Division $*$-algebras

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Introduction

Definition 0.1. A $*$-algebra over a field $\mathcal{K}$ is a vector space $\mathcal{A}$ over $\mathcal{K}$ together with a not necessarily commutive nor associative binary multiplication operator $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, and a unary involution $^* : \mathcal{A} \to \mathcal{A}$ satisfying the following axioms:

(1) Left distributivity of multiplication over addition: $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = (\mathbf{x} \cdot \mathbf{y}) + (\mathbf{x} \cdot \mathbf{z})$
(2) Right distributivity of multiplication over addition: $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = (\mathbf{x} \cdot \mathbf{z}) + (\mathbf{y} \cdot \mathbf{z})$
(3) Compatibility with scalars: $(a\mathbf{x}) \cdot (b\mathbf{y}) = (ab)(\mathbf{x} \cdot \mathbf{y})$
(4) * is an additive homomorphism: $(\mathbf{x} + \mathbf{y})^* = \mathbf{x}^* + \mathbf{y}^*$
(5) * is a multiplicative anti-homomorphism: $(\mathbf{x} \cdot \mathbf{y})^* = \mathbf{y}^* \cdot \mathbf{x}^*$

Definition 0.2. A representation of a $*$-algebra $\mathcal{A}$ over a field $\mathcal{K}$ is a $\mathcal{K}$-linear map $\pi : \mathcal{A} \to V$ to some vector space $\mathcal{V}$ together with a $\mathcal{K}$-linear function $L : \mathcal{V} \to End(V)$ and an involution $^\dagger : V \to V$ such that

(a) For all $\mathbf{x}, \mathbf{y} \in \mathcal{A}$, $\pi(\mathbf{x} \cdot \mathbf{y}) = L(\pi(\mathbf{x}))\pi(\mathbf{y})$
(b) For all $\mathbf{x} \in \mathcal{A}$, $\pi(x^*) = \pi(x)^\dagger$

Definition 0.3. A division $*$-algebra over $\mathcal{K}$ is a $*$-algebra over $\mathcal{K}$ satisfying the following two additional conditions:

(6) Right division: for any $\mathbf{x}$ and any non-zero $\mathbf{y}$, there exists a unique $\mathbf{z}$ such that $\mathbf{x} = \mathbf{z} \cdot \mathbf{y}$
(7) Left division: for any $\mathbf{x}$ and any non-zero $\mathbf{y}$, there exists a unique $\mathbf{z}$ such that $\mathbf{x} = \mathbf{y} \cdot \mathbf{z}$