Worksheet 5. Solutions

1) Two Containers and a Spring.

in equilibrium, let us consider the distance x as the displacement of the piston.

$$F = kx = \underbrace{P_1}_{\text{Pressure of the gas.}} \, S \longrightarrow \text{Transversal Section.}$$

$$P_1 = \frac{kx}{S}$$

The pressure of the gas, goes as

$$P_1 \underbrace{\left[V_A + xS \right]}_{\text{Total volume.}} = nRT_0 \qquad ; \qquad V_A = Sh.$$

$$P_1 = \frac{nRT_0}{Sh + Sx}$$

Then, we can solve for x taking;

$$\frac{nRT_0}{h+x} = kx \qquad \text{Or} \qquad \underbrace{kx^2 + khx - nRT_0 = 0}_{\text{Cuadratic equation.}}$$

$$x = \frac{-h \pm \sqrt{h^2 + 4\frac{nRT_0}{k}}}{2}$$

b) The work done by the gas during the irreversible isotherm transformation is equal to the variation of potential energy of the spring

$$W = \frac{1}{2} k x^2$$

c) The change of the entropy of the gas after the isothermal transformation goes as

$$\Delta S_{gas} = n R \ln\left(\frac{V_{final}}{V_{initial}}\right)$$

$$= n R \ln\left[\frac{V_A + x S}{V_A}\right]$$

$$= n R \ln\left[1 + \frac{x}{h}\right]$$

Then, the change of the entropy of the reservoir that at constant temperature, gives an amount of heat $|Q| = W$, goes as.

$$\Delta S_r = \frac{W}{T_0} = \frac{1}{2} \frac{k x^2}{T_0}$$

So, the change of entropy of the universe, goes as

$$\Delta S_{universe} = \Delta S_r + \Delta S_{gas}$$

$$= n R \ln\left[1 + \frac{x}{h}\right] + \frac{k x^2}{2 T_0}$$

2 Two Gases with Changing Volume

In equilibrium, we have

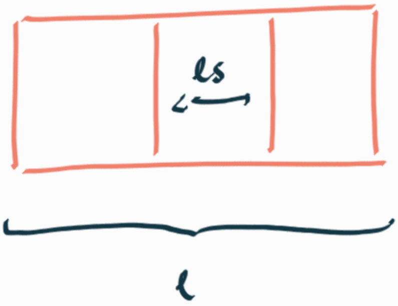
$$p_0 = \frac{\text{Force}}{\text{Area}} = \frac{k\,\boxed{\Delta l_s}}{S} \longrightarrow \Delta l_s \equiv \text{Change of length of the spring}$$

$$= \frac{k}{S}\left(|l_0 - l_s|\right)$$

and

$$V_0 = \frac{S\,(l - l_s)}{\boxed{2}}$$

Two compartments.



$$T_0 = \frac{p_0 V_0}{nR} = \frac{p_0 V_0}{R} = \frac{p_0 \cancel{S}(l - l_s)}{2R}\;\frac{k\,(|l_0 - l_s|)}{\cancel{S}}$$

Now, we have T_0, p_0 and V_0 from the given quantities.

2) After heating, the new pressure is

$$p = \frac{k\,\Delta l_s'}{S} = \frac{k\,(|l_0 - l_s'|)}{S}.$$

So that, the volume, is computed using the equation for the reversible adiabatic;

$$p_0 V_0^{\gamma} = p V_B^{\gamma} \longrightarrow V_\gamma = \left(\frac{p_0}{p}\right)^{1/\gamma} V_0.$$

and as a consequence

$$V_A = l\,S - V_B - l_s' S.$$

And from the equation of state

$$T_A = \frac{p V_A}{R}; \qquad T_B = \frac{p V_B}{R}$$

and the work

$$W_{spring} = \frac{1}{2} k (\Delta l_s')^2 - \frac{1}{2} k (\Delta l_s)^2$$

then

$$Q = \Delta U + W_{spring}$$

$$\hookrightarrow \Delta U = \Delta U_A + \Delta U_B = C_v (T_A - T_0) + C_v (T_B - T_0)$$

3. Cycles with ideal Gases

As the internal energy is an equation of state (and an exact differential), given any kind of process (Even the irreversible ones), its variation depends only on the initial and final states. Therefore.

$$\Delta U_{BC} = \underbrace{n C_v (T_C - T_B) = n C_v (T_C - T_A)}_{\text{AB is an isotherm.}}$$

$$= \frac{5}{2} n R C_v (T_C - T_A)$$

in the same sense

$$\Delta U_{cycle} = 0.$$

b) The work done during the adiabatic BC and DA is the same but with oposite signs. This, using the fact that both transformations are done between the same temperatures.

$$W_{Tot} = W_{AB} + W_{BC} + W_{CD} + W_{DA}$$

$$= W_{AB} + \cancel{W_{BC}} + W_{CD} - \cancel{W_{BC}}$$

$$= W_{AB} + W_{CD}.$$

$$= RT_A \ln\left(\frac{P_A}{P_B}\right) + RT_C \ln\left(\frac{V_D}{V_C}\right)$$

$$W_{AB} = nRT_A \ln\left(\frac{V_B}{V_A}\right)$$

$$\frac{V_A}{V_B} = \frac{\cancel{nRT_A}}{P_A} \frac{P_B}{\cancel{nRT_B}}$$

as $T_A = T_B$

$$W_{AB} = nRT_A \ln\left(\frac{P_A}{P_B}\right)$$

and as, during the reversible adiabatic holds that;

$$T_A V_A^{\gamma-1} = T_D V_D^{\gamma-1} \quad ; \qquad V_A \left(\frac{T_A}{T_D}\right)^{1/\gamma-1} = V_D.$$

