A brief guide to Differential Equations

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Abstract

This guides handles differential equations - equations with derivatives. This course aims to solve these differential equations by various methods. This particular document is meant for MATH 308 at Texas A&M University, but should work for most undergraduate-level differential equations courses.

Contents

1 Introduction

The goal of a differential equations class is to solve various differential equations. That much is obvious, but what is a **differential equation**? A differential equation is any equation that has a derivative in it. Differential equations are solved through various methods depending on the type of differential equation. This course deals with **ordinary differential equations**, which only have one independent variable.

For example,

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2 = 0$$

is an ordinary differential equation because y is *only* differentiated with respect to t. On the other hand.

$$\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2}$$

is a **partial differential equation** because y is differentiated with respect to *both* x and t. Partial differential equations are extremely difficult to solve compared to ordinary differential equations (a bit of an understatement if anything).

The **order** of a differential equation is the order of the highest derivative (also called differential coefficient). For example, the equation $\frac{dy}{dt} = 1$ has an order of 1, because the highest derivative is $\frac{dy}{dt}$. The equation $\frac{d^2y}{dt^2} + \frac{dy}{dt} + 2 = 0$ has an order of 2 because $\frac{d^2y}{dt^2}$ is the highest derivative.

Linear differential equations are differential equations where separate derivatives are added. Specifically, they must be of the form

$$a_0(x) * y + a_1(x) * y' + a_2(x) * y'' + ... = b(x)$$

where a and b are functions of x.

For example,

$$y'' + y' + 2 = 0$$

is a linear differential equation but

$$(y)(y'') = 4$$

is not a linear differential equation (since two y variables are multiplied).

1.1 Abbreviations and Notation

Here are some abbreviations and notation that are commonly used in this document:

- DEQ = Differential Equation. In other materials, you may see these refered to as ODEs (Ordinary Differential Equations) or PDEs (Partial Differential Equations).
- W(t) = Wronskian
- $\mathcal{L}$ = Laplace Transformation
- $y' = \frac{dy}{dt}$
- $y'' = \frac{d^2y}{dt^2}$ (and so on)
- Letters (especially capital letters) such as A, B_0, C_{12}, etc. are constants unless otherwise stated.
- Letters such as $A(t), B_0(t), C(x)$, etc. are functions unless otherwise stated.

1.2 Some simple differential equations

Some DEQs can be very easy to solve, however *most DEQs are hard or impossible to solve*. You may encounter some very nasty DEQs on your homework problems. Here's an easy DEQ.

A simple example is a model of a falling object with terminal velocity.

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

This DEQ can be easily solved by factoring out $-\frac{1}{5}$ and then dividing both sides by $(v - 49)$.

$$v' = 9.8 - \frac{v}{5}$$

$$v' = -\frac{1}{5}(v - 49)$$

$$\frac{v'}{v - 49} = -\frac{1}{5}$$

Now that the variables are isolated on one side, integrating both sides with respect to dt will then get you the answer.

$$\int \frac{v'}{v - 49} dt = \int -\frac{1}{5} dt$$

(Using U-Substitution)

$$u = v - 49$$

$$du = dv = v'dt$$

Note that in this problem, I replaced dv/dt with v'. It works either way, just note that if $dv/dt = v'$, then $dv = v'dt$.

$$\int \frac{du}{u} = \int -\frac{1}{5}dt$$

$$ln|u| + C_1 = -\frac{1}{5}t + C_2$$

$$ln|v - 49| = -\frac{1}{5}t + C_3$$

Here, the constants C_1 and C_2 are combined to form a constant $C_3 = C_2 - C_1$. This will be very common in some differential equation solving techniques and will be applied often.

$$|v - 49| = e^{-\frac{1}{5}t + C_3}$$

$$|v - 49| = e^{-\frac{1}{5}t}e^{C_3}$$

$$|v - 49| = e^{-\frac{1}{5}t}C_4 \qquad (C_4 = e^{C_3})$$

$$v - 49 = \pm C_4 e^{-\frac{1}{5}t}$$

The $\pm C_4$ can be reduced to a new constant C_5.

$$\boxed{v = 49 + C_5 e^{-\frac{1}{5}t}}$$

1.3 A Warning

This class will use A LOT of integrals and integration solving strategies. Be familiar with u-substitution and integration by parts. Also be familiar with logarithm and exponent rules, as those tend to show up a lot as well.

1.4 What Should I Prepare?

Depending on your professor, the work in this course can be anywhere from manageable to nearly unsolvable without a computer. For this reason, I recommend getting a differential equations solver to check your work and to save yourself some time on the nastier problems.

Free online programs such as Symbolab, Wolfram Alpha, and eMathHelp will work for the first few chapters, however they will not work when it comes to later chapters such as Laplace and series solutions. For those, you may want to get stronger software, such as Wolfram Mathematica or Matlab. To solve differential equations in Mathematica, see here.

BE FAMILIAR WITH INTEGRATION BY PARTS AND PARTIAL FRACTIONS. Being good at them will make your life much easier. And get a lot of scratch paper.

2 The Integrating Factor Method

The integrating factor method solves *linear first order differential equations.* A linear first order differential equation will have the form

$$\frac{dy}{dt} + p(t)y = g(t)$$

or

$$P(t)\frac{dy}{dt} + Q(t)y = G(t)$$

The forms above are interchangeable, to get from the second equation to the first, just divide the second equation by $P(t)$.

