Worksheet 13 - Solution

1 Bose Condensation in d-dimension

Starting with

$$Q = \prod_i \frac{1}{1-e^{\beta(\mu-\epsilon_i)}}$$

$$\Big|\ \text{taking} \quad \int V d^d k / (2\pi)^d = \frac{V S_d}{(2\pi)^d} \int k^{d-1} dk = \sum_i$$

$$\ln(Q) = \frac{V S_d}{(2\pi)^d} \int k^{d-1} \ln\left(1 - z e^{-\beta \hbar^2 k^2 / 2m}\right) dk$$

with $S_d = 2\pi^{d/2} / (d/2 - 1)!$

taking the change of variable $x = \beta\hbar^2 k^2 / 2m \longrightarrow k = \sqrt{\frac{2mx}{\beta}} / \hbar$

$$dk = \sqrt{\frac{2m}{\beta x}} / 2\hbar$$

$$\ln(Q) = -\frac{V S_d}{(2\pi)^d} \frac{1}{2} \left(\frac{2m}{\hbar^2 \beta}\right)^{d/2} \int x^{\frac{d}{2}-1} \ln\left(1 - z e^{-x}\right) dx.$$

$$\Big|\ \text{Parts integration}$$

$$= \frac{V S_d}{(2\pi)^d} \frac{1}{d} \left(\frac{2m}{\hbar^2 \beta}\right)^{d/2} \int x^{d/2} \frac{z e^{-x}}{1 - z e^{-x}} dx = \frac{V S_d}{d} \left(\frac{2m}{h^2 \beta}\right)^{d/2} \underbrace{\int dx \frac{x^{d/2}}{z^{-1} e^{x} - 1}}_{\Gamma(\frac{d}{2}+1) f_{\frac{d}{2}+1}(z)},$$

Note h not $\hbar$

Thus

$$G = -k_B T \ln(Q) = -V \frac{S_d}{d} \left(\frac{2m}{h^2 \beta}\right)^{d/2} k_B T\, \Gamma\left(\frac{d}{2}+1\right) f^+_{\frac{d}{2}+1}(z)$$

and like $x\Gamma(x) = \Gamma(x+1)$

$$G = -\frac{V}{\lambda^d} k_B T\, f^+_{\frac{d}{2}+1}(z).$$

The average number of particle

$$N = \frac{\partial}{\partial(\beta\mu)} \ln(Q) = V \frac{S_d}{d}\left(\frac{2m}{h^2\beta}\right)^{d/2} \int x^{d/2-1} \frac{z e^{-x}}{1 - z e^{-x}} dx$$

$$= V \frac{S_d}{2}\left(\frac{2m}{h^2\beta}\right)^{d/2} \Gamma\left(\frac{d}{2}\right) f^{+}_{\frac{d}{2}}(z) = \frac{V}{\lambda^d} f^{+}_{\frac{d}{2}}(z)$$

So

$$n = \frac{1}{\lambda^d} f^{+}_{\frac{d}{2}}(z)$$

b) We have that $PV = -G$, and

$$E = -\frac{\partial}{\partial\beta} \ln(Q) = \frac{d}{2} \frac{\ln(Q)}{\beta} = -\frac{d}{2} G$$

$PV = \frac{2}{d} E \rightarrow$ Classical value.

c) The Critical temperature comes from

$$n = \frac{1}{\lambda^d} f^{+}_{\frac{d}{2}}(1) = \frac{1}{\lambda^d} \zeta_{\frac{d}{2}}$$

