Certified Adversarial Defense Methods for Inverse Problems

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Abstract

In this paper, we propose a randomized smoothing approach that aims to enhance the robustness of the linear inverse problems against adversarial attacks, in particular rigorously guaranteeing an upper bound on a suitably-defined notion of sensitivity to perturbations. In addition, we propose two novel algorithms that incorporate randomized smoothing into training, where one algorithm injects random perturbations to the input data directly, and the other algorithm adds random perturbations to the gradients during backpropagation. We conduct numerical evaluations on two of the most prominent inverse problems — denoising and compressed sensing — utilizing a variety of neural network estimators and datasets. In broad scenarios, these results demonstrate a strong potential of randomized smoothing for enhancing the robustness of linear inverse problems.

1 Introduction

Linear inverse problems are fundamental in machine learning, signal processing, and statistics, with the objective of recovering an unknown vector from an (often underdetermined) set of linear measurements. Over the years, the incorporation of low-dimensional structures such as sparsity has dominated the extensive research [12, 11, 15]. More recently, advances in deep learning have led to their widespread adoption in inverse problems, including for signal modeling, decoder design, and measurement design [32, 36]. We will specifically be interested in the decoder design aspect.

While deep learning methods for decoder design have shown promising results, recent studies have suggested that they can be vulnerable to adversarial attacks that preserve the input's semantics but significantly worsen the accuracy of the output [20, 34, 3]. The study in [18] further indicates that deep learning for inverse problems can come at the cost of instabilities, and current training approaches cannot guarantee stable methods. Similar sensitivities of deep neural networks have been well-studied in classification problems, where initial defense approaches were soon broken [6, 29]. This led to the development of *certified* defenses such as Reluplex [23], AI2 [16], Provable Defenses [39], and randomized smoothing [8] have emerged, providing rigorous guarantees of robustness against norm-bounded attacks in classification. However, there is much less work on the adversarial robustness of inverse problems, which has fundamental differences from classification and even regression, notably including the fact that the output is high-dimensional.

In this work, we propose a novel method to certify the robustness of linear inverse problems against adversarial attacks using a suitably-adapted form of randomized smoothing [8]. Our approach is able to provide certified upper bounds on a suitably-defined notion of sensitivity to perturbations. To our knowledge, this kind of result has not been explored previously in the context of inverse problems.

In addition, inspired by analogies with classification and regression, we incorporate the concept of randomized smoothing into the estimator's training. We propose two versatile training algorithms that can be applied to a variety of network architectures at least up to a moderate size. Through diverse

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numerical experiments, we demonstrate that smoothing can provide significant robustness certificates against adversarial attacks, as well as being practically beneficial in both denoising and compressed sensing tasks.

1.1 Setup

A linear inverse problem consists of recovering a target signal[1] $\mathbf{x} \in \mathbb{R}^n$ from linear measurements $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$ are the measurement matrix and additive noise, respectively. Given $\mathbf{A}$ and $\mathbf{y}$, we employ a trainable estimator $f_{\boldsymbol{\theta}} : \mathbb{R}^m \to \mathbb{R}^n$ to approximate $\mathbf{x}$ as $f_{\boldsymbol{\theta}}(\mathbf{y})$, and assess correctness using the normalized mean squared error (MSE) criterion:

$$\ell(f_{\boldsymbol{\theta}}(\mathbf{y}), \mathbf{x}) = \frac{1}{n}\|f_{\boldsymbol{\theta}}(\mathbf{y}) - \mathbf{x}\|^2. \tag{1}$$

Parts of our paper are applicable to an arbitrary estimator $f_{\boldsymbol{\theta}}$, but at times we will also specifically consider training its parameters $\boldsymbol{\theta}$ given access to a data set $\mathcal{D}$ consisting of $(\mathbf{x}, \mathbf{y})$ pairs, with each $\mathbf{y}$ being a measurement of the form $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b}$ as above.

We study the adversarial robustness of $f_{\boldsymbol{\theta}}$ by assuming the existence of an unknown adversarial attack $\boldsymbol{\xi}$ on the target signal $\mathbf{x}$ at test time, such that the perturbed measurement vector is given by $\mathbf{y} = \mathbf{A}(\mathbf{x} + \boldsymbol{\xi}) + \mathbf{b}$. To limit the power of the adversary, we adopt the standard assumption that $\boldsymbol{\xi}$ is ϵ-bounded in a ℓ_2 ball (i.e., $||\boldsymbol{\xi}|| \leq \epsilon$), and in the worst-case may be chosen to maximize the error $\ell(f_{\boldsymbol{\theta}}(\mathbf{A}(\mathbf{x} + \boldsymbol{\xi}) + \mathbf{b}), \mathbf{x})$. To streamline our notation, in the rest of this paper, we will represent the clean measurement $\mathbf{A}\mathbf{x} + \mathbf{b}$ as $\mathbf{y}_{\mathbf{x}}$ and represent the perturbed measurement $\mathbf{A}(\mathbf{x} + \boldsymbol{\xi}) + \mathbf{b}$ as $\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}}$, which can be generalized to $\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}+\boldsymbol{\delta}}$ in the case of multiple perturbations $\boldsymbol{\xi}$ and $\boldsymbol{\delta}$.

We note that our focus is on linear measurement since they are by far the most commonly considered in the literature on inverse problems, and already capture many interesting sub-problems such as inpainting, super-resolution, denoising, and compressive sensing. Nevertheless, extensions to non-linear models may be an interesting direction for future work.

2 Related Work

Adversarial training is a commonly used training strategy (primarily used for classification problems [25, 30, 33, 39]) to improve the robustness of perturbations through training with adversarial examples [17]. It follows the framework of empirical risk minimization (ERM) but replaces the input data with some ℓ_2-bounded adversarial perturbations. Several methods have been proposed in prior work [25] to generate these perturbations, with the multi-step iterative method Projected Gradient Descent (PGD) being particularly effective. [22] studied the tradeoff between robustness and standard accuracy using adversarial training in linear regression problems, whereas [18] suggested that adversarial training alone may be insufficient for addressing instabilities in linear inverse problems. Nonetheless, we include adversarial training equipped with the PGD method as one of the baselines in our experiments and provide the ERM formulation as

$$\min_{\boldsymbol{\theta}} \mathbb{E}_{(\mathbf{x},\mathbf{y})\in\mathcal{D}}\big[\ell(f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}^*}), \mathbf{x})\big], \quad \text{where} \quad \boldsymbol{\xi}^* = \underset{\boldsymbol{\xi}:\|\boldsymbol{\xi}\|_2\leq\epsilon}{\operatorname{argmax}}\, l(f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}})), \mathbf{x}). \tag{2}$$

Jacobian regularization is a technique to improve robustness against adversarial attacks by penalizing large derivatives with respect to input data, specifically the norm of the input-output Jacobian matrix. The idea was first proposed by [19] but only implemented as a layer-wise regularization due to computational constraints. Later it was implemented at full scale in [37] and was made more efficient in [38, 40]. [21, 2] showed that both the Frobenius and spectral norms of the Jacobian matrix are experimentally effective in improving the robustness of neural networks in regression problems.

In the context of linear inverse problems, we can motivate the use of the Jacobian by considering the Taylor expansion of $f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}})$ with respect to the adversarial perturbation $\boldsymbol{\xi}$:

$$f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}}) = f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}} + \mathbf{A}\boldsymbol{\xi}) = f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}}) + \mathbf{J}(\mathbf{y}_{\mathbf{x}})\mathbf{A}\boldsymbol{\xi} + O(\|\boldsymbol{\xi}\|^2), \tag{3}$$

where $\mathbf{J}(\mathbf{y}_{\mathbf{x}}) \in \mathbb{R}^{n\times m}$ is the input-output Jacobian matrix. When the ℓ_2-radius of the adversarial perturbation $\boldsymbol{\xi}$ is small enough, the higher-order terms can be ignored, and the stability of the

[1] For image data, we think of $\mathbf{x}$ as being its vectorized version, with n being the total number of pixels.

estimation is mainly determined by $\mathbf{J}(\mathbf{y_x})\mathbf{A}$. Therefore, we can use Jacobian regularization by penalizing the spectral norm $\max_{(\mathbf{x},\mathbf{y})\in\mathcal{D}}\|\mathbf{J}(\mathbf{y})\mathbf{A}\|$, as proposed by [2], and control the strength of regularization using a hyperparameter β. The ERM formulation for training is provided as

$$\min_{\boldsymbol{\theta}} \mathbb{E}_{(\mathbf{x},\mathbf{y})\in\mathcal{D}}\Big[\ell(f_{\boldsymbol{\theta}}(\mathbf{y}),\mathbf{x})\Big] + \beta \max_{(\mathbf{x},\mathbf{y})\in\mathcal{D}}\|\mathbf{J}(\mathbf{y})\mathbf{A}\|. \tag{4}$$

Randomized smoothing is a technique that was first introduced for certifying a classifier against ℓ_2-bounded noise, potentially on large neural networks and on large images [26, 8, 28]. It works by adding random noise to input data and then classifying the noisy input to obtain a probability distribution over possible labels. Several works have shown that randomized smoothing guarantees that no possible attacks under a particular constraint could succeed in classification problems. [5] proposed a certificate for discrete data and demonstrated its usefulness in graph neural networks. [27] presented certified robustness against Wasserstein adversarial attacks. [13] covered parameterized transformations and certified robustness in the parameter space. [14] developed a certification method for image and point cloud segmentation. Furthermore, [35] embedded randomized smoothing into the classifier's adversarial training framework, which inspired us to propose Algorithm 1 in this paper.

