Finding the Mukhanov-Sasaki Equation

Josiel Mendonça Soares de Souza

Departamento de Física Teórica e Experimental, Universidade Federal do Rio Grande do Norte, Brazil

1.1 Perturbed FLRW Metric: Newtonian Gauge

$$dS^2 = -(1+\phi)dt^2 + 2a\beta_i dx^i dt + a^2[(1-\phi)\delta_{ij} + h_{ij}^{TT}]dx^i dx^j \tag{1.1}$$

1.2 Perturbed Energy-Momentum Tensor for a Single Scalar Field

1.2.1 Matter Action - Inflaton Scalar Field

$$S_M = \int d^4x\sqrt{-g}\left[-\frac{1}{2}(\partial^\mu\chi)(\partial_\mu\chi) - V(\chi)\right] \tag{1.2}$$

1.2.2 Energy-Momentum Tensor

$$T_{\mu\nu} = \partial_\mu\chi\partial_\nu\chi - \frac{1}{2}g_{\mu\nu}[(\partial\chi)^2 - 2V(\chi)] \tag{1.3}$$

where, $(\partial\chi)^2 \equiv (\partial^\mu\chi)(\partial_\mu\chi)$

$$\rho = \frac{1}{2}\dot{\chi}^2 + V(\chi) \quad , \quad P = \frac{1}{2}\dot{\chi}^2 - V(\chi) \quad \Rightarrow \quad \rho + P = \dot{\chi}^2 \tag{1.4}$$

1.2.3 Field Perturbations

$$\chi = \bar{\chi} + \delta\chi \quad , \quad T_{\mu\nu} = \bar{T}_{\mu\nu} + \delta T_{\mu\nu} \tag{1.5}$$

$$\delta T^0{}_0 = -\delta\rho = a^{-2}(\phi\bar{\chi}'^2 - \bar{\chi}'\delta\chi' - a^2\partial_\chi V\delta\chi) \tag{1.6}$$

$$\delta T^i{}_0 = (\bar{\rho} + \bar{P})(\nu^i + \partial^i q)/a = \frac{\bar{\chi}'}{a^3}(\bar{\chi}'\beta^i + \partial^i\delta\chi) \tag{1.7}$$

$$\delta T^i{}_j = \delta^i_j\delta P = a^{-2}\delta^i_j(\bar{\chi}'\delta\chi' - \phi\bar{\chi}'^2 - a^2\partial_\chi V\delta\chi) \tag{1.8}$$

1.3 Inflaton Scalar Field Motion Equations

$$\bar{\chi}'' + 2\mathcal{H}\bar{\chi}' + a^2\partial_\chi V = 0 \tag{1.9}$$

$$\delta\chi'' + 2\mathcal{H}\delta\chi' - \nabla^2\delta\chi + a^2(\partial_\chi^2 V)\delta\chi = 4\phi'\bar{\chi}' - 2a^2\phi\partial_\chi V \tag{1.10}$$

1.4 Friedmann's Equation

$$H^2 = \frac{8\pi G}{3}\rho \quad \Rightarrow \quad \mathcal{H}^2 = \frac{k}{2}(\bar{\chi}'^2 + 2a^2 V) \tag{1.11}$$

$$\dot{H} = -4\pi G(\rho + P) \quad \Rightarrow \quad \mathcal{H}' = k(a^2 V - \bar{\chi}'^2) \tag{1.12}$$

$$\mathcal{H}^2 - \mathcal{H}' = \frac{3}{2}k\bar{\chi}'^2 \tag{1.13}$$

where, $k \equiv (8\pi G)/3$.

1.5 Gauge Invariant Scalar Field

A **gauge invariant field** of our system is defined below (in the Newtonian Gauge):

$$\mathcal{R} \equiv \phi + aHq = \phi + \mathcal{H}q$$

In terms of the inflaton field χ:

$$\boxed{\mathcal{R} = \phi + \mathcal{H}\left(\frac{\delta\chi}{\bar{\chi}'}\right)} \tag{1.14}$$

where q comes from Eq.(1.7).