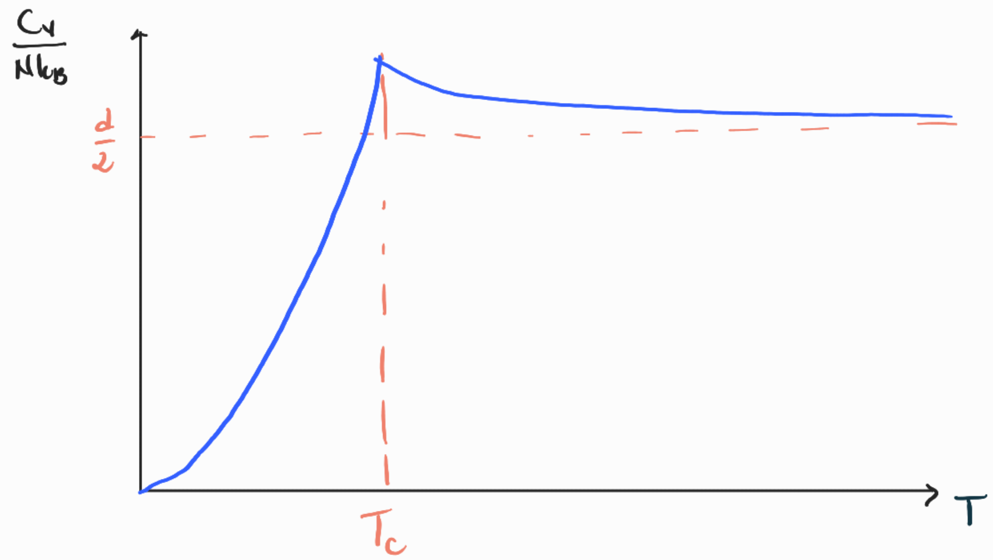
$$\downarrow$$

$$T_c = \frac{h^2}{2m k_B} \left(\frac{n}{\zeta_{\frac{d}{2}}}\right)^{2/d}$$

d) At $T < T_c$, $\mu = 0 \Rightarrow z = 1$

$$c(T) = \left.\frac{\partial E}{\partial T}\right|_{z=1} = -\frac{d}{2} \left.\frac{\partial G}{\partial T}\right|_{z=1} = -\frac{d}{2}\left(\frac{d}{2}+1\right)\frac{G}{T} = \frac{d}{2}\left(\frac{d}{2}+1\right)\frac{V}{\lambda^d} k_B \zeta_{\frac{d}{2}+1}$$

e) The heat capacity goes as



f) The maximum of $C(T)$ happens at the critical temperature.

$$C_{max} = C(T_c) = \frac{d}{2}\left(\frac{d}{2}+1\right)\frac{V}{\left(\zeta_{\frac{d}{2}}/n\right)} k_B f^{+}_{\frac{d}{2}+1}(1)$$

$$= \frac{d}{2} N k_B \left(\frac{d}{2}+1\right)\frac{\zeta_{\frac{d}{2}+1}}{\zeta_{\frac{d}{2}}}$$

Thus

$$\frac{C_{max}}{C(T\to\infty)} = \left(\frac{d}{2}+1\right)\frac{\zeta_{\frac{d}{2}+1}}{\zeta_{\frac{d}{2}}}$$

The particular value for $d=3$ is $1{,}283$.

g) for $d<3$ we have that $f^{+}_{m}(x\to 1)\to\infty$, then no condensation is possible.

Neutron Star Core

given the Fermi-Dirac Statistics

$$p[n(\epsilon)] = \frac{e^{\beta(\mu-\epsilon)n}}{1+e^{\beta(\mu-\epsilon)}} \quad \text{for} \quad n=0,1$$

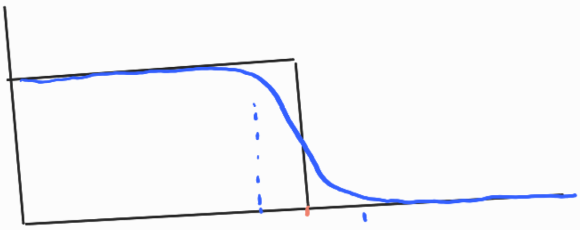
just two possible cases
Fermions !!

a shell of energies $\mu \pm \delta$ goes.

$$p[n(\mu+\delta)] = \frac{e^{\beta\delta n}}{1+e^{\beta\delta}} \quad \rightarrow p[n(\mu+\delta)=1] = \frac{e^{\beta\delta}}{1+e^{\beta\delta}} = \frac{1}{1+e^{-\beta\delta}}$$

$$p[n(\mu-\delta)] = \frac{e^{-\beta\delta n}}{1+e^{-\beta\delta}} \quad \rightarrow p[n(\mu-\delta)=0] = \frac{1}{1+e^{-\beta\delta}} = p[n(\mu+\delta)=1]$$

There is a symmetry on the occupation of states



b) we have that the excitation energy is

$$E(T)-E(0) = \sum_{k,s} \left[\langle n_+(k)\rangle \epsilon_+(k) - (1-\langle n_-(k)\rangle)\epsilon_-(k)\right]$$

Basically we take the energy of those electrons leaving the ground state and we add the energy of the correspondent excited state

$$= g\sum_k 2\langle n_+(k)\rangle \epsilon_+(k) = 2gV\int \frac{d^3\bar{k}}{(2\pi)^3} \frac{\epsilon_+(\bar{k})}{\exp(\beta\epsilon_+(\bar{k}))+1}$$

taking $k = k_F$ and setting $d^3k \approx 4\pi k_F\, dq$ with $q = |\vec{k}| - k_F$.