The integrating factor method exploits the product rule of derivatives. For a differential equation

$$\frac{dy}{dt} + p(t)y = g(t)$$

the integrating factor method multiplies every term in the equation by a function called the integrating factor μ.

$$\mu\frac{dy}{dt} + \mu p(t)y = \mu g(t) \tag{1}$$

The product rule of derivatives states that

$$\frac{d}{dt}(\mu y) = \mu y' + \mu' y$$

Note: For ordinary differential equations (differentiated w.r.t. one variable) you can assume $y' = \frac{dy}{dt}$ *A similar rule applies for other variables like* μ*.*

The integrating factor method attempts to match the product rule above with the left hand side of (1). In other words,

$$\mu' = \mu p(t)$$

$$\mu(t) = \pm e^{\int p(t)dt}$$

Note: We usually chose the positive result for convenience, but both results will work.

If this μ value is found, then the left hand side of (1) reduces to $(\mu y)'$ and then the equation becomes

$$(\mu y)' = \mu g(t) \tag{2}$$

Integrating both sides and solving for y will yield

$$y = \frac{1}{\mu(t)}\int \mu(t)g(t)dt$$

2.1 Summary

For a linear first order differential equation

$$\frac{dy}{dt} + p(t)y = g(t)$$

the answer will be

$$y = \frac{1}{\mu(t)}\int \mu(t)g(t)dt$$

if $\mu(t)$ is

$$\mu(t) = e^{\int p(t)dt}$$

2.2 Example

$$y' + 0.5y = 0.5e^{\frac{t}{3}}$$

1. Find $\mu(t)$

$$\mu = e^{\int 0.5dt} = e^{0.5t}$$

2. Solve $y = \frac{1}{\mu(t)}\int \mu(t)g(t)dt$

$$y = \frac{1}{e^{0.5t}}\int e^{0.5t}\cdot 0.5e^{\frac{t}{3}}dt$$

$$\boxed{y = \frac{3}{5}e^{\frac{1}{3}t} + Ce^{-\frac{1}{2}t}}$$

3 Separable First Order DEQs

A first order differential equation is separable if it can be written as

$$M(x)dx + N(y)dy = 0$$

$$M(x)dx = -N(y)dy$$

Note: M(x) has to be a function of x ONLY. A similar rule applies for N(y)

These can be solved by integrating both sides.

$$\int M(x)dx = -\int N(y)dy$$

. Basically if you can separate x and y to different sides of the equation, you can solve pretty easily by integrating the equation.

3.1 Example

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}$$

Isolate y and x to opposite sides of the equation (multiply by dx and 2(y-1)).

$$2(y-1)dy = (3x^2 + 4x + 2)dx$$

Integrate

$$\int 2(y-1)dy = \int (3x^2 + 4x + 2)dx$$

$$\boxed{y^2 - 2y = x^3 + 2x^2 + 2x + C}$$

The above equation is in *Implicit Form.*

$$\boxed{y = 1 - \sqrt{x^3 + 2x^2 + 2x + C_1}}$$

The above equation is in *Explicit Form.*

3.2 A Note on Implicit and Explicit Form

Implicit form is when you don't isolate y, while explicit form will always have the form of $y = ...$ Often times you will have to find both. Explicit form may need some "creative solving" to reach (usually just completion of squares but on rare occasions more complicated methods are needed). You will be asked to find both in this course.

Also see here.

4 Modeling with First Order DEQs

Differential equations can be used to create models.

The key to modeling with First Order DEQs is to translate text to math. Write out all given variables, all unknown values, and solve for the various unknowns. Treat it like the same way you would treat a really nasty algebra word problem.

4.1 Example

At t = 0, a tank has Q_0 lb of salt in 100 gallons of water. Assume that water with a concentration of 2 pounds of salt per gallon is entering the tank at a rate r gallons/min and a well-stirred mixture is draining at the same rate. What is the amount of salt in the tank at a time t?

Given values:

- Density of water entering the tank: 2 pounds per gallon.
- Total amount of water: 100 gallons

Unknown values:

- Amount of salt in the tank at time t: Q(t)
- Initial amount of salt present: Q_0

If Q(t) is the amount of salt in the tank at time t and Δt is a small time interval, then

$$Q(t+\Delta t) - Q(t) = \text{(Density of water entering)(Amount of water entering)}-$$

$$\text{(Density of water leaving)(Amount of water leaving)}$$

In math terms, this becomes

$$Q(t+\Delta t) - Q(t) = 2(r\Delta t) - \frac{Q(t)}{100}(r\Delta t)$$

If Δt is infinitely small (if we reduce the above equation to a derivative by dividing both sides by Δt), then

$$\boxed{Q'(t) = 2r - \frac{Q(t)}{100}r}$$

4.1.1 Additional Note: Clarification of the last step

$$Q(t+\Delta t) - Q(t) = 2(r\Delta t) - \frac{Q(t)}{100}(r\Delta t)$$

Divide both sides by Δt

$$\frac{Q(t+\Delta t) - Q(t)}{\Delta t} = \frac{2(r\Delta t)}{\Delta t} - \frac{\frac{Q(t)}{100}(r\Delta t)}{\Delta t}$$

The right hand side is the definition of a derivative.

$$Q'(t) = \frac{2(r\Delta t)}{\Delta t} - \frac{\frac{Q(t)}{100}(r\Delta t)}{\Delta t}$$

The left hand side cancels pretty easily.

$$Q'(t) = 2r - \frac{Q(t)}{100}r$$

5 Determining Continuity

In the equation

$$y' + p(t)y = g(t)$$

if p and g are continuous on an interval $I = \alpha < t < \beta$, then there is also a solution to the differential equation across the interval I.

