coincides with either m' or Δ'. Since all the other involved circles are the same, then by Equation (3.2.1), it follows that $\phi_\Sigma(\mathfrak{s}_o) = a\mathfrak{s}_{o_1}$ where $a = 1$ or $a = \pm 2$, so at the level of homology, we're still in the same class. Suppose now that Σ merges two components of L_0 into one component of L_1. If the two strands involved in the 1-handle move have the same orientation, then there is no compatible orientation o_1 of L_1 so the right-hand side of the equation is 0. But by Figure 3.2.1, the strands must have opposite labels and since $m'(a \otimes b) = 0$, then the $\phi_\Sigma(\mathfrak{s}_o) = 0$ and the equation holds. If the two strands instead have different orientations, then there is a unique compatible orientation o_1 on L_1, and the result follows in the same way as the previous case.

It just remains to do the induction step. If we decompose $\Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 is a composition of elementary cobordisms between L_0 and $L_{\frac{1}{2}}$ and Σ_2 is a composition of elementary cobordisms between $L_{\frac{1}{2}}$ and L_1, then $\phi'_\Sigma = \phi'_{\Sigma_2} \circ \phi'_{\Sigma_1}$, so for an element $[\mathfrak{s}_o] \in Lee(L_0)$ we have under the induction hypothesis that

$$\phi'_\Sigma([\mathfrak{s}_o]) = \phi'_{\Sigma_2}(\sum_{o_I} a_I [\mathfrak{s}_{o_I}]) = \sum_{o_I} a_I \sum_{o_J} b_J [\mathfrak{s}_{o_J}] = \sum_{(o_I, o_J)} a_I b_J [\mathfrak{s}_{o_J}]$$

It can be easily seen that the orientations (o_I, o_J) are in bijection with the orientations $\{o_K\}$, on L_1, compatible with o on L_0, so the equation holds. $\square$

Corollary 3.3.13. *Let $K_0, K_1 \subset S^3$ be knots and Σ a connected cobordism between K_0 and K_1. Then the map ϕ'_Σ is an isomorphism.*

Proof. If o is an orientation on K_0, then since Σ is connected there is a unique compatible orientation o_1 on K_1. Since $\{\mathfrak{s}_o, \mathfrak{s}_{\overline{o}}\}$ is a basis for $Lee(K_0)$, then by the previous theorem $\phi'_\Sigma(s_o) = k_1\mathfrak{s}_{o_1}$ and $\phi'_\Sigma(s_o) = k_2\mathfrak{s}_{\overline{o_1}}$ $(k_1, k_2 \neq 0)$ will form a basis for $Lee(K_1)$ and the result follows. $\square$

We are now ready to prove the main result of this section.

Theorem 3.3.14. *If $K \subset S^3$ is a knot, then $|s(K)| \leq 2g_4(K)$.*

Proof. Consider an oriented, properly embedded surface $\Sigma \subset D^4$ of genus g such that $\partial\Sigma = K \subset \partial D^4$. Remove now a 2-disk from Σ to obtain an orientable, connected cobordism $\Sigma^\circ = \Sigma - \text{int } D^2$ in $S^3 \times I$, between K and the unknot U, with Euler characteristic $\chi(\Sigma^\circ) = -2g$. Take now $[x] \in Lee(K) - \{0\}$ to be a class such that $s(x) = s_{max}(K)$. By the previous corollary, $\phi'_{\Sigma^\circ}(x)$ is a non-zero element of $Lee(U)$, and since Σ° is obtained by attaching $2g$ 1-handles to an annulus $K \times I \subset S^3 \times I$, then the induced map ϕ'_{Σ° on homology will be filtered of degree $-2g$ and thus $s(x) - 2g \leq s(\phi'_{\Sigma^\circ}(x))$. Now since $s_{max}(U) = 1$, then it follows that $s(x) - 2g \leq 1$, but since by hypothesis $s(x) = s_{max}(K) = s(K) + 1$, we conclude that that $s(K) \leq 2g$. If we apply the same reasoning to $-K$ (the mirror of K), then it follows that $s(-K) \leq 2g$, and by Proposition 3.3.9 we have that $s(K) \geq -2g$. Combining both inequalities, we get the desired result. $\square$

Recall that two knots K_1 and K_2 are said to be concordant if there is a properly embedded annulus $S^1 \times I \subset S^3 \times I$ with boundaries $K_1 \times \{0\} \subset S^3 \times \{0\}$ and $\overline{K_2} \times \{1\} \subset S^3 \times \{1\}$. An equivalent definition can be taken in terms of the sliceness of their connected sum - two knots K_1 and K_2 are said to be concordant if $K_1\# - \overline{K_2}$ is slice. Notice that this forms an abelian group (up to isotopy) whose elements are knots, the operation is given by connected sum and the inverse elements are simply the mirror of the respective knots. This group is denoted by $Conc(S^3)$ and the s-invariant induces an homomorphism between $Conc(S^3)$ and $\mathbb{Z}$.

Theorem 3.3.15. *The map s induces a homomorphism from $Conc(S^3)$ to $\mathbb{Z}$.*

Proof. If K_1 and K_2 are concordant knots in S^3, then $K_1\#-\overline{K_2}$ is slice. By the previous theorem we then have that $s(K_1\#-\overline{K_2}) = 0$ and thus by Proposition 3.3.10 and Proposition 3.3.9 we get $s(K_1)-s(K_2) = 0$. Since $Conc(S^3)$ forms an abelian group under the operation of connected sum and inverse given by the mirror, then by the same propositions we have a group homomorphism between $Conc(S^3)$ and $\mathbb{Z}$. $\square$

We finish this section by giving a proof of the Milnor Conjecture, which concerns the slice genus of the torus knots $T_{p,q}$. This was first proven in [KM93] using techniques coming from gauge theory, but now using the s-invariant, we can give a "combinatorial" proof which doesn't require all this heavy geometric machinery.

Recall that a knot is said to be positive if it admits a diagram with only positive crossings (in particular the knots $T_{p,q}$ are positive). For positive knots, the s-invariant completely determines their slice genus.

Proposition 3.3.16. *If $K \subset S^3$ is a positive knot, then $s(K) = 2g_4(K) = 2g(K)$.*

Proof. If D is a diagram for K with only positive crossings, then an oriented resolution for D will have every crossing resolved by a 0-resolution. In particular since $\mathfrak{s}_o$ is obtained from the oriented resolution of D, then it will have minimal homological grading and thus it will be the only class homologous to itself. By Corollary 3.3.6, it follows that $s_{min}(K) = s([\mathfrak{s}_0]) = q(s_0)$. Note now that if we return to the basis $\{v_-, v_+\}$ of V, then $\mathfrak{s}_o$ will equal a product of k factors $v_+ \pm v_-$, but since $q(s_0) = s_{min}(K)$ then it must have the lowest q-grading among elements of $V^{\otimes k}$, which corresponds to the product k factors of v_-. Since $deg(v_-) = -1$, it then follows that $q(\mathfrak{s}_o) = q(v_- \otimes v_- \otimes ... \otimes v_-) = -k + 0 + n_+ - 0 = -k + n$, and thus $s(K) = -k + n + 1$. Since we can always find a Seifert surface for K via the Seifert algorithm (which produces a surface with Euler characteristic $n - k$), then $2g(K) \leq k - n + 1 = s(K) \leq 2g_4(K)$. But since $g_4(K) \leq g(K)$ (we can push a Seifert surface for K into D^4), then the result follows. $\square$

Corollary 3.3.17 (Milnor Conjecture)**.** *The slice genus of the (p, q)-torus knot $T_{p,q}$ is* $\frac{(p-1)(q-1)}{2}$

Proof. It is known that $g(T_{p,q}) = \frac{(p-1)(q-1)}{2}$ and since torus knots are positive then by the previous proposition it follows that $s(T_{p,q}) = \frac{(p-1)(q-1)}{2}$. $\square$

3.4 Extension of the s-invariant to links

We have defined the s-invariant for knots, but notice how most of the work that we've done also applies to links. It turns out that it is pretty straightforward to generalize the s-invariant in this sense. To do so, we follow the work of Beliakova and Wehrli presented in [BW08].

If L is an oriented link then its Lee homology is generated by 2^n elements, where n is the number of components of L. Since L is oriented, there is an orientation o which agrees with that of L, so that by Lemma 3.3.4 $\mathfrak{s}_o + \mathfrak{s}_{\overline{o}}$ belongs to one of the summands of $C_{Lee}(L) \cong C^o_{Lee}(L) \oplus C^e_{Lee}(L)$, while $\mathfrak{s}_o - \mathfrak{s}_{\overline{o}}$ belongs to the other. This means that the q-grading of $\mathfrak{s}_o + \mathfrak{s}_{\overline{o}}$ differs from the q-grading of $\mathfrak{s}_o - \mathfrak{s}_{\overline{o}}$ by 2 mod 4. In fact, it can be shown that they differ by exactly 2, leading to the following extension of the s-invariant.

Definition 3.4.1. If $L \subset S^3$ is an oriented link with orientation o, then we define the s-invariant of L as

$$s(L) = \frac{s(\mathfrak{s}_o + \mathfrak{s}_{\overline{o}}) + s(\mathfrak{s}_o - \mathfrak{s}_{\overline{o}})}{2}$$

Notice that for a knot K we have no choice of orientation o, when it comes to the generators $\mathfrak{s}_o$ and $\mathfrak{s}_{\overline{o}}$, so indeed the s-invariant for links agrees with the one we previously defined for knots.

In a similar fashion to knots, we can relate the s-invariant of links which form the boundaries of a cobordism. Indeed, if Σ is a smooth cobordism between L_1 and L_2 with compatible orientations (such that every connected component of Σ has a boundary component in L_1), then we can proceed as in Theorem 3.3.14 and the discussion leading up to it. In this case, if Σ is a cobordism with Euler characteristic $\chi(\Sigma)$, then $s(L_2) \geq s(L_1) + \chi(\Sigma)$ and in a similar fashion we get $s(L_1) \geq s(L_2) + \chi(\Sigma)$. Combining both results, we get the following theorem.

Theorem 3.4.2. *If $L_1, L_2 \subset S^3$ are oriented links and $\Sigma \subset S^3 \times I$ is a properly embedded, oriented, smooth cobordism between L_1 and L_2, such that every connected component of Σ has a boundary component in L_1, then*

$$|s(L_2) - s(L_1)| \leq -\chi(\Sigma)$$

□

Of particular interest to us is to be able to recognize the behavior of the s-invariant under disjoint unions of links.

Proposition 3.4.3. *If $L_1, L_2 \subset S^3$ are oriented links, then $s(L_1 \sqcup L_2) = s(L_1) + s(L_2) - 1$.*

Proof. If we fix an orientation o on $L_1 \sqcup L_2$, then there are unique orientations o_1 on L_1 and o_2 on L_2 which are compatible with that of $L_1 \sqcup L_2$ under the cobordism which identifies $Lee(L_1 \sqcup L_2)$ with $Lee(L_1) \otimes Lee(L_2)$. In particular, this isomorphism will map a generator $\mathfrak{s}_o \in Lee(L_1 \sqcup L_2)$ to $\mathfrak{s}_{o_1} \otimes \mathfrak{s}_{o_2} \in Lee(L_1) \otimes Lee(L_2)$. In a similar fashion to Corollary 3.3.6, we'll have that $s(\mathfrak{s}_o) = s_{min}(L_1 \sqcup L_2) = s(L_1 \sqcup L_2) - 1$ and $s(\mathfrak{s}_{o_i}) = s_{min}(L_1) = s(L_{o_i}) - 1$ for $i = 1, 2$. We then have that $s(L_1 \sqcup L_2) - 1 = s(L_1) - 1 + s(L_2) - 1$ and thus the proposition follows. □

We can further generalize Theorem 3.3.14 to the case of links. Since we have a link and not a knot, we replace the usual notion of slice with weakly slice. In this case, a link $L \subset S^3$ is said to be weakly slice if there exists a connected, oriented, properly embedded, smooth surface $\Sigma \subset D^4$ of genus 0, such that $\partial\Sigma = L$ (L is strongly slice if instead every component bounds a disk in the 4-ball).

Proposition 3.4.4. *If $L \subset S^3$ is an n-component, weakly slice link then $|s(L)| \leq n - 1$.* □

Proof. If L is weakly slice, then we can argue as in the proof of Theorem 3.3.14 to construct a cobordism Σ of genus 0 from L to the unknot. Notice that Σ has n disks removed (since L has n components) so its Euler characteristic differs from the disk by n, this means that $\chi(\Sigma) = -n + 1$. Since $s(U) = 0$, then by Theorem 3.4.2 we have that $|s(L)| \leq n - 1$. $\square$

Chapter 4

Applications of the s-invariant to the topology of 4-manifolds

This chapter can be thought of as the application of the s-invariant to problems in 4-manifold topology. The goal is to highlight the special position that the s-invariant holds, when compared to similar invariants. This can be first seen in Section 4.2, where Piccirillo uses the s-invariant to solve the problem of determining the sliceness of the Conway knot, which was previously open for 50 years. This was possible because unlike similar invariants, the s-invariant is not a 0-trace invariant. Another place where the s-invariant might shine is in the detection of exotic structures. It is known that most invariants coming from gauge theory cannot distinguish between homotopy equivalent manifolds, but at the current time, this is not known to be true for the s-invariant. This means that there is a possibility that the s-invariant might be able to distinguish homotopy equivalent manifolds and by the work of Freedman (Theorem 2.4.11) detect exotic structures. This approach is taken in Section 4.3 to try and solve the smooth Poincaré conjecture in dimension 4. While unsucessful, it leads to an improved strategy, which we pursue in Section 4.5. This is the main focus of this thesis, where one tries to detect exotic structures not only on S^4 but also on $\#n\mathbb{CP}^2$ by using the s-invariant. For this approach we need to have a pair of knots K and K' which share a 0-surgery homeomorphism, such that K is H-slice in $\#n\mathbb{CP}^2$ (the slice disk is null-homologous) and $s(K') < 0$ - this will produce an exotic $\#n\mathbb{CP}^2$. The key ingredint is to be able to produce pairs of knots with homeomorphic 0-surgeries. A strategy to do so is developed in Section 4.5 by constructing a type of framed three component links called RBG-links. Such links will always produce two knots with homeomorphic 0-surgeries. Finally in Section 4.6, we analyze the limitations of this approach and give some conditions on the RBG-links to hope to be able to continue to pursue this strategy. Note that except for some minor corrections and improvements, none of this work is original and if one wishes to go into further detail, the original sources are given in each section.

4.1 The Trace embedding lemma and Exotic $\mathbb{R}^4$'s

As a first application of the s-invariant to problems in 4-manifold topology, we will see how we can use it to construct exotic $\mathbb{R}^4$'s.

Consider a closed, oriented, smooth 4-manifold X and a 2-disk D properly embedded in $X - \text{int } D^4$, such that $\nu D \cong D \times D^2$ is a tubular neighborhood of D. Since D is properly embedded, then its boundary is identified with a knot K, such that if we compute the self-intersection of D by taking a parallel copy $D' \times \{p\} \subset \nu D$ along some tranverse vector field, then we identify a knot $K' = \partial D'$, such that K' will be a parallel copy of K, which in turn determines a framing for K (compare with Example 2.3.10). This leads to the following definition.

Definition 4.1.1. Let X be a closed, oriented, smooth 4-manifold and $(K, k) \subset S^3$ be a framed knot. We say that (K, k) is slice in X or $X^\circ = X - \text{int } D^4$ if K is the boundary of a properly, smoothly embedded disk D in X° which induces the framing k on K.

From here on out, we might refer to a (K, k) slice knot as k-slice, and we will use X° to denote the manifold obtained from X by removing a disk as in the previous definition.

If D is a slice disk for (K, k), then the framing on K will be equal to the negative of the self-intersection number of D, that is $k = -[D] \cdot [D]$. If νD is a tubular neighborhood of D in X°, then the boundary of the exterior $X^\circ - \nu D$ coincides with the boundary of $\nu D \subset X^\circ$. Since D induces the framing k on K, then the boundary of $X^\circ - \nu D$ is identified with $S^3_k(K)$. If K bounds a null-homologous disk in X°, then we have the following notion of sliceness.

Definition 4.1.2. Let X be a closed, smooth 4-manifold. We say that a knot $K \subset S^3$ is H-slice in X if it bounds a smoothly, properly embedded disk D in $X^\circ = X - \text{int } D^4$, such that $[D] = 0 \in H_2(X^\circ, \partial X^\circ; \mathbb{Z})$.

Note that if X has definite intersection form (either positive or negative), then the only homology class with self-intersection 0 is the 0 class, so if $(K, 0)$ is slice in X°, then K must also be H-slice. If X has no homology instead, then K being slice means that K is also H-slice, so in particular the two notions agree in S^4 with the usual notion of sliceness.

When we discuss sliceness of knots in manifolds other then S^4, the first thing that comes to mind should be the knot trace $X_k(K)$. In this case the core of the 2-handle is a disk D which is bound by K in the interior of $X_k(K)$. If we now remove a 4-ball (for instance the 0-handle), we get a properly embedded disk D with self-intersection k and boundary K. After changing the orientation of the boundary S^3 so it coincides with the one coming from the usual normal-first orientation, then we see that $(-K, -k)$ bounds a slice disk with self intersection $[D] \cdot [D] = k$. Thus there is an intrinsic relationship between slice disks and knot traces, which is highlighted in the following lemma. For this lemma, we will adapt the proof of lemma 3.3 of [HP19], so as to suit our later needs.

Lemma 4.1.3 (Framed trace embedding lemma)**.** *Let X be a closed, oriented, smooth 4-manifold. A framed knot $K \subset S^3$ is slice in X with slice disk $[D] = \beta \in H_2(X^\circ, \partial X^\circ; \mathbb{Z})$ such that $\beta \cdot \beta = -k$ if and only if $-X_k(K)$ smoothly embeds in X with embedding $f\colon -X_k(K) \to X$ such that $f_*(\alpha) = \beta$, where α denotes a generator of $H_2(-X_k(K); \mathbb{Z})$.*

Proof. Take $D \subset X^\circ$ to be a slice disk for (K, k) which admits a tubular neighborhood $\nu D \cong D \times D^2$ in X°. Notice now that if we glue a 4-ball B to X° then $T = D \cup B$ will be a knot trace smoothly embedded in X. When we view the S^3 component where the gluing happens as the boundary of X° then the 2-handle realizing T is attached along K, but since knot traces are obtained by attaching a 2-handle to S^3 oriented as the boundary of D^4, then T is glued along the mirror of K, so it is actually a knot trace of $-K \subset \partial D^4$. It remains to check that the framing on the trace T is in fact $-k$. To do so, take a Seifert surface F for K and push it into D^4. Now $\Sigma = F \cup D$ will be a closed, oriented surface which under the isomorphism $H_2(X) \cong H_2(X^\circ, \partial X^\circ)$ represents β. Since $\Sigma \subset T$ is also homologically non-trivial, then it represents a generator $\alpha \in H_2(T)$, and thus under the inclusion $i_* \colon H_2(T) \to H_2(X^\circ, \partial X^\circ)$ we have an identification $i_*(\alpha) = \beta$. Since by hypothesis $[\Sigma] \cdot [\Sigma] = \beta \cdot \beta = -k$, then $\alpha \cdot \alpha = -k$ and so T is identified with $-X_k(K)$.

