Notes on the 1D Nonlinear Schrödinger Equation, its soliton solution, and its long-wave Optical Wave Turbulence modification

Jonathan Skipp

Abstract

These notes contain several results relating to the 1D NLSE, its bright soliton solution, and its modification for Optical Wave Turbulence in the long-wave regime.

1 Derivation the bright soliton solution

This first section contains a reproduction of the derivation for the single-bright-soliton solution of the 1D Nonlinear Schrödinger Equation (NLSE), based on the method outlined in the lecture notes Sergey Nazarenko for the University of Warwick course MA4L0, Spring 2016, pp.46-50. This derivation is carried out for the normalisation of the NLSE with all coefficients equal to unity. I also state how how to convert the equation, and its soliton solution, to other normalisations. In addition, I state the Fourier transform and waveaction spectrum of the soliton.

Consider the one-dimensional (1D), cubic, focusing, NLSE, written in units in which **all coefficients are unity**:

$$i\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial \mathrm{x}^2} + |\psi|^2\psi = 0. \tag{1}$$

We seek a bright soliton solution, i.e. a localised pulse of the form

$$\psi = e^{i(\beta \mathrm{x} - \gamma t)} f(\xi), \tag{2}$$

where the constants β and γ are real, and $f(\xi)$ is a real function of the travelling coordinate $\xi = \mathrm{x} - ct$. For a localised pulse we have $f(\xi) \to 0$ as $\xi \to \pm\infty$.

We substitute (2) into (1) and set $\beta = c/2$ in order to eliminate the term v'. Defining $\mu = (c/2)^2 - \gamma$ we obtain an ODE for the soliton profile

$$f'' - \mu f + f^3 = 0. \tag{3}$$

or

$$f'' = -\frac{\mathrm{d}}{\mathrm{d}f} U(f),$$

which is Newton's second law for a particle with dynamical variable ("position") $f(\xi)$ depending on "time" ξ, moving in a quartic potential $U(f) = f^4/4 - \mu f^2/2$. For such a particle the energy is conserved:

$$E = \frac{f'^2}{2} + U(f) = \text{const.}$$

Considering the dynamics of such a particle in a potential well of shape $U(f)$, most trajectories are periodic. Translating back to the solution $\psi(\mathrm{x}, t)$, this would represent a periodic nonlinear wave rather than a localised pulse. Such a pulse is given only by the homoclinic orbits of the system (3) that start and end at $f = 0$, $f' = 0$, corresponding to $E = 0$. We choose the right orbit with positive f.

The choice of trajectory with $E = 0$ can also be seen directly by going back to (3), multiplying by f', and noting that each term on the LHS can be written as a derivative with respect to ξ. Noting that f decays at infinity, we integrate immediately to obtain

$$\frac{f'^2}{2} + \frac{f^4}{4} - \mu\frac{f^2}{2} = E = 0. \tag{4}$$

Equation (4) is separable; we have

$$\int \frac{\mathrm{d}f}{\sqrt{\mu f^2 - f^4/2}} = \int \mathrm{d}\xi = \xi.$$

Changing variables to $z = \sqrt{2\mu}/f$, the LHS beomes

$$\frac{1}{\sqrt{\mu}} \int \frac{\mathrm{d}z}{\sqrt{z^2 - 1}} = \frac{1}{\sqrt{\mu}} \operatorname{arccosh}(z).$$

In terms of f, the soliton profile is

$$f = \frac{\sqrt{2\mu}}{\cosh(\sqrt{\mu}\xi)}$$

which we substitute, together with $\gamma = (c/2)^2 - \mu$, into (2) to find the form of the bright soliton

$$\psi(\mathrm{x}, t) = \sqrt{2\mu}\, \operatorname{sech}\big[\sqrt{\mu}(\mathrm{x} - ct)\big]\, e^{i\left(\frac{c}{2}\mathrm{x} + \left[\mu - \left(\frac{c}{2}\right)^2\right]t\right)}.$$

