LiMo WiSe 16/17 Sheet 4: Ex 4

Task:

Let $x = (x_1, \ldots, x_n)'$ be a random vector with $x_i \overset{iid}{\sim} N(\mu, \sigma^2)$ for $i = 1, \ldots, n$ and μ is known. The parameter σ^2 should be estimated. Look at the estimator

$$T = T(x) = \sum_{i=1}^{n} (x_i - \mu)^2$$

Is T sufficient for σ^2?

Solution:

Background

A statistic $T(x)$ is sufficient for a parameter θ if $f(x|T(x) = t, \theta) = f(x|T(x) = t)$. That means, $x|T(x) = t$ does not depend on θ. Or, all information is contained in the statistic. Equivalently we can use the factorization theorem:

$$f(x|\theta) = h(x) \cdot g(T(x)|\theta)$$

Note that,

- $h(x)$ does not depend on the parameter
- $g(T(x)|\theta)$ depends via the statistic on the parameter

Answer

$$\begin{aligned}
T(x) &= \sum_{i=1}^{n} (x_i - \mu)^2 \\
f(x|\sigma^2) &\overset{iid}{=} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(\frac{1}{2\sigma^2}(x_i - \mu)^2\right) \\
&= \underbrace{\sigma^{-n} \exp\left(\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right)}_{=g(T(x)|\sigma^2)} \underbrace{(2\pi)^{-\frac{n}{2}}}_{=h(x)}
\end{aligned}$$

According to the factorization theorem $T(x)$ is sufficient for σ^2.

More examples (not from the sheet)

Exponential distribution

Let $x = (x_1, \ldots, x_n) \overset{iid}{=} Exp(\lambda)$ and $T(x) = \sum_{i=1}^{n} x_i$. Is $T(x)$ sufficient for λ?

$$
\begin{aligned}
T(x) &= \sum_{i=1}^{n} x_i \\
f(x|\lambda) &\overset{iid}{=} \prod_{i=1}^{n} f(x_i|\lambda) \\
&= \underbrace{\lambda^n \exp(-\lambda \sum x_i)}_{=g(T(x)|\sigma^2)} \underbrace{(1)}_{=h(x)}
\end{aligned}
$$

$T(x)$ is sufficient.

Binomial distribution

Let $x_i \overset{iid}{\sim} Bin(1, p)$, $i = 1, \ldots, 4$; are $T(x) = \sum_{i=1}^{n} x_i$ and $\tilde{T}(x) = x_1$ sufficient on p?

For $T(x)$:

$$
\begin{aligned}
p((1,1,0,1)) &= pp(1-p)p \\
p((1,1,0,1)|T(x) = 3) &= \frac{1}{\binom{4}{3}} = \frac{1}{4}
\end{aligned}
$$

$x|T(x)$ does not depend on p and is hence sufficient.

For $\tilde{T}(x)$:

$$
\begin{aligned}
p((1,1,0,1)) &= pp(1-p)p \\
p((1,1,0,1)|\tilde{T}(x) = 1) &= p(1-p)p
\end{aligned}
$$

$x|\tilde{T}(x)$ depends on p and is not sufficient.