Randomized smoothing has also been extended to certain tasks beyond classification. [7] used it to certify the regression of the object detector's bounding-box coordinates. [4] applied it to certify the accuracy score in watermarking problems and proposed a training framework from which we are inspired to propose Algorithm 2. These two works are the closest to ours, but there are some significant differences. In particular, in our case, the measurements are mapped to a high-dimensional signal manifold, whereas the the prior works consider outputting a single real number or only a few real numbers. In our setting, a naive strategy of summarizing the output via its loss (a single real number) is infeasible due to the loss being unknown at the decoder, as we discuss below.

3 Proposed Methods

3.1 Randomized smoothing for inverse problems during testing

We first review the necessary background of randomized smoothing for estimation in a generic *scalar-valued output* setting. Given an ordinary real-valued estimator $F : \mathbb{R}^n \to \mathbb{R}$, randomized smoothing is a method that constructs a new, smoothed estimator G from the base estimator F at testing time. Under *mean smoothing*, the smoothed estimator G takes as input $\mathbf{x} \in \mathbb{R}^n$ and returns the expectation of F under isotropic Gaussian noise perturbation $\boldsymbol{\delta}$ on $\mathbf{x}$, i.e.,

$$G(\mathbf{x}) = \mathbb{E}_{\boldsymbol{\delta}\sim\mathcal{N}(\mathbf{0},\sigma^2\mathbf{I})}\big[F(\mathbf{x}+\boldsymbol{\delta})\big], \tag{5}$$

where the noise level σ of $\boldsymbol{\delta}$ controls the tradeoff between robustness and accuracy.

While (5) is the go-to approach for classification, its direct use for estimating *continuous-valued* quantities may result in skewed results and location-dependent bounds, as detailed in [7]. To overcome this limitation, they proposed a more suitable "percentile smoothing" (see (6) below) for one-dimensional regression tasks. In high-dimensional inverse problems, although incorporating randomized smoothing may seem feasible by regressing over the scalar estimation loss $\ell(f_{\boldsymbol{\theta}}(\mathbf{y_x}),\mathbf{x})$, we cannot access the original signal $\mathbf{x}$ and only have the measurement $\mathbf{y_x}$ at hand (or more generally $\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}}$). Hence, instead of considering the loss to $\mathbf{x}$ itself, we consider the loss between two different reconstructions, one of which has an additional perturbation. Formally, we define the following.

Definition 3.1. Given an estimator $f_{\boldsymbol{\theta}} : \mathbb{R}^m \to \mathbb{R}^n$, an adversarially perturbed measurement $\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}}$, and a Gaussian noise vector $\boldsymbol{\delta} \sim \mathcal{N}(\mathbf{0},\sigma^2\mathbf{I})$, we use $\boldsymbol{\delta}$ to perturb $\mathbf{x}$ further and obtain the measurement $\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}+\boldsymbol{\delta}} = \mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}} + \mathbf{A}\boldsymbol{\delta}$. We refer to $\ell(f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}+\boldsymbol{\delta}}), f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}}))$ as the *discrepancy*, and consider the percentile smoothing of the discrepancy as a function of $\mathbf{x}$ and $\boldsymbol{\xi}$:

$$D_p(\mathbf{x},\boldsymbol{\xi}) = \inf\left\{t \in \mathbb{R} \mid \mathbb{P}\big[\ell(f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}+\boldsymbol{\delta}}), f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}})) \le t\big] \ge p\right\}, \tag{6}$$

where p is the quantile value.

The underlying idea behind the *discrepancy* is that a small $D_p(\mathbf{x},\boldsymbol{\xi})$ indicates that the estimation retains consistency despite the presence of Gaussian noise $\boldsymbol{\delta}$. This consistency is advantageous for adversarial robustness, allowing the addition of Gaussian noise to drown out an adversarial perturbation without impacting the reconstruction too much.

For concreteness, we often focus on $p = 0.5$, i.e., the median. Although $D_p(\mathbf{x}, \boldsymbol{\xi})$ will generally not admit a closed form, we can still approximate it by applying a Monte Carlo approach with a suitably-chosen confidence level (see Algorithm 3 in Appendix C).

The preceding ideas give rise to two estimators of $\mathbf{x}$ that we will consider throughout the paper:

- For certified estimation, the Monte Carlo procedure with S samples gives an appropriate quantile value $q \in (0, 1)$; we compute the relevant S values of $\ell(f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}+\boldsymbol{\delta}}), f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}}))$ with randomly sampled $\boldsymbol{\delta}$, sort them, and take the one at position $q \cdot S$ (with integer rounding).
- We also consider median-based estimation, with the same procedure but simply $q = \frac{1}{2}$.

We emphasize that the Gaussian noise $\boldsymbol{\delta}$ for smoothing is intentionally introduced by the algorithm, whereas $\boldsymbol{\xi}$ is considered to be unknown and adversarial. We henceforth refer to them as the smoothing noise $\boldsymbol{\delta}$ and the adversarial noise $\boldsymbol{\xi}$.

3.2 Discrepancy certification

We follow Lemma 1 from [7] and propose the following corollary to bound the worst-case smoothed discrepancy (see Appendix A for the proof):

Corollary 3.2. *The median smoothed discrepancy with adversarial noise can be bounded by the smoothed discrepancy in the absence of adversarial noise as*

$$D_{0.5}(\mathbf{x}, \boldsymbol{\xi}) \leq D_{\Phi(\frac{\epsilon}{\sigma})}(\mathbf{x}, \mathbf{0}) \quad \forall\, \boldsymbol{\xi} \text{ s.t. } \|\boldsymbol{\xi}\| \leq \epsilon, \tag{7}$$

where Φ is the cumulative distribution function of unit normal distribution, and $\mathbf{0}$ is a zero vector.

Equation (7) states that a worst-case upper bound on the median smoothed discrepancy $D_{0.5}(\mathbf{x}, \boldsymbol{\xi})$ is the $\Phi(\frac{\epsilon}{\sigma})$-th percentile smoothed discrepancy $D_{\Phi(\frac{\epsilon}{\sigma})}(\mathbf{x}, \mathbf{0})$, for *arbitrary* $\boldsymbol{\xi}'$ within an ϵ-ball around zero. This corollary guarantees that the sensitivity of $\mathbf{x}$ (as measured by the discrepancy) is minimally affected by arbitrary shifts up to radius ϵ. We focus on the median for simplicity, but the proof reveals a more general form for any specified quantile value in $(0, 1)$.

It is worth noting that $\mathbf{x}$ and $\boldsymbol{\xi}$ in Corollary 3.2 represent generic vectors whose assignments may vary depending on how the corollary is applied. In particular, suppose that we are in a scenario where the true signal is $\mathbf{x}^*$, and the true adversarial noise is $\boldsymbol{\xi}^*$, so the measurement vector is $\mathbf{y}_{\mathbf{x}^*+\boldsymbol{\xi}^*}$. Then we may apply Corollary 3.2 with $\mathbf{x} = \mathbf{x}^* + \boldsymbol{\xi}^*$, meaning that we are giving a guarantee on the discrepancy for all signals sufficiently close to $\mathbf{x}^* + \boldsymbol{\xi}^*$ (rather than sufficiently close to $\mathbf{x}^*$ itself; we cannot work with $\mathbf{x}^*$ directly since $\mathbf{y}_{\mathbf{x}^*}$ is not known to the algorithm).

We also emphasize that our approach does not directly ensure "good" or "bad" estimation but rather provides a guarantee on the discrepancy. This is analogous to classification, where a certificate may show that the decision is unaffected within a certain radius, without claiming it to be the correct decision in the first place. While this is a standard limitation of smoothing methods, we still consider using discrepancy calculations, Peak Signal-to-Noise Ratio (PSNR) values (a standard measure of accuracy with the definition provided in Section 4), and Structural Similarity (SSIM) values (an indicator of perceived similarity between estimations and ground truths) for experimental evaluations.

3.3 Embedding randomized smoothing into training

The previous subsection covers the use of randomized smoothing during testing, and applies to arbitrary $f_{\boldsymbol{\theta}}$ (e.g., pre-trained). In this section, we provide algorithms that use randomized smoothing during training, i.e., selecting the parameters $\boldsymbol{\theta}$ of $f_{\boldsymbol{\theta}}$ based on training data.

