Worksheet 12 - Solutions

1) Pauli Paramagnetism.

a) The energy of the electron gas, is given by

$$E = \sum_p E_p(n_p^+, n_p^-),$$

where $n_p^{\pm} = 0$ or 1, denote the number of particles having $\pm$ spins and momentum p

$$E_p(n_p^+, n_p^-) \equiv \left(\frac{p^2}{2m} - \mu_0 B\right) n_p^+ + \left(\frac{p^2}{2m} + \mu_0 B\right) n_p^-$$

$$= (n_p^+ + n_p^-)\frac{p^2}{2m} - (n_p^+ - n_p^-)\mu_0 B.$$

The grand partition function

$$Q = \sum_{N=0}^{\infty} \exp(-\beta\mu N) \times \sum_{\{n_p^+, n_p^-\}}^{N = \sum(n_p^+ + n_p^-)} \exp\left(-\beta \sum_p E_p(n_p^+, n_p^-)\right)$$

$$= \sum_{\{n_p^+, n_p^-\}} \exp\left(\sum_p \beta\mu(n_p^+ + n_p^-) - \beta E_p(n_p^+, n_p^-)\right)$$

$$= \prod_p \sum_{\{n_p^+, n_p^-\}} \exp\left(\beta\mu(n_p^+ + n_p^-) - \beta E_p(n_p^+, n_p^-)\right)$$

$$= \prod_p \sum_{\{n_p^+, n_p^-\}} \exp\left\{\beta\left[\left(\mu - \mu_0 B - \frac{p^2}{2m}\right)n_p^+ + \left(\mu + \mu_0 B - \frac{p^2}{2m}\right)n_p^-\right]\right\}$$

$$= \prod_p \underbrace{\left\{1 + \exp\left[\beta\left(\mu - \mu_0 B - \frac{p^2}{2m}\right)\right]\right\}}_{Q_0(\mu - \mu_0 B)} \times \underbrace{\left\{1 + \exp\left[\beta\left(\mu + \mu_0 B - \frac{p^2}{2m}\right)\right]\right\}}_{Q_0(\mu + \mu_0 B)}$$

$$\boxed{Q_0(\mu) = \prod_p \left\{1 + \exp\left(\mu - \frac{p^2}{2m}\right)\right\}}$$

Then

$$\ln(Q) = \ln\left\{Q_0(\mu + \mu_0 B)\right\} + \ln\left\{Q_0(\mu - \mu_0 B)\right\}$$

each one of the terms contribute

$$\ln Q(\mu) = \sum_p \ln\left(1 + \exp\left[\beta\left(\mu + p^2/2\right)\right]\right)$$

$$\longrightarrow \sum_p = \frac{V}{(2\pi\hbar)^3}\int d^3\vec{p}$$

$$= \frac{V}{(2\pi\hbar)^3}\int d^3p \ln\left(1 + z e^{-\beta\frac{p^2}{2m}}\right) \qquad \text{where} \quad z = e^{\beta\mu}$$

$$= \frac{V}{h^3}\frac{4\pi m}{\beta}\sqrt{\frac{2m}{\beta}}\int dx \sqrt{x}\ln\left(1 + z e^{-x}\right)$$

after integration by parts

$$\ln Q_0(\mu) = V\left(\frac{2\pi m k_B T}{h^2}\right)^{3/2}\frac{2}{\sqrt{\pi}}\,\frac{2}{3}\int dx \frac{x^{3/2}}{z^{-1}e^x + 1} = \frac{V}{\lambda^3} f^-_{5/2}(z)$$

hence

$$\ln(Q(\mu)) = \frac{V}{\lambda^3}\left[f^-_{5/3}\left(z e^{\beta\mu_0 B}\right) + f^-_{5/3}\left(z e^{-\beta\mu_0 B}\right)\right]$$

So that, the total grand free energy

$$G = -k_B T\ln(Q(\mu)) = -k_B T\frac{V}{\lambda^3}\left[f^-_{5/3}\left(z e^{\beta\mu_0 B}\right) + f^-_{5/3}\left(z e^{-\beta\mu_0 B}\right)\right]$$