1.5.1 First Derivative of $\mathcal{R}$

$$\boxed{\mathcal{R}' = \phi' + (3kV\bar{\chi}' + \mathcal{H}\partial_\chi V)\left(\frac{a^2\delta\chi}{\bar{\chi}'^2}\right) + \mathcal{H}\left(\frac{\delta\chi'}{\bar{\chi}'}\right)} \tag{1.15}$$

That is of the form:

$$\mathcal{R}' = \phi' + A_1\delta\chi + A_2\delta\chi'$$

whith,

$$A_1 \equiv \left(\frac{a}{\bar{\chi}'}\right)^2 (3kV\bar{\chi}' + \mathcal{H}\partial_\chi V) \quad , \quad A_2 \equiv \frac{\mathcal{H}}{\bar{\chi}'}$$

1.5.2 Second Derivative of $\mathcal{R}$

$$\mathcal{R}'' = \phi'' + A_1'\delta\chi + (A_1 + A_2')\delta\chi' + A_2\delta\chi''$$

$$\begin{aligned}
A_1' &= \left(\frac{a^2}{\bar{\chi}'^3}\right)\{2\mathcal{H}[9kV\bar{\chi}'^2 + (a\partial_\chi V)^2] + 3k\bar{\chi}'(\bar{\chi}'^2 + 4a^2V)\partial_\chi V\} \\
&+ \left(\frac{a}{\bar{\chi}'}\right)^2 [2k(\bar{\chi}'^2 - a^2V)\partial_\chi V + \mathcal{H}\bar{\chi}'\partial_\chi^2 V - 6k\mathcal{H}V\bar{\chi}'] \\
A_2' &= \left(\frac{a}{\bar{\chi}'}\right)^2 [\mathcal{H}\partial_\chi V + 3k\bar{\chi}'V] \\
A_1 + A_2' &= 2\left(\frac{a}{\bar{\chi}'}\right)^2 (3kV\bar{\chi}' + \mathcal{H}\partial_\chi V)
\end{aligned}$$

$$\boxed{\begin{aligned}
\mathcal{R}'' &= \phi'' + \left(\frac{a^2}{\bar{\chi}'^3}\right)\{[2\mathcal{H}[9kV\bar{\chi}'^2 + (a\partial_\chi V)^2] + 3k\bar{\chi}'(\bar{\chi}'^2 + 4a^2V)\partial_\chi V] \\
&\quad + \bar{\chi}'[2k(\bar{\chi}'^2 - a^2V)\partial_\chi V + \mathcal{H}\bar{\chi}'\partial_\chi^2 V - 6k\mathcal{H}V\bar{\chi}']\}\,\boldsymbol{\delta\chi} \\
&\quad + 2\left(\frac{a}{\bar{\chi}'}\right)^2 (3kV\bar{\chi}' + \mathcal{H}\partial_\chi V)\boldsymbol{\delta\chi}' + \left(\frac{\mathcal{H}}{\bar{\chi}'}\right)\boldsymbol{\delta\chi}''
\end{aligned}} \tag{1.16}$$

Now, we are going to use:

$$\begin{aligned}
\delta\chi'' + 2\mathcal{H}\delta\chi' - \nabla^2\delta\chi + a^2(\partial_\chi^2 V)\delta\chi &= 4\phi'\bar{\chi}' - 2a^2\phi\partial_\chi V \\
\delta\chi'' = [\nabla^2\boldsymbol{\delta\chi} - 2\mathcal{H}\boldsymbol{\delta\chi}' - a^2(\partial_\chi^2 V)\boldsymbol{\delta\chi}] &+ 4\phi'\bar{\chi}' - 2a^2\phi\partial_\chi V
\end{aligned}$$

In order to eliminate $\nabla^2\boldsymbol{\delta\chi}$ we are going to use:

$$\mathcal{R} = \phi + \mathcal{H}\left(\frac{\delta\chi}{\bar{\chi}'}\right) \quad \Rightarrow \quad \nabla^2\mathcal{R} = \nabla^2\phi + \left(\frac{\mathcal{H}}{\bar{\chi}'}\right)\nabla^2\boldsymbol{\delta\chi}$$