5.1 Example

For the DEQ

$$ty' + 2y = 4t^2$$

(which reduces to)

$$y' + \frac{2}{t}y = 4t$$

$$p(t) = \frac{2}{t}, (t \neq 0)$$

$$g(t) = 4t$$

the differential equation will (be guaranteed to) have a solution at anywhere except t = 0. If it is given that $y(1) = 2$, then the maximum interval containing t=1 on which both p and g is continuous is (0, ∞)

For nonlinear equations

$$y' = f(t, y)$$

then the equation will have a solution whenever both $f(t, y)$ and $f_y(t, y)$ are continuous on 2 intervals: I and J. The interval I corresponds to the interval on which t is continuous, while J corresponds to the interval on which y is continuous.

5.2 Example

$$\frac{dy}{dt} = \frac{3t^2 + 4t + 2}{2(y+1)}$$

$$2(y-1)y' = 3t^2 + 4t + 2$$

(Equation is nonlinear)

$$y' = f(x, y) = \frac{3t^2 + 4t + 2}{2(y-1)}$$

$$f_y(x, y) = -\frac{3t^2 + 4t + 2}{2(y-1)^2}$$

Both $f(x, y)$ and $f_y(x, y)$ are continuous everywhere except for y = 1. The function is continuous on all t, so therefore the interval I is $I = (-\infty, \infty$, and the interval J is $J = (-\infty, 1)U(1, \infty)$.

If it's given that y(0) = 1, you can't prove that a unique solution exists because $f(x, y)$ and $f_y(x, y)$ is not continuous at y = 1. HOWEVER this is just a litmus test, it doesn't mean that there is no solution, it just means you can't say for certain if there is a solution. In fact, for this DEQ you can solve it directly to prove that the unique solution exists there.

6 Autonomous DEQs

An Autonomous DEQ is one where

$$\frac{dy}{dt} = f(y)$$

(as oppossted to the general equation)

$$\frac{dy}{dt} = f(t, y)$$

.

An equilibirum solution (critical point) occurs when

$$\frac{dy}{dt} = 0$$

Equilibrium solutions are considered *stable* if solutions started near it approach it as $t- > \infty$. They are *unstable* if solutions diverge as $t- > \infty$ and are *semi-stable* if some solutions converge and some solutions diverge.

You can determine the stability of equilibrium points by testing points around the equilibrium points, kinda similarly to how critical points were tested in Calc 1. If the value of the DEQ is positive at a certain (non-critical point), then all solutions of the DEQ will approach the HIGHER equilibrium point (or infinity). If the value of the DEQ is negative at a certain (non-critical point), then all solutions of the DEQ will approach the LOWER equilibirum point (or negative infinity).

6.1 Example

Determine equilibrium points and the stability of each equilibrium.

$$\frac{dy}{dt} = y(2-y)(y-5)$$

Equilibrium Points

$$y(2-y)(y-5) = 0$$

$$y = 0, 2, 5$$

To determine the stability of each equilibrium, test points around each critical point.

- At y = -1 ($y < 0$), the DEQ is positive, so solutions will approach the higher equilibrium point instead of negative infinity (approaches 0).
- At y = 1 ($0 < y < 2$), the DEQ is negative, so solutions will approach the lower equilibrium point (approaches 0).
- At y = 3 ($2 < y < 5$), the DEQ is positive, so solutions will approach the higher equilibrium point (approaches 5).
- At y = 10 ($5 < y$), the DEQ is negative, so solutions will approach the lower equilibrium point instead of infinity (approaches 5)

From this, we can tell that solutions will approach 0 on both sides and will approach 5 on both sides. It will not approach 2 on either side. Therefore, 0 and 5 are stable equilibirum points and 2 is an unstable equilibrium point.

7 Exact DEQs

If there is a DEQ

$$M(x,y) = N(x,y)y' = 0$$

then the DEQ is considered exact if

$$M_y(x,y) = N_x(x,y)$$

.

Note: ($M_y = \frac{d}{dy}M$, $N_x = \frac{d}{dx}N$)

If the DEQ is exact, then there's a function ψ such that

$$\psi_x(x,y) = M(x,y)$$

and

$$\psi_y(x,y) = N(x,y)$$

The form of the solution to the DEQ will be $\psi = C$, where C is a constant.

7.1 Example

$$ycos(x) + 2xe^y + (sin(x) + x^2e^y - 1)y' = 0$$

1. Determine M and N

$$M(x, y) = ycos(x) + 2xe^y$$

$$N(x, y) = sin(x) + x^2e^y - 1$$

2. Determine if the DEQ is exact

$$M_y(x, y) = cos(x) + 2xe^x$$

$$N_x(x, y) = cos(x) + 2xe^x$$

Because $M_x = N_y$, the equation is exact.

3. Find ψ

$$\psi = \int M dx = \int ycos(x) + 2xe^y dx$$

Note: You can also use $\psi = \int N dy$ *if that is easier.*

$$\psi = ysin(x) + x^2e^y + h(y)$$

The constant of the integral is going to be in the form of h(y), or h(x) if you chose to do $\int N dy$.

4. Solve for h(y), given that $psi_y(x, y) = N(x, y)$ (or $psi_x(x, y) = M(x, y)$ if you decided to integrate N).

$$\psi_y = sin(x) + x^2e^y + h'(y) = N(x, y)$$

$$\psi_y = sin(x) + x^2e^y + h'(y) = sin(x) + x^2e^y - 1$$

$$h'(y) = -1$$

$$h(y) = -y + C_0$$

5. Substitute h(y) back into ψ and rewrite in the form of $\psi = C$.

$$\psi = ysin(x) + x^2e^y - y + C_0 = C_1$$

$$\psi = ysin(x) + x^2e^y - y = C$$

8 Homogeneous DEQs with Constant Coefficients

Second Order Linear DEQs are of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

OR

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

A homogeneous DEQ is one where g(t) = 0

$$y'' + p(t)y' + q(t)y = 0$$

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

Similarly to first order DEQs, the above two equations are interchangeable by just dividing (or multiplying) by P(t).

