Worksheet 9 - Solutions

1. Fermion Gas Mixture

As the gases are non interacting, let us consider a single gas

$$Z = Z_{tr} \times Z_s$$

Z: full partition function

Z_{tr}: transport

Z_s: spin.

$$Z_s = \left(e^{-\beta \gamma B} + e^{\beta \gamma B} \right)^N$$

$$= 2^N \cosh^N (\beta \gamma B)$$

And

$$\mu = -\frac{k_B T}{N} \ln \left[\frac{1}{N!} \left(\frac{V \cosh(\beta \gamma B)}{\lambda_T} \right)^N \right]$$

$\frac{1}{N!}$: Stirling

$$\approx -\frac{k_B T}{\cancel{N}} \left[-\cancel{N} \ln(N) + \cancel{N} + \cancel{N} \ln \left(\frac{V \cosh(\beta \gamma B)}{\lambda_T} \right) \right]$$

$$= -k_B T \ln \left[\frac{V e \cosh(\beta \gamma B)}{N \lambda_T} \right]$$

So, equating the chemical potentials for red and green

$$\mu_{red} = \mu_{green}$$

$$\cancel{-k_B T \ln} \left[\frac{\cancel{V} e \cosh(\beta \gamma_r B)}{N_r \cancel{\lambda_T}} \right] = \cancel{-k_B T \ln} \left[\frac{\cancel{V} e \cosh(\beta \gamma_g B)}{N_g \cancel{\lambda_T}} \right]$$

$$\frac{\cosh(\beta \gamma_r B)}{N_r} = \frac{\cosh(\beta \gamma_g B)}{N_g}$$

$$\frac{N_r}{N_g} = \frac{\cosh(\beta \gamma_r B)}{\cosh(\beta \gamma_g B)}$$

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2 Molecular Oxigen.

1)

$$Z = \sum_{\mu_s} e^{-\beta \mathcal{H}(\mu_s)} = \sum_{\mu_s} e^{-\beta (p_i^2/2m - \mu B S_i^2)}$$

$$= \frac{1}{N!} \left(\frac{V}{\lambda^3} \left(e^{\beta \mu B} + 1 + e^{-\beta \mu B} \right) \right)^N$$

where

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$$

2) the probabilities go

$$P(S=-1) = \frac{e^{-\beta \mu B}}{2\cosh(\beta \mu B) + 1}$$

$$P(S=0) = \frac{1}{2\cosh(\beta \mu B) + 1}$$

$$P(S=1) = \frac{e^{\beta\mu B}}{2\cosh(\beta\mu B)+1}$$

3) The average magnetic moment;

$$\langle M\rangle = \frac{1}{\beta}\frac{\partial \ln Z}{\partial B} = \frac{1}{\beta}\frac{\partial\left(N\ln(2\cosh\beta\mu B+1)\right)}{\partial B}$$

$$= \frac{1}{\cancel{\beta}}N\frac{1}{2\cosh\beta\mu B+1}\times 2\sinh(\beta\mu B)\times\mu\cancel{\beta}$$

$$= \mu N\frac{2\sinh(\beta\mu B)}{2\cosh(\beta\mu B)+1}$$

d) And finally, the susceptibility $\chi = \left.\frac{\partial\langle M\rangle}{\partial B}\right|_{B=0}$

$$\chi = \mu N\left[\frac{2\cosh(\beta\mu B)\beta\mu}{2\cosh(\beta\mu B)+1} - \frac{2\times\sinh(\beta\mu B)\times(2\sin(\beta\mu B)\beta\mu)}{(2\cosh(\beta\mu B)+1)^2}\right]$$

$$= N\beta\mu^2\left.\left[\frac{2\cancel{\cosh(\beta\mu B)}^{1}(2\cancel{\cosh(\beta\mu B)}^{1}+1)-\cancel{[2\sinh(\beta\mu B)]^2}^{0}}{(2\cancel{\cosh(\beta\mu B)}^{1}+1)^2}\right]\right|_{B=0}$$

$$= N\beta\mu^2\left[\frac{2\times\cancel{(2+1)}}{(2+1)^{\cancel{2}}}\right] = \frac{2}{3}N\beta\mu^2$$

