Statistical Mechanics

Worksheet 13

July 13th, 2023

1 Bose condensation in d-dimensions

Consider a gas of non-interacting (spinless) bosons with an energy spectrum $\epsilon = p^2/2m$, contained in a box of "volume" $V = L^d$ in d-dimensions.

1. Calculate the grand potential $\mathcal{G} = -k_B T \ln \mathcal{Q}$, and the density $n = N/V$, at a chemical potential μ. Express your answers in terms of d, and $f_m^+(z)$, where $z = e^{\beta\mu}$, and

$$f_m^+(z) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{x^{m-1}}{z^{-1}e^x - 1} dx \tag{1}$$

   Hint Use integration by parts on the expression for $\ln \mathcal{Q}$ just as in Worksheet 11.

2. Calculate the ratio PV/E, and compare it to the classical value.
3. Find the critical temperature, $T_c(n)$, for Bose-Einstein condensation.
4. Calculate the heat capacity $C(T)$ for $T < T_c(n)$.
5. Sketch the heat capacity at all temperatures.
6. Find the ratio, $C_{\max}/C(T \to \infty)$, of the maximum heat capacity to its classical limit, and evaluate it in $d = 3$.
7. How does the above calculated ratio behave as $d \to 2$? In what dimensions are your results valid? Explain.

2 Neutron star core

Professor Rajagopal's group at MIT has proposed that a new phase of QCD matter may exist in the core of neutron stars. This phase can be viewed as a condensate of quarks in which the low energy excitations are approximately

$$\mathcal{E}\left(\vec{k}\right)_\pm = \pm\hbar^2 \frac{\left(\left|\vec{k}\right| - k_F\right)^2}{2M} \tag{2}$$

The excitations are fermionic, with a degeneracy of $g = 2$ from spin.

1. At zero temperature all negative energy states are occupied and all positive energy ones are empty, i.e. $\mu(T=0)=0$. By relating occupation numbers of states of energies $\mu+\delta$ and $\mu-\delta$, or otherwise, find the chemical potential at finite temperatures T.

2. Assuming a constant density of states near $k=k_F$, i.e. setting $d^3k \approx 4\pi k_F^2 dq$ with $q = \left|\vec{k}\right| - kF$, show that the mean excitation energy of this system at finite temperature is

$$E(T) - E(0) \approx 2gV\frac{k_F^2}{\pi^2}\int_0^\infty dq \frac{\mathcal{E}_+(q)}{\exp(\beta\mathcal{E}_+(q))+1}. \tag{3}$$

3. Give a closed form answer for the excitation energy by evaluating the above integral.

4. Calculate the heat capacity, C_V, of this system, and comment on its behavior at low temperature.

3 Relativistic bose gas in d-dimensions

Consider a gas of non-interacting (spinless) bosons with an energy $\epsilon = c\left|\vec{p}\right|$, contained in a box of "volume" $V = L^d$ in d-dimensions.

1. Calculate the grand potential $\mathcal{G} = -k_BT\ln\mathcal{Q}$, and the density $n = N/V$, at a chemical potential μ. Express your answers in terms of d, and $f_m^+(z)$, where $z = e^{\beta\mu}$, and

$$f_m^+(z) = \frac{1}{(m-1)!}\int_0^\infty \frac{x^{m-1}}{z^{-1}e^x - 1}dx \tag{4}$$

***H*int** Use integration by parts on the expression for $\ln\mathcal{Q}$ just as in Worksheet 11.

2. Calculate the pressure P, its energy E, and compare the ratio $E/(PV)$ to the classical value.

3. Find the critical temperature, $T_c(n)$, for Bose-Einstein condensation, indicating the dimensions where there is a transition.

4. What is the temperature dependence of the heat capacity $C(T)$ for $T < T_c(n)$?

5. Evaluate the dimensionless heat capacity $C(T)/(Nk_B)$ at the critical temperature $T = T_c$, and compare its value to the classical (high temperature) limit.

4 Graphene

Graphene is a single sheet of carbon atoms bonded into a two dimensional hexagonal lattice. It can be obtained by exfoliation (repeated peeling) of graphite. The band structure of graphene is such that the single particles excitations behave as relativistic Dirac fermions, with a spectrum that at low energies can be approximated by

$$\mathcal{E}_\pm\left(\vec{k}\right) = \pm\hbar v\left|\vec{k}\right| \tag{5}$$

There is spin degeneracy of $g=2$, and $v \approx 10^6 ms^{-1}$. Recent experiments on unusual transport properties of graphene were reported in *Nature* **438**, 197 (2005). In this problem, you shall calculate the heat capacity of this material.

1. If at zero temperature all negative energy states are occupied and all positive energy ones are empty, find the chemical potential $\mu(T)$.

2. Show that the mean excitation energy of this system at finite temperature satisfies

$$E(T) - E(0) = 4A \int_0^\infty \frac{d^2\vec{k}}{(2\pi)^2} \frac{\mathcal{E}_+\left(\vec{k}\right)}{\exp\left(\beta\mathcal{E}_+\left(\vec{k}\right)\right) + 1}. \tag{6}$$

3. Give a closed form answer for the excitation energy by evaluating the above integral.

4. Calculate the heat capacity, C_V , of such massless Dirac particles.

5. Explain qualitatively the contribution of phonons (lattice vibrations) to the heat capacity of graphene. The typical sound velocity in graphite is of the order of 2×10^4 ms^{-1} . Is the low temperature heat capacity of graphene controlled by phonon or electron contributions?