Warm-Up Exercises

1. Symmetries of 4-Wave Interaction Coefficient

Consider the quartic Hamiltonian $H_4 = \sum_{1234} W_{12}^{34}\, \delta_{12}^{34}\, a_1 a_2 a_3^* a_4^*$ ← $a_i = a(k_i)$

shorthand notation: $\sum_{k_1,k_2,k_3,k_4}$; $W(k_1,k_2;k_3,k_4)$; $\delta(k_3+k_4-k_1-k_2)$

Show that the 4-wave interaction coefficient has the following symmetries:

$$W_{12}^{34} \overset{①}{=} W_{12}^{43} \overset{②}{=} W_{21}^{34} \overset{③}{=} \left(W_{34}^{12}\right)^*$$

Solution ①
$$H_4 = \sum_{1234} W_{12}^{34}\, \delta_{12}^{34}\, a_1 a_2 a_3^* a_4^*$$
$$= \sum_{1243} W_{12}^{43}\, \delta_{12}^{43}\, a_1 a_2 a_4^* a_3^* \qquad \text{(exchange dummy indices } k_3 \leftrightarrow k_4\text{)}$$
$$= \sum_{1234} W_{12}^{43}\, \delta_{12}^{34}\, a_1 a_2 a_3^* a_4^* \qquad \text{(change order of summation, commutativity of multiplic'n and noting } \delta_{12}^{43} = \delta_{12}^{34}\text{)}$$

Equality holds no matter the values taken by the $a_i(t)$ ⇒ equal term by term. Comparing coefficients gives $W_{12}^{34} = W_{12}^{43}$.

② As above with $k_1 \leftrightarrow k_2$.

③ H_4 is an energy so must be real, i.e.

$$H_4 = H_4^*$$
$$\sum_{1234} W_{12}^{34}\, \delta_{12}^{34}\, a_1 a_2 a_3^* a_4^* = \sum_{1234} \left(W_{12}^{34}\right)^* \delta_{12}^{34}\, a_1^* a_2^* a_3 a_4$$
$$= \sum_{3412} \left(W_{34}^{12}\right)^* \delta_{34}^{12}\, a_3^* a_4^* a_1 a_2 \qquad (k_1,k_2 \leftrightarrow k_3,k_4)$$
$$= \sum_{1234} \left(W_{34}^{12}\right)^* \delta_{12}^{34}\, a_1 a_2 a_3^* a_4^*$$

(change order of summation, commutativity of multiplication, $\delta_{34}^{12} = \delta(k_1+k_2-k_3-k_4) = \begin{cases} 1 & k_1+k_2=k_3+k_4 \\ 0 & \text{otw} \end{cases} = \delta_{12}^{34}$.)

Equating coeffs ⇒ $W_{12}^{34} = \left(W_{34}^{12}\right)^*$.

2. Scaling of Dirac Delta Function

Show $\delta(Ax) = \frac{1}{|A|}\delta(x)$ (Dirac deltas)

Solution

$A>0$:
$$\int_{-\infty}^{\infty} dx\, f(x)\, \delta(Ax) \overset{x'=Ax}{=} \frac{1}{A}\int_{-\infty}^{\infty} dx'\, f\left(\tfrac{x'}{A}\right) \delta(x') = \frac{1}{A} f(0) = \int_{-\infty}^{\infty} dx\, f(x)\, \frac{\delta(x)}{A}$$

But $f(x)$ arbitrary so $\delta(Ax) = \delta(x)/A$.

$A<0$:
$$\int_{-\infty}^{\infty} dx\, f(x)\, \delta(Ax) \underset{x'=Ax}{=} \frac{1}{A}\int_{\infty}^{-\infty} dx'\, f\left(\tfrac{x'}{A}\right)\delta(x') = -\frac{1}{A}\int_{-\infty}^{\infty} dx'\, f\left(\tfrac{x'}{A}\right)\delta(x') = -\frac{1}{A} f(0)$$
$$= \int_{-\infty}^{\infty} dx\, f(x)\, \frac{\delta(x)}{|A|}$$

Again $f(x)$ arbitrary so $\delta(Ax) = \delta(x)/|A|$.