For a homogeneous DEQ with constant coefficients of

$$ay'' + by' + cy = 0$$

(MUST EQUAL 0)

The solution can be found by guessing its form of $y = e^{rt}$, where r is given by

$$ar^2 + br + c = 0$$

If there are 2 *distinct, real* numbers of r_1 and r_2, then the fundamental solutions of the DEQ are

$$y = C_1 e^{r_1 t}$$

and

$$y = C_2 e^{r_2 t}$$

The General Solution for the DEQ is given by

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

8.1 Continuity

Second Order DEQs of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

are continuous when p(t), q(t) and g(t) are continuous.

9 Wronskian

If you are given 2 solutions to a DEQ y_1 and y_2, then the Wronskian determinant of the solutions is

$$W(t) = W[y_1, y_2](t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

If the Wronskian is NOT equal to 0, then y_1 and y_2 are fundamental solutions and you can write the equation in the form

$$y = C_1 y_1(t) + C_2 y_2(t)$$

9.1 Example

Show that $y_1(t) = t^{0.5}$ and $y_2(t) = t^{-1}$, $t > 0$.

$$W(t) = W[y_1, y_2](t) = \begin{bmatrix} t^{0.5} & t^{-1} \\ 0.5t^{0.5} & -t^{-1} \end{bmatrix} = t^{0.5}(-t^{-1}) - 0.5t^{0.5}t^{-1} = -\frac{3}{2}t^{-\frac{3}{2}}$$

Since it is given that $t > 0$, y_1 and y_2 form a fundamental set of solutions.

10 Complex Roots

If you solve

$$ay'' + by' + c = 0$$

$$ar^2 + br + c = 0$$

and you get results of $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$

Then the Real-Valued Solutions will be

$$u = e^{\lambda t} cos(\mu t)$$

$$v = e^{\lambda t} sin(\mu t)$$

And the general solution is given by

$$C_1 u + C_2 v = C_1 e^{\lambda t} cos(\mu t) + C_2 e^{\lambda t} sin(\mu t)$$

(Also you may be asked to check that $W[u, v] \neq 0$)

11 Repeated Roots

If you solve

$$ay'' + by' + c = 0$$

$$ar^2 + br + c = 0$$

and you get results of $r = r_1 = r_2$ (if there is only 1 r value), then the roots are considered repeated.

The general solution for a repeated roots question is given by

$$y = C_1 e^{r_1 t} + C_2 t e^{r_2 t}$$

12 Nonhomogeneous Linear Diffeqs - Method of Undetermined Coefficients

A nonhomogeneous diffeq will have the form

$$y'' + p(t)y' + q(t)y = g(t)$$

where $g(t) \neq 0$.

Nonhomogeneous DEQs are solved by finding the homogeneous solution to the DEQ (y_h) and then by guessing a particular solution y_p. This is called the method of undetermined coefficients. The sum of the homogeneous and particular solutions will be the solution to the DEQ.

Note: This method is typically only effective on constant terms for $p(t)$ *and* $q(t)$*. It also only works on a few classes of* $g(t)$ *values, which you will see below.*

The homogeneous solution of the DEQ is easy enough, just set $g(t) = 0$ and solve the resulting homogeneous DEQ.

$$ay'' + by' + cy = g(t)$$

$$ar^2 + br + c = 0$$

$$y_h = C_1 y_1 + C_2 y_2$$

The particular solution is guessed using the following criteria. (Will discuss the t^s in a bit).

IF $g(t) = a_0 t^n + a_1 t^n - 1 + ... + a_n$, then

$$y_p = t^s[A_0 t^n + A_1 t^{n-1} + ... + A_n]$$

IF $g(t) = [a_0 t^n + a_1 t^n - 1 + ... + a_n]e^{\alpha t}$

$$y_p = t^s[A_0 t^n + A_1 t^{n-1} + ... + A_n]e^{\alpha t}$$

IF $g(t) = [a_0 t^n + a_1 t^n - 1 + ... + a_n]e^{\alpha t} sin(\beta t)$ (or $cos(\beta t)$)

$$y_p = t^s([A_0 t^n + A_1 t^{n-1} + ... + A_n]e^{\alpha t} sin(\beta t) + [B_0 t^n + B_1 t^{n-1} + ... + B_n]e^{\alpha t} cos(\beta t))$$

12.1 Examples for Particular Solutions

If $g(t) = 2t^2 + 3$

$$y_p = At^2 + Bt + C$$

If $g(t) = (2t^2 + 3)e^{4t}$

$$y_p = (At^2 + Bt + C)e^{4t}$$

If $g(t) = te^{2t} cos(3t)$

$$y_p = (A_0 t + A_1)e^{2t} sin(3t) + (B_0 t + B_1)e^{2t} cos(3t)$$

12.2 How to determine s

The s in t^s is determined by the following criteria

- IF r = 0 is a simple root, then s = 1
- IF r = 0 is a repeated root, then s = 2
- ELSE s = 0
- IF $r = \alpha$ is a simple root, then s = 1
- IF $r = \alpha$ is a repeated root, then s = 2
- ELSE s = 0
- IF $r = \alpha + \beta i$ is a root, then s = 1
- ELSE s = 0

