Study sheet 2, Exercise 3

Johannes, Sebastian

11 November 2016

a

$$
\begin{aligned}
L(\beta_0, \beta_1; \mathbf{x}) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2\right) \\
&\propto \prod_{i=1}^{n} \exp\left(-\frac{1}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2\right) \\
l(\beta_0, \beta_1; \mathbf{x}) &= \sum_{i=1}^{n} \left(-\frac{1}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2\right) \\
&= -\frac{1}{2\sigma^2} \sum_{i=1}^{n} \left((y_i - \beta_0 - \beta_1 x_i)^2\right)
\end{aligned}
$$

Nun müssen wir die partielle Ableitung nach β_0 bilden, nullsetzten und nach β_0 auflösen.

$$
\begin{aligned}
\frac{\partial l(\beta_0, \beta_1; \mathbf{x})}{\partial \beta_0} &= \frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) \overset{!}{=} 0 \quad | \cdot \sigma^2 \\
\sum_{i=1}^{n} \beta_0 &= \sum_{i=1}^{n} (y_i - \beta_1 x_i) \\
n\beta_0 &= \sum_{i=1}^{n} (y_i - \beta_1 x_i) \\
\hat{\beta}_0 &= \bar{y} - \beta_1 \bar{x}
\end{aligned}
$$

Somit hab wir gezeigt, dass der ML-Schätzer für β_0 gleich dem OLS-Schätzer für β_0 ist. Als nächsten leiten wir $l(\beta_0, \beta_1; \mathbf{y})$ nach β_1 ab und lösen die Ableitung nach β_1 auf.

$$
\begin{aligned}
\frac{\partial l(\beta_0, \beta_1; \mathbf{x})}{\partial \beta_1} &= \sum_{i=1}^{n} -\frac{1}{2\sigma^2} 2(y_i - \beta_0 - \beta_1 x_i) x_i \\
&= \sum_{i=1}^{n} \frac{1}{2\sigma^2} 2(y_i x_i - \beta_0 x_i - \beta_1 x_i^2) \stackrel{!}{=} 0 \quad | \cdot \sigma^2 | : 2 \\
&= \sum_{i=1}^{n} (y_i x_i - \beta_0 x_i - \beta_1 x_i^2) \quad | \beta_0 \text{ einsetzen} \\
&= \sum_{i=1}^{n} (y_i x_i - (\bar{y} - \beta_1 \bar{x}) x_i - \beta_1 x_i^2) \\
&= \sum_{i=1}^{n} (y_i x_i - \bar{y} x_i + \beta_1 \bar{x} x_i - \beta_1 x_i^2) \\
&= \sum_{i=1}^{n} y_i x_i - \bar{y} \sum_{i=1}^{n} x_i + \beta_1 \sum_{i=1}^{n} \bar{x} x_i - \beta_1 \sum_{i=1}^{n} x_i^2 \\
-\beta_1 \sum_{i=1}^{n} \bar{x} x_i + \beta_1 \sum_{i=1}^{n} x_i^2 &= \sum_{i=1}^{n} y_i x_i - \bar{y} \sum_{i=1}^{n} x_i \\
\beta_1 \left(\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} \bar{x} x_i \right) &= \sum_{i=1}^{n} y_i x_i - \bar{y} \sum_{i=1}^{n} x_i \\
\beta_1 &= \frac{\sum_{i=1}^{n} y_i x_i - \bar{y} \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} \bar{x} x_i} \\
&= \frac{\sum_{i=1}^{n} y_i x_i - n \bar{y} \bar{x}}{\sum_{i=1}^{n} x_i^2 - n \bar{x}^2} \\
\beta_1 &= \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}
\end{aligned}
$$

Somit ist gezeigt, dass der ML-Schätzer für β_1 gleich dem OLS-Schätzer für β_1 ist.

b

$$
\begin{aligned}
l(\beta_0, \beta_1, \sigma^2) &\propto n \, log(\sqrt{\sigma^2}) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 \\
\frac{\partial l(\beta_0, \beta_1; \mathbf{x})}{\partial \sigma^2} &= -\frac{n}{\sigma} \frac{1}{2} (\sigma^2)^{-\frac{1}{2}} + \frac{1}{2} \frac{1}{\sigma^4} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 \\
&= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 \\
&= -\frac{1}{2\sigma^2} \left(n - \frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 \right) \stackrel{!}{=} 0 \\
&= n - \frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 \\
n &= \frac{1}{\sigma^2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 \\
\hat{\sigma}^2_{ML} &= \frac{1}{n} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2
\end{aligned}
$$

c

$$\begin{aligned}
\hat{\beta}_1 &= \frac{\sum_{i=1}^n (x_i - \bar{x}) y_i}{\sum_{i=1}^n (x_i - \bar{x})^2} \\
Var(\hat{\beta}_1) &= Var\left[\frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}\right] \\
&= Var\left[\frac{\sum_{i=1}^n y_i (x_i - \bar{x}) - \sum_{i=1}^n \bar{y}(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}\right] \\
&= Var\left[\frac{\sum_{i=1}^n (\beta_0 + \beta_1 x_i + e_i)(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}\right] \\
&= Var\left[\frac{\sum_{i=1}^n e_i (x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}\right] \\
&= \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{(\sum_{i=1}^n (x_i - \bar{x})^2)^2} Var(e_i) \\
&= \frac{1}{\sum_{i=1}^n (x_i - \bar{x})^2} \sigma^2 \\
&= \frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{x})^2}
\end{aligned}$$

d

$$\begin{aligned}
\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} \\
&= \sum_{i=1}^n \frac{1}{n} y_i - \bar{x} \frac{\sum_{i=1}^n (x_i - \bar{x}) y_i}{\sum_{i=1}^n (x_i - \bar{x})^2} \\
Var(\hat{\beta}_0) &= Var\left[\sum_{i=1}^n \left(\frac{1}{n} - \frac{\bar{x}(x_i - \bar{x})}{2(x_i - \bar{x})^2}\right) y_i\right] \\
&= \sum_{i=1}^n \left[\frac{1}{n} - \frac{\bar{x}(x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}\right]^2 Var(y_i) \\
&= \sum_{i=1}^n \left[\frac{1}{n^2} - \frac{2\bar{x}(x_i - \bar{x})}{n \sum_{i=1}^n (x_i - \bar{x})^2} + \frac{\bar{x}^2 (x_i - \bar{x})^2}{\left(\sum_{i=1}^n (x_i - \bar{x})^2\right)^2}\right] \sigma^2 \\
&= \sigma^2 \left[\frac{n}{n^2} - \frac{2\bar{x}(x_i - \bar{x})}{n \sum_{i=1}^n (x_i - \bar{x})^2} \sum_{i=1}^n (x_i - \bar{x}) + \frac{\bar{x}^2}{\left(\sum_{i=1}^n\right)^2} (x_i - \bar{x})^2\right] \\
&= \sigma^2 \left[\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2}\right]
\end{aligned}$$