Thus, we have, for the last term of $\mathcal{R}''$:

$$\left(\frac{\mathcal{H}}{\bar{\chi}'}\right)\boldsymbol{\delta\chi}'' = \left(\frac{\mathcal{H}}{\bar{\chi}'}\right)\nabla^2\boldsymbol{\delta\chi} + \left(\frac{\mathcal{H}}{\bar{\chi}'}\right)\left\{4\phi'\bar{\chi}' - 2a^2\phi\partial_\chi V - [2\mathcal{H}\boldsymbol{\delta\chi}' + a^2(\partial_\chi^2 V)\boldsymbol{\delta\chi}]\right\}$$

$$\left(\frac{\mathcal{H}}{\bar{\chi}'}\right)\boldsymbol{\delta\chi}'' = [\nabla^2\mathcal{R} - \nabla^2\phi] + \left(\frac{\mathcal{H}}{\bar{\chi}'}\right)\left\{4\phi'\bar{\chi}' - 2a^2\phi\partial_\chi V - [2\mathcal{H}\boldsymbol{\delta\chi}' + a^2(\partial_\chi^2 V)\boldsymbol{\delta\chi}]\right\}$$

Rewriting $\mathcal{R}''$:

$$\begin{aligned}
\mathcal{R}'' - \nabla^2\mathcal{R} \;=\;& (\phi'' - \nabla^2\phi) + \left(\frac{a^2}{\bar{\chi}'^3}\right)\left\{[2\mathcal{H}[9kV\bar{\chi}'^2 + (a\partial_\chi V)^2] + 3k\bar{\chi}'(\bar{\chi}'^2 + 4a^2V)\partial_\chi V]\right. \\
&\left. +\bar{\chi}'[2k(\bar{\chi}'^2 - a^2V)\partial_\chi V + \mathcal{H}\bar{\chi}'\partial_\chi^2 V - 6k\mathcal{H}V\bar{\chi}']\right\}\boldsymbol{\delta\chi} \\
& +2\left(\frac{a}{\bar{\chi}'}\right)^2(3kV\bar{\chi}' + \mathcal{H}\partial_\chi V)\boldsymbol{\delta\chi}' + \\
& + \left(\frac{\mathcal{H}}{\bar{\chi}'}\right)\left\{4\phi'\bar{\chi}' - 2a^2\phi\partial_\chi V - [2\mathcal{H}\boldsymbol{\delta\chi}' + a^2(\partial_\chi^2 V)\boldsymbol{\delta\chi}]\right\}
\end{aligned} \tag{1.17}$$

$$\begin{aligned}
\mathcal{R}'' - \nabla^2\mathcal{R} \;=\;& \left[(\phi'' - \nabla^2\phi) + \left(\frac{\mathcal{H}}{\bar{\chi}'}\right)(4\phi'\bar{\chi}' - 2a^2\phi\partial_\chi V)\right] \\
& + \left(\frac{a^2}{\bar{\chi}'^3}\right)\left\{[2\mathcal{H}[9kV\bar{\chi}'^2 + (a\partial_\chi V)^2] + 3k\bar{\chi}'(\bar{\chi}'^2 + 4a^2V)\partial_\chi V]\right. \\
& \left. +\bar{\chi}'[2k(\bar{\chi}'^2 - a^2V)\partial_\chi V + \mathcal{H}\bar{\chi}'\partial_\chi^2 V - 6k\mathcal{H}V\bar{\chi}']\right\}\boldsymbol{\delta\chi} - \left(\frac{\mathcal{H}}{\bar{\chi}'}\right)[a^2(\partial_\chi^2 V)\boldsymbol{\delta\chi}] \\
& +2\left(\frac{a}{\bar{\chi}'}\right)^2(3kV\bar{\chi}' + \mathcal{H}\partial_\chi V)\boldsymbol{\delta\chi}' - \left(\frac{\mathcal{H}}{\bar{\chi}'}\right)2\mathcal{H}\boldsymbol{\delta\chi}'
\end{aligned}$$

