Notes on the 1D Nonlinear Schrödinger Equation, its soliton solution, and its long-wave Optical Wave Turbulence modification

Jonathan Skipp

Abstract

These notes contain several results relating to the 1D NLSE, its bright soliton solution, and its modification for Optical Wave Turbulence in the long-wave regime.

1 Derivation the bright soliton solution

This first section contains a reproduction of the derivation for the single-bright-soliton solution of the 1D Nonlinear Schrödinger Equation (NLSE), based on the method outlined in the lecture notes Sergey Nazarenko for the University of Warwick course MA4L0, Spring 2016, pp.46-50. This derivation is carried out for the normalisation of the NLSE with all coefficients equal to unity. I also state how how to convert the equation, and its soliton solution, to other normalisations. In addition, I state the Fourier transform and waveaction spectrum of the soliton.

Consider the one-dimensional (1D), cubic, focusing, NLSE, written in units in which **all coefficients are unity**:

$$i\frac{\partial\psi}{\partial t}+\frac{\partial^2\psi}{\partial \mathrm{x}^2}+|\psi|^2\psi=0. \tag{1}$$

We seek a bright soliton solution, i.e. a localised pulse of the form

$$\psi=e^{i(\beta\mathrm{x}-\gamma t)}f(\xi), \tag{2}$$

where the constants β and γ are real, and $f(\xi)$ is a real function of the travelling coordinate $\xi=\mathrm{x}-ct$. For a localised pulse we have $f(\xi)\to 0$ as $\xi\to\pm\infty$.

We substitute (2) into (1) and set $\beta=c/2$ in order to eliminate the term v'. Defining $\mu=(c/2)^2-\gamma$ we obtain an ODE for the soliton profile

$$f''-\mu f+f^3=0. \tag{3}$$

or

$$f''=-\frac{\mathrm{d}}{\mathrm{d}f}U(f),$$

which is Newton's second law for a particle with dynamical variable ("position") $f(\xi)$ depending on "time" ξ, moving in a quartic potential $U(f)=f^4/4-\mu f^2/2$. For such a particle the energy is conserved:

$$E=\frac{f'^2}{2}+U(f)=\text{const}.$$

Considering the dynamics of such a particle in a potential well of shape $U(f)$, most trajectories are periodic. Translating back to the solution $\psi(\mathrm{x}, t)$, this would represent a periodic nonlinear wave rather than a localised pulse. Such a pulse is given only by the homoclinic orbits of the system (3) that start and end at $f = 0$, $f' = 0$, corresponding to $E = 0$. We choose the right orbit with positive f.

The choice of trajectory with $E = 0$ can also be seen directly by going back to (3), multiplying by f', and noting that each term on the LHS can be written as a derivative with respect to ξ. Noting that f decays at infinity, we integrate immediately to obtain

$$\frac{f'^2}{2} + \frac{f^4}{4} - \mu\frac{f^2}{2} = E = 0. \tag{4}$$

Equation (4) is separable; we have

$$\int \frac{\mathrm{d}f}{\sqrt{\mu f^2 - f^4/2}} = \int \mathrm{d}\xi = \xi.$$

Changing variables to $z = \sqrt{2\mu}/f$, the LHS beomes

$$\frac{1}{\sqrt{\mu}} \int \frac{\mathrm{d}z}{\sqrt{z^2 - 1}} = \frac{1}{\sqrt{\mu}} \operatorname{arccosh}(z).$$

In terms of f, the soliton profile is

$$f = \frac{\sqrt{2\mu}}{\cosh(\sqrt{\mu}\xi)}$$

which we substitute, together with $\gamma = (c/2)^2 - \mu$, into (2) to find the form of the bright soliton

$$\psi(\mathrm{x}, t) = \sqrt{2\mu}\, \operatorname{sech}\big[\sqrt{\mu}(\mathrm{x} - ct)\big]\, e^{i\left(\frac{c}{2}\mathrm{x} + \left[\mu - \left(\frac{c}{2}\right)^2\right]t\right)}. \tag{5}$$