Both cases covered by $\delta(Ax) = \frac{1}{|A|}\delta(x)$

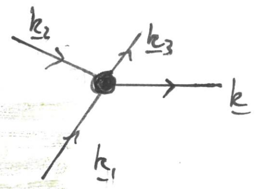
Kolmogorov – Zakharov Spectra via the Zakharov Transform

- Wave kinetic equation (WKE)

$$\dot{n}_{\underline{k}} = \mathrm{Coll}[n_{\underline{k}}] = 4\pi \int d\underline{k}_1\, d\underline{k}_2\, d\underline{k}_3\, |W^{k3}_{12}|^2\, \delta^{k3}_{12}\, \delta(\omega^{k3}_{12})\, n_1 n_2 n_3 n_{\underline{k}} \left[\frac{1}{n_k} + \frac{1}{n_3} - \frac{1}{n_1} - \frac{1}{n_2}\right]$$

$\delta(\omega_k + \omega_3 - \omega_1 - \omega_2)$ $\quad$ $\omega_i = \omega(\underline{k}_i)$ $\quad$ $n_i = n(\underline{k}_i)$

– 4-wave interactions



– Invariants: wave action $N = \int d\underline{k}\, n_{\underline{k}}$, energy $E = \int d\underline{k}\, \omega_{\underline{k}}\, n_{\underline{k}}$

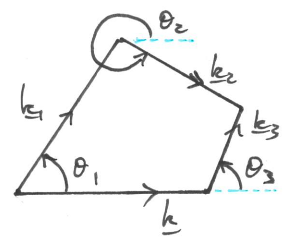
- We seek spectra that are stationary solutions of WKE that carry the flux of waveaction (to large scale) or the flux of energy (to small scale), the <u>Kolmogorov – Zakharov spectra</u> (cf Kolmogorov '41)

- <u>How?</u> By writing $\mathrm{Coll}[n_k] = \int dk_1 dk_2 dk_3\, T^{k3}_{12}\, n_1 n_2 n_3 n_k\, (k_1 k_2 k_3)^{d-1} \left[1 + \left(\frac{k_3}{k}\right)^x - \left(\frac{k_1}{k}\right)^x - \left(\frac{k_2}{k}\right)^x\right]$ and finding x s.t. $[\dots] = 0$.

- <u>Method</u> – Assume isotropy of system $\Rightarrow$ $\omega_{\underline{k}} \to \omega_k = \omega(k)$, $n_{\underline{k}} \to n_k = n(k)$ $\quad (k = |\underline{k}|)$
  – Collect angular dependence of $\mathrm{Coll}[n_k]$ into T^{k3}_{12} and establish symmetry properties $T^{k3}_{12} = T^{k3}_{21} = -T^{13}_{k2} = -T^{23}_{1k}$
  – Assume scale-invariant spectrum $n_k = A k^{\nu}$ and homogeneous $\omega_k = \lambda k^{\alpha}$: $\omega(\mu k) = \mu^{\alpha}\omega_k$; W^{k3}_{12} : $W(\mu\underline{k}_1, \mu\underline{k}_2; \mu\underline{k}_3, \mu\underline{k}) = \mu^{\beta} W^{3k}_{12}$
  – Apply Zakharov transformation to get k-scalings of each term in $\mathrm{Coll}[n_k]$

- Angular dependence of $\mathrm{Coll}[n_k]$ resides in:

– $\delta^{k3}_{12} = \delta(\underline{k} + \underline{k}_3 - \underline{k}_1 - \underline{k}_2) = \delta(k_x + k_{3x} - k_{1x} - k_{2x})\, \delta(k_y + k_{3y} - k_{1y} - k_{2y})$
$= \delta(k + k_3\cos\theta_3 - k_1\cos\theta_1 - k_2\cos\theta_2)\, \delta(0 + k_3\sin\theta_3 - k_1\sin\theta_1 - k_2\sin\theta_2)$
(manifestly angle dependent)



– $\int d\underline{k} = \int_S ds \int_0^{\infty} dk\, k^{d-1}$
integrate over (d-1) sphere

– W^{k3}_{12}

- Write $\mathrm{Coll}[n_k] = \int_0^{\infty} dk_{123}\, (k_1 k_2 k_3)^{d-1}\, n_1 n_2 n_3 n_k\, T^{k3}_{12}$, $\quad T^{k3}_{12} = 4\pi\left[\frac{1}{n_k} + \frac{1}{n_3} - \frac{1}{n_1} - \frac{1}{n_2}\right]\delta(\omega^{k3}_{12}) \int dS_{123}\, \delta^{k3}_{12}\, |W^{k3}_{12}|^2$

$dk_1\, dk_2\, dk_3$

<u>Exercise</u> $\qquad$ <u>Solution</u>

Show $T^{k3}_{12} = T^{k3}_{21} = 4\pi\left[\frac{1}{n_k} + \frac{1}{n_3} - \frac{1}{n_2} - \frac{1}{n_1}\right]\delta(\omega_k + \omega_3 - \omega_2 - \omega_1)\int dS_{123}\, \delta(\underline{k} + \underline{k}_3 - \underline{k}_2 - \underline{k}_1)\, |W^{k3}_{21}|^2 = T^{k3}_{12}$