3. Polar Rods.

1) The rotational degrees of freedom goes as

$$Z_{rot} = \frac{1}{h^2} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dp_\theta \, dp_\phi \exp\left[-\frac{\beta}{2I} \left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta} + \beta\mu\cos\theta \right) \right]$$

the momenta are Gaussian integrals, and taking $x = \cos(\theta)$

$$Z_{rot} = \left(\frac{2\pi I}{\beta h^2} \right) 2\pi \int_{-1}^{+1} dx \, e^{\beta\mu E x}$$

$$= \left(\frac{8\pi^2 I}{\beta h^2} \right) \frac{\sinh(\beta\mu E)}{\beta\mu E}$$

2) the mean polarization.

$$P = \langle \mu\cos\theta \rangle = \frac{\partial \ln(Z_{rot})}{\partial(\beta E)} = \mu \left[\coth(\beta\mu E) - \frac{1}{\beta\mu E} \right]$$

3) the zero field polarizability

$$\chi_T = \left. \frac{\partial P}{\partial E} \right|_{E=0}$$

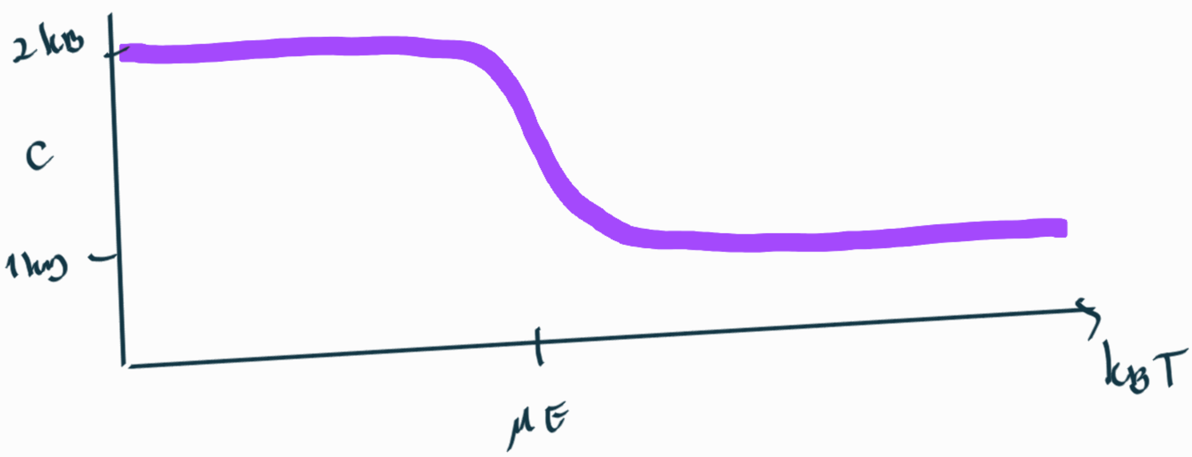
$$\chi_T = \frac{\partial P}{\partial E} = \beta\mu^2 \underbrace{\left[-\frac{1}{\sinh^2(\beta\mu E)} + \frac{1}{(\beta\mu E)^2} \right]}_{\left. \left[\frac{1}{x^2} - \frac{1}{\sinh^2(x)} \right] \right|_{x=0} = \frac{1}{3}}$$

$$\chi_T = \frac{1}{3} \mu^3 \beta$$

4) The energy stored:

$$\langle E_{rot} \rangle = -\frac{\partial \ln Z_{rot}}{\partial \beta} = 2 k_B T - \mu E \coth(\beta \mu E)$$

5)



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4 Rotation of diatomic molecules.

a) Partition function

$j(j+1)\epsilon_0 = \epsilon(j)$

$(2j+1)$ = multiplicity.