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b) The number densities of electrons with up or down spins

$$\frac{N_\pm}{V} = z\,\partial_z \ln(Q_\pm) = \frac{1}{\lambda^3} f^-_{3/2}\left(z e^{\pm\beta\mu_0 B}\right)$$

where, we used

$$z\,\partial_z f^-_n(z) = f^-_{n-1}(z)$$

then the total number

$$N = N_+ + N_- = \frac{V}{\lambda^3}\left[f^-_{3/2}\left(z\, e^{\beta\mu_0 B}\right) + f^-_{3/2}\left(z\, e^{-\beta\mu_0 B}\right)\right]$$

c) the magnetization

$$M = \mu_0 (N_+ - N_-) = \frac{V}{\lambda^3}\left[f^-_{3/2}\left(z\, e^{\beta\mu_0 B}\right) - f^-_{3/2}\left(z\, e^{-\beta\mu_0 B}\right)\right]$$

in the case of small B, $e^{\pm\beta\mu_0 B}$ can be expanded $\approx 1 \pm \beta\mu_0 B$

$$f^-_{3/2}\left(z e^{\pm\beta\mu_0 B}\right) \approx f^-_{3/2}\left(z(1 \pm \beta\mu_0 B)\right) \approx f^-_{3/2}(z) \pm z \cdot \beta\mu_0 B\, \partial_z f^-_{3/2}(z)$$

$$M = \mu_0 \frac{V}{\lambda^3}(2\beta\mu_0 B) \cdot f^-_{1/2}(z) = \frac{2\mu_0^2}{k_B T}\frac{V}{\lambda^3} B\, f^-_{1/2}(z)$$

d) The Susceptibility is

$$\chi = \left.\frac{\partial M}{\partial B}\right|_{B=0} = \frac{2\mu_0^2}{k_B T}\frac{V}{\lambda^3} f^-_{1/2}(z)$$

And from the previous part $\longrightarrow$ $N = 2\frac{V}{\lambda^3} f^-_{3/2}(z)$

let us take two cases, T_{low} and T_{high}.

low T $(\ln(z) = \beta\mu \longrightarrow \infty)$

$$f^-_n(z) = \frac{1}{\Gamma(n)}\int_0^\infty dx \frac{x^{n-1}}{1+e^{x-\ln(x)}} \underset{T\to 0}{\approx} \frac{1}{\Gamma(n)}\int_0^{\ln(z)} x^{n-1} = \frac{\ln(z)^n}{n\Gamma(n)}$$

This limit makes

$$N_{T\to 0} = 2\frac{V}{\lambda^3}\frac{4(\ln z)^{3/2}}{3\sqrt{\pi}} \quad — \quad \ln z = \left(\frac{3 N_{T\to 0}\sqrt{\pi}}{8V}\lambda^3\right)^{2/3}$$

$$\chi_{T\to 0} = \frac{4\mu_0^2 V}{\sqrt{\pi}\, k_B T \lambda^3} \cdot \left(\frac{3N\sqrt{\pi}}{8V} \lambda^3 \right)^{1/3}$$

$$= \frac{2\mu_0^2 V}{k_B T \lambda^3} \left(\frac{3N}{\pi V} \right)^{1/3} = \frac{4\pi m \mu_0^2 V}{h^2} \left(\frac{3N}{\pi V} \right)^{1/3}$$

So, the ratio

$$\left.\frac{\chi}{N}\right|_{T\to 0} = \frac{\mu_0^2}{k_B T} \frac{f_{1/2}^-}{f_{3/2}^-} = \frac{3\mu_0}{2k_B T} \frac{1}{\ln(z)} = \frac{3\mu_0}{2k_B T} \frac{1}{\beta \varepsilon_F} = \frac{3\mu_0^3}{2k_B T_F}$$

in the high temperature limit

$$f_n(z) \xrightarrow[z\to 0]{} \frac{z}{\Gamma(n)} \int_0^\infty dx\, x^{n-1} e^{-x} = z$$