The first algorithm, termed "Smoothed Adversarial Training" and building on [35], is an approach that roughly follows adversarial training (as discussed in Section 2) but uses randomized smoothing when generating the ℓ_2-bounded adversarial perturbation. Specifically, adversarial training is performed using the perturbation noise $\boldsymbol{\xi}^*$ obtained as the expected value under the smoothing noise $\boldsymbol{\delta}$. Formally, the ERM with smoothing is performed as

$$\min_{\boldsymbol{\theta}} \mathbb{E}_{(\mathbf{x},\mathbf{y})\in\mathcal{D}}\big[\ell(f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}^*}), \mathbf{x})\big], \quad \text{where } \boldsymbol{\xi}^* = \operatorname*{argmax}_{\boldsymbol{\xi}:\|\boldsymbol{\xi}\|_2\leq\epsilon} \ell\Big(\mathbb{E}_{\boldsymbol{\delta}\sim\mathcal{N}(\mathbf{0},\sigma^2\mathbf{I})}\big[f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}+\boldsymbol{\delta}})\big], \mathbf{x}\Big). \tag{8}$$

The algorithm computes $\boldsymbol{\xi}^*$ using a first-order method similar to PGD, but with the difference that the perturbed measurement $\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}}$ is further smoothed in the neighborhood around $\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}+\boldsymbol{\delta}}$ before being used in the loss function. It is difficult to evaluate $\boldsymbol{\xi}^*$ exactly, so we use Monte Carlo approximations with S i.i.d. smoothing noise samples, $\boldsymbol{\delta}_1, \ldots, \boldsymbol{\delta}_S \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$, and compute the estimated $\hat{f}_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}+\boldsymbol{\delta}})$ using the sample mean

$$\hat{f}_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}+\boldsymbol{\delta}}) = \frac{1}{S}\sum_{k=1}^{S} f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}+\boldsymbol{\delta}_k}). \tag{9}$$

See Algorithm 1 for a complete description. Although adversarial training is an empirical defense technique that generally lacks provable guarantees, we will observe promising results in our experiments. The same goes for our second algorithm introduced below.

Algorithm 1 Smoothed Adversarial Training

input Training set $\mathcal{D}$; Learning rate α; Estimator $f_{\boldsymbol{\theta}}$ and measurement matrix $\mathbf{A}$; Adversarial noise bound ϵ, iteration count T_{itr}, and step size r; Smoothing noise level σ and noise sample count S;
for $1 \le i \le |\mathcal{D}|$ **do**
  $\boldsymbol{\xi}_0 = \mathbf{0}$
  for $1 \le j \le T_{itr}$ **do**
    $\Delta = \nabla_{\mathbf{x}} \ell(\hat{f}_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}_{j-1}+\boldsymbol{\delta}}), \mathbf{x})|_{\mathbf{x}=\mathbf{x}_i}$
    (where $\hat{f}_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}_{j-1}+\boldsymbol{\delta}})$ is from (9))
    $\boldsymbol{\xi}'_j = \boldsymbol{\xi}_{j-1} + r * \Delta$
    $\boldsymbol{\xi}_j = \operatorname{argmin}_{\boldsymbol{\xi}:\|\boldsymbol{\xi}\|\le\epsilon} \|\boldsymbol{\xi} - \boldsymbol{\xi}'_j\|$
  end for
  $\boldsymbol{\xi}^* = \boldsymbol{\xi}_{T_{itr}}$
  $\boldsymbol{\theta} = \boldsymbol{\theta} - \alpha * \nabla_{\boldsymbol{\theta}} \ell(f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}_i+\boldsymbol{\xi}^*}), \mathbf{x}_i)$
end for
return $f_{\boldsymbol{\theta}}$

Algorithm 2 Smoothed Gradient Training

input: Training set $\mathcal{D}$; Learning rate α; Estimator $f_{\boldsymbol{\theta}}$ and measurement matrix $\mathbf{A}$; Smoothing noise level σ_{max}, step count T_{step}, and noise sample count S;

for $1 \le i \le |\mathcal{D}|$ **do**
  for $1 \le j \le T_{step}$ **do**
    $\mathbf{g}_{\boldsymbol{\theta}} = 0$
    $\sigma = \frac{j}{T_{step}}\sigma_{max}$
    for $1 \le k \le S$ **do**
      $\boldsymbol{\delta} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$
      $\Delta = \frac{1}{(ST_{step})} \nabla_{\boldsymbol{\theta}} \ell(f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}_i+\boldsymbol{\delta}}), \mathbf{x}_i)$
      $\mathbf{g}_{\boldsymbol{\theta}} = \mathbf{g}_{\boldsymbol{\theta}} + \Delta$
    end for
    $\boldsymbol{\theta} = \boldsymbol{\theta} - \alpha * \mathbf{g}_{\boldsymbol{\theta}}$
  end for
end for
return $f_{\boldsymbol{\theta}}$

The second algorithm, termed "Smoothed Gradient Training" and building on [4], is built upon the regular ERM training framework without a regularization term. To embed randomized smoothing, we augment the clean measurements with smoothing noise $\boldsymbol{\delta}$ and compute the estimator's gradients using an expectation with respect to $\boldsymbol{\delta}$. Specifically, for a given training pair $(\mathbf{x}, \mathbf{y})$, the gradient $\mathbf{g}_{\boldsymbol{\theta}}$ with respect to the estimator's parameter $\boldsymbol{\theta}$ is computed as

$$\mathbf{g}_{\boldsymbol{\theta}} = \mathbb{E}_{\boldsymbol{\delta}\sim\mathcal{N}(\mathbf{0},\sigma^2\mathbf{I})}\big[\nabla_{\boldsymbol{\theta}} \ell(f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\delta}}), \mathbf{x})\big]. \tag{10}$$

To approximate the gradients in the equation, we use the Monte Carlo method, averaging gradients across S draws of i.i.d. smoothing noise. However, directly adding a large amount of noise during training can make it unstable. To overcome this issue, we adopt the approach proposed by [4] and incrementally increase the noise levels within each epoch. This is achieved by introducing a step count T_{step} and injecting smoothing noise incrementally with standard deviation σ ranging from $\frac{1}{T_{step}}\sigma_{max}$ to σ_{max}, where σ_{max} is the maximum noise level. A detailed description of the algorithm can be found in Algorithm 2.

We make several observations both training algorithms described above. Firstly, the underlying idea behind our training methods shares a resemblance with their counterparts in classification: By incorporating randomized smoothing during test time, the neural network is guided to maintain stable behavior, generating similar reconstructions even in the presence of input perturbations. Secondly, for Smoothed Adversarial Training, the perturbed measurement is the one being smoothed, while for Smoothed Gradient Training, the gradients themselves are being smoothed. Thirdly, these algorithms are generally adaptable to any differentiable neural network estimator $f_{\boldsymbol{\theta}}$, allowing for a diverse range of choices in the numerical experiments. Fourthly, a possible alternative approach is to smooth the estimation loss instead. For example, in (8), $\boldsymbol{\xi}^*$ is obtained as $\operatorname{argmax}_{\boldsymbol{\xi}:\|\boldsymbol{\xi}\|_2\le\epsilon} \mathbb{E}_{\boldsymbol{\delta}\sim\mathcal{N}(\mathbf{0},\sigma^2\mathbf{I})}\big[\ell\big(f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}+\boldsymbol{\delta}}), \mathbf{x}\big)\big]$; and in (10), the gradient is computed not before

but after the expectation, i.e., $\nabla_{\boldsymbol{\theta}} \mathbb{E}_{\boldsymbol{\delta} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})} \big[\ell(f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\delta}}), \mathbf{x}) \big]$. However, loss smoothing appears to be less effective, because it takes the expected value in a one-dimensional manifold space and as a result, diminishes the "diversity" of the randomness.

4 Experiments

We conduct experiments in two different tasks: denoising, where $\mathbf{A} = \mathbf{I}$ and $\mathbf{b}$ is Gaussian noise with level $\sigma_{\mathbf{b}}$; and compressed sensing, where $\mathbf{A}$ is a random Gaussian matrix with compression ratio m/n, and the additive noise $\mathbf{b}$ is set to zero. We consider training and testing sets from both the same data domain and different data domains (i.e., transfer learning), as is common in this line of works. Specifically, for denoising, we use the network DPDNN [10] as the estimator trained on the (grayscale converted) DIV2K dataset [1] and tested on the DIV2K validation set and BSD68 dataset [31], while for compressed sensing, we use the estimator ISTA-Net^{++} [41] trained on the Train400 dataset [24] and tested on the Set11 [24] and a subset of the (grayscale converted) ImageNet [9] datasets. All timings are for a single Nvidia GeForce RTX 3090. The code for all these experiments has been uploaded as a supplementary file.