Exchange dummy indices $k_2 \leftrightarrow k_1$ + use symmetries of W

$= -T^{13}_{k2} = -4\pi\left[\frac{1}{n_1} + \frac{1}{n_3} - \frac{1}{n_k} - \frac{1}{n_2}\right]\delta(\omega_1 + \omega_3 - \omega_k - \omega_2)\int dS_{123}\, \delta(\underline{k}_1 + \underline{k}_3 - \underline{k} - \underline{k}_2)\, |W^{13}_{k2}|^2$

$k_2 \leftrightarrow k_3$ exchange dummy indices

$= -4\pi\left[\frac{1}{n_1} + \frac{1}{n_2} - \frac{1}{n_k} - \frac{1}{n_3}\right]\delta(\omega_1 + \omega_2 - \omega_k - \omega_3)\int dS_{123}\, \delta(\underline{k}_1 + \underline{k}_2 - \underline{k} - \underline{k}_3)\, |W^{12}_{k3}|^2 = T^{k3}_{12}$

$+\left[\frac{1}{n_k} + \frac{1}{n_2} - \frac{1}{n_1} - \frac{1}{n_2}\right]$ $\quad$ $\delta(\omega^{12}_{k3}) = \delta(\omega^{k3}_{12})$ $\quad$ $\delta^{12}_{k3} = \delta^{k3}_{12}$ $\quad$ $W^{12}_{k3} = (W^{k3}_{12})^*$

symmetries of W

$= -T^{23}_{1k}$ $\quad$ Exchange $k_3 \leftrightarrow k_1$

- This allows us to split Coll[n_k] into 4 identical pieces and write

$$\mathrm{Coll}[n_k] = \frac{1}{4}\int dk_{123}\; n_1 n_2 n_3 n_k\, (k_1 k_2 k_3)^{d-1}\Big[\underbrace{T^{k3}_{12}}_{①} + \underbrace{T^{k3}_{21}}_{②} - \underbrace{T^{13}_{k2}}_{③} - \underbrace{T^{23}_{1k}}_{④}\Big]$$

- We consider each piece ①–④ independently.

①: id.

②: Assume $n_k = A k^{\nu}$, $\omega_k = \lambda k^{\alpha}$, $W^{\mu k, \mu k_3}_{\mu k_1, \mu k_2} = \mu^{\beta} W^{k, k_3}_{k_1, k_2}$ (homogeneous with degree β).

Then apply Zakharov (-Kraichnan) transform (ZT): map ② onto ① with factor $(k_3/k)^x$.

ZT: $k_1 = \tilde{k}_1 \dfrac{k}{\tilde{k}_3}$, $k_2 = \tilde{k}_2 \dfrac{k}{\tilde{k}_3}$, $k_3 = \dfrac{k^2}{\tilde{k}_3} = k\dfrac{k}{\tilde{k}_3}$ (also, identically $k = \tilde{k}_3 \dfrac{k}{\tilde{k}_3}$)

Note $\displaystyle\int_{l}^{u} dk_3 \xrightarrow{ZT} \int_{\tilde{u}}^{\tilde{l}} d\tilde{k}_3$: ZT is a non-identity transform

Jacobian of ZT $(k_1, k_2, k_3) \longrightarrow (\tilde{k}_1, \tilde{k}_2, \tilde{k}_3)$

$J = \left|\det\left(\frac{\partial k_j}{\partial \tilde{k}_i}\right)\right|$ ie take abs of Jacobian determinant and keep track of orientation changes in the limits

$$= \left|\det\begin{pmatrix} \frac{\partial k_1}{\partial \tilde{k}_1} & \frac{\partial k_2}{\partial \tilde{k}_1} & \frac{\partial k_3}{\partial \tilde{k}_1} \\ \frac{\partial k_1}{\partial \tilde{k}_2} & \frac{\partial k_2}{\partial \tilde{k}_2} & \frac{\partial k_3}{\partial \tilde{k}_2} \\ \frac{\partial k_1}{\partial \tilde{k}_3} & \frac{\partial k_2}{\partial \tilde{k}_3} & \frac{\partial k_3}{\partial \tilde{k}_3} \end{pmatrix}\right| = \left|\det\begin{pmatrix} \frac{k}{\tilde{k}_3} & 0 & 0 \\ 0 & \frac{k}{\tilde{k}_3} & 0 \\ -\frac{\tilde{k}_1 k}{\tilde{k}_3^2} & -\frac{\tilde{k}_2 k}{\tilde{k}_3^2} & -\frac{k^2}{\tilde{k}_3^2} \end{pmatrix}\right| = \left|\left(\frac{k}{\tilde{k}_3}\right)\left(\frac{k}{\tilde{k}_3}\right)\left(-\frac{k^2}{\tilde{k}_3^2}\right)\right|$$