Assume now that $-X_k(K)$ smoothly embeds in X. Take $\Sigma_{pl} \subset -X_k(K)$ to be a piecewise-linear closed surface representing the generator $\alpha \in H_2(-X_k(K))$, which is constructed by gluing the cone $C(K)$ of the knot to the core of the 2-handle forming the trace. We can now find a neighborhood $B \cong D^4$ of the cone point that intersects Σ_{pl} along a disk Δ with $\partial \Delta = -K \subset S^3 = \partial B$. Since F_{pl} is smooth away from the cone point, then by removing such a neighborhood we can find a surface $\Sigma = \Sigma_{pl} - \text{int } \Delta$ smoothly embedded in $X - \text{int } B \cong X^\circ$. Since the boundary of Σ is given by $-K$ when viewed in $S^3 = \partial B$, then in a similar fashion to the previous case, the boundary of Σ will be given by K when viewed in $X - \text{int } B \cong X^\circ$. Since Σ is obtained by gluing an annulus $K \times I$ to the core disk of the 2-handle in $-X_k(K)$, then $\Sigma \subset X^\circ$ is a slice disk for (K, k). As before, the inclusion $i_* \colon H_2(-X_k(K)) \to H_2(X)$ identifies $i_*(\alpha) = \beta \in H_2(X)$. Since Σ represents a non-trivial element $\alpha' \in H_2(X^\circ, \partial X^\circ)$, then under the isomorphism $H_2(X^\circ, \partial X^\circ) \cong H_2(X)$ we have an identification of α' with β. Furthermore, since $[\Sigma] \cdot [\Sigma] = \alpha' \cdot \alpha' = -k$, then $\beta \cdot \beta = -k$ and the claim follows. ☐

By specializing to the H-slice case, we immediately get the following corollary.

Corollary 4.1.4 (H-slice trace embedding lemma)**.** *Let X be a closed, oriented, smooth 4-manifold. A knot $K \subset S^3$ is H-slice in X if and only if $-X_0(K)$ smoothly embeds in X, inducing the zero map on $H_2(X^\circ, \partial X^\circ)$.* ☐

We finally note that the 4-sphere admits an orientation reversing diffeomorphism, so by specializing to the usual sliceness case we get the following corollary.

Corollary 4.1.5 (Trace embedding lemma)**.** *A knot $K \subset S^3$ is slice if and only if $X_0(K)$ smoothly embeds in S^4.* ☐

Since our goal is to distinguish between topological and smooth structures, then we must introduce the following topological notion of sliceness.

Definition 4.1.6. Let M be a topological m-manifold embedded in another topological n-manifold N. We say that M is locally flat at a point $x \in N$ if there is a neighborhood U of x, such that the pair $(U, U \cap N)$ is homeomorphic to $(\mathbb{R}^n, \mathbb{R}^{n-m})$ or $(\mathbb{R}^n_+, \mathbb{R}^{n-m})$. We say that a topological embedding is locally flat if it is locally flat at every point. In particular if $K \subset S^3 = \partial D^4$ is a knot, then K is topologically slice if there is a disk D with $\partial D = K$, such that D is locally flat in D^4.

Note that the flatness condition is necessary, otherwise every knot is topologically slice - we can obtain a topological embedded disk bounding K by taking the cone over K, but the embedding won't be flat at the cone point. For flat embeddings, we have an analogous result to the Tubular neighborhood theorem ([Bro62]). This allows us to prove the previous theorem by changing every instance of the word smooth with topologically flat. So by virtue of the Trace embedding lemma (Corollary 4.1.5) working in both the smooth and topological flat category, then by finding a topologically slice knot which is not smoothly slice we are able to distinguish between smooth structures of $\mathbb{R}^4$.

Definition 4.1.7. We say that an exotic $\mathbb{R}^4$ is large if it contains a compact, smooth 4-manifold which cannot be smoothly embedded in the standard $\mathbb{R}^4$.

Theorem 4.1.8. *If there exists a knot K which is topologically slice but not smoothly slice, then there exists a large exotic $\mathbb{R}^4$.*

Proof. Suppose K is a topologically slice knot which is not smoothly slice, then by the Trace embedding lemma (Corollary 4.1.5), $X_0(K)$ admits a topological flat embedding in S^4. If we identify $S^4 = R^4 \cup \{\infty\}$, then $X_0(K)$ also topologically embeds in $\mathbb{R}^4$ (away from the point at infinity), and we can consider the manifold $Y = \mathbb{R}^4 - \text{int}\, X_0(K)$. Note that Y is a connected, non-compact manifold with boundary, so by Theorem A.0.6 it admits a smooth structure. We equip Y with this smooth structure, which induces a smooth structure on the boundary $\partial Y = \partial X_0(K)$ that is homeomorphic to $S^3_0(K)$, but since by Theorem A.0.10 every 3-manifold admits exactly one smooth structure, then ∂Y is also diffeomorphic to $S^3_0(K)$. If we denote such a diffeomorphism by $\varphi\colon \partial Y \to S^3_0(K)$, then since Y and $X_0(K)$ are both smooth, we have that $R = Y \cup_\varphi X_0(K)$ is a smooth manifold homeomorphic to $\mathbb{R}^4$. Making use of the Trace embedding lemma again, we know that $X_0(K)$ doesn't smoothly embedded in the standard $\mathbb{R}^4$, but it is smoothly embedded in R, so R must be a manifold which is homeomorphic but not diffeomorphic to $\mathbb{R}^4$, hence an exotic $\mathbb{R}^4$. Furthermore R is a large exotic $\mathbb{R}^4$, since $X_0(K)$ is a compact, smooth submanifold of R which doesn't embed smoothly in the standard $\mathbb{R}^4$. $\square$

We can make use of the s-invariant to obstruct the sliceness of K, so we just need to find a source of topologically slice knots. The following theorem gives us a sufficient condition for a knot to be topologically slice.

Theorem 4.1.9 (Freedman - Theorem $11.7B$ of [Fre82])**.** *A knot $K \subset S^3$ with Alexander polynomial $\Delta_K(t) = 1$ is topologically slice.* $\square$

In [Shu07] the author checked all non-alternating knots with up to 16 crossings for knots which satisfy the requirements of the previous theorem. He found 699 knots with Alexander polynomial 1, from which 82 have have non-zero s-invariant. We can then use any of these to construct an exotic $\mathbb{R}^4$.

4.2 The Conway knot is not slice

One of the most renowned applications of the s-invariant comes in the form of the recent proof that the Conway knot is not slice. This is an 11 crossing knot (Figure 4.2.1) whose sliceness has been an open question for 50 years, making it the last knot with less than 13 crossings for which we couldn't determine its sliceness. It turned out that the s-invariant was the missing ingredient. What distinguishes the s-invariant from the other popular slice obstructions (for instance the τ-invariant from knot Floer homology [OS03]) is that it is not a 0-trace invariant ([Pic19]). If we have two knots with diffeomorphic 0-traces, then by the Trace embedding lemma (Corollary 4.1.5) we can obstruct the sliceness of one by computing the s-invariant of the other (which isn't possible with the τ-invariant since if one of the knots has vanishing τ, then so will the other). This is exactly the strategy that Piccirillo employed in [Pic20] to finally prove that the Conway knot C is not slice. In this section, we follow the work of Piccirillo and give a proof of this acclaimed result.

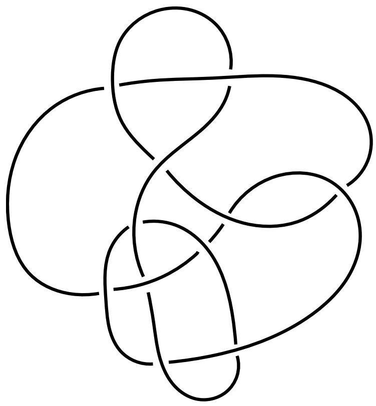

Figure 4.2.1: The Conway knot.

To do so, we start by defining a particular type of three-component link called dualizable link. The main interest in this type of link comes from the fact that hidden inside them we can always find two knots with diffeomorphic 0-trace.

Definition 4.2.1. Let $L = (R, r) \cup (B, b) \cup (G, g)$ be a three-component, integrally framed link in S^3. We say that L is a dualizable link if $r = 0$ and if we have link isotopies $B \cup R \cong B \cup \mu_B$ and $G \cup R \cong G \cup \mu_G$, where μ_B, μ_G denote meridians of B and G respectively.

Since all the framings on a dualizable link are integral, then we can always construct a 4-manifold by attaching handles along the components of the link.

Theorem 4.2.2. *Any dualizable link L has a pair of associated knots K_B and K_G and a 4-manifold X such that $X \cong X_0(K_B) \cong X_0(K_G)$.*

Proof. Define X to be the 4-manifold obtained by attaching a 1-handle to R and 0-framed 2-handles to B and G. Since R is isotopic to a meridian μ_B, then after isotopies, R will bound a disk D_R which intersects B exactly once and which intersects G along a finite number of points. Note now that $B \cup R$ form a cancelling $1/2$-handle pair, which to remove, we first need to remove the intersection points of G and D_R. This can be done by sliding G over B, which will yield a new knot K_G with framing 0, which is unknotted from $B \cup R$. We can now remove the cancelling $1/2$-pair and are left with a 0-framed 2-handle attached directly to D^4, which corresponds to the knot trace $X_0(K_G)$. This gives us a diffeomorphism between X and $X_0(K_G)$. Since we can do the same by reversing the roles of B and G, then we also have a knot K_B and diffeomorphism between X and $X_0(K_B)$. $\square$

While the previous theorem gave us a procedure to construct knots with diffeomorphic 0-traces, we still need to find a way to get a dualizable link starting from our knot of interest C. We note that C has unknotting number 1 and thus the following proposition will give us a technique to do so.

Lemma 4.2.3. *Let $K \subset S^3$ be a knot with unknotting number 1. Then there exists a dualizable link L with associated 4-manifold X, such that $X \cong X(K)$.*

Proof. Let D be a diagram for K (which will play the role of B in our dualizable link) and c a positive crossing which unknots D (the proof for negative c is identical). Construct now a link L' as depicted on the left of Figure 4.2.2 - R is a 0-framed blackboard parallel of D on the portion of the diagram that is not depicted in the figure, and G is a 0-framed meridian of R. Proceeding as in the previous theorem, we can now construct a 4-manifold X by gluing a 1-handle to R and 0-framed 2-handles to K and G. Note that $R \cup G$ forms a cancelling $1/2$-handle pair, and to remove it, we first need to slide K over G until we undo all the crossings between K and R. We can do so without changing the knot type of K, so after cancelling the $1/2$-handle pair, we are left with $X_0(K)$. It then follows that $X \cong X_0(K)$. Note that L' is not necessarily dualizable, so we still need to find a dualizable link L with the same associated 4-manifold. To do so we start by sliding B over R as depicted in Figure 4.2.2. Recall that a blackboard parallel D' of D will have $lk(D, D') = w(D)$, so since R is obtained from D' by changing a crossing and adding $-w(D) + 2$ twists with D, then $lk(K, R) = w(D) - w(D) + 2 - 1 = 1$. The slide changes the framing of K by -2 and since we remove the crossing c, then B can be isotoped to an unknot (K has unknotthing number 1). We now slide B over R again as indicated in the figure. Now B will be 0-framed with $lk(B, G) = 0$ and since R is a parallel copy of D with a crossing change (in particular it will have unknotting number 1), then we can isotope it to be an unknot meridian of B. Since G is a 0-framed meridian of R by construction, it then follows that L is a dualizable link. $\square$

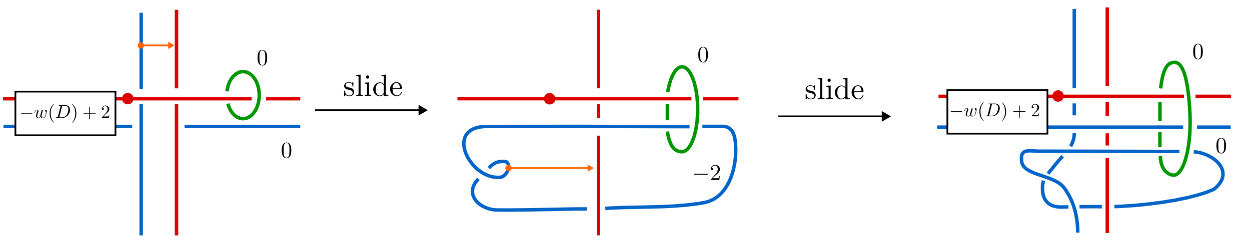

Figure 4.2.2: Constructing the dualizable link associated to a knot with unknotting number 1.

We are now ready to prove that the Conway knot is in fact not slice.

Theorem 4.2.4. *The Conway knot is not slice.*

Proof. We proceed as in Lemma 4.2.3 to construct a dualizable link L, with associated 4-manifold X such that $X \cong X(C)$ (Figure 4.2.3). By Theorem 4.2.2, we can now do isotopies and handle slides to find a Kirby diagram for K' as in the right side of Figure 4.2.3 (the full handle calculus is done in Figure 4 of [Pic20]). The goal is now to obstruct the sliceness of K' by using the s-invariant. To do so the author computes the values of $dim(Kh^{i,j}(K'))$ via Bar-natan's Kh fast routine, which we present in Table 4.1. We know that Lee homology is contained in homological degree 0, so either $s(K') = 0$ or $s(K') = 2$. To verify that $s(K')$ is in fact 2, we analyze the pages of the spectral sequence induced by the filtration. Note that the differential d^n of the n-th page, has bidegree $(1, 4(n-1))$, so by looking at the E^n pages (for $n \geq 2$), we see that there are no generators in degree $(1, 4(n-1)+3)$ or $(-1, -4(n-1)+3)$. This means that the generator in $Kh^{0,3}(K')$ must survive to E^∞, so $s_{max}(K') = 3$ and thus $s(K') = 2$. By Theorem 3.3.14 we then have that any properly embedded surface in the 4-ball with boundary K' must have genus at least 1, so in particular K' can't be slice. Now by using the the Trace embedding lemma (Corollary 4.1.5) we see that $X_0(K')$ doesn't embed in S^4, and since $X_0(K') \cong X_0(C)$, then $X_0(C)$ can't embed in S^4 and thus C can't be slice. □

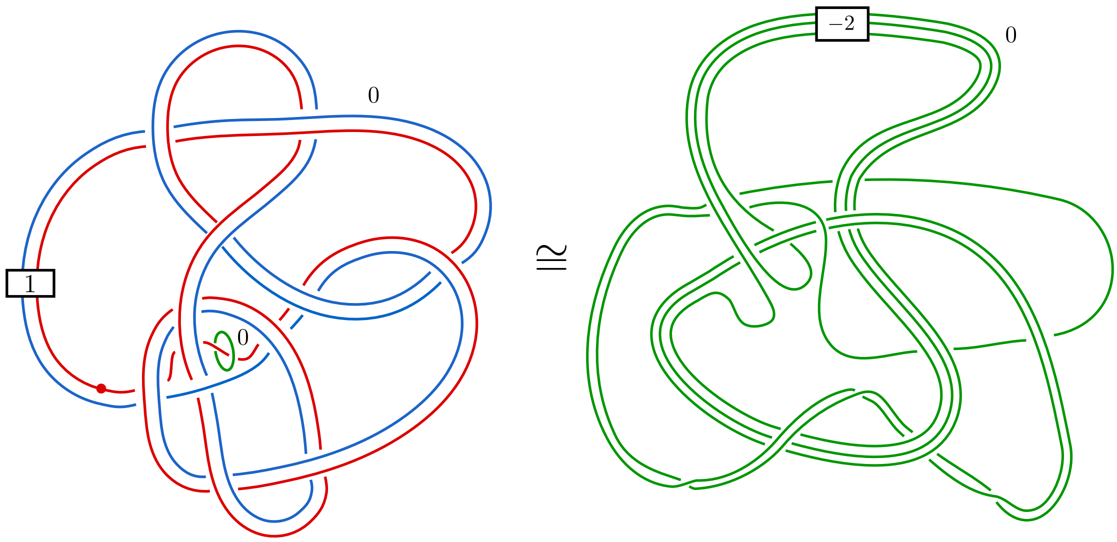

Figure 4.2.3: X is diffeomorphic to $X_0(K')$.

4.3 An approach to the smooth Poincaré conjecture

In [FGMW10] a possible strategy to disprove the Smooth Poincaré conjecture in dimension 4 was proposed - find a homotopy 4-sphere X and remove a 4-ball, so that $\partial(X - \text{int } D^4) = S^3$. If we now find a knot $K \subset S^3$ which bounds a smoothly embedded disk in X° but which is not smoothly slice (in the standard 4-ball), then by Theorem 2.4.11 X is homeomorphic to S^4 but by the Trace embedding lemma (Corollary 4.1.5) it can't be diffeomorphic. Thus X would be an exotic S^4. They propose we use the s-invariant to obstruct the sliceness of K since it is not known whether or not the s-invariant vanishes if K bounds a smoothly embedded disk in a homotopy 4-ball (unlike his cousin τ from knot Floer homology, which is known to not distinguish between homotopy 4-balls).

The authors looked at knots in two particular homotopy 4-spheres coming from the Cappel-Shaneson construction ([CS76]).

j \ i	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
49																																				1
47																																				
45																																	1	1	1	
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Table 4.1: Khovanov homology of K' (Table 1 of [Pic20]).

Definition 4.3.1. Let M be a closed, oriented 3-manifold and $\phi\colon M \to M$ a self-diffeomorphism of M. Consider now the mapping torus

$$T_\phi = \frac{M \times I}{\phi(M \times \{1\}) \sim M \times \{0\}}$$

which we can identify with an M-bundle over S^1. Notice that the 0-section sphere will have a framing $f \in \pi_0(O(3)) \cong \mathbb{Z}_2$, so we define the Cappel-Shaneson construction of the pair (M, ϕ) to be the manifold obtained by switching the framing on this sphere (i.e. by doing surgery on a circle).