Finally, we note that the NLSE is invariant to spatial translation and to $U(1)$ phase shifts, allowing us to insert constants x_0 and ϕ_0 to obtain the more general solution

$$\psi(\mathrm{x}, t) = \sqrt{2\mu}\, \operatorname{sech}\big[\sqrt{\mu}(\mathrm{x} - \mathrm{x}_0 - ct)\big]\, e^{i\left(\frac{c}{2}(\mathrm{x} - \mathrm{x}_0) + \left[\mu - \left(\frac{c}{2}\right)^2\right]t\right)}\, e^{i\phi_0}. \tag{5}$$

1.1 Conversion to the "$(1/2)\partial_{xx}$" notation

Another common notation for the NLSE has a factor of one-half in front of the kinetic term, namely

$$i\frac{\partial\psi}{\partial t} + \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + |\psi|^2\psi = 0. \tag{6}$$

Note in equation (6) I have written x in an upright font to distinguish it from the spatial coordinate x in (1).

The bright soliton solution for (6) is

$$\psi(x, t) = a\frac{e^{iv(x - x_0 - vt) + \frac{1}{2}i(a^2 + v^2)t + i\theta}}{\cosh\left[a(x - x_0 - vt)\right]}, \tag{7}$$

for example, see Gelash and Agafontsev, Phys. Rev. E 98, 042210 (2018). (Note carefully the signs of the $v^2 t$ factors in the exponentials.)

To convert between the notation of (1), (5) and that of (6), (7), we simply set

$$\mathrm{x} = \sqrt{2}\, x, \qquad \mu = \frac{a^2}{2}, \qquad c = \sqrt{2}\, v, \qquad \phi_0 = \theta.$$

1.2 Conversion to the NLS with arbitrary coefficients

More generally we can rescale space and the field amplitude in the NLSE (1) and its 1-soliton solution (5) via

$$\mathrm{x} = \frac{1}{\sqrt{C_l}} x, \qquad \psi = \sqrt{C_n}\Psi, \qquad c = \frac{v}{\sqrt{C_l}},$$

to obtain the equation

$$i\frac{\partial \Psi}{\partial t} + C_l \frac{\partial^2 \Psi}{\partial x^2} + C_n |\Psi|^2 \Psi = 0,$$

which has the soliton solution

$$\Psi(x,t) = A \operatorname{sech}[\kappa(x - x_0 - vt)] \exp\left(i \left\{ \frac{v}{2C_l}(x - x_0) + \left[\frac{A^2 C_n}{2} - \frac{1}{C_l}\left(\frac{v}{2}\right)^2 \right] t \right\} \right) \exp(i\phi_0).$$

where the soliton amplitude A, the the inverse κ of its characteristic width are

$$A = \sqrt{\frac{2\mu}{C_n}}, \qquad \text{and} \qquad \kappa = A\sqrt{\frac{C_n}{2C_l}},$$

respectively.

2 Waveaction and spatio-temporal spectra of the 1-soliton solution (“$(1/2)\partial_{xx}$” notation)

Reverting to the normalisation of section 1.1, we consider the 1-soliton solution (7), setting x_0 and θ set to 0 for convenience (nonzero values result in an overall phase shift in the Fourier transforms, which make no difference to the spectra):

$$\psi(x,t) = a\frac{e^{iv(x-vt)+i(\frac{a^2+v^2}{2})t}}{\cosh\left[a(x-vt)\right]}. \tag{8}$$

In this section we find the spatial Fourier transform and spatio-temporal Fourier transform of (8), and from them obtain the respective waveaction and spatio-temporal spectra.

To do so we need the result

$$J(\kappa; x) := \int_{-\infty}^{\infty} \frac{2e^{i\kappa x}}{e^x + e^{-x}}\, \mathrm{d}x = \pi \operatorname{sech}\left(\frac{\pi\kappa}{2}\right), \tag{9}$$

the proof of which is a pleasantly diverting exercise in contour integration, which we give in Appendix A.

