Study sheet 3, Exercise 2

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Consider a multiple linear regression model

$$y_i = \beta_0 + \beta_1 x_{i1} + ... + \beta_p x_{ip} + \varepsilon_i, i = 1, ..., n$$

$\varepsilon \sim \mathcal{N}_n(\mathbf{0}, \sigma^2 \mathbf{V})$, where $\varepsilon = (\varepsilon_1, ..., \varepsilon_n)^T, \mathbf{V} \neq \mathbf{I}_n$ is a known positive definite matrix and the design matrix $\mathbf{X}$ is of full column rank. Consider the ordinary least squares estimator $\hat{\boldsymbol{\beta}} = (\mathbf{X}\mathbf{X}^{'})^{-1}\mathbf{X}^T\mathbf{y}$ of the regression coefficients. Decide whether the following statements are true or false (in general, without any further assumptions imposed on $\mathbf{V}$ and $\mathbf{X}$):

a) $\hat{\boldsymbol{\beta}}$ is an unbiased estimator of $\boldsymbol{\beta}$. Book p. 168. *True.* $\hat{\boldsymbol{\beta}}$ is an unbiased for uncorrelated covariates (proof see p. 145) as well as for correlated covariates.

$$\begin{aligned} \mathrm{E}(y) &= \mathbf{X}\beta \\ \mathrm{E}(\beta) &= \mathrm{E}((\mathbf{X'X})^{-1}\mathbf{X'}y) \\ &= (\mathbf{X'X})^{-1}\mathbf{X'X}\beta \quad = \beta \end{aligned}$$

No variance needed.

b) $\mathrm{Cov}(\hat{\boldsymbol{\beta}}) = \sigma^2(\mathbf{X}\mathbf{X}^{'})^{-1}$. book p. 168. *False.* When a model with correlation ($\mathbf{V} \neq \mathbf{I}$) is fitted via OLS, the covariance looks as follows: $\mathrm{Var}(\hat{\boldsymbol{\beta}}) = \sigma^2(\mathbf{X'X})^{-1}\mathbf{X}^{'}\mathbf{V}\mathbf{X}(\mathbf{X}^{'}\mathbf{X})^{-\mathbf{1}}$. This is higher than the covariance, fitted via GLS

c) $\hat{\boldsymbol{\beta}}$ is normally distributed. *True.* $\beta = (\mathbf{X'X})^{-1}\mathbf{X'}y$ and $y \sim \mathcal{N}(\mathbf{X}\beta, \sigma^2\mathbf{V})$.

d) $\hat{\boldsymbol{\beta}}$ is the best linear unbiased estimator of $\boldsymbol{\beta}$. Book p. 165. *False.* The BLUE for correlated x_i is the generalized least squares estimator $\hat{\boldsymbol{\beta}}^* = (\mathbf{X}^{'}\mathbf{V}^{-\mathbf{1}}\mathbf{X})^{-1}\mathbf{X}^{'}\mathbf{V}^{-\mathbf{1}}\mathbf{y}$. Although the estimation of $\hat{\boldsymbol{\beta}}$ is unbiased, the covariance is overestimated. The OLS estimator is thus inefficient -> Not BLUE.