We're particularly interested in homotopy 4-spheres, so we shift our attention to the family coming from the 3-torus. In such a case, the Cappell-Shaneson construction is known to produce homotopy 4-spheres. This family is labeled as Σ_m and is obtained by taking the mapping torus of T^3 with diffeomorphism

$$A_m = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & m+1 \end{pmatrix}$$

for $m \in \mathbb{Z}$. One choice of framing always yields back S^4, so it is standard to only consider the other choice when talking about these examples.

In particular, the authors constructed knots in both Σ_{-1} and Σ_1, and unfortunately after many hours of computations they found that both knots had s-invariant 0, so they weren't useful for this approach. This turns out not to be a coincidence however, since only a few days later, it was shown that all homotopy spheres Σ_m were in fact standard ([Akb10]). This doesn't mean that the proposed approach isn't viable, only that we need another source of homotopy 4-spheres.

With that in mind we consider the Gluck construction. In [Glu62] Gluck showed that any orientation-preserving diffeomorphism of $S^2 \times S^1$ is either isotopic to the identity or to the map $\varphi\colon S^2 \times S^1 \to S^2 \times S^1$, given by $\varphi(x, \theta) = (r_\theta(x), \theta)$, where r_θ denotes the rotation by an angle θ about the z-axis of S^2, i.e r_θ is the generator of $\pi_1(SO(3)) \cong \mathbb{Z}_2$. This remarks leads to the following definition.

Definition 4.3.2. Let X be a smooth 4-manifold and $S \subset X$ and embedded 2-sphere with trivial normal bundle. The operation of removing a tubular neighborhood $\nu S \cong S^2 \times D^2$ of S from X and gluing in $D^2 \times S^2$ via the non-trivial diffeomorphism of the boundary given before, is called a Gluck twist of X along S and denoted by $G_S(X)$.

In particular, for a simply connected, closed 4-manifold X, a Gluck twist along a null-homologous S will always produce a manifold $G_S(X)$ with the same intersection form as X (see [Glu62]), so by Freedman's theorem (Theorem 2.4.11) it will also be homeomorphic X. If $X = S^4$ then this gives us a source of homotopy 4-spheres which we may use in the strategy outlined before.

Note that a Gluck twist is equivalent to doing surgery on S (removing a tubular neighborhood $\nu S \cong S^2 \times D^2$ and gluing in $S^1 \times D^3$), changing the framing on this new circle (framings of S^1 in a 4-manifold are in correspondence with $\pi_0(O(3)) \cong \mathbb{Z}_2$) and then doing surgery again on this circle to glue in $D^2 \times S^2$. We make use of this remark, to draw a Kirby diagram for $G_S(X)$ as follows - the embedded sphere S with trivial normal bundle, will be a 0-framed unknot in a diagram for X, so applying a Gluck twist on S, amounts to changing the unknot to a dotted circle, applying a ± 1 twist to all 2-handles running through this dotted circle (equivalent to blowing down a ∓ 1-framed unknot parallel to the dotted circle), and then changing the dotted circle back to a 0-framed unknot (Figure 4.3.1).

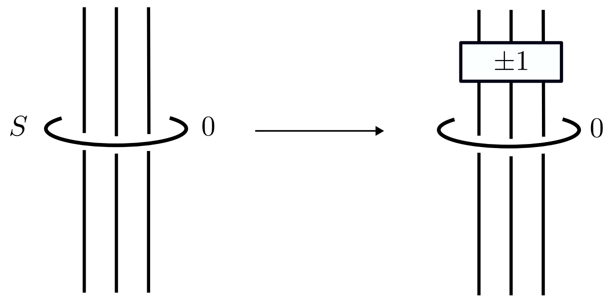

Figure 4.3.1: Gluck twist.

An alternative way to see this is by adding a cancelling $1/2$-handle pair as in the beggining of Figure 4.3.2, surgering the 0-framed unknot (corresponding to νS) to a dotted circle and surgering the introducted dotted circle to a 0-framed unknot. We claim this operation describes a Gluck twist and it is equivalent to the previous description.

Proposition 4.3.3. *The 4-manifold obtained by the previous operation is diffeomorphic to $G_S(X)$. In particular, the two previous descriptions are equivalent.*

Proof. When we do surgery on the 0-framed unknot describing νS, we are removing a tubular neighborhood of S and gluing in an $S^1 \times D^3$, and by doing surgery on the dotted circle coming from the cancelling $1/2$-handle pair we are removing an $S^1 \times D^3$ and gluing in a $D^2 \times S^2$. Since after the surgeries we are still left with a cancelling $1/2$-handle pair, then after slides we will be able to cancel it, so this operation corresponds to removing νS and gluing in $D^2 \times S^2$. This is depicted in Figure 4.3.2, along with a diffeomorphism to the previous description. Note that the 2-handles belonging to X, only change framing in the first 2-handle slides (since all others correspond to sliding over 0-framed unknots or dotted circles which don't link the 2-handles) and it corresponds to the same type of slides which we use to derive the blow-up operation (as in Figure 2.5.8) so the framing changes in the same way. It remains to check that we are not simply removing νS and gluing it back in by the trivial diffeomorphism of the boundary. To do so, we apply the operation to one of the 0-framed unknots in the diagram of $S^2 \times S^2$ as in Figure 4.3.3. Since we obtain a diagram for $S^2 \tilde{\times} S^2$, which is not homeomorphic to $S^2 \times S^2$ (see Example 2.5.1), then the gluing is not trivial and thus it corresponds to a Gluck twist. □

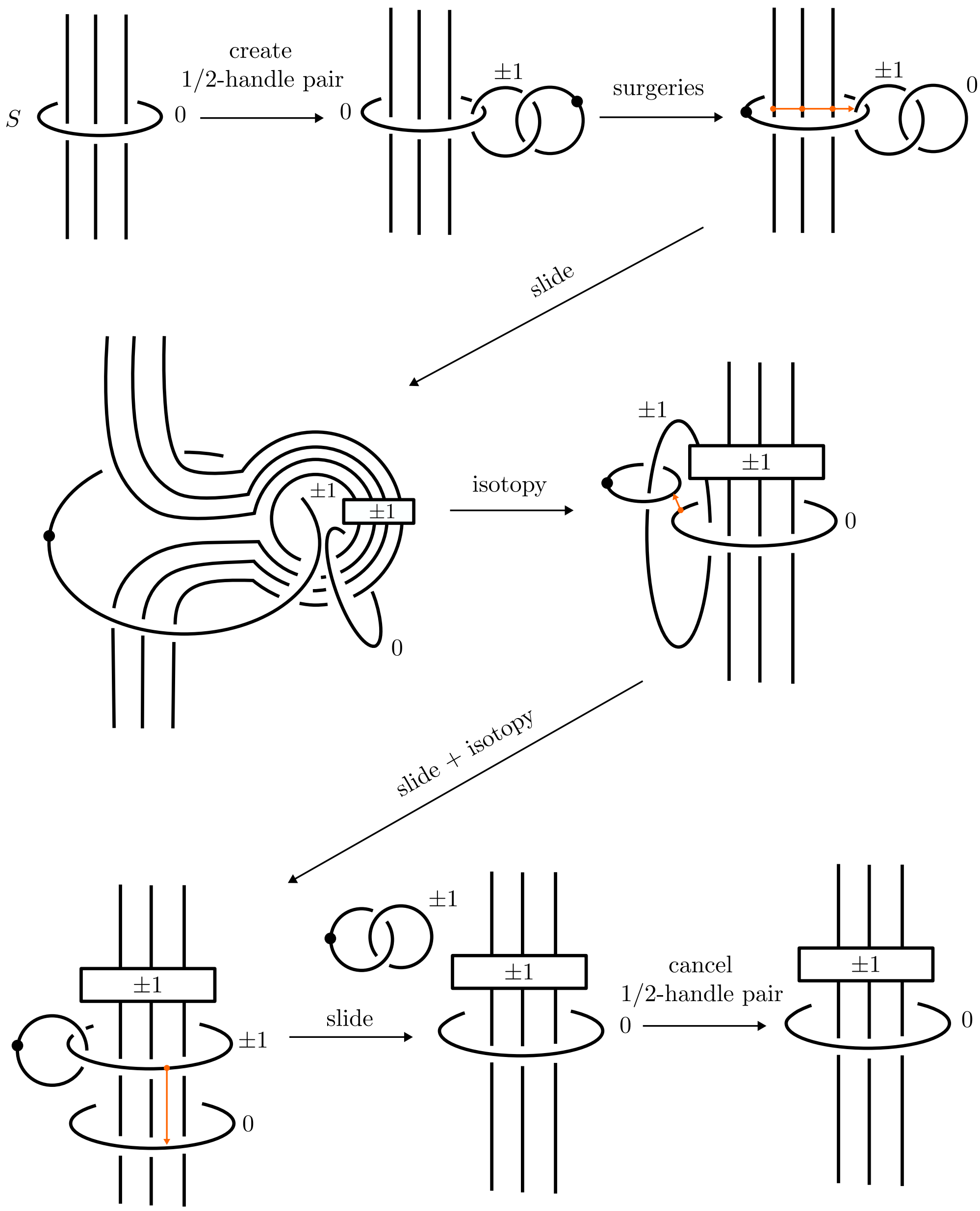

Figure 4.3.2: Alternative description of the Gluck twist.

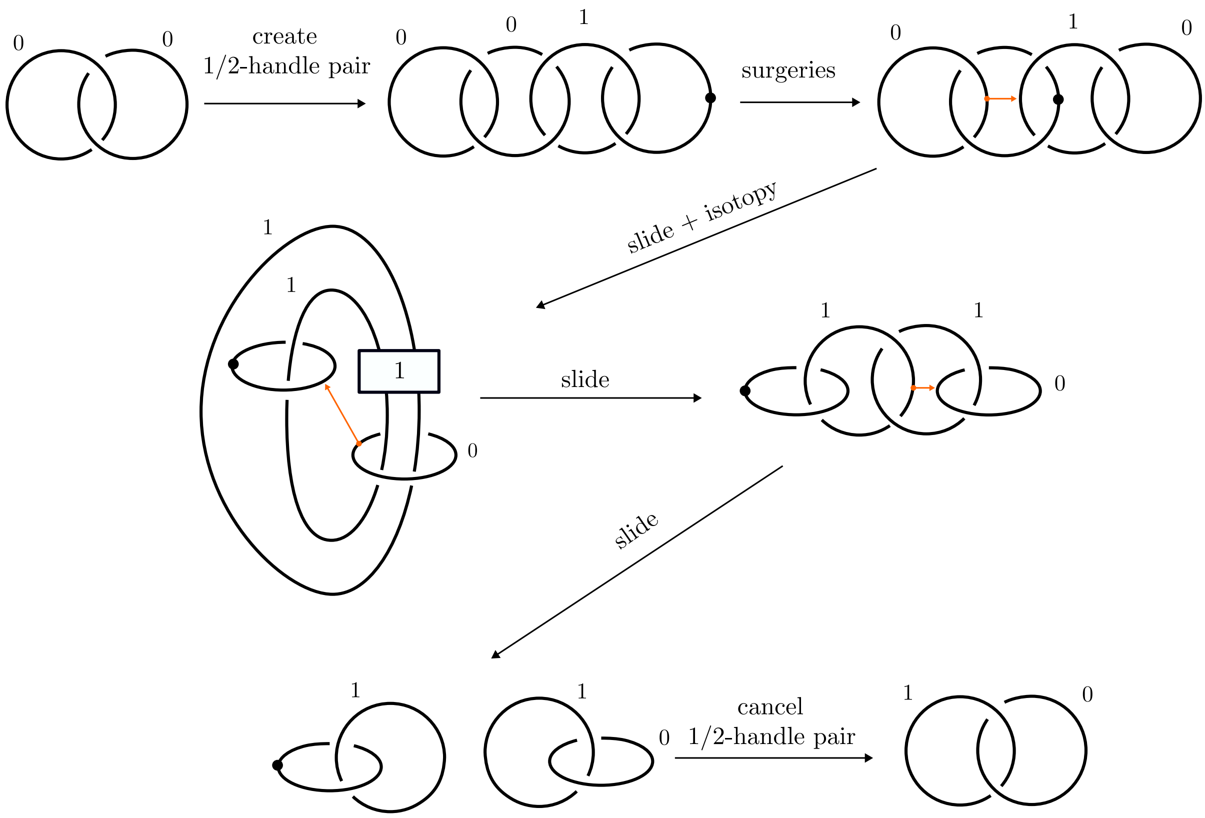

Figure 4.3.3: The gluing map is not trivial.

Example 4.3.4. If S is an unknotted 2-sphere embedded in S^4, then Gluck twisting S won't change the diffeomorphism type of S^4. To see this consider the standard handle diagram for S^4 given by a 0-handle and a 4-handle, and add a canceling $2/3$-handle pair. Since S^4 is closed, then the 3-handle is uniquely attached and so the Kirby diagram for this handle description will simply be given by a 0-framed unknot U representing the 2-handle. This unknot represents a tubular neighborhood $\nu S \cong S^2 \times D^2$ of the unknotted 2-sphere S, and since no other handles go through U, then Gluck twisting along S will yield the same diagram. We can cancel the $2/3$-handle pair to recover the standard diagram for S^4.

In general it is not known whether Gluck twisting a knotted S^2 in S^4 will recover a manifold diffeomorphic to S^4, but it turns out that for a generic 2-sphere in a smooth 4-manifold X, the manifold obtained by Gluck twisting along S becomes diffeomorphic to X after taking a connected sum with $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$.

Theorem 4.3.5. *Let S be a 2-sphere smoothly embedded in a smooth 4-manifold X. Then $G_S(X)\#\mathbb{CP}^2$ is diffeomorphic to $X\#\mathbb{CP}^2$ and $G_S(X)\#\overline{\mathbb{CP}^2}$ is diffeomorphic to $X\#\overline{\mathbb{CP}^2}$.*

Proof. The proof is given in Figure 4.3.4. A generic diagram for S in X is given locally by a 0-framed unknot with possibly some other 2-handles of X going through it. We start by blowing up a ± 1-framed unknot parallel to S. After that we do the Gluck twist along S by adding ± 1 twists to the strands going through S. Note that blowing up and doing the Gluck twist changes the framing of the strands going through S by ± 1 and ∓ 1 respectively, so after both operations the framing on the handles of X doesn't change. After handle-sliding the ± 1-framed unknot over S we recover $X\#\mathbb{CP}^2$ or $X\overline{\#\mathbb{CP}^2}$ depending on the sign. ☐

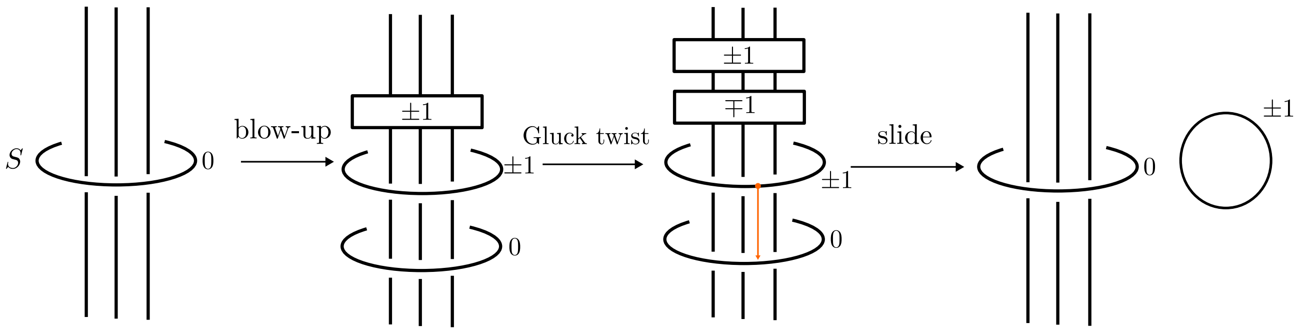

Figure 4.3.4: Diffeomorphism between a blow-up of X and a blow-up of $G_S(X)$.

4.4 An Adjunction inequality in $\#n\mathbb{CP}^2$

If one wishes to employ the strategy of [FGMW10] to a homotopy 4-sphere $G_S(S^4)$ obtained by doing a Gluck twist, then we need to find a way to obstruct the sliceness of a knot $K \subset G_S(S^4) - \text{int } D^4$. Note that by Theorem 4.3.5, both S^4 and $G_S(S^4)$ become diffeomorphic to $\mathbb{CP}^2$ (or $\overline{\mathbb{CP}^2}$) after taking a connected sum with $\mathbb{CP}^2$ (or $\overline{\mathbb{CP}^2}$), so to be able to employ their strategy, we need to find an inequality similar to Theorem 3.3.14 which holds in $\mathbb{CP}^2$ (or $\overline{\mathbb{CP}^2}$). In this direction, Manolescu, Marengon, Sarkar and Willis were able to prove an adjunction inequality for knots in $\#n\mathbb{CP}^2$ and $\#n\overline{\mathbb{CP}^2}$. In this section, we follow their work developed in [MMSW23]. This will not only be able to prove that the FGMW approach can't work for Gluck twists, but also pave the way for a new approach to the smooth Poincaré conjecture (and more) which we will discuss in the following sections.

Theorem 4.4.1. *Let $X = \#n\overline{\mathbb{CP}^2}$, $K \subset \partial X^\circ = S^3$ be a knot and $\Sigma \subset X$ be a smoothly embedded, oriented surface such that $\partial\Sigma = K$ and $[\Sigma] = 0 \in H_2(X^\circ, \partial X^\circ; \mathbb{Z})$. Then*

$$s(K) \leq 1 - \chi(\Sigma)$$

where $\chi(\Sigma)$ denotes the Euler characteristic of Σ.

