Statistical Mechanics

Worksheet 12

June 6th, 2023

1 Pauli Paramagnetism

Calculate the contribution of electron spin to its magnetic susceptibility as follows. Consider non-interacting electrons, each subject to a Hamiltonian

$$\mathcal{H}_1 = \frac{\vec{p}^2}{2m} - \mu_0 \vec{\sigma} \cdot \vec{B} \tag{1}$$

where $\mu_0 = e\hbar/2mc$ and the eigenvalues of $\vec{\sigma} \cdot \vec{B}$ is $\pm B$ (The orbital effect, $\vec{p} \to \vec{p} - e\vec{A}$ has been ignored.)

1. Calculate the grand potential $\mathcal{G}_- = -k_B T \ln(\mathcal{G}_-)$ at a chemical potential μ.
2. Calculate the densities $n_+ = N_+/V$, and $n_- = N_-/V$, of electrons pointing parallel and anti-parallel to the field.
3. Obtain the expression for the magnetization $M = \mu_0(N_+ - N_-)$, and expand the result for small B.
4. Sketch the zero-field susceptibility $\chi(T) = \partial M/\partial B|_{B=0}$, and indicate its behavior at low and high temperatures.
5. Estimate the magnitude of χ/N for a typical metal at room temperature.

2 Boson Magnetism

Consider a gas of non-interacting spin 1 bosons, each subject to a Hamiltonian

$$\mathcal{H}_1 = \frac{\vec{p}^2}{2m} - \mu_0 s_z B, \tag{2}$$

where $\mu_0 = e\hbar/2mc$ and s_z takes *three* possible values of $(-1, 0, +1)$. (The orbital effect, $\vec{p} \to \vec{p} - e\vec{A}$ has been ignored.)

1. In a grand canonical ensemble of chemical potential μ, what are the average occupation numbers $\left\{ \left\langle n_+(\vec{k}) \right\rangle, \left\langle n_0(\vec{k}) \right\rangle, \left\langle n_-(\vec{k}) \right\rangle \right\}$ of one-particle states of wavenumber $\vec{k} = \vec{p}/\hbar$?

2. Calculate the average total numbers $\{N+, N_0, N_-\}$ of bosons with the three possible values of s_z in terms of the functions $f_m^+(z)$.

3. Write down the expression for the magnetization $M(T, \mu) = \mu_0(N_+ - N_-)$, and by expanding the result for small B find the *zero-field susceptibility* $\chi(T\mu) = \partial M/\partial B|_{B=0}$.

   To find the behavior of $\chi(T\mu)$, where $n = N/V$ is the total density, proceed as follows:

4. For $B = 0$, find the high-temperature expansion for $z(\beta, n) = e^{\beta\mu}$, correct to second order in n. Hence obtain the first correction from quantum statistics to $\chi(T\mu)$ at high temperatures.

5. Find the temperature $T_c(n, B = 0)$ of Bose-Einstein condensation. What happens to $\chi(T\mu)$ on approaching $T_c(n)$ from the high-temperature side?

6. What is the chemical potential μ for $T < T_c(n)$, at a small but finite value of B? Which one-particle state has a macroscopic occupation number?

7. Using the result in (f), find the spontaneous magnetization

$$\overline{M}(T, n) = \lim_{B \to 0} M(T, n, B) \tag{3}$$

3 Dirac Fermions

Are non-interacting particles of spin 1/2. The one-particle states come in pairs of positive and negative energies,

$$\mathcal{E}_\pm(\vec{k}) = \pm\sqrt{m^2c^4 + \hbar^2k^2c^2} \tag{4}$$

independent of spin.

1. For any fermionic system of chemical potential μ, show that the probability of finding an occupied state of energy $\mu + \delta$ is the same as that of finding an unoccupied state of energy $\mu + \delta$. (δ is any constant energy.)

2. At zero temperature all negative energy Dirac states are occupied and all positive energy ones are empty, that is, $\mu(T = 0) = 0$. Using the result in (1) find the chemical potential at finite temperature T.

3. Show that the mean excitation energy of this system at finite temperature satisfies

$$E(T) - E(0) = 4V \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{\mathcal{E}_+(\vec{k})}{\exp\left(\beta\mathcal{E}_+(\vec{k})\right) + 1} \tag{5}$$

4. Evaluate the integral in part (3) for massless *Dirac particles* (i.e., for $m = 0$).

5. Calculate the heat capacity, C_V, of such massless *Dirac particles*.

6. Describe the qualitative dependence of the heat capacity at low temperature if the particles are massive.