2.1 Spatial FT and waveaction spectrum

The spatial Fourier transform of (8) is

$$
\begin{aligned}
\hat{\psi}(k,t) &= \int_{-\infty}^{\infty} a \frac{e^{iv(x-vt)+i(\frac{a^2+v^2}{2})t}}{\cosh\left[a(x-vt)\right]} e^{-ikx}\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{2e^{iv(x-vt)}e^{-ik(x-vt)}}{e^{a(x-vt)}+e^{-a(x-vt)}}\,\mathrm{d}x \cdot a e^{i(\frac{a^2+v^2}{2})t} e^{-ikvt} \\
&= \int_{-\infty}^{\infty} \frac{2e^{i(v-k)\xi}}{e^{a\xi}+e^{-a\xi}}\,\mathrm{d}\xi \cdot a e^{i(\frac{a^2+v^2}{2}-kv)t}
\end{aligned}
$$

where in the last step we have made a Galilean transform into the frame moving at the soliton speed, with coordinates $\xi = x - vt$ and $t' = t$ and immediately dropped the prime on the time coordinate (in particular for the Galilean transform $\mathrm{d}x = \mathrm{d}\xi$). Using (9) with $(v-k)/a \to \kappa$ and $a\xi \to x$ we then obtain the spatial Fourier transform of the 1-soliton solution

$$
\hat{\psi}(k,t) = \pi \,\mathrm{sech}\left(\frac{\pi(k-v)}{2a}\right) e^{i(\frac{a^2+v^2}{2}-kv)t} \tag{10}
$$

. The waveaction spectrum follows:

$$
n_k(t) \propto |\hat{\psi}_k|^2 = \pi^2 \,\mathrm{sech}^2\left(\frac{\pi(k-v)}{2a}\right). \tag{11}
$$

WHAT ABOUT THE NORMALISATION ?? .

2.2 Spatio-temporal transform and spectrum

The temporal Fourier transform of (10) is trivial, but note that the sign in the exponent of the transform kernel $e^{i\omega t}$ is positive, as the full spatio-temporal decomposition into plane waves goes (schematically) as $\psi(x,t) \sim \tilde{\psi}(k,\omega)\exp[i(kx-\omega t)]$. The spatio-temporal Fourier coefficient is thus

$$
\begin{aligned}
\tilde{\psi}(k,\omega) &= \int_{-\infty}^{\infty} \hat{\psi}(k,t) e^{i\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} e^{i(\frac{a^2+v^2}{2}-kv)t} e^{i\omega t}\,\mathrm{d}t \cdot \pi \,\mathrm{sech}\left(\frac{\pi(k-v)}{2a}\right).
\end{aligned}
$$

Using the identity $\int_{\mathbb{R}} \exp[i(\omega+\omega_0)t]\mathrm{d}t = 2\pi\delta(\omega+\omega_0)$ we obtain the spatio-temporal Fourier transform

$$
\tilde{\psi}(k,\omega) = 2\pi^2 \delta\left(\omega + \frac{a^2+v^2}{2} - kv\right) \mathrm{sech}\left(\frac{\pi(k-v)}{2a}\right) \tag{12}
$$

and the spatio-temporal spectrum varies as

$$
n_{k\omega} \propto |\tilde{\psi}|^2 = 4\pi^4 \left[\delta\left(\omega + \frac{a^2+v^2}{2} - kv\right)\right]^2 \mathrm{sech}^2\left(\frac{\pi(k-v)}{2a}\right). \tag{13}
$$

In (13) the argument of the delta function implies that the spatio-temporal spectrum of the soliton will be zero except on the line

$$\omega = vk - \frac{a^2 + v^2}{2}, \tag{14}$$

i.e. in the k–ω plane the soliton spectrum will have a gradient of v. The dependence of the sech profile with $(k - v)$ implies that in the k–ω plane, the soliton spectrum will be centred horizontally on $k_c = v$. Thus from (14) the soliton spectrum will be centred vertically on $\omega_c = (v^2 - a^2)/2$.