Proof. Consider the $\overline{\mathbb{CP}^1}$ sitting inside $\overline{\mathbb{CP}^2}$ with tubular neighborhood $\nu\overline{\mathbb{CP}^1}$. From our previous discussion, it is clear that $\nu\overline{\mathbb{CP}^1}$ is a D^2-bundles over S^2 with Euler number -1, so it has boundary $\partial(\nu\overline{\mathbb{CP}^1}) \cong S^3_{-1}(U) \cong S^3$, where U is the unknot ($\nu\overline{\mathbb{CP}^1}$ is represented by the (-1)-framed unknot in the handle description of $\overline{\mathbb{CP}^2}$ given in Example 2.3.10). Notice that the projection map $S^3 \longrightarrow \overline{\mathbb{CP}^1}$ describes the Hopf fibration. Consider now the surface $\Sigma \subset X$. For each copy C_i of $\overline{\mathbb{CP}^1} \subset \#n\overline{\mathbb{CP}^2}$ (for $i = 1..., n$) we can assume that Σ intersects each C_i transversely along a finite number of points. Since $[\Sigma] = 0 \in H_2(X^\circ, \partial X^\circ)$, then $[\Sigma] \cdot [\overline{\mathbb{CP}^1}] = 0$ and thus Σ must intersect each C_i in $2p_i$ points, p_i positively and p_i negatively. For each C_i, remove now a copy of $\nu\overline{\mathbb{CP}^1} \subset X$. Since $\overline{\mathbb{CP}^2} - \nu\overline{\mathbb{CP}^1} \cong D^4$ (if we remove the 0- and 2-handle from $\overline{\mathbb{CP}^2}$ we are left with a 4-ball from the 4-handle) then what remains is $Y = \#nD^4_i$. Since Σ sits inside X°, then when we remove the neighborhoods of each C_i, we also remove neighborhoods of the points of intersection of Σ with C_i. This yields a surface Σ° which is obtained from Σ by removing $2p_i$ disks (for $i = 1, ..., n$). Note that $\Sigma^\circ \subset Y^\circ$ intersects each boundary component $S^3_i = \partial D^4_i \subset \#nD^4_i$

along $2p_i$ fibers of the Hopf fibration, such that p_i are oriented in one way and p_i are oriented in the other. This corresponds to the link $F_{(2p_i,2p_i)}(1)$ with p_i strands oriented in one way and p_i strands oriented in the other (Figure 4.4.1). By further removing the 4-ball from X (to obtain X°) and removing neighborhoods of arcs connecting the boundary components S^3_i, we get $Y^\circ \cong S^3 \times I$. We then have a cobordism Σ° in $S^3 \times I$, from a knot K to a link $F_{(2p_1,2p_1)}(1) \sqcup F_{(2p_2,2p_2)}(1) \sqcup \cdots \sqcup F_{(2p_n,2p_n)}(1)$, so we can use the functoriality of Khovanov homology under cobordisms (as we did to prove Theorem 3.3.14) to get the following inequality

$$s(K) + \chi(\Sigma^\circ) \leq s(F_{(2p_1,2p_1)}(1) \sqcup F_{(2p_2,2p_2)}(1) \sqcup \cdots \sqcup F_{(2p_n,2p_n)}(1))$$

Since Σ° is obtained from Σ by removing $2p_i$ disks (for $i = 1, ..., n$), then $\chi(\Sigma^\circ) = \chi(\Sigma) - (2p_1 + 2p_2 + \ldots + 2p_n)$. Furthermore by Proposition 3.4.3 we have that $s(F_{(2p_1,2p_1)}(1) \sqcup F_{(2p_2,2p_2)}(1) \sqcup \cdots \sqcup F_{(2p_n,2p_n)}(1)) = s(F_{(2p_1,2p_1)}(1)) \sqcup s(F_{(2p_2,2p_2)}(1)) \sqcup \cdots \sqcup s(F_{(2p_n,2p_n)}(1)) - n$. Now we just need to compute $s(F_{(2p,2p)}(1))$. This turns out to be a really hard task, but luckily the majority of [MMSW23] is dedicated to it, and thus by Theorem 1.7 of the aforementioned article we know that $s(F_{(2p,2p)}(1)) = 1 - 2p$. It then follows that

$$\begin{aligned}
s(K) &\leq s(F_{(2p_1,2p_1)}(1) \sqcup F_{(2p_2,2p_2)}(1) \sqcup \cdots \sqcup F_{(2p_n,2p_n)}(1)) - \chi(\Sigma^\circ) \\
&= (s(F_{(2p_1,2p_1)}(1)) \sqcup s(F_{(2p_1,2p_1)}(1)) \sqcup \cdots \sqcup s(F_{(2p_n,2p_n)}(1)) - (n-1)) - (\chi(\Sigma) - (2p_1 + 2p_2 + \ldots + 2p_n)) \\
&= ((1 - 2p_1) + (1 - 2p_2) + \ldots + (1 - 2p_n) - (n-1)) - \chi(\Sigma) + (2p_1 + 2p_2 + \ldots + 2p_n) \\
&= 1 - \chi(\Sigma)
\end{aligned}$$

thus $s(K) \leq 1 - \chi(\Sigma)$. $\square$

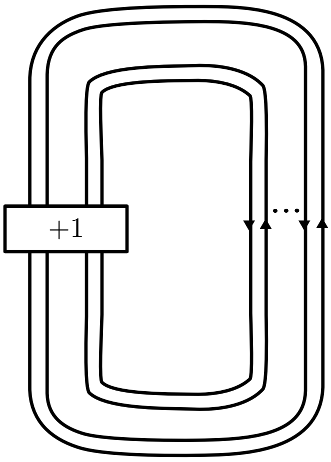

Figure 4.4.1: $F_{(p,p)}(1)$.

Note that if K bounds a null-homologous surface in $\#n\mathbb{CP}^2 - \text{int } D^4$, then its mirror $-K$ will bound a null-homologous surface in $\#n\overline{\mathbb{CP}^2} - \text{int } D^4$. Since by Proposition 3.3.9 we have that $s(-K) = -s(K)$, then we can apply the previous theorem to $-K$ and deduce a similar result for $\#n\mathbb{CP}^2$.

Corollary 4.4.2. *Let $X = \#n\mathbb{CP}^2$, $K \subset \partial X^\circ = S^3$ be a knot and $\Sigma \subset X$ be a smoothly embedded, oriented surface such that $\partial\Sigma = K$ and $[\Sigma] = 0 \in H_2(X^\circ, \partial X^\circ; \mathbb{Z})$. Then*

$$s(K) \geq \chi(\Sigma) - 1$$

where $\chi(\Sigma)$ denotes the Euler characteristic of Σ. □

To determine the H-sliceness of a knot in $\#n\mathbb{CP}^2$ or $\#n\overline{\mathbb{CP}^2}$, we need to see what happens when Σ is a disk. In this case, the Euler characteristic equals 1 and thus we have the follow corollary.

Corollary 4.4.3. *Let $K \subset S^3$ be a knot.*

(a) *If K is H-slice in $\#n\overline{\mathbb{CP}^2}$, then $s(K) \leq 0$.*

(b) *If K is H-slice in $\#n\mathbb{CP}^2$, then $s(K) \geq 0$.* □

Equipped with these inequalities, we are finally able to prove that the [FGMW10] strategy can't work on homotopy 4-spheres obtained by Gluck twisting.

Theorem 4.4.4. *Let $G_S(S^4)$ be a homotopy 4-sphere obtained from S^4 by doing a Gluck twist. If $K \subset S^3$ is H-slice in $G_S(S^4)$, then $s(K) = 0$.*

Proof. Recall that by Theorem 4.3.5, $G_S(S^4)\#\mathbb{CP}^2 \cong \mathbb{CP}^2$ and $G_S(S^4)\#\overline{\mathbb{CP}^2} \cong \overline{\mathbb{CP}^2}$, so if K is H-slice in $G_S(S^4)$ then it is also H-slice in both $\mathbb{CP}^2$ and $\overline{\mathbb{CP}^2}$. By Corollary 4.4.3 we then have that $s(K) \leq 0$ and $s(K) \geq 0$, so $s(K) = 0$. □

As with many other similarities between the s-invariant and the τ-invariant, this adjunction inequality provides yet another one. For negative definite 4-manifolds (in particular $\#n\overline{\mathbb{CP}^2}$) the following theorem is known to hold for the τ-invariant.

Theorem 4.4.5 (Theorem 1.1 of [OS03])**.** *Let X be oriented, smooth 4-manifold X with negative definite intersection form, $b_1(X) = 0$ and $\partial X = S^3$. If $K \subset \partial X$ is a knot and Σ a smooth, properly embedded surface in X with $\partial\Sigma = K$, then*

$$2\tau(K) \leq 1 - \chi(\Sigma) - |[\Sigma]| - [\Sigma] \cdot [\Sigma]$$

where $|[\Sigma]|$ denotes the L^1-norm of $[\Sigma] \in H_2(X, \partial X; \mathbb{Z})$. □

Compare this theorem with Theorem 4.4.1. For the s-invariant we are limited to $X = \#n\overline{\mathbb{CP}^2}$ and Σ null-homologous, but notice that in such conditions, the previous theorem will produce the same adjunction inequality as theorem 4.4.1 by simply replacing $2\tau(K)$ with $s(K)$. We however have no general formula for when Σ is a surface with non-trivial homology, yet it is conjectured that a similar result holds.

Conjecture 4.4.6 (Conjecture 9.8 of [MMSW23])**.** *Let $X = \#n\overline{\mathbb{CP}^2}$, K be a knot in $\partial X^\circ = S^3$ and Σ a smooth, properly embedded surface in X° such that $\partial\Sigma = K$. Then*

$$s(K) \leq 1 - \chi(\Sigma) - |[\Sigma]| - [\Sigma] \cdot [\Sigma]$$

In the same vein as Corollary 4.4.2, it is also expected that a similar result holds in $\#n\mathbb{CP}^2$.

Conjecture 4.4.7. *Let $X = \#n\mathbb{CP}^2$, K be a knot in $\partial X^\circ = S^3$ and Σ a smooth, properly embedded surface in X° such that $\partial\Sigma = K$. Then*

$$s(K) \geq \chi(\Sigma) - 1 + |[\Sigma]| + [\Sigma] \cdot [\Sigma]$$

4.5 From 0-surgeries to exotic definite 4-manifolds

As we've seen in the previous section, the idea of using the s-invariant to obstruct the sliceness of knots in the boundary $\partial X^\circ = S^3$, where X is a homotopy 4-sphere obtained by Gluck twisting, doesn't work, so a large class of potential counterexamples to the smooth Poincaré conjecture has been eliminated. To continue to pursue this idea, we need not only need to find another source of homotopy 4-spheres, but also a source of slice knots in them, so that we can apply our techniques. The following proposition gives us some direction.

Proposition 4.5.1. *Let K and K' be knots and $\phi\colon S^3_0(K) \to S^3_0(K')$ a homeomorphism between their 0-surgeries. If K is slice and K' is not slice then there exists an exotic 4-sphere.*

Proof. Since K is smoothly slice, then by the Trace embedding lemma (Corollary 4.1.5) $X_0(K)$ smoothly embeds in S^4. If D is a smooth slice disk for K and νD a tubular neighborhood for D in S^4, then we have a decomposition $S^4 = X_0(K) \cup_{S^3_0(K)} (D^4 - \text{int } \nu D)$ given by removing such an open tubular neighborhood. By hypothesis $S^3_0(K)$ is homeomorphic to $S^3_0(K')$, and thus by Theorem A.0.10 they are also diffeomorphic, so we can create a smooth 4-manifold $S = X_0(K') \cup_{\phi^{-1}} (D^4 - \text{int } \nu D)$ by removing $X_0(K)$ and gluing in $X_0(K')$ along the boundary. This new 4-manifold S will be homeomorphic to S^4, but since by the Trace embedding lemma (Corollary 4.1.5) $X_0(K')$ doesn't smoothly embed in S^4, then S can't be diffeomorphic to the standard 4-sphere, and thus is an exotic sphere. □

Note that the knot K' bounds a smoothly embedded disk in $S - \text{int } D^4$, given by the core of the 2-handle in $X_0(K')$. So if we start with a slice knot K and a 0-surgery homeomorphism between K and another knot K', then K' will be a slice knot in a homotopy 4-sphere, which we can then obstruct from being slice by using the s-invariant. So instead of trying to produce both homotopy 4-spheres and then slice knots in them, we can do both at the same time by finding knots with homeomorphic 0-surgeries - this will give us a homotopy 4-sphere with a slice knot in it. This slight tweak to the strategy of Section 4.3 also benefits from the fact that it may be possible to apply it to other closed, simply connected 4-manifolds. We just need to have a result similar to the previous proposition, and be able to obstruct the sliceness of knots in these manifolds by using the s-invariant. Due to the adjunction inequality given by Corollary 4.4.2, we can already do this in $\#n\mathbb{CP}^2$ by instead focusing on obstructing their H-sliceness. Furthermore by Donaldson's theorem (Theorem 2.4.13) every simply connected, positive definite, closed, smooth 4-manifold has intersection form equivalent to that of $\#n\mathbb{CP}^2$, and thus by Freedman's theorem (Theorem 2.4.11) they are in fact homeomorphic to $\#n\mathbb{CP}^2$. So we may be able to use the s-invariant to obstruct exotic structures on a large class of 4-manifolds. The following theorem, which generalizes the idea of the previous paragraph and proposition, will give us the last ingredient we need in order to pursue such a goal.

Theorem 4.5.2. *Let X be a closed, oriented, simply connected, smooth 4-manifold. If K is an H-slice knot in X and there exists a 0-surgery homeomorphism $\phi\colon S^3_0(K) \to S^3_0(K')$, then K' is an H-slice knot in a 4-manifold X' homotopy equivalent to X.*

Proof. Since K is H-slice in X, then it admits a smooth, slice disk $D \subset X^\circ$. If we now take a tubular neighborhood νD of D in X°, then we can define a 4-manifold $Y = X^\circ - \text{int } \nu D$, which will have boundary $\partial Y = S^3_0(K)$. Since by hypothesis $\phi\colon S^3_0(K) \to S^3_0(K')$ is a homeomorphism, then after identifying $S^3_0(K')$ with $-\partial(-X_0(K'))$ we get a homeomorphism $\phi^{-1}\colon -\partial(-X_0(K')) \to S^3_0(K) = \partial Y$ and we can construct another 4-manifold $X' = -X_0(K') \cup_{\phi^{-1}} Y$ by gluing along such a map. We can give X' a handle description as follows - give Y a handle description and think about gluing the dual of $-X_0(K')$. Since $-X_0(K')$ is given by a 0-handle along with a single 2-handle, then we glue the dual 2-handle along a 0-framed meridian $\phi^{-1}(\mu_{K'})$ of $K' \subset S^3_0(K')$, and we cap it off with a 4-handle (corresponding to the 0-handle). Now we can see that $\pi_1(X')$ is generated by $\pi_1(Y)$ with an aditional relation given by the dual 2-handle, i.e $\pi_1(X') \cong \pi_1(Y)/\langle[\phi^{-1}(\mu_{K'})]\rangle$. Note now that $\pi_1(Y)$ is normally generated by the inclusion $i\colon \partial Y \to Y$, but $\pi_1(\partial Y)$ is simply $\pi_1(S^3_0(K'))$. Since $\pi_1(S^3_0(K')) = \langle \pi_1(S^3 - K') \mid \mu^0 \rangle = \pi_1(S^3 - K')$, then it is generated by a meridian $\mu_{K'}$ of K' and thus $\pi_1(X') \cong \pi_1(Y)/\langle[\phi^{-1}(\mu_{K'})]\rangle \cong 1$. By using the Mayer-Vietoris sequence on the pair $-X_0(K') \cup Y$, one can further check that the homology of X' agrees with that of X. Since both X and X' are simply connected with the same homology, then by Theorem 2.4.10 it follows that X' is homotopy equivalent to X. It remains to see that K' is H-slice in X'. Note that K' always admits a smooth slice disk D' in $-X_0(K')$ given by the core of the 2-handle, so since $-X_0(K')$ smoothly embeds in X' by construction, then K' admits a slice disk in X'. Notice now that $-X_0(K')$ embeds in X' inducing the zero map on $H_2(X^\circ)$ (by the previous Mayer-Vietoris computation) and since by the long exact sequence on homology we have that $H_2(X'^\circ) \cong H_2(X'^\circ, \partial X'^\circ)$, then $-X_0(K')$ embeds in X' inducing the zero map on $H_2(X'^\circ, \partial X'^\circ)$, so by the H-slice trace embedding lemma (Corollary 4.1.4) K' is H-slice in X' and the result follows. $\square$

This program was started by Manolescu and Piccirillo in [MP21], setting their goals on finding pairs of knots K and K' with homeomorphic 0-surgeries, such that K is H-slice in some $\#n\mathbb{CP}^2$, while K' is H-slice in a homotopy $\#n\mathbb{CP}^2$. If we can show that K' is not H-slice in the standard $\#n\mathbb{CP}^2$ (by showing that $s(K') < 0$), then the previous theorem gives us an exotic $\#n\mathbb{CP}^2$. In this section we follow their approach, which leads to a collection of 23 knots, such that if any of them is shown to be H-slice in $\#n\mathbb{CP}^2$, then we have found an exotic $\#n\mathbb{CP}^2$.

The first step is to produce pairs of knots with homeomorphic 0-surgeries. To do so, we use the idea of dualizable links introduced in Section 4.2. As before, these will be three component links $L = R \cup B \cup G$, which earn their name from the three components involved - Red, Blue and Green.

Definition 4.5.3. An RBG-link $L = (R, r) \cup (B, b) \cup (G, g) \subset S^3$ is a three component link with rational coefficients r, b and g respectively, such that there are homeomorphisms $\psi_B\colon S^3_{r,g}(R, G) \to S^3$ and $\psi_G\colon S^3_{r,b}(R, B) \to S^3$ and such that $H_1(S^3_{r,g,b}(L); \mathbb{Z}) \cong \mathbb{Z}$.

The main motivation for working with such links is that hidden inside every RBG-link is a pair of knots with homeomorphic 0-surgeries, and in fact any 0-surgery homeomorphism can be realized as an RBG-link. Recall that dualizable links had instead an associated 0-trace diffeomorphism, and as we'll shortly see, this is a situation we'll want to avoid.

Theorem 4.5.4 (Theorem 1.2 of [MP21]). *Any RBG-link L has a pair of associated knots K_B and K_G and a 0-surgery homeomorphism $\phi_L\colon S^3_0(K_B) \to S^3_0(K_G)$. Conversely, if K and K' are knots then for any 0-surgery homeomorphism $\phi\colon S^3_0(K) \to S^3_0(K')$ there is an associated RBG-link L_ϕ such that $K_B = K$ and $K_G = K'$.*

We'll delay the proof of this theorem to focus on a more restrictive class of RBG-links. These are called special RBG-links and they have the "special" property that the homeomorphisms ψ_B and ψ_G are induced by the slam-dunk operation.

Definition 4.5.5. Let $L = (R, r) \cup (B, b) \cup (G, g) \subset S^3$ be an RBG-link. We say that L is special if $r \in \mathbb{Z}$, $b = g = 0$, $H_1(S^3_{r,b,g}(L); \mathbb{Z}) \cong \mathbb{Z}$ and if μ_R denotes a meridian of R, then there are link isotopies $R \cup B \cong R \cup \mu_R \cong R \cup G$.

