Separability of mixed states:
necessary and sufficient conditions

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Abstract

We provide necessary and sufficient conditions for separability of mixed states. As a result we obtain a simple criterion of separability for 2×2 and 2×3 systems. Here, the positivity of the partial transposition of a state is necessary and sufficient for its separability. However, it is not the case in general. Some examples of mixtures which demonstrate the utility of the criterion are considered.

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Quantum inseparability, first recognized by Einstein, Podolsky and Rosen [1] and Schrödinger [2], is one of the most astonishing features of quantum formalism. After over sixty years it is still a fascinating object from both theoretical and experimental points of view. Recently, together with a dynamical development of experimental methods, a number of possible practical applications of the quantum inseparable states has been proposed including quantum computation [3] and quantum teleportation [4]. The above ideas are based on the fact that the quantum inseparability implies, in particular, the existence of the pure entangled states which produce nonclassical phenomena. However, in laboratory one deals with mixed states rather than pure ones. This is due to the uncontrolled interaction with the enviroment. Then it is very important to know which mixed states can produce quantum effects. The problem is much more complicated than in the pure states case. It may be due to the fact that mixed states apparently possess the ability to behave classically in some respects but quantum mechanically in others [5].

In accordance with the so-called generalized inseparability principle [6] we will call a mixed state of a compound quantum system inseparable if it cannot be written as a convex combination of product states. The problem of the inseparability of mixed states was first raised by Werner [7], who constructed a family of inseparable states which admit the local hidden variable model. It has been pointed out [8] that, nevertheless, some of them are non-local and this "hidden" nonlocality can be revealed by subjecting them to more complicated experiments than single von Neumann measurements considered by Werner (see also Ref. [9] in this context). This shows that it is hard to classify the mixed states as purely classical or quantum ones.

Recently the separable states have been investigated within the information-theoretic approach [6,10–12]. It has been shown that they satisfy a series of the so-called quantum α-entropy inequalities (for $\alpha = 1, 2$ [10,11] and $\alpha = \infty$ [12]). Moreover, the separable two spin-$\frac{1}{2}$ states with maximal entropies of subsystems have been completely characterized in terms of the α-entropy inequalities [6]. It is remarkable that there exist inseparable states which do *not* reveal nonclassical features under the entropic criterion [11].

Then the fundamental problem of an "operational" characterization of the separable states arises. So far only some necessary conditions of separability have been found [6,7,10,11,13]. An important step is due to Peres [14], who has provided a very strong condition. Namely, he noticed that the separable states remain positive if subjected to partial transposition. Then he conjectured that this is also sufficient condition.

In this Letter we present two necessary and sufficient conditions for separability of mixed states. It provides a complete, operational characterization of separable states for 2×2 and 2×3 systems. It appears that Peres' conjecture is valid for those cases. However, as we show in the Appendix, the conjecture is not valid in general. We also illustrate our results by means of some examples and discuss them in the context of the α-entropy inequalities.

To make our considerations more clear, we start from the following notation and definitions. We will deal with the states on the finite dimensional Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. An operator ϱ acting on $\mathcal{H}$ is a state if $\mathrm{Tr}\varrho = 1$ and if it is a positive operator i.e.

$$\mathrm{Tr}\varrho P \geq 0 \tag{1}$$

for any projectors P. A state is called separable [1] if it can be approximated in the trace norm by the states of the form

$$\varrho = \sum_{i=1}^{k} p_i \varrho_i \otimes \tilde{\varrho}_i \tag{2}$$

where ϱ_i and $\tilde{\varrho}_i$ are states on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. By $\mathcal{A}_1$ and $\mathcal{A}_2$ we will denote the set of the operators acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Recall that $\mathcal{A}_i$ constitute a Hilbert space (so-called Hilbert-Schmidt space) with scalar product $\langle A, B \rangle = \mathrm{Tr}B^\dagger A$. The space of the linear maps from $\mathcal{A}_1$ to $\mathcal{A}_2$ is denoted by $\mathcal{L}(\mathcal{A}_1, \mathcal{A}_2)$. We say that a map $\Lambda \in \mathcal{L}(\mathcal{A}_1, \mathcal{A}_2)$ is positive if it maps positive operators in $\mathcal{A}_1$ into the set of positive operators i.e. if $A \geq 0$ implies $\Lambda(A) \geq 0$. Finally we need the definition of completely positive map. One says [15] that a map $\Lambda \in \mathcal{L}(\mathcal{A}_1, \mathcal{A}_2)$ is completely positive if the induced map

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[1]The presented definition of separable states is due to Werner [7] who called them classically correlated states.

$$\Lambda_n = \Lambda \otimes I : \mathcal{A}_1 \otimes \mathcal{M}_n \to \mathcal{A}_2 \otimes \mathcal{M}_n \qquad (3)$$

is positive for all n. Here $\mathcal{M}_n$ stand for the set of the complex matrices $n \times n$ and I is the identity map[2].