A Proof of eq (9)

Consider the contour integral

$$\underbrace{\oint_C \frac{2e^{i\kappa z}}{e^z + e^{-z}}\,\mathrm{d}z}_{I_C} = \underbrace{\int_{-R}^{R} \frac{2e^{i\kappa x}}{e^x + e^{-x}}\,\mathrm{d}x}_{I_R} + \underbrace{\int_\Gamma \frac{2e^{i\kappa z}}{e^z + e^{-z}}\,\mathrm{d}z}_{I_\Gamma}$$

where C is the contour formed by the length along the real axis $-R \leq x \leq R$ and a semicircular closure Γ of radius R. If $\kappa > 0$ (respectively if $\kappa < 0$) we take Γ to lie in the upper (lower) half-plane, so that in the limit $R \to \infty$ we have $I_\Gamma \to 0$; also $I_R \to J(\kappa; x)$, and therefore

$$J(\kappa; x) = \lim_{R\to\infty} I_C.$$

The integrand in I_C has simple poles at $z_n = i\pi(n+1/2)$ with $n \in \{0, 1, 2, \ldots\}$. Defining $\delta z = z - z_n$, the n-th residue is

$$\begin{aligned}
\mathrm{Res}_n &= \lim_{\delta z\to 0} \delta z\, \frac{2e^{i\kappa(z_n+\delta z)}}{e^{z_n+\delta z} + e^{-z_n-\delta z}} \\
&= \lim_{\delta z\to 0} \delta z\, \frac{2e^{-\pi\kappa(n+1/2)}}{\underbrace{e^{i\pi n}}_{(-1)^n} \underbrace{e^{i\pi/2}}_{i} e^{\delta z} + \underbrace{e^{-i\pi n}}_{(-1)^n} \underbrace{e^{-i\pi/2}}_{-i} e^{-\delta z}} \\
&= \lim_{\delta z\to 0} \delta z\, \frac{2(-i)(-1)^n e^{-\pi\kappa(n+1/2)}}{e^{\delta z} - e^{-\delta z}} \\
&= \lim_{\delta z\to 0} \delta z\, \frac{2(-i)(-1)^n e^{-\pi\kappa(n+1/2)}}{[1 + \delta z + \mathcal{O}(\delta z^2)] - [1 - \delta z + \mathcal{O}(\delta z^2)]}. \\
&= \lim_{\delta z\to 0} \delta z\, \frac{2(-i)(-1)^n e^{-\pi\kappa(n+1/2)}}{2\,\delta z + \mathcal{O}(\delta z^2)}. \\
&= (-i)(-1)^n e^{-\pi\kappa(n+1/2)}.
\end{aligned}$$

Thus by the residue theorem the n-th contribution to I_C due to Res_n is

$$(2\pi i)\,\mathrm{Res}_n = (2\pi i)\left[(-i)(-1)^n e^{-\pi\kappa(n+1/2)}\right] = 2\pi e^{-\pi\kappa/2}\left(-e^{-\pi\kappa}\right)^n.$$

Taking the $R \to \infty$ limit we sum an infinite number of these contributions, which is easy to evaluate as the sum is geometric:

$$
\begin{aligned}
\lim_{R\to\infty} I_C &= \sum_{n=0}^{\infty} 2\pi e^{-\pi\kappa/2} \left(-e^{-\pi i\kappa}\right)^n \\
&= 2\pi e^{-\pi\kappa/2} \frac{1}{1+e^{-\pi\kappa}} \\
&= \frac{2\pi}{e^{\pi\kappa/2} + e^{-\pi\kappa/2}}.
\end{aligned}
$$

Equating this to $J(\kappa; x)$ we finally we obtain (9):

$$
J(\kappa; x) = \pi \,\mathrm{sech}\left(\frac{\pi\kappa}{2}\right).
$$