Special RBG-links still have the same condition on homology that $H_1(S^3_{r,b,g}(L)) \cong \mathbb{Z}$, but since they are integrally framed, then they are boundaries of 4-dimensional 2-handlebodies and we can make use of Corollary 2.6.14 to see what this restriction entails. Indeed, $H_1(S^3_{r,b,g}(L))$ will have a presentation matrix given by the linking matrix of L,

$$A = \begin{pmatrix} r & 1 & 1 \\ 1 & 0 & lk(G,B) \\ 1 & lk(B,G) & 0 \end{pmatrix}$$

where the entries with 1 come from the fact that B and G are meridians to R. In particular by the same corollary, we have that $H_1(S^3_{r,b,g}(L))$ is finite if and only if det $A \neq 0$, so by the condition on homology the determinant of A must be zero. This means that $-lk(B,G)(r \cdot lk(B,G) - 1) + lk(B,G) = -r \cdot lk(B,G)^2 + 2lk(B,G)$ must be zero, so either $l = 0$ or $r \cdot lk(B,G) = 2$.

We now prove Theorem 4.5.4 for special RBG-links. Instead of proving the theorem in its full generality, we focus on the special case since the technique used in the proof to construct the pair of knots with homeomorphic 0-surgeries will be particularly useful in the remainder of this thesis.

Proof of Theorem 4.5.4 for special RBG-links. Since $L = R \cup B \cup G$ is a special RBG-link, then $(G, 0)$ will be isotopic to a meridian of (R, r), so we can perform a slam-dunk on the pair $R \cup G$, which will induce a homeomorphism $\psi'_B\colon S^3_{r,0}(R, G) \to S^3_{r-\frac{1}{0}}(R)$, and since we can remove any component with infinite coefficient, then the homeomorphism is simply $\psi'_B\colon S^3_{r,0}(R, G) \to S^3$. Note that in the link diagram of L, the diagram for $(B, 0)$ might intersect $(G, 0)$, i.e. since $(G, 0)$ is isotopic to a meridian of (R, r), then we can isotope it to such a meridian, where it will bound a disk D_G which will intersect $(B, 0)$ along some points. We can't perform the slam-dunk in such conditions, but we can slide $(B, 0)$ over (R, r) such that $(B, 0)$ and D_G no longer intersect, so that we are in a situation where we can perform the slam-dunk (Figure 4.5.1). Since this can always be done for special RBG-links, then this situation is standard and we will refer to this whole process in $S^3_{r,b,g}(R, B, G)$ as simply slam-dunking. From this description, then ψ'_B induces a slam-dunk homeomorphism $\psi_B\colon S^3_{r,0,0}(R, B, G) \to S^3_{f_b}(K_B)$, where $K_B = \psi_B(B)$. It remains to check the framing f_B on K_B, but since $H_1(S^3_{r,b,g}(L)) \cong \mathbb{Z}$, then by Corollary 2.6.13 f_b must be 0, so we get a homeomorphism $\psi_B\colon S^3_{r,0,0}(R, B, G) \to S^3_0(K_B)$. Note that we can do the exact same thing but with $(B, 0)$ and $(G, 0)$ exchanging roles, so we also get a knot K_G and a homeomorphism

$\psi_G\colon S^3_{r,0,0}(R,B,G)\to S^3_0(K_G)$. The desired homeomorphism is then given by $\phi_L=\psi_G\circ\psi_B^{-1}\colon S^3_0(K_B)\to S^3_0(K_G)$.

Suppose now that there is a 0-surgery homeomorphism $\phi\colon S^3_0(K)\to S^3_0(K')$. Let $(B,b)=(K',0)$ and $(R,r)=\phi((\mu_K,0))$, where μ_K is a meridian of K. Since μ_K is a meridian of K, then we can slam-dunk the pair to get a homeomorphism $\psi\colon S^3_{0,0}(K,\mu_K)\to S^3$. Note now that ϕ identifies the pair $(B,0)\cup(R,r)$ with $(K,0)\cup(\mu_K,0)$, so by composing with the previous slam-dunk we get a homeomorphism $\psi\circ\phi^{-1}\colon S^3_{r,0}(R,B)\to S^3$. If we now take G to be a 0-framed meridian of R, then we also have a slam-dunk $\psi_B\colon S^3_{r,0}(R,G)\to S^3$. By further letting $\psi_G=\psi\circ\phi^{-1}\colon S^3_{r,0}(R,B)\to S^3$, we can define the associated RBG-link to be $L=(R,r)\cup(B,0)\cup(G,0)$. $\square$

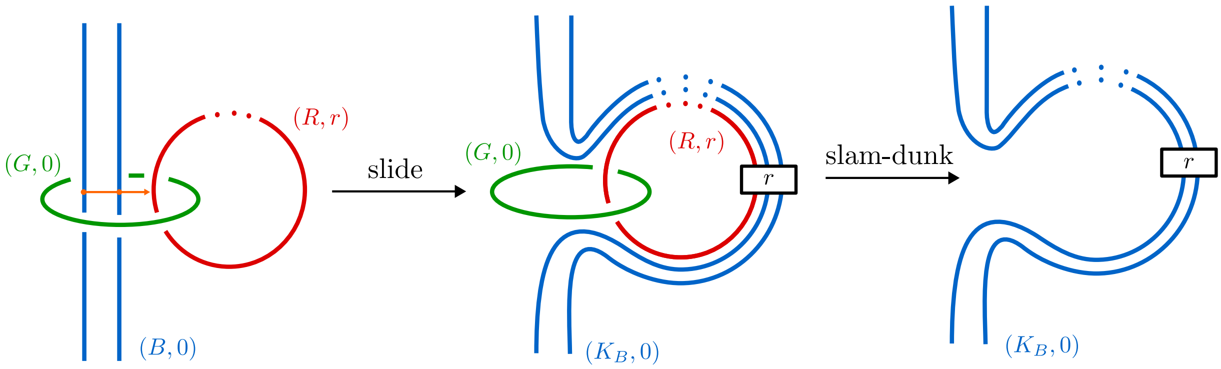

Figure 4.5.1: Sliding $(B,0)$ over (R,r) to obtain K_B.

As we hinted at before, it is useless to have 0-surgery homeomorphisms which have diffeomorphic 0-trace since by the H-slice trace embedding lemma (Corollary 4.1.4), the H-sliceness is preserved under trace diffeomorphisms, so that if K is H-slice in $\#n\mathbb{CP}^2$ and $X_0(K)\cong X_0(K')$ then K' will also be H-slice in $\#n\mathbb{CP}^2$. If one wishes to produce an exotic $\#n\mathbb{CP}^2$, then this is a situation which we must avoid. In general determining if homeomorphic 0-surgeries have diffeomorphic 0-traces is a really difficult task, and in fact even the problem of determining whether a 0-surgery homeomorphism extends to a 0-trace diffeomorphism is open in general. So, instead we focus on the homeomorphism case, that is if a 0-surgery homeomorphism extends to a 0-trace homeomorphism. This is a well-studied problem and while it isn't as fruitful as the diffeomorphism case, it still gives us some insight into this problem since if a 0-surgery homeomorphism doesn't extend to a 0-trace homeomorphisms then it obviously also doesn't extend to a 0-trace diffeomorphism.

Definition 4.5.6. If K and K' are knots and $\phi\colon S^3_0(K)\to S^3_0(K')$ is a 0-surgery homeomorphisms, then we say that ϕ is even if the 4-manifold $X_0(K')\cup_\phi -X_0(K)$ has even intersection form, and odd otherwise. We say that an RBG-link is even (odd) if the associated 0-surgery homeomorphisms is even (odd).

Theorem 4.5.7. *For knots K and K', a 0-surgery homeomorphism $\phi\colon S^3_0(K)\to S^3_0(K')$ extends to a 0-trace homeomorphism $\Phi\colon X_0(K)\to X_0(K')$ if and only if ϕ is even.*

Proof. We only prove the only if direction, as the other one goes beyond the scope of this thesis (for a proof see [Boy86]). Consider the double $D(X_0(K'))=X_0(K')\cup_{id_{\partial X_0(K')}}-X_0(K')$, which is obtained from $X_0(K')$ by attaching a 0-framed 2-handle to a meridian of K', followed by a 4-handle. Since the framings on the 2-handles are both 0, then $D(X_0(K'))$ is even. Now note that since $\phi\colon S^3_0(K) \to S^3_0(K')$ extends to a 0-trace homeomorphism $\Phi\colon X_0(K) \to X_0(K')$, then the manifold obtained from $X_0(K')$ by gluing along the identity will be homemorphic to the one obtained by gluing along ϕ, so $D(X_0(K'))$ is homeomorphic to $X_0(K') \cup_\phi -X_0('K)$ and thus ϕ is even. $\square$

Note that the previous theorem only guarantees that a 0-surgery homeomorphism doesn't extend to a 0-trace homeomorphism, not that their 0-traces aren't homeomorphic. For special RBG-links, it turns out that their parity is especially easy to determine.

Proposition 4.5.8. *A special RBG-link $L = (R, r) \cup (G, 0) \cup (B, 0)$ is even if and only if r is even.*

Proof. Let $\phi_L^{-1}(\mu_{K_G}) = \gamma \subset S^3_0(K_B)$ be the framed knot given by the image of a 0-framed meridian μ_{K_G} of K_G under the associated 0-surgery homeomorphism $\phi_L\colon S^3_0(K_B) \to S^3_0(K_G)$. Note that γ will represent the belt-sphere of the 2-handle in $X_0(K_G)$, so we get a handle decomposition for $X_0(K_B) \cup_{\phi_L} -X_0(K_G)$ as follows - $X_0(K_B)$ will have a handle decomposition given by 0-framed 2-handle glued to D^4 along K_B, to which we glue a 2-handle along γ and a 4-handle (both coming from the dual handle decomposition of $X_0(K_G)$). Since K_B has framing 0, then the parity of ϕ_L is determined by the parity of γ. We claim that if r is even then γ is even. Note that the slam-dunk homeomorphism $\psi_G\colon S^3_{r,0,0}(R, B, G) \to S^3_0(K_G)$, from the proof of Theorem 4.5.4, is obtained by first sliding $(G, 0)$ over (R, r), so in particular if μ_G is a 0-framed meridian for G in $S^3_{r,0,0}(R, B, G)$ then $\mu_{K_G} = \psi_G(\mu_G)$ will be a 0-framed meridian in $S^3_0(K_G)$. Since γ is by definition the image of μ_{K_G} under the 0-surgery homeomorphism ϕ_L, then by the previous remark we just need to see what happens to μ_G under the slam-dunk homeomorphism $\psi_B\colon S^3_{r,0,0}(R, B, G) \to S^3_0(K_B)$. Under ψ_B, we slide $(B, 0)$ over (R, r), but now μ_G is a meridian of $(G, 0)$ (it bounds a disk which intersects $(G, 0)$ once) which we also need to slide over (R, r) to be able to slam-dunk $R \cup G$ (Figure 4.5.2). After the slide, μ_G will represent $\gamma \subset S^3_0(K_B)$ and since μ_G is 0-framed and doesn't link (R, r), then γ has framing r. $\square$

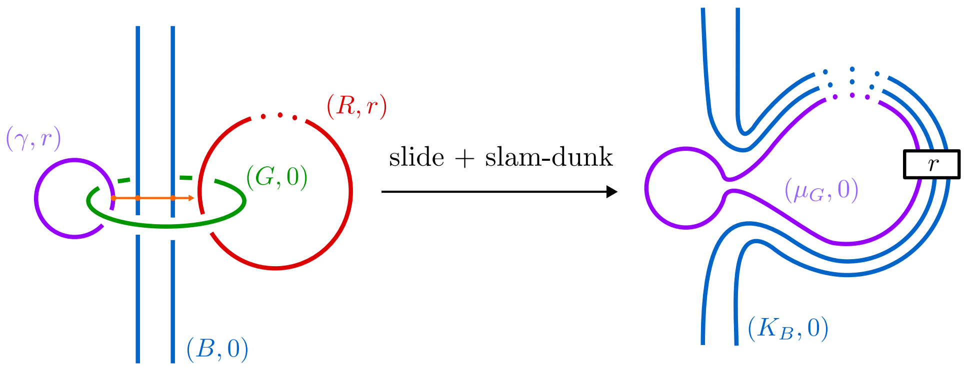

Figure 4.5.2: Sliding $(B, 0)$ along with $(\mu_G, 0)$ over (R, r).

Definition 4.5.9. A special RBG-link L is said to be small if

- B bounds a properly embedded disk Δ_B that intersects R transversely in exactly one point, and intersects G transversely in at most two points.

– G bounds a properly embedded disk Δ_G that intersects R transversely in exactly one point, and intersects B transversely in at most two points.

If L is a small RBG-link, then Δ_B intersects G transversely in at most two points and so by Theorem 4.5.4 it takes at most 2 slides of G over R to be able to slam-dunk $R \cup B$ and recover K_G. Since this condition also holds for Δ_G, then the same follows for K_B. This is useful in practice since it keeps K_G and K_B from getting too large. It turns out that for our purposes, it suffices to consider the case where Δ_B and G (or Δ_G and B) intersect in exactly two points.

Proposition 4.5.10 (Proposition 4.11 of [MP21]). *If L is a small RBG-link such that Δ_B intersects G in less than two points, then $K_G = K_B$.* □

With the goal of generating pairs of knots with homeomorphic 0-surgeries, Manolescu and Piccirillo looked at a particular family of RBG-links, presented in Figure 4.5.3. The Manolescu-Piccirillo family is given by a link depending on six parameters $L(a, b, c, d, e, f)$ corresponding to the number of full twists on some strands of the link. The a corresponds to the twists between R and a parallel copy, representing the framing of a 2-handle attached along R. Similarly, b also represents twists between R and a parallel copy, along with twists on B, so in particular the framing on R is given by $r = a + b$. Note that B and G are both 0-framed and clearly isotopic to meridians of R (in the link diagram of $L(a, b, c, d, e, f)$ without G or B respectively), and since $lk(B, G) = 0$ then $H_1(S^3_{r,b,g}(L)) \cong \mathbb{Z}$, so it is a family of special RBG-links. Notice that the computation for the linking number involves 4 crossings between B and G, so in particular there are disks Δ_B and Δ_G for B and G respectively, such that $|\Delta_B \cap G| = |\Delta_G \cap B| = 2$. This means that all the links $L(a, b, c, d, e, f)$ are also small RBG-links - the disks intersect the opposing color knots in exactly 2 points so as to avoid the previous proposition.

Through the slam-dunk homeomorphism, this family of links gives rise to pairs of knots $K_B(a, b, c, d, e, f)$ and $K_G(a, b, c, d, e, f)$ with homeomorphic 0-surgeries (Figure 4.5.4). By restricting the parameters $a, c, e \in [-2, 2]$ and $b, d \in [-1, 1]$ they found 3375 such pairs.

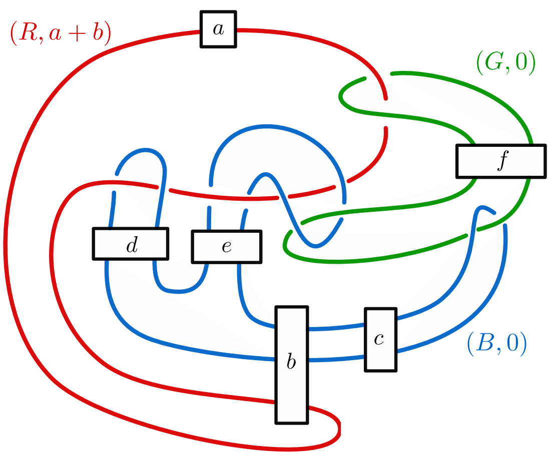

Figure 4.5.3: Small RBG-link that gives rise to the Manolescu-Piccirillo family.

Not all these pairs were however suited for our goals. For instance there are pairs where both knots are equivalent and pairs with diffeomorphic 0-traces - two situations which we have to avoid (for more

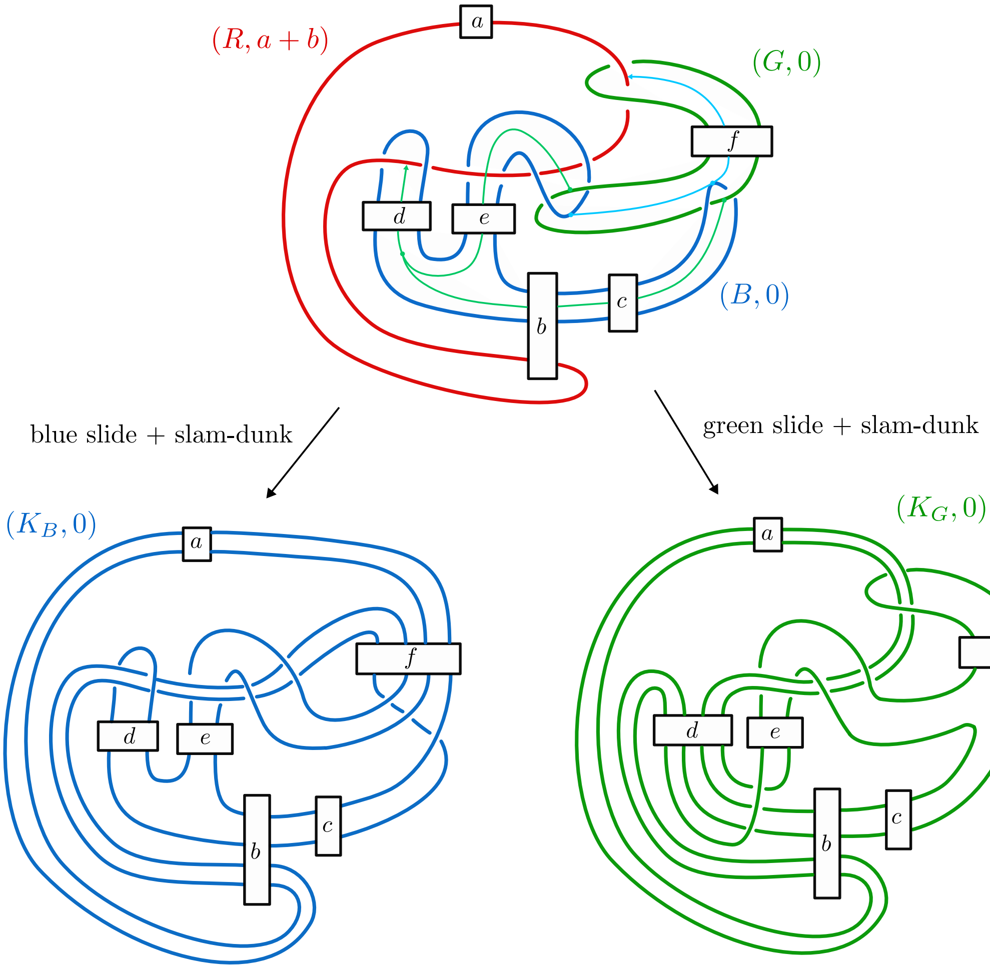

Figure 4.5.4: $K_B(a, b, c, d, e, f)$ and $K_G(a, b, c, d, e, f)$.

details see the original article [MP21]). After removing all links with the previous properties, we are left with 1800 pairs of knots. The authors now looked at pairs such that one knot has $s < 0$ (making it not H-slice in $\#n\mathbb{CP}^2$) while the other has $s = 0$ (making it a candidate to being H-slice in $\#n\mathbb{CP}^n$). This leads to them finding 23 pairs of knots, one with $s = -2$ and the other with $s = 0$. Some information of the knots K_1, ... K_{23} with $s = 0$ is given in Table 4.2, and if any of them is shown to be H-slice in $\#n\mathbb{CP}^2$, then we have an exotic $\#n\mathbb{CP}^2$.