$$T_D = T_C \ , \quad V_A = \frac{R T_A}{p_A} \quad \gamma = 1{,}4$$

$$W_{tot} = R\, T_A \ln\left(\frac{p_A}{p_B}\right) + R\, T_C \ln\left(\frac{V_A}{V_C} \left(\frac{T_A}{T_D}\right)^{1/0{,}4}\right)$$

c) The efficiency

$$\eta = \frac{W}{Q_{AB}} = \frac{R\, T_A \ln\left(\frac{p_A}{p_B}\right) + R\, T_C \ln\left(\frac{V_A}{V_C} \left(\frac{T_A}{T_D}\right)^{1/0{,}4}\right)}{R\, T_A \ln\left(\frac{p_A}{p_B}\right)}$$

d) The Carnot efficiency

$$\eta_c = 1 - \frac{T_{cold}}{T_{hot}} = 1 - \frac{T_C}{T_A}$$

e) the work done by the Carnot cycle with the same isotherm AB

$$\eta_c = \frac{W_c}{Q_{AB}} \longrightarrow W_c = \eta_c\, Q_{AB}$$

$$W_c = \left[1 - \frac{T_C}{T_A}\right] R\, T_A \ln\left(\frac{p_A}{p_B}\right)$$

Considering the correspondent reversible adiabatic expansion of the Carnot cycle BC' goes to a Volume $V_{C'}$, so

$$T_B V_B^{\gamma-1} = T_C V_{C'}^{\gamma-1}$$

$$T_A V_B^{\gamma-1} = T_C V_{C'}^{\gamma-1}$$

$$\longrightarrow V_B = \frac{RT_A}{P_B} \; ; \; V_B^{\gamma-1} = \left(\frac{RT_A}{P_B}\right)^{\gamma-1}$$

The Volume $V_{C'}$ gets

$$V_{C'}^{\gamma-1} = \frac{T_A}{T_C}\left(\frac{RT_A}{P_B}\right)^{\gamma-1} \longrightarrow V_{C'} = \left(\frac{T_A}{T_C}\right)^{1/\gamma-1} \frac{RT_A}{P_B}.$$

$V_{C'}$ is such that $\Delta S_{BC'} = 0$ compared to the other case in which $\Delta S_{BC} > 0$ as is nonreversible

f) The change of entropy over the irreversible adiabatic BC

$$\Delta S_{BC} = C_V \ln\left(\frac{T_C V_C^{\gamma-1}}{T_B V_B^{\gamma-1}}\right) > 0.$$

and during the whole cycle $\Delta S_{tot} = 0$ as it is an equation of state.

4. Perverse initial conditions.

The idea of ergodicity requires that almost every point in phase space mixes with the others under dynamics.

in the case of the initial conditions here described, the trajectory on phase space (P.S) reduces to a lower dimensional surface compared with the whole P.S, therefore it has measure zero.

Indeed, it is an unstable equilibrium state, any small perturbation would make the system mix with the other regular states.

5 Counting

1) Three people refuse ride facing backwards, then, they must be placed looking forwards.

$$\# \text{ Must Forward} = \frac{(\text{Seats forwards})!}{(\text{Seats for.} - \text{People forward})!} = \frac{9!}{(9-3)!}$$

$$= \frac{9 \cdot 8 \cdot 7 \cdot 6!}{6!} = 9 \cdot 8 \cdot 7$$

similarly

$$\# \text{ Must backwards} = 8 \cdot 7.$$

Then $17-5 = 12$ seats are left. and we have $7-3-2=2$ people, hence

$$(9 \cdot 8 \cdot 7) \times (8 \cdot 7) \times (12 \cdot 11) = 3725568.$$

are the number of ways the passengers can be placed

2) We have 8 people. so, in the room A can be 3,4 or 5 people

$$\overset{\text{3 people}}{\frac{8!}{(8-3)!}} + \overset{\text{4 people}}{\frac{8!}{(8-4)!}} + \overset{\text{5 people.}}{\frac{8!}{(8-5)!}}$$

$$= 8 \cdot 7 \cdot 6 + 8 \cdot 7 \cdot 6 \cdot 5 + 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 = 8736.$$

4.

3. let us start with the red balls at the end.

we have $5+4+4-2 = 11$ balls before the end.

we must compute the permutations

then we have

$$\underbrace{\frac{11!}{3!\ \underbrace{4!\ \ 4!}_{\text{blue and white bolls.}}}}_{}= 11550 \text{ ways.}$$

$$3! \rightarrow \text{red bolls}$$

and the case of non red ones at the end.

$$\frac{11!}{5!\cdot 4!\cdot 2!} = 6930 \text{ ways}$$

Hence the total number of way s.

$$\underset{\text{red}}{11550} + \underset{\text{white and blue}}{2 \times 6930} = \boxed{25410 \text{ ways.}}$$

4) Since the number must be even, the last digit must be a 6.
there are 4 of them. 5 numbers are left.

$$5! = \#\ \text{permutations}$$
$$3! = \#\ \text{permutations of } 6's$$
$$2! = \#\ \text{permutations of } 5's.$$

then the combinatios we want are

$$4\left(\frac{5!}{3!\ 2!}\right) = \overset{2}{\cancel{4}} \cdot \frac{5\cdot 4\cdot \cancel{3!}}{\cancel{3!} \times \cancel{2} \times 1} = 40.$$

having 2 5's together results on the same number of permuting 4 units

$$4\cdot \frac{4!}{3!} = 4\cdot 4 = 16$$

Thus, the probability of two 5's together and the number being even.

$$P = 16/40 = 2/5 = 0{,}4.$$