This comes as the highest contrib. is $|\vec{k}| = k_F$.

$$E(T) - E(0) \approx 2gV \frac{\cancel{4\pi} k_F^2}{\cancel{8}\pi^{\cancel{3}2}} \cdot \cancel{2} \int_0^\infty dq \frac{\mathcal{E}_+(q)}{\exp(\beta\mathcal{E}_+(q))+1} = \frac{2gV\, k_F^2}{\pi^2}\int_0^\infty dq \frac{\mathcal{E}_+(q)}{\exp(\beta\mathcal{E}_+(q))+1}$$

c) for $\mathcal{E}_+(q) = \hbar^2 q^2/2M$ and taking $\frac{\beta\hbar^2 q^2}{2M} = x$

$$E(T) - E(0) = \frac{gV\, k_F^2}{\pi^2} k_B T \left(\frac{2M k_B T}{\hbar^2}\right) \underbrace{\int_0^\infty dx \frac{x^{1/2}}{e^x+1}}$$

$$= \frac{gV\, k_F^2}{\pi^2} k_B T \left(\frac{2M k_B T}{\hbar^2}\right) \frac{\sqrt{\pi}}{2}\left(1 - \frac{1}{\sqrt{2}}\right)\zeta_{3/2}$$

$$= \left(1 + \frac{1}{\sqrt{2}}\right)\frac{\zeta_{3/2}}{\pi}\frac{V\, k_F^2}{\lambda_T} k_B T. \quad \text{with } \lambda_T = \frac{h}{\sqrt{2Mk_BT\pi}}$$

d) The heat capacity

$$C_V = \left.\frac{\partial E}{\partial T}\right|_V = \frac{3\,\zeta_{3/2}}{2\pi}\left(1 - \frac{1}{\sqrt{2}}\right)\frac{V\, k_F^2}{\lambda} k_B = \frac{3}{2}\frac{E}{T} \propto \sqrt{T}$$

Relativistic Bose gas in d-dimensions.

Starting with

$$Q = \prod_i \frac{1}{1-e^{\beta(\mu-\epsilon_i)}}$$

$\Big\downarrow$ taking $\displaystyle\int V d^d k/(2\pi)^d = \frac{V S_d}{(2\pi)^d}\int k^{d-1}\,dk = \sum_i$

$$\ln(Q) = \frac{V S_d}{(2\pi)^d}\int k^{d-1}\ln\left(1 - z\,e^{-\beta\hbar c k}\right)dk$$

with $S_d = 2\pi^{d/2}/(d/2-1)!$

taking the change of variable $x = \beta\hbar c k$

$$\ln(Q) = -\frac{V S_d}{(2\pi)^d}\left(\frac{k_B T}{\hbar c}\right)^d \int_0^\infty x^{d-1}\ln\left(1 - z e^{-x}\right)dx.$$

$\Big\downarrow$ Parts integration

$$= \frac{V S_d}{(2\pi)^d}\frac{1}{d}\left(\frac{k_B T}{\hbar c}\right)^d \int x^d \frac{z e^{-x}}{1-z e^{-x}}dx = \frac{V S_d}{d}\left(\frac{k_B T}{h c}\right)^d \underbrace{\int dx \frac{x^d}{z^{-1}e^x - 1}}_{d!\ \ f_{d+1}(z)},$$

Note h not $\hbar$

Thus

$$G = -k_B T\ln(Q) = -V\frac{S_d}{d}\left(\frac{k_B T}{h c}\right)^d k_B T\, d!\, f^+_{d+1}(z)$$

result that can be written as;

$$G = -\frac{k_B T\, V}{\lambda_c^d}\frac{\pi^{d/2}\, d!}{(d/2)!} f^+_{d+1}(z)$$

where $\lambda_c = hc/k_B T$

The average number of particles.