thus

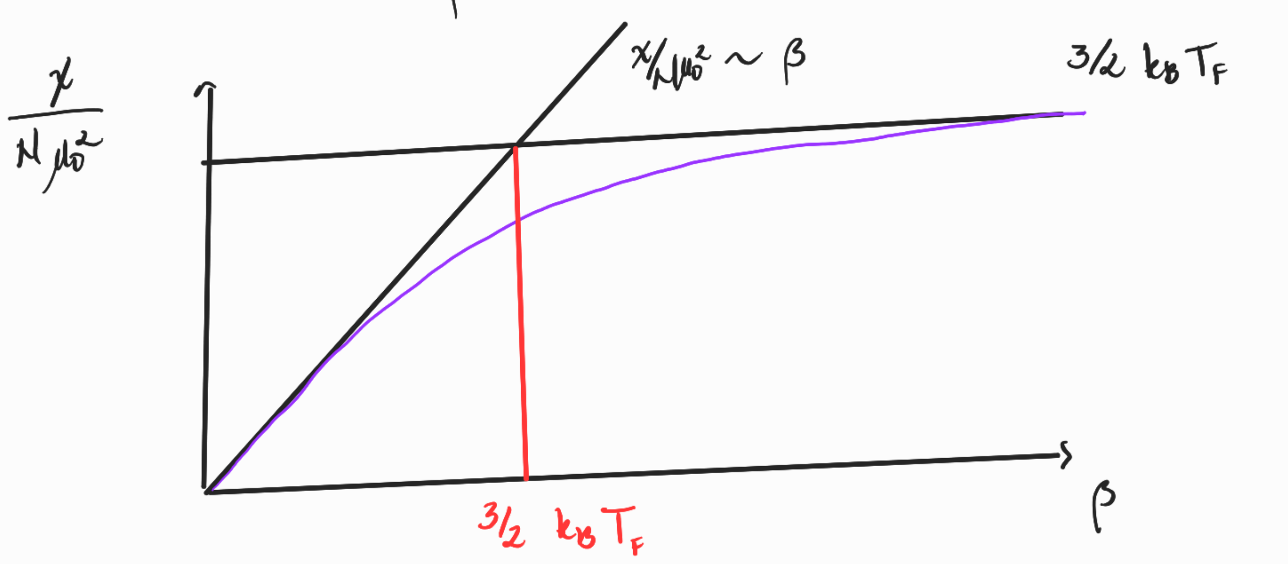
$$N \xrightarrow[\beta\to 0]{} \frac{2V}{\lambda^3} \times z \to \qquad z \simeq \frac{N}{2V} \cdot \lambda^3 = \frac{N}{2V} \left(\frac{\beta h^2}{2\pi m} \right)^{3/2} \longrightarrow 0$$

and

$$\chi \simeq \frac{2\mu_0^2 V}{k_B T \lambda^3} \times z = \frac{N\mu_0^2}{k_B T}$$

So

$$\left.\frac{\chi}{N}\right|_{\substack{T\to\infty \\ \beta\to 0}} = \frac{\mu_0^2}{k_B T} \longrightarrow \text{Curie Susceptibility}$$



e) At room temperature

$T_{room} << T_F \approx 10^4 k$ so we can use our lowT limit results.

we can use that

$$\mu_0 = \frac{e\hbar}{2mc} \approx 9.3 \times 10^{24} \, J/T$$

so

$$\frac{\chi}{N} = \frac{3\mu_0^2}{2k_B T_F} \approx \frac{3\times(9.3\times10^{-24})^2}{2\times(1.38\times10^{-23})} \approx 9.4 \times 10^{-24} \, J/T^2$$

Boson Magnetism.

1) The Bose-Einstein Distribution goes as

$$n_s(\vec{k}) = \frac{1}{e^{\beta[\mathcal{H}(s)-\mu]} - 1} \qquad \text{for} \quad S = -1, 0, 1.$$

$$n_s(\vec{k}) = \frac{1}{e^{\left[\beta\left(\frac{p^2}{2m} - \mu_0 s B\right) - \beta\mu\right]} - 1}$$

b) The total numbers of particles with spin s are given by

$$N_s = \sum_{\{\vec{k}\}} n_s(\vec{k}) \longrightarrow = \frac{V}{(2\pi)^3}\int d^3k \, \frac{1}{\exp\left[\beta\left(p^2/2m - \mu_0 s B\right) - \beta\mu\right] - 1}$$

taking the change of variables $k = x^{1/2}\sqrt{2mk_BT}/\hbar$, we can express the integral as

$$N_s = \frac{V}{\lambda^3} f^+_{3/2}\left(z\, e^{\beta\mu_0 s B}\right)$$

with

$$f^+_m = \frac{1}{\Gamma(m)}\int_0^\infty \frac{dx\, x^{m-1}}{z^{-1}e^x - 1} \quad , \quad \lambda \equiv \frac{h}{\sqrt{2\pi m k_B T}} \quad ; \quad z = e^{\beta\mu}$$