$$\begin{aligned}
\mathcal{R}'' - \nabla^2\mathcal{R} \;=\;& \left[(\phi'' - \nabla^2\phi) + \left(\frac{\mathcal{H}}{\bar{\chi}'}\right)(4\phi'\bar{\chi}' - 2a^2\phi\partial_\chi V)\right] \\
& + \left(\frac{a^2}{\bar{\chi}'^3}\right)\left\{[2\mathcal{H}[9kV\bar{\chi}'^2 + (a\partial_\chi V)^2] + 3k\bar{\chi}'(\bar{\chi}'^2 + 4a^2V)\partial_\chi V]\right. \\
& \left. +\bar{\chi}'[2k(\bar{\chi}'^2 - a^2V)\partial_\chi V - 6k\mathcal{H}V\bar{\chi}']\right\}\boldsymbol{\delta\chi} \\
& +[k\bar{\chi}'(4a^2V - \bar{\chi}'^2) + 2a^2\mathcal{H}\partial_\chi V]\left(\frac{\boldsymbol{\delta\chi}'}{\bar{\chi}'^2}\right)
\end{aligned}$$

$$\boxed{\begin{aligned}
\mathcal{R}'' - \nabla^2\mathcal{R} \;=\;& \left[(\phi'' - \nabla^2\phi) + \left(\frac{\mathcal{H}}{\bar{\chi}'}\right)(4\phi'\bar{\chi}' - 2a^2\phi\partial_\chi V)\right] \\
& + \left(\frac{a^2}{\bar{\chi}'^3}\right)[2\mathcal{H}(6kV\bar{\chi}'^2 + (a\partial_\chi V)^2) + 5k\bar{\chi}'(\bar{\chi}'^2 + 2a^2V)\partial_\chi V]\boldsymbol{\delta\chi} \\
& +[k\bar{\chi}'(4a^2V - \bar{\chi}'^2) + 2a^2\mathcal{H}\partial_\chi V]\left(\frac{\boldsymbol{\delta\chi}'}{\bar{\chi}'^2}\right)
\end{aligned}} \tag{1.18}$$

Rebember the $\mathcal{R}'$ equation:

$$\boxed{\mathcal{R}' = \phi' + (3kV\bar{\chi}' + \mathcal{H}\partial_\chi V)\left(\frac{a^2}{\bar{\chi}'^2}\right)\boldsymbol{\delta\chi} + \left(\frac{\mathcal{H}}{\bar{\chi}'}\right)\boldsymbol{\delta\chi}'}$$

The next step is to use the Einstein Equations.

1.6 Perturbed Einstein's Equations: Scalar Part

$$\begin{aligned}
\delta G^0{}_0: &\quad 3\mathcal{H}(\phi' + \mathcal{H}\phi) - \nabla^2\phi &&= -\frac{3}{2}k[(\bar{\chi}'\delta\chi' - \phi\bar{\chi}'^2) + a^2(\partial_\chi V)\delta\chi] && (1.19)\\
\delta G^i{}_0: &\quad \phi' + \mathcal{H}\phi &&= \frac{3}{2}k\bar{\chi}'\delta\chi && (1.20)\\
\delta G^i{}_j: &\quad \phi'' + 3\mathcal{H}\phi' + (2\mathcal{H}' + \mathcal{H}^2)\phi &&= \frac{3}{2}k[(\bar{\chi}'\delta\chi' - \phi\bar{\chi}'^2) - a^2(\partial_\chi V)\delta\chi] && (1.21)
\end{aligned}$$

Here, I want to express the following quantities in terms of $\phi,\ \delta\chi,\ \delta\chi'$:

$$\begin{aligned}
f_1(\phi,\ \delta\chi,\ \delta\chi') &\equiv \phi'' - \nabla^2\phi && (1.22)\\
f_2(\phi,\ \delta\chi,\ \delta\chi') &\equiv \phi' && (1.23)
\end{aligned}$$