$$Z = \sum_{j=0}^{\infty} (2j+1) e^{-j(j+1)\epsilon_0/\tau}$$

$$= \sum_{j=0} \frac{d}{dj}\left[e^{-(j^2+j)\epsilon_0/\tau}\right]\left(\frac{-\tau}{\epsilon_0}\right)$$

$$= \frac{-\tau}{\epsilon_0} \sum_{j=0} \frac{d}{dj}\left(e^{-\epsilon_0/\tau}\right)^{j^2+j}$$

2) for $1 \gg \frac{\epsilon_0}{\tau}$

$$Z_R(\tau) = -\frac{\tau}{\epsilon_0}\int_0^{\infty} \frac{d}{dx}\left(e^{-\epsilon_0/\tau}\right)^{x^2+x} = -\frac{\tau}{\epsilon_0}\left(\left(e^{-\epsilon_0/\tau}\right)^{x^2+x} - \left(e^{-\epsilon_0/\tau}\right)^0\right)$$

$$\approx \frac{\tau}{\epsilon_0}$$

3) for $\frac{\tau}{\epsilon_0} \ll 1 \rightarrow \frac{\epsilon_0}{\tau} \gg 1$

$$Z = (2\times 0+1)\, e^{-0(0+1)\epsilon_0/\tau}$$

$$+ (2\times 1+1)\, e^{-1(1+1)\epsilon_0/\tau}$$

$$+ \mathcal{O}(e^{-3})$$

$$= 1 + e^{-2\epsilon_0/\tau}$$

4) The internal energy

$$U = \tau^2 \frac{\partial \ln Z}{\partial \tau}$$

$1 \gg \frac{\epsilon_0}{\tau}$ $\qquad U = \tau^2 \partial_\tau \left(\ln \frac{\tau}{\epsilon_0} \right) = \tau$

$1 \ll \frac{\epsilon_0}{\tau}$

$$U = \tau^2 \left(\frac{1}{1+3e^{-2\epsilon_0/\tau}} \right) \times (3e^{-2\epsilon_0/\tau}) \left(\frac{2\epsilon_0}{\tau^2} \right)$$

$$= \frac{6\epsilon_0 e^{-2\epsilon_0/\tau}}{1+3e^{-2\epsilon_0/\tau}}$$

$$C_V = \left(\frac{\partial u}{\partial \tau}\right)_V \longrightarrow 1 \gg \frac{\epsilon_0}{\tau} \qquad C_V = 1$$

$$\frac{\tau}{\epsilon_0} \ll 1 \qquad C_V = 6\,\epsilon_0 \left(\frac{e^{-2\epsilon_0/\tau}\left(\frac{2\epsilon_0}{\tau^2}\right)\left(1+3e^{-2\epsilon_0/\tau}\right) - \left(3e^{-2\epsilon_0/\tau}\right)\left(\frac{2\epsilon_0}{\tau^2}\right)e^{-2\frac{\epsilon_0}{\tau}}}{\left(1+3e^{-2\epsilon_0/\tau}\right)^2} \right)$$

$$= 12\,\epsilon_0^2\,\frac{e^{-2\epsilon_0/\tau}}{\tau^2} \times \frac{1}{\left(1+3e^{-2\epsilon_0/\tau}\right)^2}$$

$$\approx 12 \left(\frac{e^{-2\epsilon_0/\tau}}{(\tau/\epsilon_0)^2} \right)$$

5 zipper problem.