$= \dfrac{k^4}{\tilde{k}_3^4}$ (Exercise) (Solution)

Exercise Apply ZT to each part of ② to show

$$\frac{1}{4}\int dk_{123}\; n_1 n_2 n_3 n_k\, (k_1 k_2 k_3)^{d-1}\, T^{k3}_{21} = \frac{1}{4}\int dk_{123}\; n_1 n_2 n_3 n_k\, (k_1 k_2 k_3)^{d-1}\, T^{k3}_{12}\left(\frac{\tilde{k}_3}{k}\right)^{x}$$

(change dummy variables $(\tilde{k}_1, \tilde{k}_2, \tilde{k}_3) \to (k_1, k_2, k_3)$)

$x = -4 -4\nu -4d +4 +\nu +\alpha +d -2\beta$
$= \alpha - 3\nu - 3d - 2\beta$

Solution

$dk_{123} \overset{ZT}{=} J\, d\tilde{k}_{123} = \left(\dfrac{\tilde{k}_3}{k}\right)^{-4} d\tilde{k}_{123}$, with $d\tilde{k}_{123} = d\tilde{k}_1\, d\tilde{k}_2\, d\tilde{k}_3$

Using $n_k = A k^{\nu}$,

$$A^4 (k_1 k_2 k_3 k_k)^{\nu} (k_1 k_2 k_3)^{d-1} \overset{ZT}{=} A^4 (\tilde{k}_1 \tilde{k}_2 k \tilde{k}_3)^{\nu} \left(\tilde{k}_1 \tilde{k}_2 k \frac{k}{\tilde{k}_3}\right)^{d-1}\left(\frac{\tilde{k}_3}{k}\right)^{-4\nu - 3d + 3}$$

$$= \tilde{n}_1 \tilde{n}_2 \tilde{n}_3 n_k\, (\tilde{k}_1 \tilde{k}_2 \tilde{k}_3)^{d-1}\left(\frac{\tilde{k}_3}{k}\right)^{-4\nu - 4d + 4}$$

Note swap of $(k_3, k) \to (k, \tilde{k}_3)$ under ZT!

$$T^{k3}_{21} = 4\pi\underbrace{\left[n_k^{-1} + n_3^{-1} - n_2^{-1} - n_1^{-1}\right]}\;\underbrace{\delta(\omega^{k3}_{21})}\;\underbrace{\int dS_{123}}\;\underbrace{\delta^{k3}_{21}}\;\underbrace{|W^{k3}_{21}|^2} = T^{k3}_{12}\left(\frac{\tilde{k}_3}{k}\right)^{y}$$

$y = \nu + \alpha + d - 2\beta$

$n_k = A k^{\nu}$:

$$\frac{1}{A}\left[k^{-\nu} + k_3^{-\nu} - k_2^{-\nu} - k_1^{-\nu}\right]$$
$$\overset{ZT}{=} \frac{1}{A}\left[\tilde{k}_3^{-\nu}\left(\frac{k}{\tilde{k}_3}\right)^{-\nu} + k^{-\nu}\left(\frac{k}{\tilde{k}_3}\right)^{-\nu} - \tilde{k}_2^{-\nu}\left(\frac{k}{\tilde{k}_3}\right)^{-\nu} - \tilde{k}_1^{-\nu}\left(\frac{k}{\tilde{k}_3}\right)^{-\nu}\right]$$
$$= \frac{1}{A}\left[\tilde{k}_3^{-\nu} + k^{-\nu} - \tilde{k}_2^{-\nu} - \tilde{k}_1^{-\nu}\right]\left(\frac{\tilde{k}_3}{k}\right)^{\nu}$$
$$= \left[n_k^{-1} + n_3^{-1} - n_1^{-1} - n_2^{-1}\right]\left(\frac{\tilde{k}_3}{k}\right)^{\nu}$$

$\omega = \lambda k^{\alpha}$:

$$\delta\left(\lambda(k^{\alpha} + k_3^{\alpha} - k_2^{\alpha} - k_1^{\alpha})\right)$$
$$\overset{ZT}{=} \delta\left(\lambda\left(\tilde{k}_3^{\alpha}\left(\frac{k}{\tilde{k}_3}\right)^{\alpha} + k^{\alpha}\left(\frac{k}{\tilde{k}_3}\right)^{\alpha} - \tilde{k}_2^{\alpha}\left(\frac{k}{\tilde{k}_3}\right)^{\alpha} - \tilde{k}_1^{\alpha}\left(\frac{k}{\tilde{k}_3}\right)^{\alpha}\right)\right)$$
$$= \delta\left((\lambda\tilde{k}_3^{\alpha} + \lambda k^{\alpha} - \lambda\tilde{k}_2^{\alpha} - \lambda\tilde{k}_1^{\alpha})\left(\frac{k}{\tilde{k}_3}\right)^{\alpha}\right)$$
$$= \delta(\omega_k + \tilde{\omega}_3 - \tilde{\omega}_1 - \tilde{\omega}_2)\left(\frac{\tilde{k}_3}{k}\right)^{\alpha}$$

scaling of $\delta(Ax) = \delta(x)/|A|$.