Combining Eq.(1.20) and Eq.(1.21) in order to eliminate ϕ' from Eq.(1.21) we get:

$$\phi'' = \frac{3k}{2}[\bar{\chi}'^2\phi - (3\mathcal{H}\bar{\chi}' + a^2\partial_\chi V)\boldsymbol{\delta\chi} + \bar{\chi}'\boldsymbol{\delta\chi}'] \tag{1.24}$$

Making Eq.(1.19) - $3\mathcal{H}\cdot$ Eq.(1.20) we get:

$$-\nabla^2\phi = \left(\frac{3k}{2}\right)[\bar{\chi}'^2\phi - (3\mathcal{H}\bar{\chi}' + a^2\partial_\chi V)\boldsymbol{\delta\chi} - \bar{\chi}'\boldsymbol{\delta\chi}'] \tag{1.25}$$

Combining Eq.(1.24) and Eq.(1.25) we get:

$$\begin{aligned}
\phi'' - \nabla^2\phi &= \frac{3k}{2}[\bar{\chi}'^2\phi - (3\mathcal{H}\bar{\chi}' + a^2\partial_\chi V)\boldsymbol{\delta\chi} + \bar{\chi}'\boldsymbol{\delta\chi}']\\
&+ \frac{3k}{2}[\bar{\chi}'^2\phi - (3\mathcal{H}\bar{\chi}' + a^2\partial_\chi V)\boldsymbol{\delta\chi} - \bar{\chi}'\boldsymbol{\delta\chi}']\\
\Rightarrow f_1 \equiv \phi'' - \nabla^2\phi &= 3k[\bar{\chi}'^2\phi - (3\mathcal{H}\bar{\chi}' + a^2\partial_\chi V)\boldsymbol{\delta\chi}] && (1.26)
\end{aligned}$$

f_2 comes directely from Eq.(1.20):

$$f_2 \equiv \phi' = \frac{3}{2}k\bar{\chi}'\boldsymbol{\delta\chi} - \mathcal{H}\phi \tag{1.27}$$

Subistituting f_1 and f_2 in Eq.1.18 we get:

$$\begin{aligned}
\mathcal{R}'' - \nabla^2\mathcal{R} &= 3k[\bar{\chi}'^2\phi - (3\mathcal{H}\bar{\chi}' + a^2\partial_\chi V)\boldsymbol{\delta\chi}] \\
&+ \left(\frac{\mathcal{H}}{\bar{\chi}'}\right)\left[4\bar{\chi}'\left(\frac{3}{2}k\bar{\chi}'\boldsymbol{\delta\chi} - \mathcal{H}\phi\right) - 2a^2\phi\partial_\chi V\right] \\
&+ \left(\frac{a^2}{\bar{\chi}'^3}\right)[2\mathcal{H}(6kV\bar{\chi}'^2 + (a\partial_\chi V)^2) + 5k\bar{\chi}'(\bar{\chi}'^2 + 2a^2V)\partial_\chi V]\boldsymbol{\delta\chi} \\
&+ [k\bar{\chi}'(4a^2V - \bar{\chi}'^2) + 2a^2\mathcal{H}\partial_\chi V]\left(\frac{\boldsymbol{\delta\chi}'}{\bar{\chi}'^2}\right) \qquad (1.28)
\end{aligned}$$

$$\boxed{\begin{aligned}
\mathcal{R}'' - \nabla^2\mathcal{R} &= -[k\bar{\chi}'(4a^2V - \bar{\chi}'^2) + 2a^2\mathcal{H}\partial_\chi V]\frac{\phi}{\bar{\chi}'} \\
&+ [3k\mathcal{H}\bar{\chi}'^2(4a^2V - \bar{\chi}'^2) + 2ka^2\bar{\chi}'(\bar{\chi}'^2 + 5a^2V)(\partial_\chi V) + 2\mathcal{H}(a^2\partial_\chi V)^2]\frac{\boldsymbol{\delta\chi}}{\bar{\chi}'^3} \\
&+ [k\bar{\chi}'(4a^2V - \bar{\chi}'^2) + 2a^2\mathcal{H}\partial_\chi V]\left(\frac{\boldsymbol{\delta\chi}'}{\bar{\chi}'^2}\right) \qquad (1.29)
\end{aligned}}$$