12.3 Example with s

Find the form of the particular solution of the DEQ

$$y'' + 2y' - 3y = 5e^{-3t}$$

Homogeneous

$$y'' + 2y' - 3y = 0$$
$$r^2 + 2r - 3 = 0$$
$$(r+3)(r-1) = 0$$
$$r = -3, 1$$
$$y = C_1 e^{-3t} + C_2 e^t$$

Particular

$$g(t) = 5e^{-3t}$$
$$r = -3 = \alpha, \text{(Simple Root)}; s = 1$$
$$y_p = tAe^{-3t}$$

13 Variation of Parameters

For the DEQ

$$y'' + p(t)y' + q(t)y = g(t)$$

Variation of Parameters if performed by first finding the homogeneous solution to the diffeq

$$y'' + p(t)y' + q(t)y = 0$$

$$y = C_1 y_1(t) + C_2 y_2(t)$$

The General Solution via Variation of Parameters is given by

$$y(t) = U_1(t)y_1(t) + U_2(t)y_2(t)$$

where

$$U_1(t) = -\int \frac{y_2 g(t)}{W} dt$$
$$U_2(t) = \int \frac{y_1 g(t)}{W} dt$$

and W is the wronskian of y_1 and y_2 found in the homogeneous solution.

General Solution

$$y(t) = (-\int \frac{y_2 g(t)}{W} dt) y_1 + (\int \frac{y_1 g(t)}{W} dt) y_2$$

NOTE: The integrals will have added constants of C_1 and C_2 respectively. Make sure to include them. Alternately, if you don't want to deal with the constants, you can ignore the constants and add the homogeneous solution to your final result when you do variation of parameters (at least I think this works).

13.1 Exmaple

$$y'' - 3y' - 4y = 3e^{2t}$$

Homogeneous Solution

$$y'' - 3y' - 4y = 0$$

$$r^2 - 3r - 4 =$$

$$r = 4, -1$$

$$y_h = C_1 e^{4t} + C_2 e^{-t}$$

Wronskian

$$W = \begin{bmatrix} e^{4t} & e^{-t} \\ 4e^{4t} & -e^{-t} \end{bmatrix} = -5e^{3t}$$

Variation of Parameters

$$U_1 = -\int \frac{e^{-t} 3e^{2t}}{-5e^{3t}} dt = -\frac{3}{10e^{-2t} + C_1}$$

$$U_2 = \int \frac{e^{4t} 3e^{2t}}{-5e^{3t}} dt = -\frac{1}{5} e^{3t} + C_2$$

$$y = U_1 y_1 + U_2 y_2 = (-\frac{3}{10} e^{-2t} + C_1) e^{4t} + (-\frac{1}{5} e^{3t} + C_2) e^{-t}$$

$$y = C_1 e^{4t} + C_2 e^{-t} - \frac{1}{2} e^{2t}$$

14 Laplace Transform

Definition of the Laplace Transform

$$\mathcal{L}[f(t)] = F(t) = \int_0^\infty e^{-st} f(t) dt$$

Laplace only exists if the function can be integrated from $[0, \infty)$.*

**It's a little more complicated than that, but it's not terribly important for the scope of this class. You won't really be assessed on if the Laplace transform exists for a function.*

Laplace is linear - it can be split if functions are added. Also constants can be taken out of the Laplace.

$$\mathcal{L}[C_1 f_1(t) + C_2 f_2(t)] = C_1 \mathcal{L} f_1(t) + C_2 \mathcal{L} f_2(t)$$

The Laplace integral itself can also be separated to deal with piecewise functions. See the example below:

14.1 Example

Note: If I ever say "Laplace the function" it means to take the Laplace transform.

Laplace the function f(t):

$$f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & t > 1 \end{cases}$$

$$\mathcal{L}f(t) = F(t) = \int_0^\infty e^{-st} f(t)dt = \int_0^1 e^{-st} \cdot 1dt + \int_1^\infty e^{-st} \cdot 0dt$$

$$F(t) = -\frac{1}{s}e^{-s} + \frac{1}{s}$$

Here you integrate over each interval separately.

The laplace transform changes the "space of functions". In the function space of $f(t)$, something like e^t is expressed as is. However in the function space of $F(s)$, e^t changes to the form $\frac{1}{s-1}$.

The idea of the Laplace transform is that we change the space of the function $f(t)$ to the function space $F(s)$. From there, we isolate cases of $\mathcal{L}y$ and take the inverse Laplace (reverting back to the function space of $f(t)$ to solve for y. A similar approach is used for other "transformations" such as the Fourier transformation.

15 Solving DEQs with Laplace

1. Apply Laplace Transform
2. Isolate $\mathcal{L}y$ (often represented simply by "$\mathcal{L}y = Y$")
3. Apply the Inverse Laplace Transform

USE THE LAPLACE TABLES

WARNING: Inverse Laplace is, in most cases, SIGNIFICANTLY harder to do than the Laplace transform. You will often have to do partial fraction decompositions and unconventional completions of squares to turn the equation into a solvable form.