Consider N links. $\epsilon_s = 0$ are closed and ϵ open.

$$Z = \sum_{s=0}^{N} \exp(-s\epsilon/\tau) \qquad x = \epsilon/\tau$$

$$= e^{-0x} + e^{-x} + \cdots\cdots + e^{-xN}$$

and take

$$e^{-\epsilon/\tau} Z = \sum_{s=0}^{N} \exp(-s\epsilon/\tau) \times e^{-\epsilon/\tau}$$

$$= \sum_{s=0}^{N} \exp(-(s+1)\epsilon/\tau)$$

$$Z - e^{-\epsilon/\tau} Z = (1 + \cancel{e^{-x}} + \cancel{\cdots\cdots} + \cancel{e^{-Nx}}) - (\cancel{e^{-x}} + \cancel{\cdots\cdots} + \cancel{e^{-Nx}} + e^{-(N+1)x})$$

$$= 1 - e^{-(N+1)x}$$

$$Z = \frac{1 - e^{-(N+1)\epsilon/\tau}}{1 - e^{-\epsilon/\tau}}$$

2)

$$\langle s \rangle = \frac{-1}{\epsilon} \frac{\sum (-s\epsilon)\, e^{-s\epsilon\beta}}{z} = -\frac{1}{\epsilon} \partial_\beta \ln(z) = \frac{-1}{\epsilon} \frac{\partial_\beta z}{z}$$

let us compute this derivative.

$$Z = \frac{1 - e^{-(N+1)\epsilon/\tau}}{1 - e^{-\epsilon/\tau}}$$

$$= \frac{e^{\epsilon\beta} - e^{-N\beta\epsilon}}{e^{\epsilon\beta} - 1}$$

$$\partial_\beta Z = \frac{e^{\epsilon\beta}\epsilon - e^{-N\beta\epsilon}(-N\beta\epsilon)}{(e^{\epsilon\beta}-1)} - \frac{(e^{\epsilon\beta} - e^{-N\beta\epsilon})}{(e^{\epsilon\beta}-1)^2}(e^{\epsilon\beta})\epsilon$$

$$= \epsilon\,\frac{(e^{\epsilon\beta}+Ne^{-N\beta\epsilon})(e^{\epsilon\beta}-1) - e^{\epsilon\beta}(e^{\epsilon\beta} - e^{-N\beta\epsilon})}{(e^{\epsilon\beta}-1)^2}$$

$$= \epsilon\,\frac{e^{\epsilon\beta}(\cancel{e^{\epsilon\beta}}+Ne^{-N\beta\epsilon} - \cancel{e^{\epsilon\beta}} + e^{-N\beta\epsilon}) - e^{\epsilon\beta} - Ne^{-N\beta\epsilon}}{(e^{\epsilon\beta}-1)^2}$$

$$= \epsilon\,\frac{e^{\epsilon\beta}(N+1)e^{-N\beta\epsilon} - e^{\epsilon\beta} - Ne^{-N\beta\epsilon}}{(e^{\epsilon\beta}-1)^2}$$

$$\frac{\partial_\beta Z}{Z} = \epsilon\,\frac{e^{\epsilon\beta}(N+1)e^{-N\beta\epsilon} - e^{\epsilon\beta} - Ne^{-N\beta\epsilon}}{\underbrace{(e^{\epsilon\beta}-1)^{\cancel{2}} \times (e^{\epsilon\beta} - e^{-N\epsilon\beta})}_{(e^{\epsilon\beta}-1)(e^{\epsilon\beta} - e^{-N\epsilon\beta})}} \times \cancel{(e^{\epsilon\beta}-1)}$$

$$(e^{\epsilon\beta}-1)(e^{\epsilon\beta} - e^{-N\epsilon\beta})$$

$$e^{2\epsilon\beta} - e^{\epsilon\beta} - e^{-(N-1)\epsilon\beta} + e^{-N\epsilon\beta} \approx e^{2\epsilon\beta} + O(e^{\epsilon\beta})$$

$$\frac{\partial_\beta Z}{Z} = \epsilon\,\frac{(N+1)e^{-(N-1)\beta\epsilon} - e^{\epsilon\beta} - Ne^{-N\beta\epsilon}}{e^{2\epsilon\beta}}$$

$$\approx \epsilon\,(-e^{-\epsilon\beta})$$

$$\langle S \rangle = e^{-\epsilon/\tau}$$