Substituting f_2 in $\mathcal{R}'$ equation:

$$\mathcal{R}' = \left[\frac{3}{2}k\bar{\chi}'\boldsymbol{\delta\chi} - \mathcal{H}\phi\right] + (3kV\bar{\chi}' + \mathcal{H}\partial_\chi V)\left(\frac{a^2}{\bar{\chi}'^2}\right)\boldsymbol{\delta\chi} + \left(\frac{\mathcal{H}}{\bar{\chi}'}\right)\boldsymbol{\delta\chi}'$$

$$\boxed{\mathcal{R}' = \mathcal{H}\left[-\phi + (3\bar{\chi}'\mathcal{H} + a^2\partial_\chi V)\frac{\boldsymbol{\delta\chi}}{\bar{\chi}'^2} + \left(\frac{\boldsymbol{\delta\chi}'}{\bar{\chi}'}\right)\right]}$$

1.7 Concluding the Computations

Now we have:

$$\boxed{\begin{aligned}
\mathcal{R}'' - \nabla^2\mathcal{R} &= -\frac{\mathbf{1}}{\bar{\chi}'}[k\bar{\chi}'(4a^2V - \bar{\chi}'^2) + 2a^2\mathcal{H}\partial_\chi V]\phi \\
&+ [3k\mathcal{H}\bar{\chi}'^2(4a^2V - \bar{\chi}'^2) + 2ka^2\bar{\chi}'(\bar{\chi}'^2 + 5a^2V)(\partial_\chi V) + 2\mathcal{H}(a^2\partial_\chi V)^2]\frac{\boldsymbol{\delta\chi}}{\bar{\chi}'^3} \\
&+ \frac{\mathbf{1}}{\bar{\chi}'}[k\bar{\chi}'(4a^2V - \bar{\chi}'^2) + 2a^2\mathcal{H}\partial_\chi V]\left(\frac{\boldsymbol{\delta\chi}'}{\bar{\chi}'}\right) \qquad (1.30) \\
-\frac{\mathcal{R}'}{\mathcal{H}} &= \phi - (3\bar{\chi}'\mathcal{H} + a^2\partial_\chi V)\frac{\boldsymbol{\delta\chi}}{\bar{\chi}'^2} - \left(\frac{\boldsymbol{\delta\chi}'}{\bar{\chi}'}\right) \qquad (1.31)
\end{aligned}}$$

Now, we are going to multiply the equation Eq.(1.31) the the orange coefficient show in Eq.(1.30):

$$
\begin{aligned}
-\frac{1}{\mathcal{H}\bar{\chi}'}[k\bar{\chi}'(4a^2V - \bar{\chi}'^2) + 2a^2\mathcal{H}\partial_\chi V]\mathcal{R}' = \frac{1}{\bar{\chi}'}[k\bar{\chi}'(4a^2V - \bar{\chi}'^2) + 2a^2\mathcal{H}\partial_\chi V]\phi \\
-[3k\mathcal{H}\bar{\chi}'^2(4a^2V - \bar{\chi}'^2) + 2ka^2\bar{\chi}'(\bar{\chi}'^2 + 5a^2V)(\partial_\chi V) + 2\mathcal{H}(a^2\partial_\chi V)^2]\frac{\delta\chi}{\bar{\chi}'^3} \\
-\frac{1}{\bar{\chi}'}[k\bar{\chi}'(4a^2V - \bar{\chi}'^2) + 2a^2\mathcal{H}\partial_\chi V]\left(\frac{\delta\chi'}{\bar{\chi}'}\right)
\end{aligned}
\tag{1.32}
$$

