Division $*$-algebras

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Introduction

Solèr's theorem gives an a priori reason to do physics over either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. In this document we explore some concepts in the neighborhood of Solèr's theorem.

1 Basic Definitions

Definition 1.1. A **$*$-algebra** over a field $\mathcal{K}$ is a vector space $\mathcal{A}$ over $\mathcal{K}$ together with a not necessarily commutive nor associative binary multiplication operator $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, and a unary involution $^* : \mathcal{A} \to \mathcal{A}$ satisfying the following axioms:

(1) Left distributivity of multiplication over addition: $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{z})$
(2) Right distributivity of multiplication over addition: $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = (\mathbf{x} \cdot \mathbf{z}) + (\mathbf{y} \cdot \mathbf{z})$
(3) Compatibility with scalars: $(a\mathbf{x}) \cdot (b\mathbf{y}) = (ab)(\mathbf{x} \cdot \mathbf{y})$
(4) * is an additive homomorphism: $(\mathbf{x} + \mathbf{y})^* = \mathbf{x}^* + \mathbf{y}^*$
(5) * is a multiplicative anti-homomorphism: $(\mathbf{x} \cdot \mathbf{y})^* = \mathbf{y}^* \cdot \mathbf{x}^*$

Definition 1.2. A **presentation** of a $*$-algebra over a field $\mathcal{K}$ is a vector space V over $\mathcal{K}$ together with a $\mathcal{K}$-linear function $L : V \to End(V)$ and a linear involution $^\dagger : V \to V$ such that for all $x, y \in V$ $(L(x)y)^\dagger = L(y^\dagger)x^\dagger$.

We say $(V, L, ^\dagger)$ is a **presentation of** $\mathcal{A}$ if there is a linear isomorphism $\pi : \mathcal{A} \to V$ such that:

(a) For all $\mathbf{x}, \mathbf{y} \in \mathcal{A}$, $\pi(\mathbf{x} \cdot \mathbf{y}) = L(\pi(\mathbf{x}))\pi(\mathbf{y})$
(b) For all $\mathbf{x} \in \mathcal{A}$, $\pi(\mathbf{x}^*) = \pi(\mathbf{x})^\dagger$

Observation 1.3. *Every $*$-algebra $\mathcal{A}$ over a field $\mathcal{K}$ has a presentation where π is the forgetful map from $\mathcal{A}$ to the underling vector space V of $\mathcal{A}$, $L(x)y := \pi(\pi^{-1}(x) \cdot \pi^{-1}(y))$, and $x^\dagger := \pi(\pi^{-1}(x)^*)$.*

Proof. Properties (a) and (b) are satisfied by definition. It's clear $\dagger$ is an involution. $(x, y) \mapsto L(x)y$ is $\mathcal{K}$-bilinear since it is the pushforward of multiplication in $\mathcal{A}$, so $x \mapsto L(x)$ is $\mathcal{K}$-linear. It suffices to show then that for all $x, y \in V$, $(L(x)y)^\dagger = L(y^\dagger)x^\dagger$. We can see this as follows:

$$\begin{aligned} (L(x)y)^\dagger &= \pi(\pi^{-1}(L(x)y)^*) = \pi((\pi^{-1}(x) \cdot \pi^{-1}(y))^*) \\ &= \pi(\pi^{-1}(y)^* \cdot \pi^{-1}(x)^*) = \pi(\pi^{-1}(y^\dagger) \cdot \pi^{-1}(x^\dagger)) = L(y^\dagger)x^\dagger \end{aligned}$$

. ☐

Observation 1.4. *Given a presentation of a $*$-algebra over $\mathcal{K}$, $(V, L, ^\dagger)$, we can turn V into a $*$-algebra by defining $\cdot : V^2 \to V$ by $x \cdot y = L(x)y$ and defining $^* : V \to V$ by $x^* = x^\dagger$.*

Proof. Left as an exercise for the reader. ☐

Definition 1.5. A **division $*$-algebra** over $\mathcal{K}$ is a $*$-algebra over $\mathcal{K}$ satisfying the following two additional conditions:

(6) Right division: for any $\mathbf{x}$ and any non-zero $\mathbf{y}$, there exists a unique $\mathbf{z}$ such that $\mathbf{x} = \mathbf{z} \cdot \mathbf{y}$

(7) Left division: for any $\mathbf{x}$ and any non-zero $\mathbf{y}$, there exists a unique $\mathbf{z}$ such that $\mathbf{x} = \mathbf{y} \cdot \mathbf{z}$

Observation 1.6. *A presentation of a $*$-algebra over $\mathcal{K}$ is of a division $*$-algebra if and only if $\forall x \in V$, $L(x) \in GL(V)$.*