Theorem 4.5.11. *If any of the 23 knots in Table 4.2 is H-slice in $\#n\mathbb{CP}^2$ for some n ($n = 0$ corresponds to S^4), then an exotic $\#n\mathbb{CP}^2$ exists.*

Proof. For $i = 1, ..., 23$, the knots K_i have companion knots K_i' such that $s(K_i') = -2$ and such that there are homeomorphisms $\phi\colon S_0^3(K_i) \to S_0^3(K_i')$. If K_i would be H-slice in $\#n\mathbb{CP}^2$, then by Theorem 4.5.2, K_i' would be H-slice in a 4-manifold X homotopy equivalent to $\#n\mathbb{CP}^2$ (hence homeomorphic to $\#n\mathbb{CP}^2$ by Theorem 2.4.11), but since $s(K_i') = -2$ then K_i' isn't H-slice in any $\#n\mathbb{CP}^2$, so X can't be diffeomorphic to any $\#n\mathbb{CP}^2$, and thus must be an exotic $\#n\mathbb{CP}^2$. $\square$

Name	Identifier	# crossings	Alexander polynomial
K_1	$K_B(0,1,0,1,2,-1)$	29	1
K_2	$K_B(1,1,0,1,1,-1)$	29	1
K_3	$K_G(1,1,0,-1,1,1)$	32	1
K_4	$K_B(2,1,-1,1,1,-1)$	29	1
K_5	$K_G(2,1,-1,-1,1,1)$	32	1
K_6	$K_B(1,1,-1,1,2,-1)$	29	$t^2-2t+3-2t^{-1}+t^{-2}$
K_7	$K_B(1,1,1,1,0,-1)$	31	$t^2-6t+11-6t^{-1}+t^{-2}$
K_8	$K_B(1,1,2,1,-1,-1)$	36	$4t^2-20t+33-20t^{-1}+4t^{-2}$
K_9	$K_G(1,1,-1,-1,2,1)$	32	$t^2-2t+3-2t^{-1}+t^{-2}$
K_{10}	$K_G(1,1,1,-1,0,1)$	32	$t^2-6t+11-6t^{-1}+t^{-2}$
K_{11}	$K_G(1,1,2,-1,-1,1)$	41	$4t^2-20t+33-20t^{-1}+4t^{-2}$
K_{12}	$K_G(1,1,2,0,-1,1)$	20	$2t^2-12t+21-12t^{-1}+2t^{-2}$
K_{13}	$K_B(2,1,-2,1,2,-1)$	35	$2t^2-6t+9-6t^{-1}+2t^{-2}$
K_{14}	$K_B(2,1,0,1,0,-1)$	31	$-2t+5-2t^{-1}$
K_{15}	$K_B(2,1,1,1,-1,-1)$	33	$2t^2-12t+21-12t^{-1}+2t^{-2}$
K_{16}	$K_B(2,1,2,1,-2,-1)$	37	$6t^2-30t+49-30t^{-1}+6t^{-2}$
K_{17}	$K_G(2,1,-2,-1,2,1)$	37	$2t^2-6t+9-6t^{-1}+2t^{-2}$
K_{18}	$K_G(2,1,-2,0,2,1)$	16	$t^2-2t+3-2t^{-1}+t^{-2}$
K_{19}	$K_G(2,1,0,-1,0,1)$	34	$-2t+5-2t^{-1}$
K_{20}	$K_G(2,1,1,-1,-1,1)$	36	$2t^2-12t+21-12t^{-1}+2t^{-2}$
K_{21}	$K_G(2,1,2,-1,-2,1)$	42	$2t^2-12t+21-12t^{-1}+2t^{-2}$
K_{22}	$K_G(2,1,1,0,-1,1)$	18	$t^2-8t+15-8t^{-1}+t^{-2}$
K_{23}	$K_G(2,1,2,0,-2,1)$	22	$5t^2-26t+43-26t^{-1}+5t^{-2}$

Table 4.2: If any of these 23 knots is H-slice in any $\#n\mathbb{CP}^2$, then we have an exotic $\#n\mathbb{CP}^2$ (Table 1 of [MP21]).

By looking at Table 4.2 we see that the knots K_1, ..., K_5 have alexander polynomial 1, so by Theorem 4.1.9 they are topologically slice, and thus prime candidates to being smoothly slice. Note that the knots K_1, K_4, K_5 and K_{13}, ..., K_{23} have $r = a + b \equiv 1 \bmod 2$, so by Proposition 4.5.8 the 0-surgery homeomorphisms between the knots and their companions don't extend to 0-trace homeomorphisms and thus they can't extend to 0-trace diffeomorphisms. This means that we cannot immediately obstruct the H-sliceness of these knots by the s-invariant of their companions, so they are potential candidates to construct an exotic $\#n\mathbb{CP}^2$.

4.6 An obstruction to the Manolescu-Piccirillo approach

We now follow the work of Nakaramu in [Nak22] to investigate the limitations of the previous approach. We start by proving that even though we can't tell if the knots K_i share a 0-trace with their partners, we can show that their 0-traces will become diffeomorphic after taking a connected sum. The proof of this fact is very similar, in spirit, to the one used in Section 4.2, to construct the 0-trace diffeomorphisms - we give a Kirby diagram description to a special RBG-link by putting a dot on R (describing a 1-handle) and attach 0-framed 2-handles to B and G, and we proceed to slide $(B, 0)$ (or $(G, 0)$) over R, so that we can cancel the $1/2$-handle pair. Note that the slides we need to do in order to be able to cancel the $1/2$-handle pair, will be the same as the slides we need to do to perform a slam-dunk on the boundary (R is identified with a 0-framed 2-handle on the boundary). This leads us to the conclusion that the diffeomorphism induced by such slides, along with cancelling the $1/2$-handle pair, will extend the slam-dunk homeomorphism on the boundary, which is described in the proof of Theorem 4.5.4 (see Figure 4.6.1). In our case R will be r-framed so we can't really add a dot to R and interpret it as a 1-handle, we however note that adding the 1-handle is the same as removing a tubular neighborhood of a disk from the manifold, so we can instead remove an r-slice disk and a similar technique will still result.

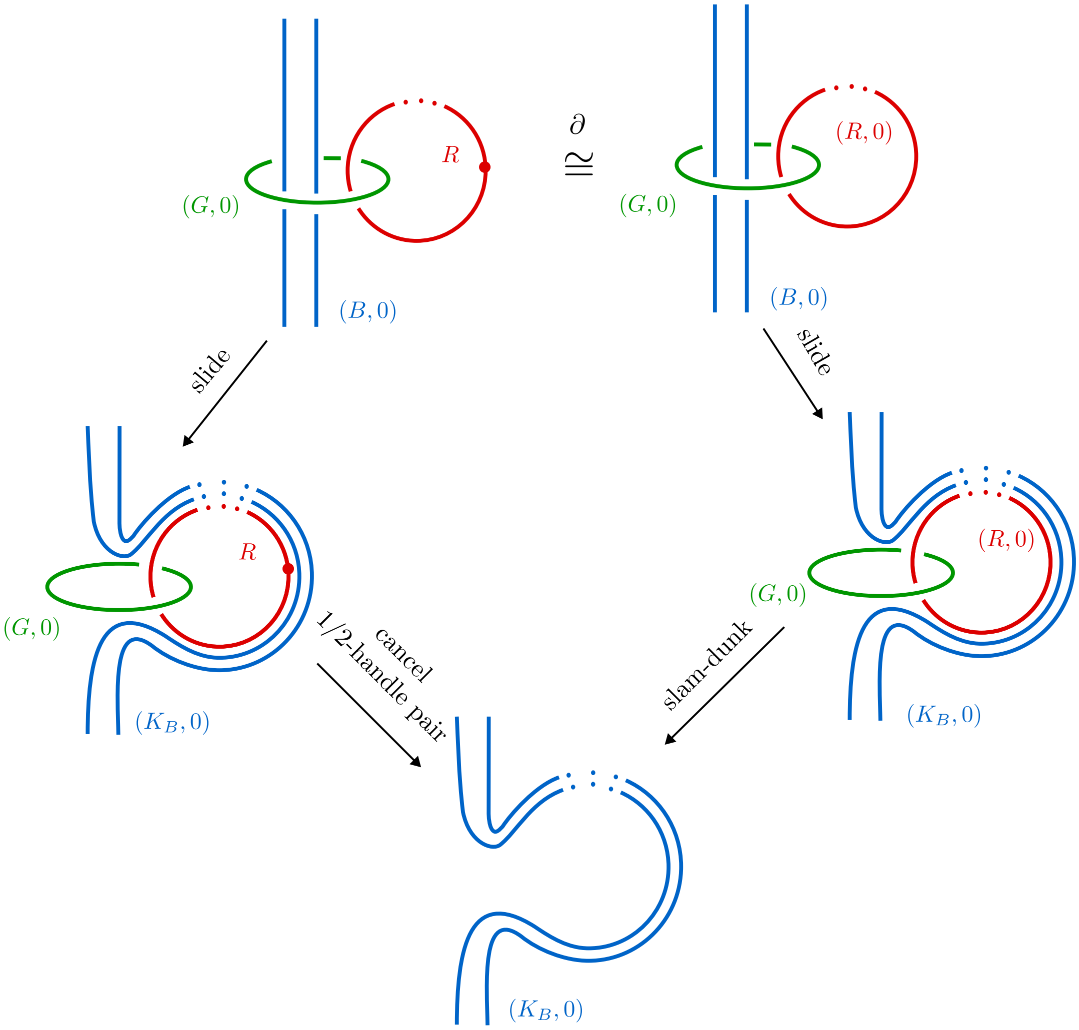

Figure 4.6.1: A $1/2$-handle cancel extends the slam-dunk homoeomorphism.

Lemma 4.6.1. *Let W be a closed, oriented, smooth 4-manifold and $L = (R,r) \cup (B,0) \cup (G,0)$ a special RBG-link. If (R,r) is slice in W, then the associated 0-surgery homeomorphism $\phi_L\colon S^3_0(K_B) \to S^3_0(K_G)$ extends to a diffeomorphism $\Phi_L\colon X_0(K_B)\#W \to X_0(K_G)\#W$.*

Proof. Since (R,r) is slice in W, then it admits an r-slice disk D in W°. If νD is a tubular neighborhood of D in W°, then define Y to be the 4-manifold obtained from W° by removing νD and attaching 0-framed 2-handles h_B and h_G along B and G respectively. On the boundary this amounts to doing 0-surgeries on B and G and an r-surgery on R, which means that $\partial Y \cong S^3_{r,0,0}(R,B,G)$. Since L is a special RBG-link, then $(G,0)$ is isotopic to a meridian of (R,r), so there will be a diffeomorphism $\Psi'_B\colon W^\circ - \nu D \cup_{(G,0)} h_G \to W^\circ$

which by the previous remark extends the slam-dunk homeomorphism $\psi'_B\colon S^3_{r,0}(R,G) \to S^3$. The 2-handle h_B is attached to $(B,0)$, so after acting with Ψ'_B it will be attached to $(K_B,0)$. Note that this is very similar to the proof of Theorem 4.5.4 except now all surgery homeomorphisms are induced by diffeomorphisms of the 4-manifold itself. In a similar fashion, Ψ'_B will induce a diffeomorphism $\Psi_B\colon Y \to W^\circ \cup_{(K_B,0)} h_B$ which extends the slam-dunk homeomorphism $\psi_B\colon S^3_{r,0,0}(R,B,G) \to S^3_0(K_B)$. Notice now that $W^\circ = W - D^4 \cong W\#D^4$ (taking the connected sum in the interior of D^4 means that we're left with a boundary component S^3), so in particular $W^\circ \cup_{(K_B,0)} h_B \cong X_0(K_B)\#W$. Thus rewriting the diffeomorphism we get that $\Psi_B\colon Y \to X_0(K_B)\#W$ extends $\psi_B\colon S^3_{r,0,0}(R,B,G) \to S^3_0(K_B)$. Since L is a special RBG-link, then $(B,0)$ will also be isotopic to a meridian of (R,r) and we can repeat the process to get a diffeomorphism $\Psi_G\colon Y \to X_0(K_G)\#X$ extending the homeomorphism $\psi_G\colon S^3_{r,0,0}(R,B,G) \to S^3_0(K_G)$. Composing both, we get a diffeomorphism $\Phi_L = \Psi_G \circ \Psi_B^{-1}\colon X_0(K_B)\#W \to X_0(K_G)\#W$ which extends the associated 0-surgery homeomorphism $\phi_L\colon S^3_0(K_B) \to S^3_0(K_G)$. □

We recall that it is problematic for our strategy if the associated 0-trace homeomorphism extends to a 0-trace diffeomorphism. The previous theorem allows us instead to extend to a 0-trace diffeomorphism after stabilizing. Using the H-slice trace embeddling lemma (Corollary 4.1.4), we can then extend the H-sliceness of the associated knots K_B and K_G to a connected sum.

Corollary 4.6.2. *Let X, W be closed, oriented, smooth 4-manifolds and $L = (R,r) \cup (B,0) \cup (G,0)$ a special RBG-link, such that the associated knot K_B (or K_G) is H-slice in X. If (R,r) is slice in W, then there is a 4-manifold X' with the homotopy type of X such that $X'\# - W$ is diffeomorphic to $X\# - W$. In particular K_G (or K_B) is H-slice in $X\# - W$.*

Proof. Since L is a special RBG-link, then there is an associated 0-surgery homeomorphism $\phi_L\colon S^3(K_B) \to S^3(K_G)$ and since K_B is H-slice in X, then it admits a null-homologous slice disk D in X°. By Theorem 4.5.2, this means that K_G is H-slice in a 4-manifold $X' = X^\circ - \nu D \cup_{\phi_L} -X_0(K_G)$ with the same homotopy type as X. Notice that $X = X^\circ - \nu D \cup -X_0(K_B)$ but since (R,r) is slice in W, then by the previous theorem ϕ_L extends to a diffeomorphism $\Phi_L\colon X_0(K_B)\#W \to X_0(K_G)\#W$, so after taking a connected sum with W we have that $X\# - W$ is diffeomorphic to $X'\# - W$. Since K_G is H-slice in X' then it is also H-slice in $X'\# - W$ and hence in $X\# - W$. Note that exchanging the roles of K_B and K_G leads to the same result, so the result follows. □

We wish to obstruct the H-sliceness of the knots K_i in $\#n\mathbb{CP}^2$. While the s-invariant vanished for all of them, we can make use of the previous corollary to obstruct their H-sliceness by analyzing their partner knots in a different connected sum $\#m\mathbb{CP}^2$.

Theorem 4.6.3. *The knots K_1, ... K_{23} given by Table 4.2 are not H-slice in any $\#n\mathbb{CP}^2$.*

Proof. For $i = 1, ..., 23$, each K_i arises from a special RBG-link with $R = U$ and $r = a + b \geq 0$. Note that $(U,1)$ is slice in $\overline{\mathbb{CP}^2}$ with slice disk given by the core of the 2-handle, so (U,r) will be slice in $\#r\overline{\mathbb{CP}^2}$ (the case $(U,0)$ is equivalent to regular sliceness, and the slice disk is given by a Seifert surface). The proof then follows by contradiction - suppose the knots K_i were to be H-slice in $\#n\mathbb{CP}^2$, then by the previous corollary their partner knots K'_i would be H-slice in $\#(n+r)\mathbb{CP}^2$. However, all K'_i have $s(K'_i) = -2$, so by the adjunction inequality (Corollary 4.4.2) they can't be H-slice in any $\#m\mathbb{CP}^2$ and thus the knots K_i couldn't have been H-slice in any $\#n\mathbb{CP}^2$ in the first place. $\square$

Note that for the previous proof to work we required that the framing of R in the associated RBG-link had $r \geq 0$. If we had $r < 0$ then (R, r) would be slice in $\#r\mathbb{CP}^2$ and we wouldn't have a contradiction since K_i' having s-invariant -2 wouldn't obstruct it from being H-slice in $\#n\mathbb{CP}^2\#|r|\overline{\mathbb{CP}^2}$. Recall that the links $L(a, b, c, d, e, f)$ coming from the Manolescu-Piccirillo family, all had $r = a + b$ with $a \in [-2, 2]$ and $b \in [-1, 1]$, so we weren't restricted to links with $r \geq 0$, however as we've seen they didn't find any links with negative r that produced viable knots (such that one of them had $s < 0$ and the other $s = 0$). We will see that this is no coincide and in fact if we had $r < 0$, then the knots $K_B(a, b, c, d, e, f)$ and $K_G(a, b, c, d, e, f)$ associated with an RBG-link coming from the Manolescu-Piccirilo family, would have $s(K_B), s(K_G) \geq 0$. As we'll shortly see, to achieve such a task we will need to explicitly construct framed slice disks for both K_B and K_G in $\#n\mathbb{CP}^2$, so we take a short detour to give some techniques on how one might do so.

Lemma 4.6.4. *Let X be a closed, oriented, smooth 4-manifold and $(K, k) \subset S^3$ a slice knot in X with slice disk $D \subset X^\circ$. If $\Delta \subset S^3$ is a disk intersecting K transversely in l points counted with sign, and K_+ the knot obtained from K by performing a positive full twist to the strands going through Δ, then $(K_+, k + l^2)$ is slice in $X\#\overline{\mathbb{CP}^2}$.*

Proof. Consider X° and construct the 4-manifold Y° by adding a (-1)-framed 2-handle along $\partial\Delta$ (Figure 4.6.2). This is equivalent to taking a connected sum of X with $\overline{\mathbb{CP}^2}$ – int D^4 (Remove the 0-handle to perform the connected sum), so we can identify Y° with $X\#\overline{\mathbb{CP}^2}$ – int $D^4 = (X\#\overline{\mathbb{CP}^2})^\circ$. Note that if we now blow-down the boundary of the 2-handle, then we identify the boundary $\partial Y^\circ = S^3_{-1}(\partial\Delta)$ with S^3. Since (K, k) intersects Δ in l points counted with sign, then the blow-down has the effect of adding a full positive twist to K and increasing its framing by l^2. This is just the knot $(K_+, k + l^2) \subset S^3$, so it follows that $(K_+, k + l^2)$ is slice in $W\#\overline{\mathbb{CP}^2}$. $\square$

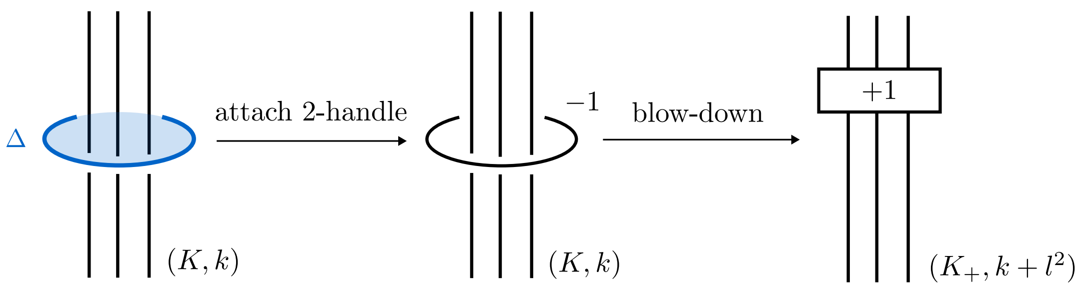

Figure 4.6.2: Constructing a slice disk for K_+.