15.1 Example

$$y'' - y' - 2y = 0, y(0) = 1, y'(0) = 0$$

Laplace

$$\mathcal{L}[y'' - y' - 2y = 0]$$

$$(s^2Y - sy(0) - y'(0)) - (sYy(0)) - 2Y = 0$$

$$s^2Y - s - sY + 1 - 2Y = 0$$

Isolate Y

$$(s^2 - s - 2)Y - s + 1 = 0$$

$$Y = \frac{s-1}{s^2 - s - 2}$$

Inverse Laplace

$$Y = \mathcal{L}(y), y = \mathcal{L}^{-1}(Y)$$

$$y = \mathcal{L}^{-1}\frac{s-1}{s^2 - s - 2}$$

Partial Fraction Decomposition

$$\frac{s-1}{s^2 - s - 2} = \frac{A}{s-2} + \frac{B}{s+1}$$

$$A = \frac{1}{3}, B = \frac{2}{3}$$

$$y = \mathcal{L}^{-1}[\frac{\frac{1}{3}}{s-2} + \frac{\frac{2}{3}}{s+1}]$$
$$y = \frac{1}{3}\mathcal{L}^{-1}[\frac{1}{s-2}] + \frac{2}{3}\mathcal{L}^{-1}\frac{1}{s+1}]$$
$$y = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

16 Heaviside Step Functions

A Step Function is a means of representing piecewise functions using a switch-like function. The step function can be thought of as turning functions "on" or "off" at certain times

The Definition of the Heaviside is given as follows:

$$u_c(t) = \begin{cases} 1 & t \geq c \\ 0 & t < c \end{cases}$$

For example, the function

$$f(x) = u_3(t)x$$

will be $f(x) = 0$ before $x = 3$ (switch is off) and $f(x) = x$ at and after $x = 3$ (switch is on). Likewise, the function

$$f(x) = u_3(t)x + u_7(t)x^2$$

will be $f(x) = 0$ before $x = 3$ (all switches are off), $f(x) = x$ from $3 \leq x < 7$ (first switch is on), and $f(x) = x + x^2$ for $x \geq 7$ (both switches are on).

The laplace of a heaviside function (by itself) is

$$\mathcal{L}u_c(t) = \frac{1}{s}e^{-sc}$$

To convert a piecewise function into a heaviside form, use heavisides to turn functions on one at a time. Subtract the previous function to deactivate it. For example,

$$f(t) = \begin{cases} t & \geq t < 3 \\ 3t^2 & 3 \geq t < 5 \\ 4t^3 & 5 \geq t < 7 \\ 5t^4 & 7 \geq t < 9 \\ 6t^5 & 9 \geq t \end{cases}$$

will be represented as

$$h(t) = u_1(t)t + u_3(t)(3t^2 - t) + u_5(t)(4t^3 - 3t^2) + u_7(t)(5t^4 - 4t^3) + u_9(t)(6t^5 - 5t^4)$$

Notice how you can "turn off" previous Heavisides by negating the previous function. For example, in the above function, the $u_1(t)$ function is "turned off" placing a $-t$ in $u_3(t)$

The Laplace of a Heaviside function (with a function attached) is given by

$$\mathcal{L}u_c(t)f(t) = e^{-cs}\mathcal{L}f(t+c)$$

For example, the Laplace of the function

$$f(t) = u_4(t)(t-4) - u_8(t)(t-8)^2 + u_8(t)68$$

is given by

$$\mathcal{L}f(t) = \mathcal{L}u_4(t)(t-4) - \mathcal{L}u_8(t)(t-8)^2 + \mathcal{L}u_8(t)68$$
$$F(t) = e^{-4s}\mathcal{L}(t-4+4) - e^{-8s}\mathcal{L}(t-8+8)^2 + \frac{68e^{-8s}}{s}$$
$$F(t) = e^{-4s}\mathcal{L}(t) - e^{-8s}\mathcal{L}(t)^2 + \frac{68e^{-8s}}{s}$$
$$F(t) = \frac{e^{-4s}}{t^2} - \frac{2e^{-8s}}{s^3} + \frac{68e^{-8s}}{s}$$

When the forcing function (the nonhomogeneous part) of the function is a piecewise function, represent it as a Heaviside and then Laplace both sides.

17 Impulse (Dirac/Delta) Functions

An impulse function is one that's equal to infinity at a certain point, and equal to 0 at all other points. Sometimes called the Dirac Delta or Delta function.

$$\delta(t-t_0) = \begin{cases} 0 & t \neq t_0 \\ \infty & t = t_0 \end{cases}$$

For a real number t_0,

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t) dt = f(t_0)$$

and

$$\mathcal{L}[\delta(t-t_0) f(t)] = e^{-st_0} f(t_0)$$

18 The Convolution Integral

For

$$\mathcal{L}f(t) = F(s), \mathcal{L}g(t) = G(s)$$

$$\mathcal{L}^{-1}[F(s)G(s)] = f * g = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

18.1 Example (Laplace)

$$h(t) = \int_0^t (t-\tau)^2 cos(2\tau) d\tau$$

$$f(t) = t^2, F(s) = \frac{2}{s^3}$$

$$g(t) = cos(2t), G(s) = \frac{s}{s^2+4}$$

$$\mathcal{L} = \{\, * \,\} = F(s)G(s) = \frac{2}{s^3} \cdot \frac{s}{s^2+4}$$

18.2 Example (Inverse Laplace)

$$H(s) = \frac{1}{s^4(s^2+1)} = \frac{1}{s^4} \cdot \frac{1}{s^2+1}$$

$$F(s) = \frac{1}{s^4}, f(t) = \frac{1}{3!} t^3$$

$$G(s) = \frac{1}{s^2+1}, g(t) = sin(t)$$

$$\mathcal{L}^{-1}H(s) = f * g = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$$

$$\mathcal{L}^{-1}H(s) = f * g = \int_0^t \frac{(t-\tau)^3}{6} sin(\tau) d\tau = \int_0^t \frac{\tau^3}{6} sin(t-\tau) d\tau$$

19 Series Solutions of 2nd Order Linear Equations

For series solutions, you assume the solution will be

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

where x_0 is an ordinary point (see below).