4.1 Experimental settings

To assess the impact of randomized smoothing during training, we compare three baselines with our proposed training algorithms:

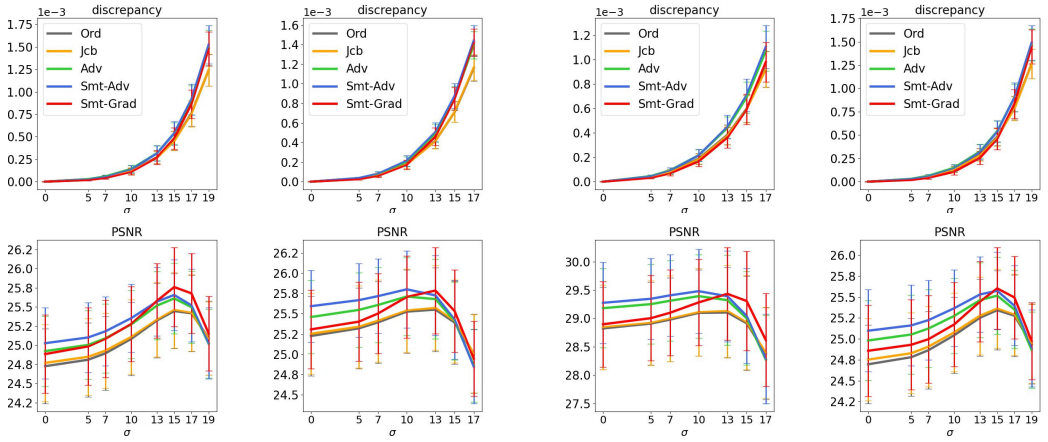
1. **Ordinary Training (Ord)**: the standard training algorithm that does not incorporate any perturbation defense mechanisms, with the ERM formulation simply presented as $\min_{\boldsymbol{\theta}} \mathbb{E}_{(\mathbf{x},\mathbf{y}) \in \mathcal{D}} \big[\ell(f_{\boldsymbol{\theta}}(\mathbf{y}), \mathbf{x}) \big]$;
2. **Adversarial Training (Adv)**: the adversarial training algorithm equipped with PGD method as introduced in (2);
3. **Jacobian Regularization (Jcb)**: the Jacobian regularization training algorithm using the spectral norm for penalization as detailed in (4), where we determine the value of hyperparameter β using Algorithm 4 from [2];
4. **Smoothed Adversarial Training (Smt-Adv)** from Algorithm 1;
5. **Smoothed Gradient Training (Smt-Grad)** from Algorithm 2.

Training parameters such as the number of epochs, batch size, and learning rate, are shared across all algorithms, whereas other parameters such as smoothing noise levels and adversarial attack iteration count are exclusive to certain algorithms (see Appendix B for further details). To evaluate the effectiveness of randomized smoothing at test time, we explore different choices of adversarial attack radii and various smoothing noise levels. Specifically, at test time, we apply the PGD method to produce adversarial attacks using a step size and number of iterations matching those used in the Adv and Smt-Adv algorithms during training. These hyperparameters (specified in Appendix B) are selected to ensure that the adversarial noise reaches the boundary of its ℓ_2-radius ball. For the denoising task, we consider ℓ_2 radii $\epsilon < 15$, while for the compressed sensing task, we consider $\epsilon < 7$, avoiding using extreme values and preventing the estimates from being excessively poor. For smoothing, we examine noise levels ranging from 0 to 45 and use Monte Carlo approximation with 50 samples (see Appendix B for a discussion on other choices and the sensitivity to varying S) and confidence level $c = 0.99$. Additionally, since our training process involves an additive noise level of $\sigma_{\mathbf{b}} \leq 15$ and a compression ratio of $m/n \leq 0.5$, we restrict our testing to additive noise levels and compression ratios within these ranges.

We follow a common practice of rescaling images to have pixel values in $[0, 1]$. We then compute and analyze both the discrepancy (introduced in Section 3.1), PSNR, and SSIM. The general definition of PSNR depends on the maximum possible pixel value, but our $[0, 1]$-normalization simplifies this to $\text{PSNR} = 10 \log_{10} \frac{1}{\text{MSE}}$, where the MSE is as defined in (1). The definition of SSIM is more complicated to state but is well-known, so is omitted here.

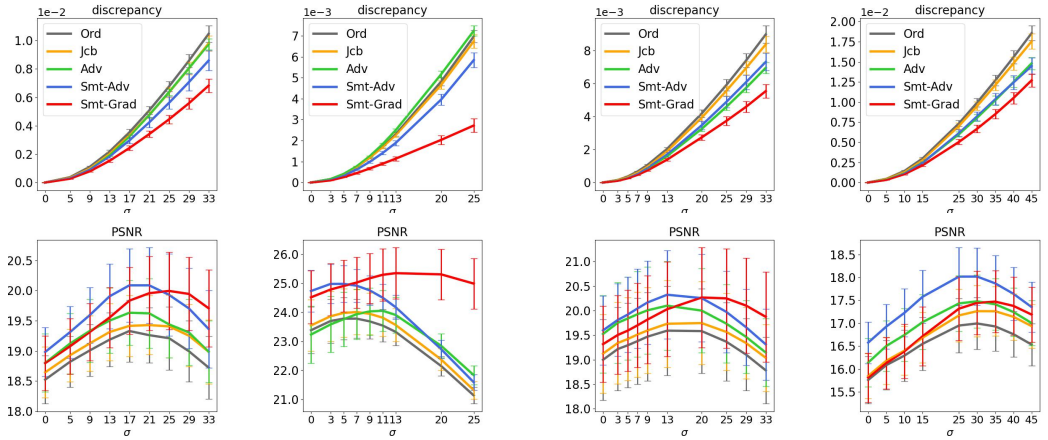
4.2 Certificate evaluation

We first examine the certificates of all types of trained estimators under varying ℓ_2 radii of the adversarial noise. Afterwards, we will investigate how adjusting the smoothing noise used in training can impact the certificate.



$\sigma_{\mathbf{b}} = 2, \epsilon = 10$ $\sigma_{\mathbf{b}} = 10, \epsilon = 10$ $\sigma_{\mathbf{b}} = 5, \epsilon = 5$ $\sigma_{\mathbf{b}} = 5, \epsilon = 15$

(a) BSD68 (denoising) (b) DIV2K (denoising)



$m/n = 0.1, \epsilon = 5$ $m/n = 0.3, \epsilon = 5$ $m/n = 0.1, \epsilon = 3$ $m/n = 0.1, \epsilon = 7$

(c) Set11 (compressed sensing) (d) ImageNet (compressed sensing)

Figure 1: Certified smoothed discrepancy and its corresponding smoothed PSNR. The experimental settings for each pair of resulted discrepancy (top) and PSNR (bottom) are displayed in the footer.

Figure 1 presents a comparison of the certified results for both tasks. We display the results as the average over the corresponding dataset and include error bars representing half a standard deviation. We first observe that the certified upper bound of smoothed discrepancy increases with the smoothing level, which is expected as higher smoothing levels amount to more deviations from the original signal, as seen in (6). Regarding the PSNR of the reconstructions when smoothing is used, we consistently observe an "arc" shape in which the performance first improves with σ but then worsens, with the peak representing an ideal trade-off between σ being large enough to improve robustness, but not so large as to over-smooth. Naturally, the trend is that the best choice of σ increases with ϵ, with the two generally being comparable. However, ϵ is not the only factor at play; when the estimation problem is made "more difficult" via increased $\sigma_{\mathbf{b}}$ (denoising) or fewer measurements (compressed sensing), the best choice of σ also tends to increase.

In comparing the five training algorithms, we find that, at the optimal smoothing noise level, Smt-Grad consistently produces the lowest certified discrepancy and the highest corresponding estimation accuracy across all datasets and tasks. This suggests that Smt-Grad-trained estimators are the least

susceptible to perturbations, at least for the attacks we consider. Moreover, when little or no test-time smoothing is applied (small σ, seen in the left parts of the plots in Figure 1), Smt-Adv is generally seen to achieve the highest PSNR. On the other hand, when significant smoothing is applied, Smt-Grad can be preferable, particularly in the case of compressed sensing, with lower discrepancy and higher PSNR. Overall, we see that both of our proposed training algorithms can effectively improve the robustness of the estimator against adversarial attacks.

We now explore how different levels of smoothing noise σ used for training can affect performance in denoising and compressed sensing tasks. Table 1 provides a comparison of the performance of algorithms trained with different smoothing noise levels (shown as σ_{tr} to distinguish from the test-time smoothing noise σ) in terms of their performance metrics. These metrics include the PSNR of the clean estimate $f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}})$, the PSNR of the perturbed estimate $f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}})$, and the PSNR of the smoothed estimate $f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}+\boldsymbol{\delta}})$ that provides the certified smoothed discrepancy. For each ℓ_2-radius ϵ of the perturbation $\boldsymbol{\xi}$, we use a zero test-time smoothing noise level to obtain the PSNR of the perturbed estimate and an empirically optimal test-time σ to obtain the smoothed PSNR. For comparison, we also include a baseline vanilla scenario that showcases the performance of the algorithm without any robustness considerations, i.e., Ord. The results demonstrate that a significant level of σ_{tr} (e.g., $\sigma_{\text{tr}} = 10$, which we use as the default training setting) can, in most cases, result in improved test-time robustness, indicated by higher PSNR values. However, as the estimator is trained more robustly against stronger adversarial attacks (when $\epsilon > 0$), the accuracy of the estimator's clean estimation drops, as shown in the second column. An analogous trade-off between robustness and accuracy has also been observed in the adversarial robustness literature, e.g., [30], [4].