3) The Magnetization

$$M(T,\mu) = \mu_0 (N_+ - N_-)$$

$$= \mu_0 \frac{V}{\lambda^3} \left(f^+_{3/2}(z e^{\beta \mu_0 B}) + f^+_{3/2}(z e^{\beta S \mu_0 B}) \right)$$

So expanding for small B.

$$f^+_{3/2}(z e^{\beta \mu_0 S B}) \approx f^+_{3/2}(z[1 \pm \beta \mu_0 B]) \approx f^+_{3/2}(z) \pm z \cdot \beta \mu_0 B \frac{\partial}{\partial z} f^+_{3/2}(z)$$

and using that $\frac{z \, df^+_m(z)}{dz} = f^+_{m-1}(z)$ we obtain;

$$M = \mu_0 \frac{V}{\lambda^3} (2\beta \mu_0 B) \cdot f^+_{1/2}(z) = \frac{2\mu_0^2}{k_B T} \frac{V}{\lambda^3} \cdot B \cdot f^+_{1/2}(z)$$

and the susceptibility

$$\chi = \left.\frac{\partial M}{\partial B}\right|_{B=0} = \frac{2\mu_0^2}{k_B T} \frac{V}{\lambda^3} \cdot f^+_{1/2}(z)$$

To find $\chi(T,n)$

4) in the high temperature limit $z \to$ Small. so we can use a Taylor exponsion around 0

$$n(B=0) = \left.\frac{N_+ + N_0 + N_-}{V}\right|_{B=0} = \frac{3}{\lambda} f^+_{3/2}(z)$$

$$= \frac{3}{\lambda^3} \left(z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \cdots \right)$$

this expression can be inverted (look for Series Reversion)

$$z = \left(\frac{n\lambda^3}{3}\right) - \frac{1}{2^{3/2}} \left(\frac{n\lambda^3}{3}\right)^2 + \cdots$$

Then

$$\chi = \frac{2\mu_0}{k_B T} \frac{V}{\lambda^3} \cdot f^{+}_{1/2}(z)$$

$$\frac{\chi}{N} = \frac{2\mu_0^2}{k_B T} \cdot \frac{1}{n\lambda^3} \left(z + \frac{z^2}{2^{1/2}} + \cdots \right)$$

$$= \frac{2\mu_0^2}{3k_B T} \left[1 + \left(-\frac{1}{2^{3/2}} + \frac{1}{2^{1/2}} \right) \left(\frac{n\lambda^3}{3} \right) + \mathcal{O}(n^2) \right]$$

5) The condensation occurs at $z=1$

$$n = \frac{3}{\lambda^3} f^{+}_{1/2}(1) \longrightarrow \text{Density}$$

$$\downarrow$$

$$\zeta_{3/2} = f^{+}_{3/2}(1) \approx 2{,}61$$

$$T_c(n) = \frac{h^2}{2\pi m k_B} \left(\frac{n}{3\,\zeta_{3/2}} \right)^{2/3}$$

and Since $\lim_{z \to 1} f^{+}_{1/2}(z) = \infty \longrightarrow \chi(T,n)$ diverges at $T_c(n)$

6) The Chemical potential

$$n_s(\vec{k}, B) = \left[z^{-1} e^{\beta \varepsilon_s(\vec{k}, B)} - 1 \right]^{-1} > 0 \text{ for all } \vec{k}, S_z$$

Then for $T < T_c$, $B < \infty$ $\quad z\, e^{\beta\mu_0 B} = 1 \longrightarrow \mu = -\mu_0 B$

macroscopically occupied state is $\vec{k} = 0$ and $s = \pm 1$.