For the DEQ

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

x_0 is a singular point if $P(x_0) = 0$ and is an ordinary point if $P(x_0) \neq 0$. Essentially, if x_0 is an ordinary point, you can write the solution to the DEQ in the form of

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1 + a_1 y_2$$

where a_1 and a_2 are constants and y_1 and y_2 are series solutions.

The power series solution to the DEQ will only exsist "in between orindary points". For example, for the DEQ

$$(x^2 - 2)y'' + (x + 3)y' + (x^2 + 1)y = 0, \text{centered at } x_0$$

$$P(x) = x^2 - 2 = 0$$

$$x = \pm\sqrt{2}$$

the power series solution will exsist on an interval of $(-\sqrt{2}, \sqrt{2})$. The radius of convergence R is given by the distance from x_0 to x (in this case, from 0 to 2, or $R = 2 - 0 = 2$), and a function will converge on an interval of $(x_0 - R, x_0 + R)$. In this case, as $x_0 = 0$, the interval of convergence is $0 - \sqrt{2}, 0 + \sqrt{2}$.

In the case of $x_0 = 1$, the interval of convergence is going to be $\sqrt{2} - 1$, as that is the distance from $x = \sqrt{2}$ to $x_0 = 1$ is $\sqrt{2} - 1$. It follows that the function will converge on $1 - (\sqrt{2} - 1), 1 + (\sqrt{2} - 1)$.

NOTE: For this class you generally don't have to know the ramifications of this interval, you just need to know how to find it.

To solve a DEQ with a power series (assuming it's centered at $x_0 = 0$, first plug in

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

then "synchronize" the powers of x - make them all equal to the same power. This can be done by using shift of index, for example

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

$$m = n - 2$$

$$n = m + 2$$

$$n = 2, m = 2 - 2 = 0$$

$$n = \infty, m = \infty$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = \sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1) x^m$$

Next equalize the index of summation (the stuff below the sum). This is done by taking out terms from the summation. For example, if you wanted to convert $n = 0$ to $n = 2$ in the equation

$$\sum_{n=0}^{\infty}(a_n)x^n$$

you would pull out the first two terms

$$a_0x^0 + a_1x^1 + \sum_{n=2}^{\infty}(a_n)x^n$$

Finally you find the recurrence relations. Factor out any x terms which are outside of the summations and combine all sums. Then set all terms equal to zero.

19.1 Example

Find a series solution to Airy's Equation at $x_0 = 0$

$$y'' - xy = 0$$

1. Substitute Series Solution

$$y = \sum_{n=0}^{\infty} a_nx^n$$

$$y' = \sum_{n=1}^{\infty} a_nnx^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_nn(n-1)x^{n-2}$$

$$\sum_{n=2}^{\infty} a_nn(n-1)x^{n-2} - x\sum_{n=0}^{\infty} a_nx^n = 0$$

$$\sum_{n=2}^{\infty} a_nn(n-1)x^{n-2} + \sum_{n=0}^{\infty} -a_nx^{n+1} = 0$$

2. Shift of Index - Two separate shifts of index are done here, one for the term x^{n-2} and one for the term x^{n+1}

$$\sum_{n=2}^{\infty} a_nn(n-1)x^{n-2} + \sum_{n=0}^{\infty} -a_nx^{n+1} = 0$$

$$\sum_{n=0}^{\infty} a_{m+2}(m+2)(m+1)x^m + \sum_{n=1}^{\infty} -a_{m-1}x^m = 0$$

3. Equalize the Index of Summation

$$2a_2 + \sum_{n=1}^{\infty} a_{m+2}(m+2)(m+1)x^m + \sum_{n=1}^{\infty} -a_{m-1}x^m = 0$$

4. Combine Sums

$$2a_2 + \sum_{n=1}^{\infty} a_{m+2}(m+2)(m+1)x^m - a_{m-1}x^m = 0$$

$$2a_2 + \sum_{n=1}^{\infty}(a_{m+2}(m+2)(m+1) - a_{m-1})x^m = 0$$

5. Find Recurrence Relations. Set respective terms equal to zero.

$$2a_2 = 0$$

$$a_2 = 0$$

$$(a_{m+2}(m+2)(m+1) - a_{m-1}) = 0$$

$$a_{m+2} = \frac{a_{m-1}}{(m+2)(m+1)}$$

6. List Recurrence

$$a_0 = a_0$$

$$a_1 = a_1$$

$$a_2 = 0$$

$$a_3 = \frac{a_0}{3*2}$$

$$a_4 = \frac{a_1}{4*3}$$

$$a_5 = \frac{a_2}{5*4}$$

$$a_6 = \frac{a_3}{6*5} = \frac{a_0}{6*5*3*2}$$

$$a_7 = \frac{a_4}{7*6} = \frac{a_1}{7*6*4*3}$$

7. Start listing out terms and combine

$$y = a_0(1 + \frac{1}{3*2} + \frac{1}{6*5*3*2} + ...) + a_1(x + \frac{1}{4*3} + \frac{1}{7*6*4*3} + ...)$$

$$y_1 = 1 + \frac{1}{3*2} + \frac{1}{6*5*3*2} + ...$$

$$y_2 = x + \frac{1}{4*3} + \frac{1}{7*6*4*3} + ...$$

20 Decomposition of Higher Order DEQs

Higher order DEQs can be decomposed by turning them into a system of first order equations. For example

$$2y''' + y'' + (t^2+1)y' + 3y = 2t$$

can be changed into a system of first order equations by setting $y = x_1$, $y' = x_2$, $y'' = x_3$, etc. The highest order term of the DEQ isn't assigned to an x value, rather it is going to be expressed in terms of the other x values.

$$y = x_1, x_1' = x_2$$

$$y' = x_2, x_2' = x_3$$

$$y'' = x_3, x_3' =?$$

$$2y''' + y'' + (t^2+1)y' + 3y = 2t$$

$$2x_3' + x_3 + (t^2+1)x_2 + 3x_1 = 2t$$

$$x_3' = \frac{-x_3 - (t^2+1)x_2 - 3x_1 + 2t}{2}$$

Thus the DEQ has been decomposed into a system of first order equations.