Unchanged, ZT only affects $|\underline{k}_i|$

$$\underbrace{\delta(k_x + k_{3x} - k_{2x} - k_{1x})\,\delta(k_y + k_{3y} - \dots)\dots}_{d \text{ components}}$$
$$\overset{ZT}{=} \delta\left((\tilde{k}_{3x} + k_x - \tilde{k}_{2x} - \tilde{k}_{1x})\left(\frac{k}{\tilde{k}_3}\right)\right)\delta\left((\dots)\left(\frac{k}{\tilde{k}_3}\right)\right)\dots$$
$$= \delta^{k_{3x}\, k_x}_{k_{2x}\, k_{1x}}\,\delta^{k_{3y}\, k_y}_{k_{2y}\, k_{1y}}\dots\left(\frac{\tilde{k}_3}{k}\right)^{d}$$
$$= \delta^{3k}_{21}\left(\frac{\tilde{k}_3}{k}\right)^{d}$$
$$= \delta^{k3}_{12}\left(\frac{\tilde{k}_3}{k}\right)^{d}$$

(Scaling of $\delta(Ax) = \delta(x)/|A|$)

$$|W(\underline{k}_2, \underline{k}_1; \underline{k}, \underline{k}_3)|^2$$
$$\overset{ZT}{=} \left|W\left(\tilde{k}_2\frac{k}{\tilde{k}_3}, \tilde{k}_1\frac{k}{\tilde{k}_3}; \tilde{k}_3\frac{k}{\tilde{k}_3}, k\frac{k}{\tilde{k}_3}\right)\right|^2$$
$$= \left|W^{3k}_{21}\left(\frac{\tilde{k}_3}{k}\right)^{-\beta}\right|^2 \quad (W \text{ homogeneous w/ degree } \beta)$$
$$= |W^{k3}_{12}|^2\left(\frac{\tilde{k}_3}{k}\right)^{-2\beta} \quad (\text{Symmetries of } W)$$

③ Similarly we want to map $T^{13}_{k2} = T^{k3}_{12}\left(\frac{\tilde{k}_1}{k}\right)^y$

Exercise What ZT to use? Solution Aim to swap $k \leftrightarrow k_1$ so choose

$$k_1 = \frac{k^2}{\tilde{k}_1}, \quad k_2 = \tilde{k}_2 \frac{k}{\tilde{k}_1}, \quad k_3 = \tilde{k}_3 \frac{k}{\tilde{k}_1}$$

④ Likewise we want $T^{23}_{1k} = T^{k3}_{12}\left(\frac{\tilde{k}_2}{k}\right)^y$

Exercise ZT? Solution Want $k \leftrightarrow k_2$, so

$$k_1 = \tilde{k}_1 \frac{k}{\tilde{k}_2}, \quad k_2 = \frac{k^2}{\tilde{k}_2}, \quad k_3 = \tilde{k}_3 \frac{k}{\tilde{k}_2}$$

- After all the ZTs we obtain

$$\dot{n}_k = \frac{1}{4} \int dk_1 dk_2 dk_3\, n_1 n_2 n_3 n_k (k_1 k_2 k_3)^{d-1} T^{k3}_{12} \left[1 + \left(\frac{k_3}{k}\right)^x - \left(\frac{k_1}{k}\right)^x - \left(\frac{k_2}{k}\right)^x\right]$$

$$x = \alpha - 3y - 3d - 2\beta$$

- Find stationary KZ spectra by requiring $\dot{n} = 0$ so examine x s.t. $[\cdots] = 0$.

Exercise what x will make $[\ldots]$ vanish?

Solution
- $x = 0$ as $[\cdots]$ becomes $[1+1-1-1] = 0$.
  It turns out that this is the waveaction flux spectrum
  $n_k = A k^{\nu_N}$, $\quad \nu_N = \frac{-2\beta}{3} - d + \frac{\alpha}{3}$
- $x = \alpha$ as $[\ldots] = \left[\frac{k^\alpha + k_3^\alpha - k_1^\alpha - k_2^\alpha}{k^\alpha}\right] = 0$ by $\delta(\omega_k + \omega_3 - \omega_1 - \omega_2)$ (remember $\omega_k = \lambda k^\alpha$).
  This is the energy flux spectrum
  $n_k = A k^{\nu_E}$, $\quad \nu_E = -\frac{2\beta}{3} - d$

Locality of KZ Spectra

ZT gives us stationary solutions of WKE but no guarantee that these solutions are physically meaningful.

