# Statistical Mechanics

Worksheet 8

June 8th, 2023

## 1 Relativistic particles

Consider $N$ indistinguishable relativistic particles move in one dimension subject to a Hamiltonian

$$\mathcal{H}(\{p_i, q_i\}) = \sum_{i=1}^{N} \left[c|p_i| + U(q_i)\right], \tag{1}$$

with $U(q_i) = 0$ for $0 \leq q_i \leq L$, and $U(q_i) = \infty$ otherwise. Consider a microcanonical ensemble of total energy $E$.

1. Compute the contribution of the coordinates $q_i$ to the available volume in phase space $\Omega(E, L, N)$.
2. Compute the contribution of the momenta $p_i$ to $\Omega(E, L, N)$.

   ***Hint*** The volume of the hyperpyramid defined by $\sum_{i=1}^{d} x_i \leq R$, and $x_i \geq 0$, in $d$ dimensions is $R^d/d!$.
3. Compute the entropy $S(E, L, N)$.
4. Calculate the one-dimensional pressure $P$.
5. Obtain the heat capacities $C_L$ and $C_P$.
6. What is the probability $p(p_1)$ of finding a particle with momentum $p_1$?

## 2 Molecular adsorption

$N$ diatomic molecules are stuck on a metal surface of square symmetry. Each molecule can either lie flat on the surface, in which case it must be aligned to one of two directions, $x$ and $y$, or it can stand up along the $z$ direction. There is an energy cost of $\epsilon > 0$ associated with a molecule standing up, and zero energy for molecules lying flat along $x$ or $y$ directions.

1. How many microstates have the smallest value of energy? What is the largest microstate energy?

2. For microcanonical macrostates of energy $E$, calculate the number of states $\Omega(E, N)$, and the entropy $S(E, N)$.

3. Calculate the heat capacity $C(T)$ and sketch it.

4. What is the probability that a specific molecule is standing up?

5. What is the largest possible value of the internal energy at any positive temperature?

# 3 Curie susceptibility

Consider $N$ non-interacting quantized spins in a magnetic field $\vec{B} = B\hat{z}$, and at a temperature $T$. The work done by the field is given by $BM_z$, with a magnetization $M_z = \mu \sum_{i=1}^{N} m_i$. For each spin, $m_i$ takes only the $2s+1$ values $-s,\ -s+1,\ \cdots,\ s-1,\ s$.

1. Calculate the Gibbs partition function $\mathcal{Z}(T, B)$. (Note that the ensemble corresponding to the macrostate $(T, B)$ includes magnetic work.)

2. Calculate the Gibbs free energy $G(T, B)$, and show that for small $B$,

$$G(B) = G(0) - \frac{N\mu^2 s(s+1)B^2}{6k_B T} + \mathcal{O}(B^4) \tag{2}$$

3. Calculate the zero field susceptibility $\chi = \partial M_z/\partial B|_{B=0}$ , and show that it satisfies *Curie's law*

$$\chi = c/T. \tag{3}$$

4. Show that $C_B - C_M = cB^2/T^2$ , where $C_B$ and $C_M$ are heat capacities at constant $B$ and $M$, respectively.

# 4 Two-dimensional Solid

Consider a solid surface to be a two-dimensional lattice with $N_s$ sites. Assume that $N_a$ atoms ($N_a \ll N_s$) are adsorbed on the surface, so that each site has either zero or one adsorbed atom. An adsorbed atom has energy $E = -\varepsilon$, where $\varepsilon > 0$. Assume that atoms on the surface do not interact with each other.

1. If the surface is at temperature T, compute chemical potential of the adsorbed atoms as a function of $T$, $\varepsilon$, and $N_a/N_s$ (use the canonical ensemble).

2. If the surface is in equilibrium with an ideal gas of similar atoms at temperature $T$, compute the ratio $N_a/N_s$ as a function of pressure, $P$ of the gas. Assume the gas has number density $n$.

   ***Hint*** Equate chemical potentials of the adsorbed atoms and the gas.

## 5 Mixture of indistinguishable Gases

An ideal gas is composed of $N$ *red* atoms of mass $m$, $N$ *blue* atoms of mass $m$, and $N$ *green* atoms of mass $m$. Atoms of the same color are indistinguishable. Atoms of different color are distinguishable.

1. Use the canonical ensemble to compute the entropy of this gas.

2. Compute the entropy of ideal gas of $3N$ *red* atoms of mass $m$. Does if differ from that of the mixture?. If so, by how much?.