		ℓ_2 radius ϵ at test time					
algorithm trained with σ_{tr}	ϵ=0	ϵ=5		ϵ=11		ϵ=15	
		$\sigma = 0$	$\sigma = 10$	$\sigma = 0$	$\sigma = 14$	$\sigma = 0$	$\sigma = 18$
Ord	32.71	27.48	28.15	24.27	24.90	22.71	23.18
Smt-Adv, $\sigma_{\text{tr}} = 10$	31.87	27.85	28.44	24.56	25.13	22.97	23.36
Smt-Adv, $\sigma_{\text{tr}} = 5$	32.21	27.69	28.39	24.43	25.07	22.84	23.30
Smt-Adv, $\sigma_{\text{tr}} = 1$	32.40	27.48	28.27	24.26	24.97	22.71	23.23
Smt-Grad, $\sigma_{\text{tr}} = 10$	31.74	27.61	28.49	24.48	25.18	22.91	23.40
Smt-Grad, $\sigma_{\text{tr}} = 5$	31.87	27.61	28.39	24.43	25.11	22.86	23.36
Smt-Grad, $\sigma_{\text{tr}} = 1$	31.92	27.54	28.07	24.43	24.87	22.85	23.16

(a) $\sigma_{\mathbf{b}} = 2$ on BSD68 (denoising)

		ℓ_2 radius ϵ at test time					
algorithm trained with σ_{tr}	ϵ=0	ϵ=1		ϵ=5		ϵ=9	
		$\sigma = 0$	$\sigma = 3$	$\sigma = 0$	$\sigma = 21$	$\sigma = 0$	$\sigma = 35$
Ord	27.79	24.64	24.72	18.52	19.28	15.56	16.79
Smt-Adv, $\sigma_{\text{tr}} = 10$	26.91	24.63	24.69	18.98	20.04	16.33	17.83
Smt-Adv, $\sigma_{\text{tr}} = 5$	27.16	24.41	24.52	18.62	19.70	15.67	17.11
Smt-Grad, $\sigma_{\text{tr}} = 10$	27.30	24.85	24.90	18.79	19.99	15.78	17.15
Smt-Grad, $\sigma_{\text{tr}} = 5$	27.48	24.77	24.83	18.65	19.31	15.56	16.51

(b) $m/n = 0.1$ on Set11 (compressed sensing)

Table 1: Trade-off between smoothing noise used during training and the resulting certificates during testing. The second column corresponds to the PSNR of the clean estimate, while the remaining cells indicate the PSNR of the perturbed estimate and the smoothed PSNR of the smoothed estimate.

4.3 Empirical smoothed median estimate

In practice, one may not specifically require certificates, and accordingly, it is of interest to study the performance when one instead adopts the simpler approach of using the median, as described in Section 3.1. To address this, we present Table 2 demonstrating the similarity between the experimental median and its corresponding certificate across various scenarios. These scenarios encompass different tasks and datasets, and we note that the rows of the table are not meant to be directly compared with one another. Overall, our results reveal that when the ideal test-time smoothing level is used, the evaluated discrepancy and PSNR values of the smoothed median estimate are not substantially different from those with certification.

We visually compare the smoothed estimate $f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}+\boldsymbol{\delta}})$ corresponding to the smoothed median discrepancy with the perturbed estimate $f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}})$ in Figure 2. In addition, We showcase the vanilla scenario where Ord is utilized and neither perturbation nor test-time smoothing is incorporated. Notably, we observe significant improvements in estimation accuracy and feature clarity in both denoising and compressed sensing tasks when smoothing is applied. Even in the absence of test-time smoothing ($\sigma = 0$), estimators trained with Smt-Adv or Smt-Grad consistently outperform the other three baselines. Further visual comparisons can be found in Figures 3 and 4 in Appendix D.

algorithm	task details (dataset)	smoothed median	smoothed certificate
Ord	$\sigma_{\mathbf{b}} = 10, \epsilon = 10, \sigma = 13$ (BSD68)	25.523 (4.165×1e-4)	25.515 (4.174×1e-4)
Jcb	$m/n = 0.1, \epsilon = 5, \sigma = 17$ (Set11)	19.424 (3.295×1e-3)	19.410 (3.303×1e-3)
Adv	$\sigma_{\mathbf{b}} = 5, \epsilon = 15, \sigma = 15$ (DIV2K)	25.514 (5.402×1e-4)	25.516 (5.410×1e-4)
Smt-Adv	$m/n = 0.1, \epsilon = 7, \sigma = 25$ (ImageNet)	18.116 (6.135×1e-3)	18.011 (6.138×1e-3)
smt-Grad	$\sigma_{\mathbf{b}} = 2, \epsilon = 10, \sigma = 15$ (BSD68)	25.759 (4.812×1e-4)	25.716 (4.821×1e-4)

Table 2: Samples of the evaluated PSNR (and discrepancy in brackets) when using the smoothed median estimate vs. the smoothed estimate with certificate, for various tasks.

4.4 Note on computation

Our experiments have shown that the utilization of randomized smoothing during both training and testing phases of linear inverse problems effectively improves the robustness of the estimator against adversarial attacks. A slight caveat is that smoothing can increase the computation time, as is also the case with smoothing in other contexts. For training, we found that Smt-Grad was not too much slower than its unsmoothed counterpart, in part due to performing will with relatively few smoothing samples. On the other hand, the smoothing operation made Smt-Adv noticeably slower than Adv. Regarding test-time performance, determining the empirically ideal smoothing noise level, as depicted in Figure 1, can also be a time-consuming process.

To support the above discussion, we present the training time per batch across all five algorithms in two tasks in Table 3 in Appendix B, where we observe that the training time required for Smt-Adv and Smt-Grad is higher than the baselines but still reasonable. We also provide Table 4 in Appendix B showcasing the testing time per data sample, where we observe that reducing smoothing samples from 1000 to 50 leads to a significant speedup of around $15\times$, while maintaining comparable performance.

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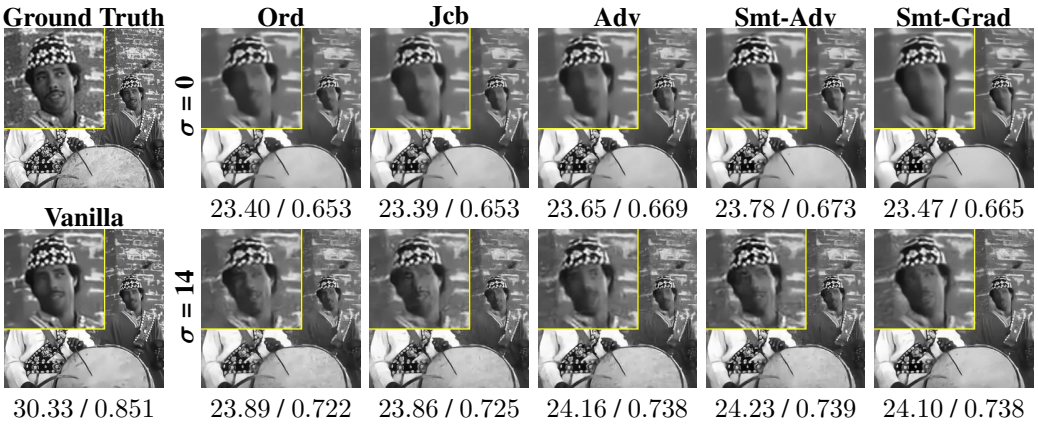
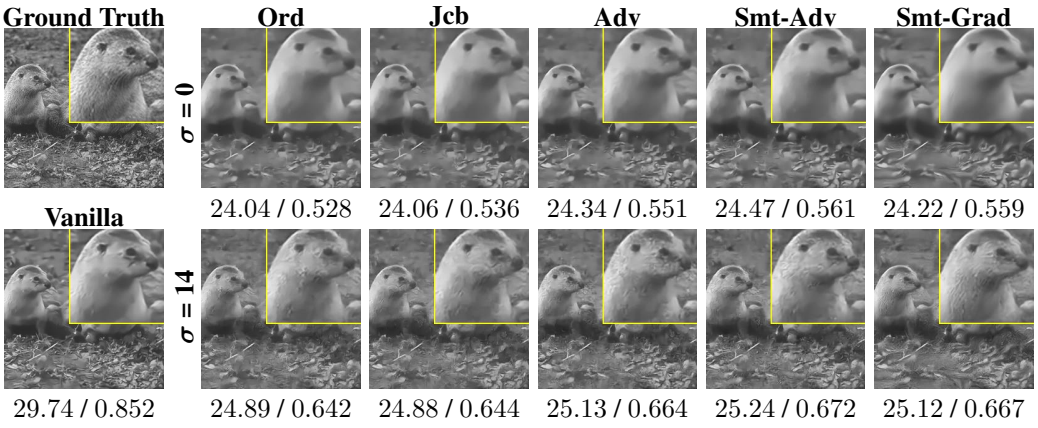
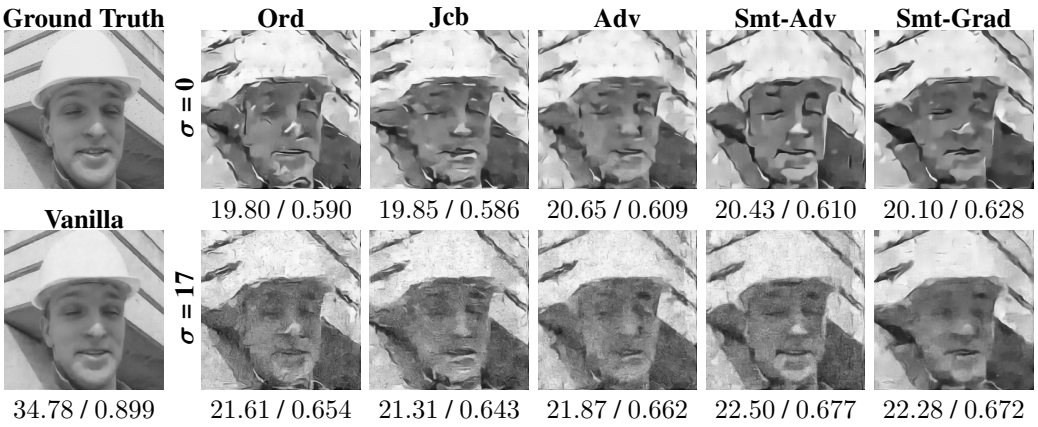