In particular $\mathrm{Coll}[n_k]$ could diverge when the KZ spectra are substituted.

This divergence is masked when carrying out the ZT as ZT is non identity, so $0 \leftrightarrow \infty$ in the limits.

So you can have cancellation of divergences when subtracting terms due to $[1 + (\frac{k_3}{k})^x - \ldots - \ldots]$.

Interpretation of this case is that the flux is non-local (interactions at a given scale are dominated by far-away scales, ie. those around the divergence).

Why this interpretation? We assumed the flux was local when we assumed the scale-free spectrum $n_k = Ak^\nu$.

(Interactions at every scale are dominated by nearby scales).

Therefore need to check that the KZ spectra obtained via ZT is local by plugging back into the original WKE and checking it converges.

Interaction Hamiltonian – Elimination of Nonresonant Terms via Canonical Transform

- Wave equations of motion (like the NLS equation) are usually ecuivalent to Hamilton's eq

$$\partial_t a_k = -i \frac{\partial H}{\partial a_k^*}$$

for Hamiltonian $H = H_2 + H_3 + H_4 + \ldots$

where the quadratic Hamiltonian

$$H_2 = \sum_k \omega_k |a_k|^2 \longrightarrow \text{linear waves, dispersion relation } \omega_k$$

and the interaction Hamiltonians $\longrightarrow$ nonlinear wave-wave interactions

$$H_3 = \sum_{123} \left(V_{12}^{3} \delta_{12}^{3} a_1 a_2 a_3^* + \text{c.c.} \right)$$ 3-wave interactions of decay (2→1)/confluence (1→2) type

$$H_4^{2\to2} = \frac{1}{2} \sum_{1234} W_{12}^{34} \delta_{12}^{34} a_1 a_2 a_3^* a_4^*$$ 4-wave 2 → 2 interactions

and/or

$$H_4^{3\to1} = \sum_{1234} \left(U_{123}^{4} \delta_{123}^{4} a_1 a_2 a_3 a_4^* + \text{c.c.} \right)$$ 4-wave 3→1 / 1→3 interactions.

Note N→M process can be read off by considering combinations of a_k and a_k^*, cf creation and anihilation operators in quantum field theory

- WT is a weakly nonlinear theory so $H_2 \gg H_3 \gg H_4 \gg \ldots \longrightarrow$ usually stop at the lowest order interaction Hamiltonian.

- <u>However</u> consider H_3.
  - Even if we can write H_3 down it does not guarantee that the leading-order interaction is 3-wave.
  - We also need the 3-wave <u>resonance conditions</u>

$$\underline{k}_3 = \underline{k}_1 + \underline{k}_2 \quad , \quad \omega_3 = \omega_1 + \omega_2$$

  to have non-trivial solutions.

  <u>Why?</u> Spatio-temporal synchronisation required to transfer energy (ω_k) momentum ($\underline{k}$) between different modes a_1, a_2, a_3.

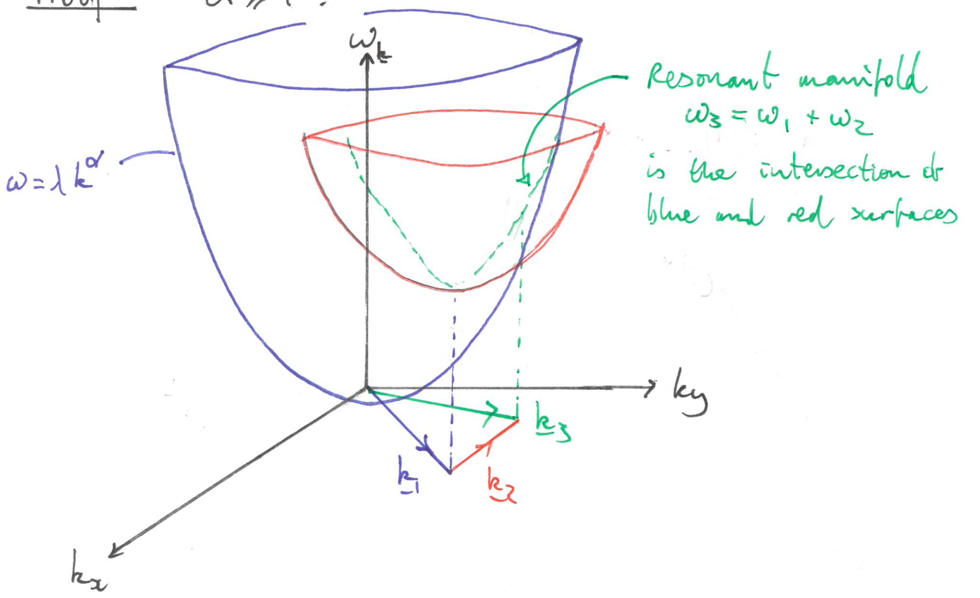
- <u>What if resonant conditions have no non-trivial solution?</u>

  Can make a <u>canonical transform</u> of variables to eliminate the non-resonant Hamiltonian.