7) The Contribution of the excited states to M vanishes as $B \to 0$

Then, for $T < T_c$, the total magnetization is given by the macroscopic occupation $(k=0,\ S_z = \pm 1)$, so

$$\bar{M}(T,n) = \mu_0 V n_+(k=0) = \mu_0 V (n - n_{excited}) = \mu_0 \left(N - \frac{3V}{\lambda^3} \zeta_{3/2} \right)$$

Dirac Fermions

1) The probability of occupation of a state of energy $\mathcal{E}$ is

Fermi-Dirac : $$p[n(\mathcal{E})] = \frac{e^{\beta(\mu-\mathcal{E})n}}{1+e^{\beta(\mu-\mathcal{E})}}, \quad \text{for } n=0,1.$$

So for $\mathcal{E}=\mu+\delta$

$$p[n(\mu+\delta)] = \frac{e^{\beta\delta n}}{1+e^{\beta\delta}} \longrightarrow \text{occupied} \Rightarrow p(n(\mu+\delta)=1) = \frac{e^{\beta\delta}}{1+e^{\beta\delta}} = \frac{1}{1+e^{-\beta\delta}}$$

similarly, for $\mathcal{E}=\mu-\delta$

$$p[n(\mu+\delta)] = \frac{e^{-\beta\delta n}}{1+e^{-\beta\delta}} \to \text{Not occupied} \Rightarrow p(n(\mu-\delta)=0) = \frac{1}{1+e^{-\beta\delta}} = p(n(\mu+\delta)=1)$$

The prob. of finding the state with energy $\mu+\delta$ is the same as one non occupied with energy $\mu-\delta$

2)

That means that for $\mu=0$

$$\langle n(\mathcal{E})\rangle + \langle n(-\mathcal{E})\rangle \Rightarrow \text{is constant for all temperatures}$$

The Particle-hole symmetry enforces

$$\mu(T)=0$$

3)

$$E(T)-E(0) = \sum_{k,S_z}\left[\langle n_+(k)\rangle \mathcal{E}_+(k) - (1-\langle n_-(k)\rangle)\mathcal{E}_-(k)\right]$$

$$= 2\sum_{k} 2\langle n_+(k)\rangle \mathcal{E}_+(k) = 4V\int \frac{d^3\vec{k}}{(2\pi)^3}\,\frac{\mathcal{E}_+(\vec{k})}{\exp(\beta\mathcal{E}_+(\vec{k}))+1}$$

4) for $m=0 \to \mathcal{E}_+(k) = \hbar c|\vec{k}|$, so

$$E(T)-E(0) = 4V\int_0^V \frac{4\pi k^2}{8\pi^3}\,\frac{\hbar c k}{e^{\beta\hbar c k}+1}\,dk = \frac{2V}{\pi^2}k_BT\int_0^\infty dx\,\frac{x^3}{e^x+1} \quad \left(\text{with } \beta\hbar c k = x\right)$$

$$E(T)-E(0) = \frac{7\pi^2}{60} V k_B T \left(\frac{k_B T}{\hbar c}\right)^3$$

Noting that

$$\int_0^x dx \frac{x^3}{e^x+1} = 3!\, f_4^-(1) = 3!\, \frac{7\pi^4}{720}$$

5)

$$C_V = \left.\frac{\partial E}{\partial T}\right|_V = \frac{7\pi^2}{15} V k_B \left(\frac{k_B T}{\hbar c}\right)^3.$$

6) in the case of $m \neq 0$ there is an energy gap between occupied and empty states, we may expect the contribution contributing with an exponential

for low energy

$$\mathcal{E}_+(k) \approx mc^2 + \frac{\hbar^2 k^2}{2m} + \cdots\cdots$$

so

$$E(T)-E(0) \approx \frac{2V}{\pi^2} mc^2 e^{-\beta mc^2} \frac{4\pi\sqrt{\pi}}{\lambda^3} \int_0^\infty dx\, x^2 e^{-x} = \frac{48}{\sqrt{\pi}} \frac{V}{\lambda^3} mc^2 e^{-\beta mc^2}$$

so the heat capacity

$$C(T) \propto k_B \frac{V}{\lambda^3} (\beta mc^2)^2 e^{-\beta mc^2}$$