The previous proof also works for the case of a knot K_- obtained by adding a negative full twist to K along Δ, except now $(K_-, k - l^2)$ will admit a slice disk in $X\#\mathbb{CP}^2$. Indeed, in this case the proof follows in the exact same manner except we attach a 1-framed 2-handle (representing a connected sum with $\mathbb{CP}^2$) and the blow-down will instead induce a negative full twist to K and a decrease in framing.

Corollary 4.6.5. *Let X be a closed, oriented, smooth 4-manifold and $(K,k) \subset S^3$ a slice knot in X with slice disk $D \subset X^\circ$. If $\Delta \subset S^3$ is a disk intersecting K transversely in l points counted with sign, and K_- the knot obtained from K by performing a negative full twist to the strands going through Δ, then $(K_-, k - l^2)$ is slice in $X\#\mathbb{CP}^2$.* □

Any crossing can be reversed by adding a positive or negative full twist on the two strands involved (see Figure 2.5.10), so the previous lemma and corollary guarantee that for any knot K, after doing enough positive (or negative) full twists, we will get a framed disk in some $\#n\overline{\mathbb{CP}^2}$ (or $\#n\mathbb{CP}^2$). This leads us to the following definition.

Definition 4.6.6. If $K \subset S^3$ is a knot, then we define the positive projective slice framing $PF_+(K)$ of K to be the smallest framing k such that (K,k) is slice in some $\#n\overline{\mathbb{CP}^2}$. Analogously, we define the negative projective slice framing $PF_-(K)$ to be the largest framing k such that (K,k) is slice in some $\#n\mathbb{CP}^2$.

Definition 4.6.7. A knot $K \subset S^3$ is called biprojectively H-slice (BPH-slice) if it is H-slice in both $\#n\mathbb{CP}^2$ and $\#n\overline{\mathbb{CP}^2}$, for some $n \geq 0$.

Lemma 4.6.8. *Let $K \subset S^3$ be a knot. Then*

(a) *$PF_+(K) = 0$ if and only if K is H-slice in some $\#n\overline{\mathbb{CP}^2}$*

(b) *$PF_-(K) = 0$ if and only if K is H-slice in some $\#n\mathbb{CP}^2$*

(c) *$PF_+(K) = PF_-(K) = 0$ if and only if K is BPH-slice.*

Proof. A knot (K,k) being slice in some 4-manifold amounts to having a slice disk D with self-intersection number $k = -[D]\cdot[D]$. This means that if $k < 0$, then (K,k) can't be slice in any negative definite 4-manifold. In particular it can't be slice in any $\#n\overline{\mathbb{CP}^2}$, so $PF_+(K) \geq 0$. Since the only homology class with self-intersection number 0 is the 0 class, then it follows that $PF_+(K) = 0$ if and only if $k = 0$, i.e. if and only if K is H-slice in some $\#n\overline{\mathbb{CP}^2}$. Applying the same argument for $k > 0$, we have that (K,k) can't be slice in any positive defitive 4-manifold, and thus can't be slice in any $\#n\mathbb{CP}^2$. We conclude that $PF_-(K) \leq 0$ and $PF_-(K) = 0$ if and only if K is H-slice in some $\#n\mathbb{CP}^2$. By combining both results we get the last item. □

Note that once we have a framing k for which K is slice in some $\#n\overline{\mathbb{CP}^2}$, then we can use Lemma 4.6.4 to construct slice disks with any framing larger than k. In a similar fashion, we can do the same in $\#n\mathbb{CP}^2$, but now instead we can construct slice disks with any framing smaller than k.

Corollary 4.6.9. *If $(K,k) \subset S^3$ is a knot such that $k \geq PF_+(K)$, then (K,k) is slice in some $\#n\overline{\mathbb{CP}^2}$. If $k \leq PF_-(K)$ instead, then (K,k) is slice in some $\#n\mathbb{CP}^2$.*

Proof. Assume that (K,k) is slice in $\#m\overline{\mathbb{CP}^2}$ and consider a meridian μ_K of K, which bounds a disk Δ that intersects K exactly once. By Lemma 4.6.4 we then have that $(K_+, k+1)$ is slice in $\#(m+1)\overline{\mathbb{CP}^2}$, but since Δ only intersects K geometrically once, then $K_+ = K$. We can repeat this process to achieve any framing we want at the expense of increasing the size of the connected sum. Since we can do the same for a slice knot (K,k) in $\#m\mathbb{CP}^2$ at the expense of decreasing the framing, then the other case follows. □

So now that we have a technique for constructing slice disks in $\#n\mathbb{CP}^2$ (or $\#n\overline{\mathbb{CP}^2}$) the next step is to bound these by our favorite invariant. The adjunction inequality given in Theorem 4.4.1 would be great but it requires the slice disks to be null-homologous, so it doesn't agree with our goals. We have to take a leap of faith and assume Conjecture 4.4.6 to be true.

Proposition 4.6.10. *Let $K \subset S^3$ be a knot and assume Conjecture 4.4.6 to be true. Then*

$$PF_-(K) + \sqrt{|PF_-(K)|} \leq s(K) \leq PF_+(K) - \sqrt{PF_+(K)}$$

Proof. Suppose (K, k) is a knot in $X = \#n\overline{\mathbb{C}P^2} - \text{int } D^4$ with slice disk D. By Conjecture 4.4.6 we have that $s(K) \leq 1 - \chi(D) - |[D]| - [D] \cdot [D]$, and since $\chi(D) = 1$ and $[D] \cdot [D] = -k$, then $s(K) \leq |[D]| + k$. Assume that the class $[\alpha] \in H_2(\#n\overline{\mathbb{C}P^2}) \cong H_2(X^\circ, \partial X^\circ)$ which represents the disk D is given by $\alpha = s_1e_1 + s_2 + e_2 + \cdots + s_ne_n$, where $\{e_1, e_2, ..., e_n\}$ represents the standard basis for $H_2(\#n\overline{\mathbb{C}P^2})$. Then

$$k = s_1^2 + s_2^2 + \cdots + s_n^2 \leq (|s_1| + |s_2| + \cdots + |s_n|)^2 = |[D]|^2$$

means that $\sqrt{k} = |[D]|$, so it follows that $s(K) \leq k - \sqrt{k}$. Since $k \geq PF_+(K)$, then it follows that $s(K) \leq PF_+(K) - \sqrt{PF_+(K)}$. Note that we can do the same if (K, k) is slice in some $\#n\mathbb{C}P^2 - \text{int } D^4$, by applying Conjecture 4.4.7. This time we'll have that $s(K) \geq k + \sqrt{k}$ and thus $s(K) \geq PF_-(K) + \sqrt{|PF_-(K)|}$. By combining both results the conclusion follows. $\square$

Notice that if we were to work with the τ-invariant, then the proposition is proven by Theorem 4.4.5 and this conjecture would really be a theorem.

Equipped with Lemma 4.6.4, we are now ready to produce the framed disks claimed before. The following lemma gives us a way to construct (-1)-slice disks for K_B and K_G in $\#|r|\mathbb{CP}^2$ when the associated RBG-link is both small and has $R = U$ the unknot. Recall that the links $L(a, b, c, d, e, f)$ coming from the Manolescu-Piccirillo family satisfy both these properties, so this lemma will gives us a way to treat the cases where the techniques used in Theorem 4.6.3 don't apply (that is when $r < 0$).

Lemma 4.6.11. *Let L be a small RBG-link with $R = U$ and $r < 0$. Then both K_B and K_G are (-1)-slice in $\#|r|\mathbb{CP}^2$ with slice disks $D_B, D_G \subset (\#|r|\mathbb{CP}^2)^\circ$ such that they intersect one of the spheres $\mathbb{CP}^1$ in exactly 3 points and the remaining $|r| - 1$ spheres $\mathbb{CP}^1$ null-homologously.*

Proof. Notice that if either $|\Delta_B \cap G|$ or $|\Delta_G \cap B|$ is less than two, then we can safely skip these cases, since by Proposition 4.5.10 we will have that $K_B = K_G$ and thus we produce knots which are not relevant for our purposes. For the case of $|\Delta_B \cap G| = |\Delta_G \cap B| = 2$, we proceed by induction on r with base case $r = -1$ and decreasing induction step, i.e. the case r follows from the $r + 1$ case. We start with the induction step since, as we'll see, the base case is significantly more challenging.

Assume L is a small RBG-link with $R = U$ and $r < -1$. This means that L is in particular a special RBG-link, so as we've previously seen, the condition on homology implies that either $r = 0$ or $r \cdot lk(B, G) = 2$. Since $r < -1$, then we must have that $lk(B, G) = 0$, but since this is just the signed number of intersections between B and Δ_G, then it follows that B will have two strands with opposite direction running through $\partial\Delta_G$. If we now slide B over R, as if we were slam-dunking $R \cup G$, then we end up with $(K_B, 0)$ running around (R, r) with two strands going through an r-twist box (induced by the framing) with opposite orientations (Figure 4.6.3). Consider now the RBG-link L^* obtained from L by switching r to $r+1$. Then K_B differs from K_B^* by having an extra twist going through the box. Note that we can recover K_B by adding a full negative twist to the 2 strands of K_B^* going through the box (Figure 4.6.3). So if $(K_B^*, -1)$ bounds a slice disk in $\#|r+1|\mathbb{CP}^2$ intersecting a $\mathbb{CP}^1$ in exactly 3 points, then by Lemma 4.6.4 $(K_B, -1)$ also bounds such a slice disk in $\#|r|\mathbb{CP}^2$ (the framing doesn't change since the strands run in opposite directions). Since the same follows by changing the roles of B and G, then $(K_G, -1)$ also bounds a slice disk in $\#|r|CP^2$ intersecting a $\mathbb{CP}^1$ in exactly 3 points.

We now focus on the base case where $r = -1$. Start with the link $L = (R, -1) \cup (B, 0) \cup (G, 0)$ in S^3 and slide $(B, 0)$ over $(R, -1)$ until we obtain $(R, -1) \cup (K_B, 0) \cup (G, 0)$. Now decrease the framing of K_B to -1 and completely remove $(G, 0)$ to obtain $L' = (R, -1) \cup (K_B, -1)$ as in Figure 4.6.4. Consider now extending S^3 by the trivial cobordism, such that $L' \subset S^3 \times \{0\}$ and $L' \times I \subset S^3 \times I$. Since $R = U$, then we can attach a (-1)-framed 2-handle along $(R, -1) \subset S^3 \times \{1\}$ to obtain $W = \overline{\mathbb{CP}^2} - \text{int}(D^4 \sqcup D^4)$ such that $L' \subset S^3 \times \{0\} = \partial_- W$. We can then find a cobordism $\Sigma = K_B \times I \subset W$. Note that L' is obtained from L in a similar fashion to L^* in the induction step, so $(K_B, -1)$ will have two strands with opposite orientation running along $(R, -1)$. Since B is a meridian of R in $B \cup R$, then after performing the slides that yield K_B, we have three intersections between K_B and a Seifert disk for R - two coming from the slides and one coming from the fact that B was a meridian. This then means that $lk(K_B, R) = 1$ and so Σ is non-trivial in homology (it has intersection number 1 with a generator of $H_2(W)$). Note that after isotopies, the only self-intersections of K_B come from the twist box with R, so $lk(K_B, K_B) = 1$ and thus Σ has self-intersection number 1, thus the framing $(K_B, -1)$ is well-defined when we think for K_B in $S^3 \times \{1\} = \partial_+ W$. We can now modify the boundary component $\partial_+ W$ by blowing down the boundary of the (-1)-framed 2-handle. Since, as we saw before, $lk(K_B, R) = 1$, then the blow-down increases the framing of K_B by 1. Note that this operation equivalent to sliding the strands of K_B over $(R, -1)$ as if we we're reversing the slam-dunk moves we previously did, and then separating $(K_B, -1)$ from $(R, -1)$. Indeed, we can think of this as sliding the cobordism Σ over the 2-handle so that we reverse the slides done to go from L to L' (Figure 4.6.4). As we've seen before, the strands of $(K_B, -1)$ have opposite orientations, so the first handle slides have no effect on the framing, meaning that we recover $(B, -1)$ as a meridian of $(R, -1)$. We can now slide $(B, -1)$ over $(R, -1)$ as in Figure 4.6.4, to obtain two unlinked unknots $(R, -1) \cup (B, 0)$ (since B is a meridian of R, then the framing changes to 0). Since we now have an unknotted, 0-framed component, the idea is to dualize W and cap off $(B, 0)$ with a disk, so that we obtain a (-1)-slice disk D_B for $K_B \subset \mathbb{CP}^2 - \text{int } D^4$. To turn W upside down, start by taking the double of W, which amounts to attaching a 0-framed unknot to a meridian of $(R, -1)$, since there are no 0-,1-,3- or 4-handles. Now to remove W from DW we simply interpret it has the boundary component $\partial_+ W$, to which we attach the dual handle. The 2-handle of W attached along $(R, -1)$ will now be identified with its boundary, and since the dual handle is a meridian of $(R, -1)$, then by blowing down along $(R, -1)$ we identify the dual handle with $(U, 1)$ and thus the dual of W is identified with $\mathbb{CP}^2 - \text{int}(D^4 \sqcup D^4)$. Now we are left with an unknot $(U, 0)$, which we can cap with a disk in the 4-handle, to construct a slice disk D_B for $(K_B, -1)$ in $\mathbb{CP}^2 - \text{int } D^4$ and thus in $\mathbb{CP}^2$. Now it only remains to check that D_B intersects

$\mathbb{CP}^1$ geometrically in three points. To do so, notice that to obtain $(B, 0)$ from $(K_B, -1)$ we had to slide K_B three times over $(R, -1)$. This means that the cobordism Σ, and hence the disk D_B, intersects the cocore of the 2-handle attached to $(R, -1)$ in three points. When we turned W upside down, the cocore of the 2-handle became the core of the 2-handle generating its dual. Recall that the core of the 2-handle, along with a Seifert disk for the attaching sphere, form the $\mathbb{CP}^1$ sitting inside $\mathbb{CP}^2$. Since in the dual decomposition $(B, 0)$ is unlinked from the attaching sphere of the dual 2-handle, then we can cap it off without adding any extra intersections. This means that D_B intersects $\mathbb{CP}^1$ exactly three times, so K_B bounds a (-1)-slice disk with the desired property. Since we can do the same for K_G by exchanging the roles of B and G, then the result follows. $\square$

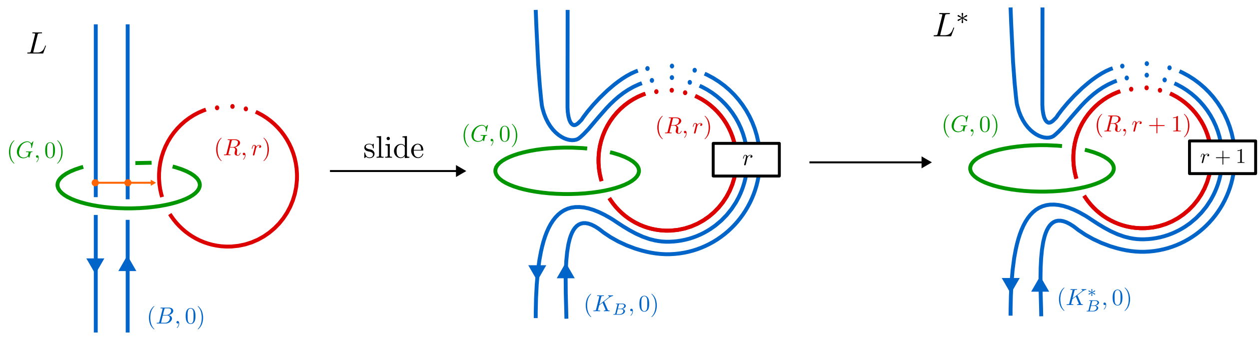

Figure 4.6.3: Constructing a slice disk for $(K_B, -1)$ in $\#|r|\mathbb{CP}^2$.

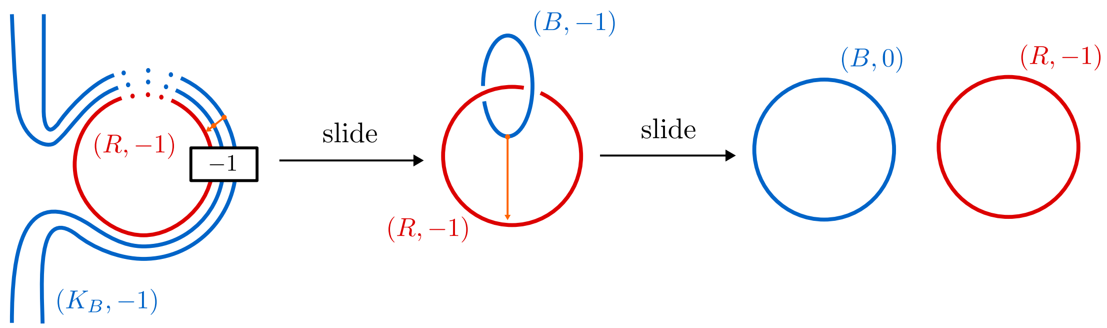

Figure 4.6.4: Constructing a slice disk for $(K_B, -1)$ in $\mathbb{CP}^2$.

The previous lemma not only produces slice disks for K_B and K_G but it also specifies how these interact with the classes generating the homology of $\#|r|\mathbb{CP}^2$. Recall that in the proof of Theorem 4.4.1, this intersection data was crucial to construct a cobordism for our knot and thus find a bound for the s-invariant. This means that we can proceed in the same manner as before and recover a bound for the s-invariant in such conditions.