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = \frac{-x_3 - (t^2+1)x_2 - 3x_1 + 2t}{2}$$

21 Basic Matrix Review

- A matrix m x n is m rows by n columns.
- For a matrix A, a_{mn} represents the item in the matrix at row m and column n.
- If A is a m x n matrix, then A^τ is a n x m matrix.
- A column vector is a m x 1 matrix.
- A row vector i a 1 x n matrix.
- A zero matrix is one where all the elements of the matrix are equal to zero.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- Matrices with the same dimensions can be added - you just add each term to it's corresponding term in the other matrix.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix}$$

- Multiplying a matrix by something multiplies all things in the matrix by that thing.

$$2\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2*1 & 2*2 \\ 2*3 & 2*4 \end{pmatrix}$$

- When multiplying matricies (A x B), each row of A is multiplied by each column of B.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1*5+2*7 & 1*6+2*0 \\ 3*5+4*7 & 3*6+4*0 \end{pmatrix}$$

$$\begin{pmatrix} \text{1st row x 1st column} & \text{1st row x 2nd column} \\ \text{2nd row x 1st column} & \text{2nd row x 2nd column} \end{pmatrix}$$

- The identity matrix is given by

$$\begin{pmatrix} 1 & 0 & 0 & ... \\ 0 & 1 & 0 & ... \\ 0 & 0 & 1 & ... \\ ... & ... & ... & ... \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- The determinant of a 2x2 matrix is

$$det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

- To integrate or differentiate a matrix, just integrate/differentiate every term in the matrix.

22 Systems of Linear Equations as Matrices

$$x_1 - 2x_2 = 7$$

$$-x_1 + x_2 = -5$$

can be represented by

$$\begin{bmatrix} x_1 - 2x_2 \\ -x_1 + x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

If there is a n x n matrix A and nonzero vectors $\vec{x}$ such that

$$A\vec{x} = \lambda\vec{x}$$

then the λs are called the eigenvalues of A, and the $\vec{x}$s are called the corresponding eigenvectors of A to λ.

$$A\vec{x} - \lambda\vec{x} = \vec{0}$$

$$(A - \lambda I)\vec{x} = \vec{0}$$

*NOT $(A - \lambda)\vec{x}$

This kind of stuff can be solved by finding the eigenvalues and then solving for the corresponding eigenvectors (x).

23 Solving Matrix DEQs

For a DEQ

$$\vec{x'} = A\vec{x}$$

where A is a 2 by 2 constant matrix, the solution of

$$\vec{x} = \xi e^{rt}$$

is guessed. Here, ξ is going to be the eigenvector of A and r is going to be the eigenvalue.

Steps

1. Solve $det(A - rI) = 0$ for r
2. Find the corresponding eigenvectors (ξ_1, ξ_2 for r_1 r_2.
3. The general solution will be
$$\vec{x} = C_1\xi_1 e^{r_1 t} + C_2\xi_2 e^{r_2 t}$$

23.1 Example

$$\vec{x'} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}$$

Step 1. Finding r

$$det(A - rI) = 0$$

$$det \begin{bmatrix} 1-r & 1 \\ 4 & 1-r \end{bmatrix} = (1-r)^2 - 4 = r^2 - 2r - 3$$

$$r = -3, r = -1$$

Step 2. Eigenvectors of r = -3

$$(A - r_1 I)\vec{\xi_1} = \vec{0}$$

$$\begin{pmatrix} 1-(-3)^1 & \\ 4 & 1-(-3) \end{pmatrix} \vec{\xi_1} = \vec{0}$$

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2x_1 + x_2 = 0$$

$$4x_1 - 2x_2 = 0$$

$$x_2 = 2x_1$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_1 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \vec{\xi_1}$$

Step 3. Eigenvectors of r = -1

Repeat the process in step 2, just use a different r value.

$$\vec{\xi_1} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Step 4. General Solution

$$\vec{x} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$$

24 Complex-Valued Eigenvalues

For Complex-Valued Eigenvavlues (if your r value(s) are complex numbers) you do the following steps.

1. $det(A - rI) = 0$, $r_1 = \lambda + i\mu$
2. $(A - r_1 I)\vec{\xi_1} = 0$, $\vec{\xi_1} = \vec{a} + i\vec{b}$
3. $\vec{x} = C_1[\vec{a}cos(\mu t) - \vec{b}sin(\mu t)]e^{\lambda t} + C_2[\vec{a}sin\mu t + \vec{b}cos(\mu t)]e^{\lambda t}$

*Note that you only take one of the roots

25 Repeated Real Roots

For repeated real roots (if the r values are the same), then you do the following steps.

1. $det(A - rI) = 0$, find r.
2. $(A - rI)\vec{\xi} = \vec{0}$, find $\vec{\xi}$
3. $(A - rI)\vec{\eta} = \vec{\xi}$, find $\vec{\eta}$
4. $\vec{x} = C_1(\xi e^{rt}) + C_2(\xi te^{rt} + \vec{\eta}e^{rt})$

26 Changelog

v1.0 - Original Document

v1.1 - Made some spelling corrections and minor fixes (Summer 2023)

v1.2