Finally, we note that the NLSE is invariant to spatial translation and to $U(1)$ phase shifts, allowing us to insert constants x_0 and ϕ_0 to obtain the more general solution

$$\psi(\mathrm{x}, t) = \sqrt{2\mu}\, \operatorname{sech}\big[\sqrt{\mu}(\mathrm{x} - \mathrm{x}_0 - ct)\big]\, e^{i\left(\frac{c}{2}(\mathrm{x} - \mathrm{x}_0) + \left[\mu - \left(\frac{c}{2}\right)^2\right]t\right)}\, e^{i\phi_0}. \tag{6}$$

1.1 Conversion to the "$(1/2)\partial_{xx}$" notation

Another common notation for the NLSE has a factor of one-half in front of the kinetic term, namely

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + |\psi|^2\psi = 0. \tag{7}$$

Note in equation (7) I have written x in an upright font to distinguish it from the spatial coordinate x in (1).

The bright soliton solution for (7) is

$$\psi(x, t) = a\frac{e^{iv(x - x_0 - vt) + \frac{1}{2}i(a^2 + v^2)t + i\theta}}{\cosh\left[a(x - x_0 - vt)\right]}, \tag{8}$$

for example, see Gelash and Agafontsev, Phys. Rev. E 98, 042210 (2018).

To convert between the notation of (1), (6) and that of (7), (8), we simply set

$$\mathrm{x} = \sqrt{2}\,x, \qquad \mu = \frac{a^2}{2}, \qquad c = \sqrt{2}\,v, \qquad \phi_0 = \theta.$$

1.2 Conversion to the NLS with arbitrary coefficients

More generally we can rescale space and the field amplitude in the NLSE (1) and its 1-soliton solution (6) via

$$\mathrm{x} = \frac{1}{\sqrt{C_l}}x, \qquad \psi = \sqrt{C_n}\Psi, \qquad c = \frac{v}{\sqrt{C_l}},$$

to obtain the equation

$$i\frac{\partial \Psi}{\partial t} + C_l\frac{\partial^2 \Psi}{\partial x^2} + C_n|\Psi|^2\Psi = 0,$$

which has the soliton solution

$$\Psi(x,t) = A\ \mathrm{sech}[\kappa(x - x_0 - vt)]\ \exp\left(i\left\{\frac{v}{2C_l}(x - x_0) + \left[\frac{A^2C_n}{2} - \frac{1}{C_l}\left(\frac{v}{2}\right)^2\right]t\right\}\right)\ \exp(i\phi_0).$$

where the soliton amplitude A, the the inverse κ of its characteristic width are

$$A = \sqrt{\frac{2\mu}{C_n}}, \qquad \text{and} \qquad \kappa = A\sqrt{\frac{C_n}{2C_l}},$$

respectively.

2 Waveaction spectrum of the 1-soliton solution ("$(1/2)\partial_{xx}$" notation)

Reverting to the normalisation of section 1.1 , we consider the soliton (8) centred at $x_0 = 0$ at $t = 0$, and with phase $\theta = 0$:

$$\psi(x,t) = a\frac{e^{ivx}}{\cosh(ax)} = a\frac{2e^{ivx}}{e^{ax} + e^{-ax}}. \tag{9}$$

Taking the Fourier transform of (9), we seek

$$\hat{\psi}(k,t) = \int_{-\infty}^{\infty} \mathrm{d}x\, a\frac{2e^{ivx}}{e^{ax} + e^{-ax}}e^{-ikx} = \int_{-\infty}^{\infty} \mathrm{d}\xi\, \frac{2e^{-i\kappa\xi}}{e^{\xi} + e^{-\xi}} = \pi\,\mathrm{sech}\left(\frac{\pi\kappa}{2}\right),$$

where in the second step we have used $\xi = ax$ and $\kappa = (k - v)/a$, and the final step follows from contour integration. Thus, in terms of the original variables, we obtain

$$\hat{\psi}_k = \pi\,\mathrm{sech}\left(\frac{\pi(k - v)}{2a}\right). \tag{10}$$

From (10), the waveaction spectrum follows:

$$n_k(t) \propto |\hat{\psi}_k|^2 = \pi^2\,\mathrm{sech}^2\left(\frac{\pi(k - v)}{2a}\right). \tag{11}$$