(a) $\boldsymbol{\sigma_b = 2}$, $\boldsymbol{\epsilon = 11}$ on BSD68 (denoising)



(b) $\boldsymbol{\sigma_b = 5}$, $\boldsymbol{\epsilon = 9}$ on DIV2K (denoising)



(c) $\boldsymbol{m/n = 0.1}$, $\boldsymbol{\epsilon = 5}$ on Set 11 (compressed sensing)

Figure 2: Estimations (with the performance shown in the form of PSNR / SSIM) from the perturbed measurements in denoising and compressed sensing tasks. **Vanilla** represents the baseline unperturbed scenario where $\boldsymbol{\epsilon = 0}$, $\boldsymbol{\sigma = 0}$, and **Ord** is employed.

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Appendix

A Proof of Corollary 3.2

Corollary 3.2. (Restated.) *The median smoothed discrepancy with adversarial noise can be bounded by the smoothed discrepancy in the absence of adversarial noise as*

$$D_{0.5}(\mathbf{x}, \boldsymbol{\xi}) \leq D_{\Phi(\frac{\epsilon}{\sigma})}(\mathbf{x}, \mathbf{0}) \quad \forall\, \boldsymbol{\xi} \text{ s.t. } \|\boldsymbol{\xi}\| \leq \epsilon, \tag{11}$$

where Φ is the cumulative distribution function of unit normal distribution, and $\mathbf{0}$ is a zero vector.

Proof. According to Lemma 2.2 in [7], given a real-valued function $f : \mathbb{R}^m \to \mathbb{R}$, its percentile smoothing function is defined as

$$\overline{h}_p(\mathbf{x}) = \inf\{t \in \mathbb{R} \mid \mathbb{P}\big[\, f(\mathbf{x} + \boldsymbol{\delta}) \leq t\big] \geq p\}, \tag{12}$$

and it is guaranteed that

$$\overline{h}_p(\mathbf{x} + \boldsymbol{\xi}) \leq \overline{h}_{\overline{p}}(\mathbf{x}) \quad \forall\, \|\boldsymbol{\xi}\|_2 \leq \epsilon, \tag{13}$$

where $\overline{p} = \Phi(\Phi^{-1}(p) + \frac{\epsilon}{\sigma})$.

Applying this result to our setting with any given estimator $f_{\boldsymbol{\theta}} : \mathbb{R}^m \to \mathbb{R}^n$, given our definition of the smoothed discrepancy

$$D_p(\mathbf{x}, \boldsymbol{\xi}) = \inf\left\{t \in \mathbb{R} \mid \mathbb{P}\big[\ell(f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}+\boldsymbol{\delta}}), f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}})) \leq t\big] \geq p\right\}, \tag{14}$$

we also have the guarantee

$$D_p(\mathbf{x}, \boldsymbol{\xi}) \leq D_{\overline{p}}(\mathbf{x}, \mathbf{0}) \quad \forall \boldsymbol{\xi} \text{ s.t. } \|\boldsymbol{\xi}\|_2 \leq \epsilon. \tag{15}$$

Then, by the definition of $\overline{p}$, we have

$$D_p(\mathbf{x}, \boldsymbol{\xi}) \leq D_{\Phi(\Phi^{-1}(p)+\frac{\epsilon}{\sigma})}(\mathbf{x}, \mathbf{0}) \quad \forall \boldsymbol{\xi} \text{ s.t. } \|\boldsymbol{\xi}\|_2 \leq \epsilon. \tag{16}$$

In particular, substituting 0.5 for p, we obtain

$$\begin{aligned} D_{0.5}(\mathbf{x}, \boldsymbol{\xi}) &\leq D_{\Phi(\Phi^{-1}(0.5)+\frac{\epsilon}{\sigma})}(\mathbf{x}, \mathbf{0}) \\ &= D_{\Phi(\frac{\epsilon}{\sigma})}(\mathbf{x}, \mathbf{0}) \end{aligned} \tag{17}$$

whenever $\|\boldsymbol{\xi}\|_2 \leq \epsilon$. □

B Further Details on the Experiments

We perform the experimental evaluation on both denoising and compressed sensing tasks. For each task, we use a suitably-chosen neural network estimator equipped with one training set and two testing sets. We consider training using all five algorithms discussed earlier. At test time, we apply the PGD method to produce adversarial attacks using a step size and the number of iterations matching those used in the Adv and Smt-Adv algorithms during training.

In addition, we select the training and testing parameters with minimal manual tuning, noting that further fine-tuning may improve the performance. The details are given as follows:

- **DPDNN** & **DIV2K, BSD68** (denoising). We use the DPDNN model proposed in [10] as the estimator $f_{\boldsymbol{\theta}}$. For training, we use the (grayscale converted) DIV2K dataset [1], which consists of 1000 images of size 256×256, and is further cropped to generate 32000 patches of size 40×40. For testing, we use the (grayscale converted) DIV2K validation set, which contains 100 images cropped to a size of 384×384, as well as the BSD68 dataset [31], which consists of 68 images cropped to the 256×256 pixels in the center.
  - Training parameters:
    - (Shared across all algorithms) We follow the settings in [10] and set the number of training epochs as 100, batch size as 32, learning rate $\alpha = 0.0005$, measurement matrix $\mathbf{A} = \mathbf{I}$, and additive noise level $\sigma_{\mathbf{b}} = 15$.

* (For Adv and Smt-Adv only) We set the iteration count $T_{itr} = 100$, step size $r = 0.4$, and adversarial noise ℓ_2 radius $\epsilon = 5$.
* (For Smt-Adv and Smt-Grad only) We set the step count $T_{step} = 6$, sample count $S = 15$, and the smoothing noise level σ in Smt-Adv to be equal to the maximum smoothing noise level σ_{max} in Smt-Grad. Specifically, $\sigma = \sigma_{\text{max}} = 10$ (in Table 1 we also use 1 and 5 for comparison).

– Testing parameters:

* (For measurement) $\mathbf{A} = \mathbf{I}$, and $\sigma_{\mathbf{b}}$ ranges from 0 to 15.
* (For PGD attack) We set the attack iteration count $T_{itr} = 100$, step size $r = 0.4$, and consider multiple adversarial noise ℓ_2 radii with ϵ ranging from 0 to 15.
* (For smoothing) We set the smoothing sample count $S = 50$ and consider multiple smoothing noise levels with σ ranging from 0 to 19.

• **ISTA-Net^{++} & Train400, Set11, ImageNet** (compressed sensing). The ISTA-Net^{++} model [41] is used as the estimator $f_{\boldsymbol{\theta}}$. We use the Train400 dataset [24], consisting of 400 natural photos of size 180×180 for training, and the Set11 dataset [24] consisting of 11 grayscale images of size 256×256 for testing. Additionally, we use 300 randomly selected images sampled from the "tall building", "open country", and "street" categories from ImageNet [9] for testing. For each of these 300 images, the Y-channel luminance is extracted from the RGB color to form a greyscale-intensity image, and each is cropped to the central 256×256 pixels.