$$N = -\frac{\partial G}{\partial \mu} = -\beta z\frac{\partial G}{\partial z}$$

$$N = \frac{V}{\lambda_c^d} \frac{\pi^{d/2} d!}{(d/2)!} f_d^+(z) \longrightarrow n = \frac{1}{\lambda_c^d} \frac{\pi^{d/2} d!}{(d/2)!} f_d^+(z)$$

b) Taking $PV = -G$

$$E = -\left.\frac{\partial \ln Q}{\partial \beta}\right|_z = \frac{d \ln Q}{\beta} = -dG$$

so $E/PV = d \Rightarrow$ the same value as the classical relativistic gas

c) The critical temperature is

$$n = \frac{1}{\lambda_c^d} \frac{\pi^{d/2} d!}{(d/2)!} f_d^+(z=1) = \frac{1}{\lambda_c^d} \frac{\pi^{d/2} d!}{(d/2)!} \zeta_d$$

This leads to

$$T_c = \frac{hc}{k_B} \left(\frac{n (d/2)!}{\pi^{d/2} d! \zeta_d} \right)^{1/d}$$

as ζ_d is finite for $d > 1$, the transition only exists for $d > 1$.

d) At $T < T_c$, $z = 1$

$$C(T) = \left.\frac{\partial E}{\partial T}\right|_{z=1} = (d+1) \frac{E}{T} = -d(d+1) \frac{G}{T} = d(d+1) \frac{V}{\lambda_c^d} k_B \frac{\pi^{d/2} d!}{(d/2)!} \zeta_{d+1}$$

So;

$$\frac{C(T_c)}{N k_B} = \frac{d(d+1) \zeta_{d+1}}{\zeta_d}.$$

Graphene

given the Fermi-Dirac Statistics

$$p[n(\epsilon)] = \frac{e^{\beta(\mu-\epsilon)n}}{1+e^{\beta(\mu-\epsilon)}} \quad \text{for} \quad n = 0, 1$$

just two possible cases
Fermions !!

a shell of energies $\mu \pm \delta$ goes.

$$p[n(\mu+\delta)] = \frac{e^{\beta\delta n}}{1+e^{\beta\delta}} \quad \rightarrow p[n(\mu+\delta)=1] = \frac{e^{\beta\delta}}{1+e^{\beta\delta}} = \frac{1}{1+e^{-\beta\delta}}$$

$$p[n(\mu-\delta)] = \frac{e^{-\beta\delta n}}{1+e^{-\beta\delta}} \quad \rightarrow p[n(\mu-\delta)=0] = \frac{1}{1+e^{-\beta\delta}} = p[n(\mu+\delta)=1]$$

b) we have that the excitation energy is

$$E(T)-E(0) = \sum_{k,s}\left[\langle n_+(k)\rangle \mathcal{E}_+(k) - (1-\langle n_-(k)\rangle)\mathcal{E}_-(k)\right]$$

$$= 2\sum_{k} 2\langle n_+(k)\rangle \mathcal{E}_+(k) = 4A\int \frac{d^2\vec{k}}{(2\pi)^2}\frac{\mathcal{E}_+(\vec{k})}{\exp(\beta\mathcal{E}_+(\vec{k}))+1}$$

c) for $\mathcal{E}_+(k) = \hbar\nu|k|$

$$E(T)-E(0) = 4A\int_0^\infty \frac{2\pi k\,dk}{4\pi^2}\frac{\hbar\nu k}{e^{\beta\hbar\nu k}+1} \quad \rightarrow \text{taking } \beta\hbar c k = x$$

$$= \frac{2A}{\pi} k_B T\left(\frac{k_B T}{\hbar\nu}\right)^2 \underbrace{\int_0^\infty dx\,\frac{x^2}{e^x+1}}_{2!\, f_3^-(1) = 2!\,\frac{3\zeta_3}{4}}$$

$$= \frac{3\zeta_3}{\pi} A\, k_B T\left(\frac{k_B T}{\hbar\nu}\right)^2$$

The heat capacity

$$C_V = \frac{\partial E}{\partial T}\bigg|_V = \frac{9\zeta_3}{\pi} A k_B \left(\frac{k_B T}{\hbar \omega}\right)^2$$

The contribution of phonons is actually negligible as the typical velocities for the Dirac fermions is way larger