Corollary 4.6.12. *Let L be a small RBG-link with $R = U$ and $r < 0$. Then $s(K_B), s(K_G) \geq 0$.*

Proof. We will proceed as in Theorem 4.4.1. Consider $D_B \subset (\#|r|\mathbb{CP}^2)$ to be the slice disk for $(K_B, -1)$ produced by the previous lemma. Since D_B intersects one of the $\mathbb{CP}^1$ in exactly 3 points, then if $\nu\mathbb{CP}^1$ is a tubular neighborhood for such a sphere, we will have that D_B will intersects $\partial(\nu\mathbb{CP}^1)$ along the link $-F_{2,1}(1)$ (the link is mirrored since we are working in $\mathbb{CP}^2$). Since D_B intersects the other spheres $\mathbb{CP}^1$

null-homologously, then D_B will intersect the boundaries of their tubular neighborhoods along the links $-F_{(p_i,p_i)}(1)$ (for $i = 1, ..., r-1$). As before we can remove such tubular neighborhoods and connect their boundaries, to get a cobordism Σ in $S^3 \times I$, from $-F_{2,1}(1) \sqcup -F_{(p_1,p_1)}(1) \sqcup \cdots \sqcup -F_{(p_{r-1},p_{r-1})}(1)$ to K_B. Thus we can use the functoriality of Khovanov homology under cobordisms to obtain

$$s(K) \geq s(-F_{(2,1)}(1) \sqcup -F_{(p_1,p_1)}(1) \sqcup \cdots \sqcup -F_{(p_{r-1},p_{r-1})}(1)) + \chi(\Sigma)$$

Since Σ is obtained from D_B by removing 3 disks for the first intersection and $2p_i$ for the other $r-1$ intersections, then $\chi(\Sigma) = \chi(D) - 3 - 2p_1 - 2p_2 - ... - 2p_{r-1}$. Furthermore we have that $s(F_{p,p}) = 1 - 2p$, and by equation 9.3 of [MMSW23] we'll have $s(F_{p,q}(1)) = (p-q)^2 - 2p + 1$ (for $p + q \leq 4$). Finally, by using Proposition 3.4.3 and Proposition 3.3.9 to simplify the computations of the s-invariant of the link, it follows that

$$\begin{aligned}
s(K) &\geq s(-F_{(2,1)}(1) \sqcup -F_{(p_1,p_1)}(1) \sqcup \cdots \sqcup -F_{(p_{r-1},p_{r-1})}(1)) + \chi(\Sigma) \\
&= -s(F_{(2,1)}(1)) + -s(F_{(p_1,p_1)}(1)) - ... - s(F_{(p_{r-1},p_{r-1})}(1)) + (r-1) + \chi(\Sigma) \\
&= 2 - (1 - 2p_1) - ... - (1 - 2p_{r-1}) + (r-1) - 2 - 2p_1 - ... - 2p_{r-1} \\
&= 0
\end{aligned}$$

and thus $s(K_B) \geq 0$. Since by Lemma 4.6.11, K_G also admits such a slice disk, then we also have that $s(K_G) \geq 0$. □

Theorem 4.6.13. *Let L be a small RBG-link with $R = U$ and associated 0-surgery homeomorphism $\phi_L \colon S^3_0(K) \to S^3_0(K')$.*

(a) *If K is H-slice in some #$n\mathbb{CP}^2$, then $s(K') \geq 0$.*

(b) *If K is H-slice in some #$n\overline{\mathbb{CP}^2}$, then $s(K') \leq 0$.*

(c) *If K is BPH-slice, then $s(K') = 0$.*

Proof. Suppose K is H-slice in some #$n\mathbb{CP}^2$. If $r \geq 0$, then the proof follows as in Theorem 4.6.3. We have that (R, r) is slice in #$r\overline{\mathbb{CP}^2}$, so since K is H-slice in #$n\mathbb{CP}^2$, then by Corollary 4.6.2 K' must be H-slice in #$(n+r)\mathbb{CP}^2$. By Corollary 4.4.2, this means that $s(K') \geq 0$. If instead $r < 0$, then by the previous corollary we must have that $s(K') \geq 0$. We have thus proved **(a)**. If instead K is H-slice in #$n\overline{\mathbb{CP}^2}$, then the mirror $-K$ will be H-slice in #$n\mathbb{CP}^2$, and thus **(b)** follows. The remaining case is obtained by combining both previous results. □

Recall that for our approach to work, we need to have K be H-slice in #$n\mathbb{CP}^2$ while K' is H-slice in a homotopy #$n\mathbb{CP}^2$, then if one could show that K' is not H-slice in the standard #$n\mathbb{CP}^2$, we would have shown that this homotopy #$n\mathbb{CP}^2$ would actually be exotic. To obstruct the H-sliceness of K' we chose the s-invariant, with the hopes of having $s(K') < 0$ and by Theorem 4.4.1 this would immediately imply that K' can't be H-slice in #$n\mathbb{CP}^2$. But the previous theorem tells us that if K is H-slice then $s(K') \geq 0$, so we get no extra information from the s-invariant and thus we can't use it to obstruct the H-sliceness of K'. This means that if one wishes to construct an exotic #$n\mathbb{CP}^2$ by using the s-invariant, then the Manolescu-Piccirillo family doesn't produce viable candidates. Note that this doesn't mean that the Manolescu-Piccirillo family can't produce knots with such properties, only that the s-invariant can't obstruct them.

The links $L(a,b,c,d,e,f)$ are a very restrictive class of RBG-links when compared to special RBG-links (or even more generally to generic RBG-links), so there might still be some hope in using the techniques of Section 4.5 along with the s-invariant, to construct an exotic $\#n\mathbb{CP}^2$. The remaining of this thesis is dedicated to this goal - if $L = (R,r) \cup (B,0) \cup (G,0)$ is a special RBG-link, then what conditions do we need to have on (R,r) so that we don't run into the problems we had with the Manolescu-Picirillo family.

For our approach to work, we need to find conditions on (R,r) such that if K_B (or K_G) is H-slice in $\#n\mathbb{CP}^2$, then we can't have the s-invariant of their partner K_G (or K_B) vanishing. If it does, then we can't obstruct the H-sliceness of the partner in the original $\#n\mathbb{CP}^2$ by using the s-invariant. With this in mind, we introduce the following technical lemma, which constructs slice disks for K_B and K_G in a connected sum, when we have a sliceness condition on R. The goal is to use this lemma to construct slice disks for K_B and K_G in $\#n\mathbb{CP}^2$ such that we can use the conjectured adjunction inequality (Conjecture 4.4.7) to give us some bounds on the s-invariant.

Lemma 4.6.14. *Let L be a special RBG-link and X a closed, smooth 4-manifold. If $(R, r+1)$ is slice in X, then K_B and K_G are both (-1)-slice in $X\#\mathbb{CP}^2$.*

Proof. This proof is very similar to the proof of Lemma 4.6.11 when $r=-1$, so we proceed as in there. Start with L and slide $(B,0)$ over (R,r) until we can slam-dunk $R \cup G$ (so that the knot $(K_B,0)$ appears). Now change the framing of K_B to -1 and remove $(G,0)$ to obtain $L' = (R,r) \cup (K_B,-1)$. Consider now the 4-manifold Y obtained by attaching an r-framed 2-handle to R and a (-1)-framed 2-handle to K_B. We again reverse the slides previously done to recover $(R,r) \cup (B,-1)$ as in Figure 4.6.4. Now slide (R,r) over $(B,-1)$ to obtain the unlinked knots $(R,r+1) \sqcup (U,-1)$ (note that we're sliding R over B, instead of the other way around). Since the two knots are disjoint, then we have a connected sum between a knot trace for R and $\overline{\mathbb{CP}^2}$. We thus have produced a diffeomorphism between Y and $X_{r+1}(R)\#\overline{\mathbb{CP}^2}$. By hypothesis $(R,r+1)$ is slice in X, so by the Framed trace embedding lemma (Lemma 4.1.3), $-X_{r+1}(R)$ smoothly embeds in X. In particular we then have that $-Y = -X_{r+1}(R)\#\overline{\mathbb{CP}^2}$ smoothly embeds in $X\#\mathbb{CP}^2$. Notice now that if $-Y$ embeds in $X\#\mathbb{CP}^2$, then so does $X_{-1}(K_B)$ since one of the two handles of Y is attached along $(K_B,-1)$. By the Framed trace embedding lemma, it again follows that $(K_B,-1)$ is slice in $X\#\mathbb{CP}^2$. Since we can do the same for K_G by exchanging the roles of B and G, then the result follows. □

Notice that by following the proof with a few modifications we can find 1-slice disks for K_B and K_G in $X\#\overline{\mathbb{CP}^2}$ instead. Indeed, take $(R,r-1)$ to be slice in X, and construct a 4-manifold by attaching handles to L as before, except now we take the framing on K_B to be 1 (and attach a 1-framed 2-handle along it). After doing the slides which reverse the slam-dunk (and don't change the framings), we are left to slide (R,r) over $(B,1)$, this time leading to the unknotted link $(R,r-1) \sqcup (U,1)$. The rest of the proof follows in the same way and we can recover the following result.

Corollary 4.6.15. *Let L be a special RBG-link and X a smooth, closed 4-manifold. If $(R, r-1)$ is slice in X, then K_B and K_G are both 1-slice in $X\#\overline{\mathbb{CP}^2}$.* □

The previous lemma provides us with slice disks for K_B and K_G which we now want to relate with the s-invariant. If we take $X = \#n\mathbb{CP}^2$, then these are (-1)-framed disks in $\#(n+1)\mathbb{CP}^2$. Since they aren't null-homologous then we can't use Corollary 4.4.2 to obtain an inequality, and so we must assume Conjecture 4.4.7 to proceed. In this case we have that $-1 \leq PF_-(K_B), PF_-(K_G)$, so by Proposition 4.6.10 it follows that $0 = -1 + \sqrt{|-1|} \leq s(K_B),\ s(K_G)$. In a similar fashion, we can do the same for the 1-slice disks for K_B and K_G in $\#(n+1)\overline{\mathbb{CP}^2}$, except this time we have the bound $s(K_B),\ s(K_G) \leq 0$. The following conjecture then follows.

Conjecture 4.6.16. *If a knot K is (-1)-slice in some $\#n\mathbb{CP}^2$, then $s(K) \geq 0$. If instead K is 1-slice in some $\#n\overline{\mathbb{CP}^2}$, then $s(K) \leq 0$.*

Combining both previous results and restating them in terms of the projective slice framing gives us the following corollary.

Corollary 4.6.17. *Let $L = (R, r) \cup (B, 0) \cup (G, 0)$ be a special RBG-link with $r < PF_-(R)$ and associated 0-surgery homeomorphism $\phi\colon S^3_0(K) \to S^3_0(K')$. If we assume Conjecture 4.6.16 to be true, then the knots K and K' are (-1)-slice in some $\#n\mathbb{CP}^2$ and $s(K), s(K') \geq 0$. If $r > PF_+(R)$ instead, then K and K' are 1-slice in some $\#n\overline{\mathbb{CP}^2}$ and $s(K), s(K') \leq 0$.*

Proof. If $r < PF_-(R)$, then $r + 1 \leq PF_-(R)$ and thus by Corollary 4.6.9 we have that $(R, r+1)$ is slice in some $\#m\mathbb{CP}^2$. By applying the previous lemma, it then follows that K and K' are then both (-1)-slice in $\#(m+1)\mathbb{CP}^2$. Furthermore, if the previous conjecture is true, then we immediately have that $s(K), s(K') \geq 0$. The proof follows in the same exact manner if $r > PF_+(R)$, except now the previous lemma produces a 1-slice disk in $\#(m+1)\overline{\mathbb{CP}^2}$ and the previous conjecture yields $s(K),\ s(K') \leq 0$. □

We are finally ready to give conditions on the framing of R, so as to avoid running into a situation where the s-invariant is not able to distinguish between homotopy equivalent manifolds.

Theorem 4.6.18. *Let $L = (R, r) \cup (B, 0) \cup (G, 0)$ be a special RBG-link with associated 0-surgery homeomorphism $\phi\colon S^3_0(K) \to S^3_0(K')$, and suppose that Conjecture 4.6.16 is true.*

(a) *If $r < PF_-(R)$ or $r \geq PF_+(R)$, and K is H-slice in some $\#n\mathbb{CP}^2$, then $s(K') \geq 0$.*

(b) *If $r \leq PF_-(R)$ or $r > PF_+(R)$, and K is H-slice in some $\#n\overline{\mathbb{CP}^2}$, then $s(K') \leq 0$.*

(c) *If $r < PF_-(R)$ or $r > PF_+(R)$ or R is BPH-slice, and K is BPH-slice, then $s(K') = 0$.*

Proof. If $r < PF_-(R)$ then by the previous corollary K is (-1)-slice in some $\#n\mathbb{CP}^2$ and $s(K) \geq 0$ (the condition that K is H-slice is not necessary in this case). If instead $r \geq PF_+(R)$, then by Corollary 4.6.9 (R, r) is slice in some $\#m\overline{\mathbb{CP}^2}$. Since K is H-slice in some $\#n\mathbb{CP}^2$, then by Corollary 4.6.2 we have that $K_G = K'$ is H-slice in $\#(m+n)\mathbb{CP}^2$. It then follows by the adjunction inequality (Corollary 4.4.2) that $s(K') \geq 0$. This proves statement **(a)**. Note that we can prove statement **(b)** in a similar fashion. Indeed, if $r \leq PF_-(R)$ then by Corollary 4.6.9 (R, r) is slice in some $\#m\mathbb{CP}^2$. Assuming that K is H-slice in some

#$n\overline{\mathbb{CP}^2}$, then by Corollary 4.6.2 we have that K' is H-slice in #$(n+m)\overline{\mathbb{CP}^2}$. Once again, by the adjunction inequality (Theorem 4.4.1) it then follows that $s(K') \leq 0$. If $r > PF_+(R)$ then, as before, the previous corollary guarantees that $s(K') \leq 0$. The final statement follows from the previous two. Indeed, if K is BPH-slice, then K is H-slice in #$n\mathbb{CP}^2$ and #$n\overline{\mathbb{CP}^2}$, for some n. If $r < PF_-(R)$, then by **(a)** we have that $s(K') \geq 0$ and by **(b)** that $s(K') \leq 0$, so $s(K') = 0$. In the same way, if $r > PF_+(R)$ then $s(K') = 0$. Finally, if R is BPH-slice then by Lemma 4.6.8 we have that $PF_-(R) = PF_+(R) = 0$, so by combining the previous two statements, we have that $s(K') = 0$. □

Notice that all the links $L(a, b, c, d, e, f)$ from the Manolescu-Piccirillo family have $R = U$ the unknot. This means that R admits a null-homologous slice disk (given by a Seifert disk for the unknot) in both #$n\mathbb{CP}^2$ and #$n\overline{\mathbb{CP}^2}$, making it biprojectively H-slice. By the previous theorem we can then see that we would always have vanishing s-invariant for K', meaning that the Manolescu-Piccirillo family is ill-equipped to construct an exotic #$n\mathbb{CP}^2$ by using the s-invariant. Maybe this could be done by using another invariant where our methods do not apply. If one wishes to continue to pursue this strategy while sticking to the s-invariant, we hope that the previous theorem gives them some direction. Even if it turns out that the s-invariant cannot distinguish between homotopy #$n\mathbb{CP}^2$'s for the remaining values of r, there is still some hope that by extending this approach to generic RBG-links, one might be able to achieve some results, even if these are more complicated to deal with.

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Appendix A

Reference theorems

The objective of this appendix is to present some reference theorems which will be necessary throughout the thesis.

Theorem A.0.1 (Isotopy extension theorem - Chapter 8, theorem 1.3 of [Hir12]). *Let $f\colon M \to N$ be an embedding of a compact, smooth manifold M in smooth manifold N, and $F\colon M \times I \to N$ an isotopy of M. If either $F(M \times I) \subset \partial N$ or $F(M \times I) \subset$ int N, then F extends to an ambient isotopy of N having compact support.* □

Theorem A.0.2 (Corollary $IV.2.4$ of [Kos07]). *Let V be a compact submanifold of M, U an open neighborhood of V in M and $f\colon N \to M$ a smooth map. Then there is an ambient isotopy $h\colon M \times I \to M$ of M, that is the identity outside of U and such that $f \pitchfork h_1(V)$.* □

Theorem A.0.3 (Alexander trick). *Let $f\colon S^{n-1} \to S^{n-1}$ be an orientation preserving homeomorphism. Then there is an orientation preserving homeomorphism $F\colon D^n \to D^n$ such that $F|_{S^{n-1}} = f$.*

Proof. Let $f\colon S^{n-1} \to S^{n-1}$ be such a homeomorphism. Then the radial extension

$$\tilde{f}(x) = \begin{cases} |x| f(\frac{x}{|x|}) & \text{if } x \in D^n - \{0\} \\ 0 & \text{if } x = 0 \end{cases}$$

gives a homeomorphism $\tilde{f}\colon D^n \to D^n$ which agrees with f on S^{n-1}. □

Theorem A.0.4 (Laudenbach-Poénaru - [LP72]). *Let $f\colon \#kS^1 \times S^2 \to \#kS^1 \times S^2$ be a diffeomorphism. Then there is a diffeomorphism $F\colon \natural kS^1 \times D^3 \to \natural kS^1 \times D^3$ such that $F|_{\#kS^1 \times S^2} = f$.* □

Theorem A.0.5 (Milnor - [Mil62]). *Every non-trivial compact 3-manifold M is isomorphic to a sum $P_1 \# \cdots \# P_k$ of prime manifolds. The summands P_i are uniquely determined up to order and isomorphism.* □

Theorem A.0.6 (Theorem 8.2 of [FQ90]). *A 4-manifold has a smooth structure in the complement of any closed set with at least one point in each compact component. In particular a connected, non-compact 4-manifold admits a smooth structure.* □

Theorem A.0.7 (Moise - Theorems 1, 3 and 4 of [Moi52]). *Every topological 3-manifold admits a PL structure. Furthermore, if M and N are two homeomorphic 3-manifolds, then they are PL homeomorphic.* $\square$

Theorem A.0.8 (Radó - Theorems 3 and 5 in Section 8 of [Moi13]). *Every topological 2-manifold admits a PL structure. Furthermore, if M and N are two homemorphic 2-manifolds, then they are PL homeomorphic.* $\square$

Theorem A.0.9 (Theorems $3.10.8$ and $3.10.9$ of [Thu97]). *For $n \leq 3$, every PL n-manifold admits a smooth structure. Furthermore, if M and N are two PL homeomorphic n-manifolds, then they are diffeomorphic.* $\square$

Combining the three previous results, the uniqueness of smooth structures on manifolds of up to dimension 3 follows.

Theorem A.0.10. *For $n \leq 3$, every n-manifold admits a unique smooth structure.* $\square$