– Training parameters:

* (Shared across all algorithms) We follow the settings in [41] and set the number of training epoch as 100, batch size as 64, learning rate $\alpha = 0.0001$, additive noise level $\sigma_{\mathbf{b}} = 0$, and Gaussian measurement matrices $\mathbf{A}$ used in each batch with a randomly selected compression ratio from the set $\{0.1, 0.2, 0.3, 0.4, 0.5\}$. This randomization is done so that the neural network can learn to reconstruct at a variety of compression ratios.
* (For Adv and Smt-Adv only) We set the iteration count $T_{itr} = 150$, step size $r = 0.2$, and adversarial noise ℓ_2 radius $\epsilon = 1$.
* (For Smt-Adv and Smt-Grad only) We set the step count $T_{step} = 6$, step size $S = 15$, and the smoothing noise level σ in Smt-Adv to be equal to the maximum smoothing noise level σ_{max} in Smt-Grad. Specifically, $\sigma = \sigma_{max} = 10$ (in Table 1 we also use 5 for comparison).

– Testing parameters:

* (For measurement) We use a Gaussian matrix $\mathbf{A}$ with compression ratio ranging from 0.1 to 0.4 and no noise, i.e., $\mathbf{b} = \mathbf{0}$.
* (For PGD attack) We set the iteration count $T_{itr} = 150$, step size $r = 0.2$, and consider multiple adversarial noise ℓ_2 radii with ϵ ranging from 0 to 7.
* (For smoothing) We set the smoothing sample count $S = 50$ and consider multiple smoothing noise levels with σ ranging from 0 to 45.

Regarding the training time per batch across all five algorithms in two tasks, we present Table 3 below and observe that the training time required for Smt-Adv and Smt-Grad is higher than the baselines but still reasonable.

Estimator & Training Set	Ord	Jcb	Adv	Smt-Adv	Smt-Grad
DPDNN & DIV2K	0.49s	0.83s	3.53s	46.81s	45.48s
ISTA-Net^{++} & Train400	0.08s	0.15s	0.27s	2.49s	5.41s

Table 3: Training time per batch in denoising (top, batch size 32) and compressed sensing (bottom, batch size 64), where sample count of $S = 15$ for Smt-Adv and Smt-Grad is utilized in both tasks.

Regarding the choice of the number of Monte Carlo samples (S) during testing, Table 4 showcases the certified discrepancy and corresponding PSNR with S ranging from 50 to 1000 in chosen tasks. The results reveal that PSNR, in fact, shows very little degradation between smaller vs. higher values of S. For certified discrepancy, a downward trend is observed as expected (i.e., decreasing S gives a worse guarantee), yet not a particularly drastic one. Notably, lower S values significantly reduce the computation time as we see that reducing S from 1000 to 50 can give a roughly $15\times$ speedup.

Consequently, we opt for $S = 50$ in all experiments due to its consistent effectiveness, particularly with respect to PSNR (higher S may still be preferred if one specifically needs a strong certified guarantee).

	S=50 (t=2.20s)		S=500 (t=16.93s)		S=1000 (t=33.52s)	
	discrepancy	*PSNR*	discrepancy	*PSNR*	discrepancy	*PSNR*
Ord	5.085(±0.948)	*25.549(±1.039)*	5.083(±0.945)	*25.552(±1.032)*	5.072(±0.923)	*25.552(±1.036)*
Jcb	4.961(±0.822)	*25.574(±1.045)*	4.958(±0.812)	*25.572(±1.045)*	4.955(±0.812)	*25.573(±1.044)*
Adv	4.626(±0.770)	*25.671(±0.980)*	4.611(±0.766)	*25.679(±0.971)*	4.613(±0.766)	*25.678(±0.972)*
Smt-Adv	4.191(±0.585)	*25.724(±0.987)*	4.179(±0.581)	*25.727(±0.981)*	4.180(±0.567)	*25.727(±0.982)*
Smt-Grad	4.184(±0.595)	*25.783(±1.052)*	4.183(±0.592)	*25.781(±1.047)*	4.166(±0.593)	*25.781(±1.050)*

(a) discrepancy (1e-4) and PSNR with $\sigma_{\mathbf{b}} = 10$, $\epsilon = 10$, $\sigma = 12$ on BSD68 (denoising)

Ord	4.328(±0.463)	*19.334(±1.017)*	4.299(±0.465)	*19.343(±0.940)*	4.289(±0.461)	*19.341(±0.955)*
Jcb	4.039(±0.440)	*19.462(±1.068)*	4.022(±0.379)	*19.472(±1.029)*	4.017(±0.387)	*19.478(±0.986)*
Adv	3.614(±0.317)	*19.561(±1.102)*	3.590(±0.312)	*19.593(±1.089)*	3.589(±0.320)	*19.598(±1.082)*
Smt-Adv	4.030(±0.560)	*20.081(±1.279)*	3.995(±0.589)	*20.163(±1.233)*	3.988(±0.563)	*20.161(±1.223)*
Smt-Grad	2.929(±0.404)	*19.904(±1.114)*	2.917(±0.407)	*19.911(±1.105)*	2.910(±0.356)	*19.943(±1.110)*

(b) discrepancy (1e-3) and PSNR with $m/n = 0.1$, $\epsilon = 5$, $\sigma = 19$ on Set11 (compressed sensing)

Table 4: Estimated certified discrepancy and PSNR (± std.) with different testing samples (S) and the corresponding testing time per data sample (t) in two selected scenarios (a) and (b), where the corresponding best choices of σ (12 and 19) are used.

C Details of Algorithms for Randomized Smoothing at Test Time

Similar to [4], Algorithm 3 below samples S random realizations of $\boldsymbol{\delta} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$ and computes the corresponding S discrepancies $\ell(f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}^*+\boldsymbol{\delta}}), f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}^*}))$. Note that these can be computed because $\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}^*}$ is given as an input and $\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}^*+\boldsymbol{\delta}} = \mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}^*} + \mathbf{A}\boldsymbol{\delta}$ (with known $\mathbf{A}$ and $\boldsymbol{\delta}$). The sequence of discrepancies is sorted, with the $\lfloor \frac{S}{2} \rfloor$-th item being used to approximate the median smoothed discrepancy $D_{0.5}(\mathbf{x}, \boldsymbol{\xi})$ and the $\lfloor \mathrm{EMPIRICAL}(c, \sigma, S, \epsilon) \rfloor$-th item to approximate its certified upper bound $D_{\Phi(\frac{\epsilon}{\sigma})}(\mathbf{x}, \mathbf{0})$.

The functionality of EMPIRICAL is described in Algorithm 4. This algorithm is essentially taken from [7], and despite its rather technical description, is conceptually simple: Given a confidence level c, perform a binary search to find a value K such that the probability of the event $\mathrm{Binomial}(S, p) \leq K$ is roughly c. Here p is set to $\Phi(\frac{\epsilon}{\sigma})$ in accordance with Corollary 3.2. The idea is then that if we sort S binomial samples and take the K-th one, we can be confident (up to the confidence level c) that it is bounded by the actual p-quantile we are interested in. When performing smoothing, we are not directly using binomial samples, but the indicator variable of whether $\ell(f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}+\boldsymbol{\delta}}), f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}}))$ is below a fixed value (see (6)) indeed follows a binomial distribution. It is worth noting that when the ratio $\frac{\epsilon}{\sigma}$ is large (e.g., greater than 2), the value of p approaches 1, causing the EMPIRICAL function to return null. To ensure reliable results, in this paper, we carefully choose ϵ and σ values that maintain a sufficiently small $\frac{\epsilon}{\sigma}$ ratio to avoid this. Additionally, we adopt the choice of the Monte Carlo sample size $S = 1000$ and confidence level $c = 0.99$ through out the paper.

Algorithm 3 Find median and certified discrepancy values and corresponding estimates

input Perturbed testing pair $(\mathbf{x}, \mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}^*})$, where $\boldsymbol{\xi}^*$ is the adversarial noise produced by PGD method (but unknown to the algorithm); Estimator $f_{\boldsymbol{\theta}}$ and measurement matrix $\mathbf{A}$; Smoothing noise level σ and noise sample count S; Confidence level c; Adversarial noise ℓ_2 radius ϵ; Function $\mathrm{EMPIRICAL}(\cdot)$ from Algorithm 4;
initiate Sorting function $\mathrm{sort}()$; Empty set $\mathrm{discrepancy\_set} = \{\}$;

for $1 \leq k \leq S$ **do**
  $\boldsymbol{\delta}_k \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$
  $\mathrm{discrepancy\_set} \xleftarrow{add} \ell(f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}^*+\boldsymbol{\delta}_k}), f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}^*}))$
end for
$\mathrm{discrepancy\_set\_sorted} = \mathrm{sort}(\mathrm{discrepancy\_set})$
Set $\hat{D}_{0.5}(\mathbf{x}, \boldsymbol{\xi}) = \mathrm{discrepancy\_set\_sorted}_{\lfloor 0.5S \rfloor}$
if $\mathrm{EMPIRICAL}(c, \sigma, S, \epsilon)$ returns non-null **then**
  Set $\hat{D}_{\Phi(\frac{\epsilon}{\sigma})}(\mathbf{x}, \mathbf{0}) = \mathrm{discrepancy\_set\_sorted}_{\lfloor \mathrm{EMPIRICAL}(c,\sigma,S,\epsilon) \rfloor}$ (Algorithm 4)
  return the estimated median discrepancy $\hat{D}_{0.5}(\mathbf{x}, \boldsymbol{\xi}^*)$, the estimated certificate discrepancy $\hat{D}_{\Phi(\frac{\epsilon}{\sigma})}(\mathbf{x}, \mathbf{0})$, and the two corresponding estimates $\hat{\mathbf{x}} = f_{\boldsymbol{\theta}}(\mathbf{y}_{\mathbf{x}+\boldsymbol{\xi}^*+\boldsymbol{\delta}_k})$, where $k = \lfloor 0.5S \rfloor$ and $k = \lfloor \mathrm{EMPIRICAL}(c, \sigma, S, \epsilon) \rfloor$
else
  return median discrepancy and corresponding estimate as above (certified result is unavailable due to overly large ratio $\frac{\epsilon}{\sigma}$)
end if