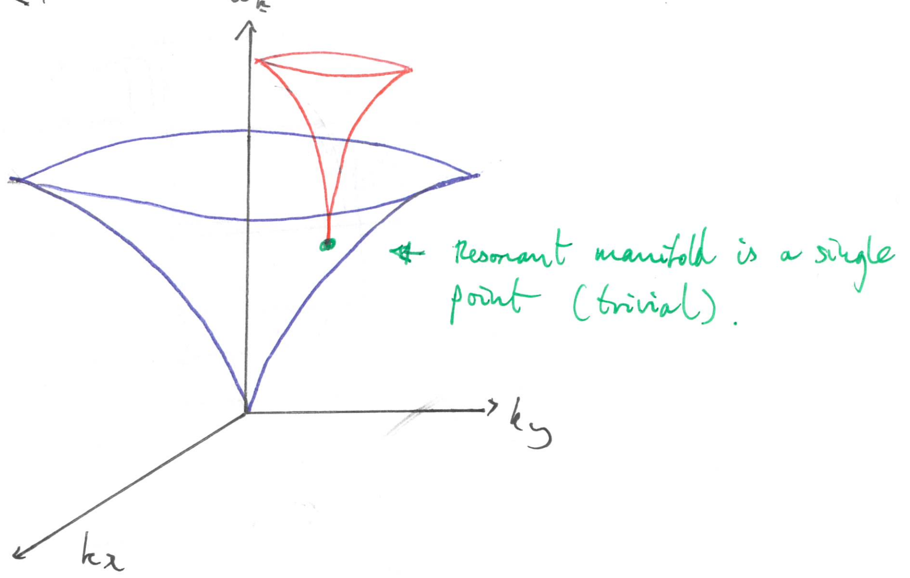
3-Wave Resonances in $d=2$: Graphical Construction

In 2D systems with power-law dispersion relation $\omega = \lambda k^\alpha$, 3-wave resonances can only exist for $\alpha \geq 1$

<u>Proof</u> $\alpha \geq 1$:



$\alpha < 1$



Eliminating the Cubic Hamiltonian

- Suppose no nontrivial solution to 3-wave resonance conditions. (Eg deep water surface gravity waves $\omega_{\underline{k}} = \sqrt{g k}$).
- Make a canonical change of variables to eliminate V_{12}^3. (Canonical $\Rightarrow$ preserve structure of Hamilton's eq.)
- Want to eliminate H_3 without changing H_2 $\longrightarrow$ transform should preserve the identity.
- Time translation is a canonical transformation (Lie transform):

$$a_{\underline{k}}(t) \longrightarrow c_{\underline{k}}(t+\tau) = a_{\underline{k}}(t)$$

$$\underbrace{c_{\underline{k}}(t)}_{\substack{\text{preserves identity} \\ (\tau\to 0)}} + \tau \partial_t c_{\underline{k}}(t) + \frac{\tau^2}{2!}\partial_{tt} c_{\underline{k}}(t) + \ldots = a_{\underline{k}}(t)$$

- Canonicity preserved by requiring $c_{\underline{k}}$ to evolve via Hamilton's eq $\partial_t c_{\underline{k}} = -i \dfrac{\partial \tilde{H}}{\partial c_{\underline{k}}^*}$

where $\tilde{H}$ is an auxilliary Hamiltonian whose parameters $\tilde{V}_{12}^3$ we choose in order to kill V_{12}^3 of original H.

$$\tilde{H} = \sum_{123} \left(\tilde{V}_{12}^3 \delta_{12}^3 c_1 c_2 c_3^* + (\tilde{V}_{12}^3)^* \delta_{12}^3 c_1^* c_2^* c_3 \right)$$

Exercise: Show

$$\frac{\partial \tilde{H}}{\partial c_{\underline{k}}^*} = \sum_{12} \Big(\tilde{V}_{12}^{\underline{k}} \delta_{12}^{\underline{k}} c_1 c_2 + \underbrace{\overset{\text{Dummy } 3\to 1}{(\tilde{V}_{\underline{k}2}^1)^* \delta_{\underline{k}2}^1 c_2^* c_1} + \overset{\text{Dummy } 3\to 2}{(\tilde{V}_{1\underline{k}}^2)^* \delta_{1\underline{k}}^2 c_1^* c_2}}_{2 V_{\underline{k}2}^1 \delta_{\underline{k}2}^1 c_1 c_2^*} \Big)$$

by permuting $2\leftrightarrow 1$ in second term and using symmetry $\tilde{V}_{\underline{k}2}^1 = V_{2\underline{k}}^1$.