The negative of this equation is exactly what we have in Eq.(1.30). Therefore we can add Eq.(1.30) to Eq.(1.32) in order to find the motion equation for $\mathcal{R}$:

$$
\boxed{\mathcal{R}'' + \zeta\mathcal{R}' - \nabla^2\mathcal{R} = 0}
\tag{1.33}
$$

where, we have defned the coeffitient ζ as:

$$
\zeta \equiv -\frac{1}{\mathcal{H}\bar{\chi}'}[k\bar{\chi}'(4a^2V - \bar{\chi}'^2) + 2a^2\mathcal{H}\partial_\chi V]
\tag{1.34}
$$

1.8 Parameter Z

Let us handle with the coeffitient ζ in order to put it in a more useful form. For this we are going to use the unperturbed Friedmann's equations and the unperturbed inflaton field equation:

$$
\begin{aligned}
2\mathcal{H}^2 &= k(\bar{\chi}'^2 + 2a^2V) && (1.35) \\
\mathcal{H}' &= k(a^2V - \bar{\chi}'^2) && (1.36) \\
a^2\partial_\chi V &= -(\bar{\chi}'' + 2\mathcal{H}\bar{\chi}') && (1.37)
\end{aligned}
$$

Working with Eq.(1.34):

$$
\begin{aligned}
\zeta &\equiv -\frac{1}{\mathcal{H}\bar{\chi}'}[k\bar{\chi}'(4a^2V - \bar{\chi}'^2) + 2a^2\mathcal{H}\partial_\chi V] \\
&= -\frac{1}{\mathcal{H}\bar{\chi}'}[3ka^2V\bar{\chi}' + k(a^2V - \bar{\chi}'^2)\bar{\chi}' + 2a^2\mathcal{H}\partial_\chi V] \\
&= -\frac{1}{\mathcal{H}\bar{\chi}'}[(2\mathcal{H}^2 + \mathcal{H}')\bar{\chi}' + \mathcal{H}'\bar{\chi}' + 2a^2\mathcal{H}\partial_\chi V] \\
&= -\frac{2}{\mathcal{H}\bar{\chi}'}[(\mathcal{H}^2 + \mathcal{H}')\bar{\chi}' + a^2\mathcal{H}\partial_\chi V] \\
&= -\frac{2}{\mathcal{H}\bar{\chi}'}[(\mathcal{H}^2 + \mathcal{H}')\bar{\chi}' - \mathcal{H}(\bar{\chi}'' + 2\mathcal{H}\bar{\chi}')] \\
&= -\frac{2}{\mathcal{H}\bar{\chi}'}[(\mathcal{H}' - \mathcal{H}^2)\bar{\chi}' - \mathcal{H}\bar{\chi}''] \\
&= \frac{2}{\bar{\chi}'}\left[\bar{\chi}'' + \mathcal{H}\bar{\chi}'\left(\frac{\mathcal{H}}{\mathcal{H}} - \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\right] \\
&= \frac{2}{\bar{\chi}'}\left[\bar{\chi}'' + \frac{\mathcal{H}\bar{\chi}'}{a}\frac{d}{d\eta}\left(\frac{a}{\mathcal{H}}\right)\right] \\
&= \frac{2\mathcal{H}}{a\bar{\chi}'}\frac{d}{d\eta}\left(\frac{a\bar{\chi}'}{\mathcal{H}}\right) = \boldsymbol{\frac{2Z'}{Z}}
\end{aligned} \tag{1.38}
$$

where we have defined Z as:

$$
Z \equiv \frac{a\bar{\chi}'}{\mathcal{H}} \tag{1.39}
$$

Finally we find the motion equation of $\mathcal{R}$ (in Fourrier space) using the parameter Z:

$$
\boxed{\tilde{\mathcal{R}}'_\kappa + 2\frac{Z'}{Z}\tilde{\mathcal{R}}'_\kappa - \kappa^2\tilde{R}_\kappa = 0} \tag{1.40}
$$