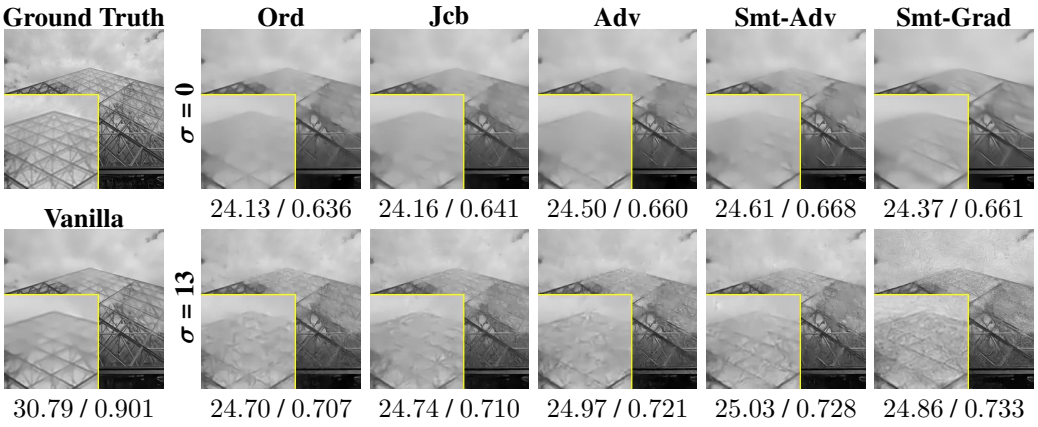
Algorithm 4 Empirical order statistics generation

input Confidence level c; Attack radius ϵ; Smoothing noise level σ; noise sample count S;
define Cumulative distribution function of binomial, $F(S, K, p) = \sum_{i=1}^{K} \binom{S}{i} p^i (1-p)^{S-i}$;

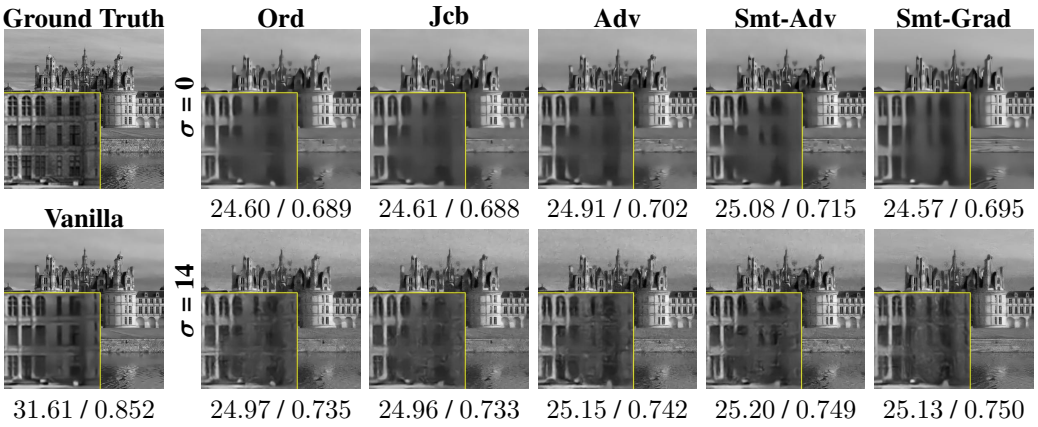
function EMPIRICAL(c, σ, S, ϵ)
  $p = \Phi(\frac{\epsilon}{\sigma})$
  $\hat{\underline{K}}, \hat{\overline{K}} = \lceil S \cdot p \rceil, S$
  while $\hat{\overline{K}} - \hat{\underline{K}} < 1$ **do**
    $\dot{K} = \lfloor \frac{(\hat{\overline{K}} - \hat{\underline{K}})}{2} \rfloor$
    if $F(S, \dot{K}, p) > c$ **then**
      $\hat{\overline{K}} = \dot{K}$
    else
      $\hat{\underline{K}} = \dot{K}$
    end if
  end while
  if $\hat{\overline{K}} < S$ **then**
    return $\hat{\overline{K}}$
  else
    return null
  end if
end function

D Supplementary Experimental Results

Here we provide two additional sets of visual comparisons, Figures 3 and 4, and once again observe that applying randomized smoothing at test time leads to visible visual enhancements in both denoising and compressed sensing tasks across a variety of datasets.

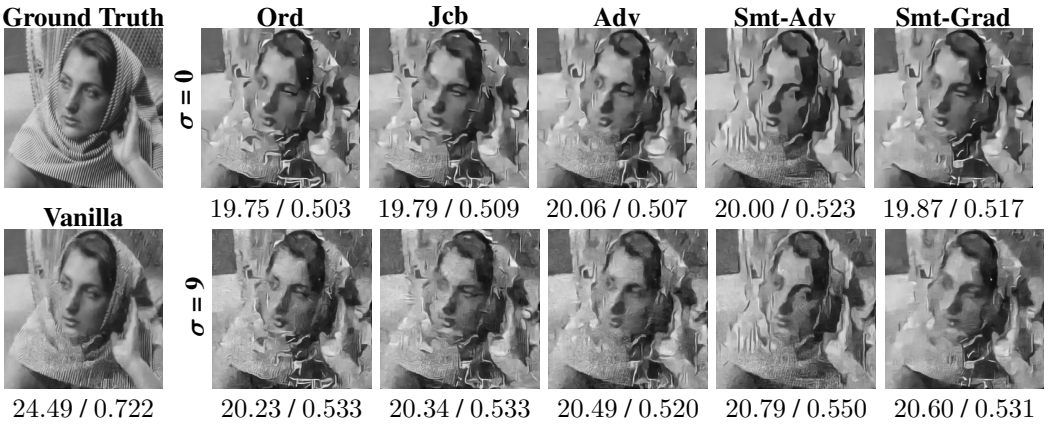


(a) $\sigma_{\mathbf{b}} = \mathbf{8}$, $\epsilon = \mathbf{10}$ on BSD68 (denoising)

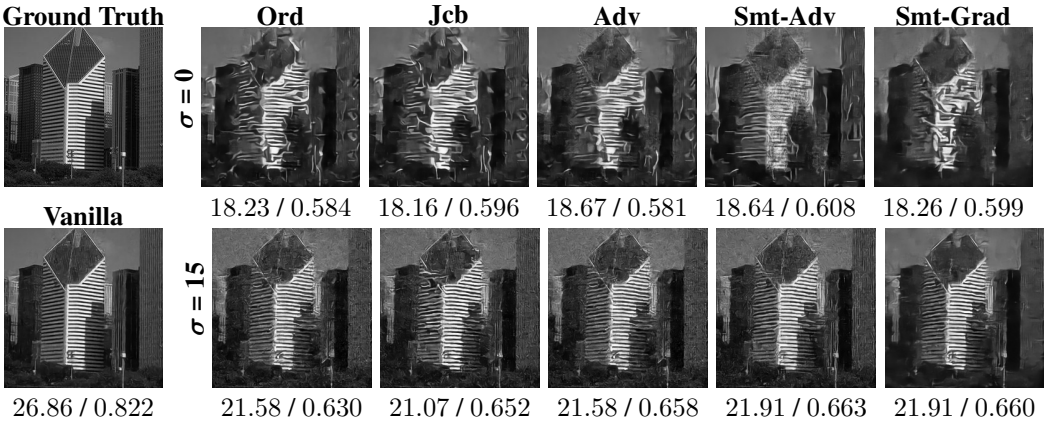


(b) $\sigma_{\mathbf{b}} = \mathbf{5}$, $\epsilon = \mathbf{11}$ on DIV2K (denoising)

Figure 3: Additional estimations (with the performance shown in the form of PSNR / SSIM) from the perturbed measurements on BSD68 and DIV2K in denoising task.



(a) $m/n = \mathbf{0.1}$, $\epsilon = \mathbf{3}$ on Set11 (compressed sensing)



(b) $m/n = \mathbf{0.2}$, $\epsilon = \mathbf{4}$ on ImageNet (compressed sensing)

Figure 4: Additional estimations (with the performance shown in the form of PSNR / SSIM) from the perturbed measurements on Set11 and ImageNet in compressed sensing task.