$\partial/\partial c_{\underline{k}}^*$ kills all terms in the sum, except the one containing $c_{\underline{k}}^*$. Need to replace dummy index 1,2, or 3 by free index $\underline{k}$. Remaining sum over 2 dummy indices. We choose these as $\underline{k}_1, \underline{k}_2$.

- Now set $\tau = 1$ in $a_{\underline{k}} = c_{\underline{k}} + \tau \partial_t c_{\underline{k}}$ and use Hamilton's eq:

$$a_{\underline{k}} = c_{\underline{k}} - i \sum_{12} \left(\tilde{V}_{12}^k \delta_{12}^k c_1 c_2 + 2 (V_{k2}^1)^* \delta_{k2}^1 c_1 c_2^* \right) + \text{h.o.t.}$$

- Original Hamiltonian (I drop underline for k from here. Remember it's a vector though!) becomes

$$H = \overbrace{\sum_k \omega_k a_k a_k^*}^{H_2} + \overbrace{\sum_{123} \left(V_{12}^3 \delta_{12}^k a_1 a_2 a_3^* + c.c. \right)}^{H_3}$$

$$= \sum_k \omega_k \Big[c_k - i \sum_{12} \big(\tilde{V}_{12}^k \delta_{12}^k c_1 c_2 + 2 (\tilde{V}_{k2}^1)^* \ldots \big) \Big] \Big[c_k^* + i \sum_{12} \big((\tilde{V}_{12}^k)^* \ldots + \tilde{V}_{k2}^1 \delta_{k2}^1 c_1^* c_2 + \tilde{V}_{1k}^2 \delta_{1k}^2 c_1 c_2^* \big) \Big] + H_3 + \text{h.o.t.}$$

$$= \sum_k \omega_k |c_k|^2 - i \sum_{123} \Big(\tilde{V}_{12}^3 \delta_{12}^3 c_1 c_2 c_3^* \omega_3 - \tilde{V}_{32}^1 \delta_{32}^1 c_1^* c_2 c_3 \omega_3 - \tilde{V}_{13}^2 \delta_{13}^2 c_1 c_2^* c_3 \omega_3 \Big) + \text{cc} + H_3 + \text{h.o.t.}$$

(Dummy $k\to 3$; $1\leftrightarrow 3$; $2\leftrightarrow 3$)

$$= \sum_k \omega_k |c_k|^2 - i \sum_{123} \big(\tilde{V}_{12}^3 \delta_{12}^3 c_1 c_2 c_3^* (\omega_3 - \omega_1 - \omega_2) + \text{cc} \big) + \sum_{123} \big(V_{12}^3 \delta_{12}^3 c_1 c_2 c_3^* + \text{cc} \big) + \text{h.o.t}$$

$\therefore$ we can eliminate H_3 from original Hamiltonian by choosing

$$\tilde{V}_{12}^3 = -i \frac{V_{12}^3}{\omega_3 - \omega_1 - \omega_2}.$$

$\longleftarrow$ Canonical transformation can only be done if resonance condition cannot be satisfied (nontrivially).

- What will the interaction Hamiltonian be after the canonical transform?

Exercise: Convince yourself that after the CT the remaining interaction will be some new H_4. Where does this come from? Higher order terms in the Taylor expansion, i.e. $\frac{\tau^2}{2!}\partial_{tt} c_k(t)$.

Solution (partial)

$$\partial_{tt} c_k = \partial_t (\partial_t c_k) = -i \partial_t \frac{\partial \tilde{H}}{\partial c_k^*} = -i \sum_{12} \Big(\tilde{V}_{12}^k \delta_{12}^k \underbrace{\partial_t (c_1 c_2)}_{\dot{c}_1 c_2 + c_1 \dot{c}_2} + 2 V_{k2}^1 \delta_{k2}^1 \underbrace{\partial_t (c_1 c_2^*)}_{\dot{c}_1 c_2^* + c_1 \dot{c}_2^*} \Big) \sim O(c^3)$$

So $H_2(c) \to O(c).O(c^3)$, $H_3(c) \to O(c).O(c).O(c^2)$, if original H has an H_4 this will give $H_4(c) = \frac{1}{2} \sum_{1-4} W_{34}^{12} \delta_{34}^{12} c_1 c_2 c_3^* c_4^*$, to leading order.

CHALLENGE Carry this out for our original $H = H_2 + H_3$. Please send to j.skipp@aston.ac.uk!