Thus the tensor product of a completely positive map and the identity maps positive operators into positive ones. It is remarkable that there are positive maps that do not possess this property. This fact is crucial for the problem we discuss here. Indeed, trivially, the product states are mapped into positive operators by the tensor product of a positive map and identity: $(\Lambda \otimes I)\varrho \otimes \tilde{\varrho} = (\Lambda \varrho) \otimes \tilde{\varrho} \geq 0$. Of course, the same holds for the separable states. Then our main idea is that this property of the separable states is essential i.e., roughly speaking, if a state ϱ is inseparable, then there exists a positive map Λ such that $\Lambda \otimes I \varrho$ is *not* positive. This means that we can recognize the inseparable states by means of the positive maps. Now the point is that not all the positive maps can help us to determine whether a given state is inseparable. In fact, the completely positive maps do not "feel" the inseparability. Thus the problem of characterization of the set of the separable states reduces to the following: one should extract from thet set of all positive maps some essential ones. As we will see further, it is possible in some cases. Namely it appears that for the 2×2 and 2×3 systems the transposition is the *only* such map.

We will start from the following

Lemma 1 *For any inseparable state $\tilde{\varrho} \in \mathcal{A}_1 \otimes \mathcal{A}_2$ there exists Hermitian operator $\tilde{A}$ such that*

$$Tr(\tilde{A}\tilde{\varrho}) < 0 \quad and \quad Tr(\tilde{A}\sigma) \geq 0 \qquad (4)$$

for any separable σ.

Proof.-From the definition of the set of separable states it follows that it is both convex and closed set in $\mathcal{A}_1 \otimes \mathcal{A}_2$. Thus we can apply a theorem (conclusion from the Hahn-Banach

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[2]Of course a completely positive map is also a positive one.

theorem) [16] which, for our purposes, can be formulated as follows. If W_1, W_2 are convex closed sets in a real Banach space and one of them is compact, then there exists a continous functional f and $\alpha \in R$ such that for all pairs $w_1 \in W_1$, $w_2 \in W_2$ we have

$$f(w_1) < \alpha \le f(w_2) \tag{5}$$

This theorem says, in particular, that a closed convex set in the Banach space is completely described by the inequalities involving continous functionals.

Noting that one-element set is compact we obtain that there exists a real functional g on the real space $\tilde{\mathcal{A}}$ generated by Hermitian operators from $\mathcal{A}_1 \otimes \mathcal{A}_2$ such that

$$g(\tilde{\varrho}) < \beta \le g(\sigma), \tag{6}$$

for all separable σ. It is a well known fact that any contionous functional g on a Hilbert space can be represented by a vector from this space. As $\tilde{\mathcal{A}}$ is a (real) Hilbert space we obtain that the functional g can be represented as

$$g(\varrho) = Tr(\varrho A), \tag{7}$$

where $A = A^\dagger$. Now, if I stands for identity then for any states ϱ, σ one has obviously $Tr(\beta I \varrho) = Tr(\beta I \sigma) = \beta$. Thus it sufficies only to take

$$\tilde{A} = A - \beta I \tag{8}$$

to complete the proof of the lemma. The lemma allows us to prove the following

Theorem 1 *A state $\varrho \in \mathcal{A}_1 \otimes \mathcal{A}_2$ is separable iff*

$$Tr(A\varrho) \ge 0 \tag{9}$$

for any Hermitian operator A satisfying $Tr(AP \otimes Q) \ge 0$, where P and Q are projections acting on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.

Proof .- If ϱ is separable then obviously it satisfies the condition (9). To prove the converse statement, suppose that ϱ satisfies the condition (9) and is inseparable. Then,

due to inseparability, it would be possible (on the strength of the lemma) to find some Hermitian operator A for which $Tr(A\varrho) < 0$ although $Tr(A\sigma) \geq 0$ for any separable state σ or, equivalently, for any product projector $P \otimes Q$ which is a contradiction.

Remark 1.- Note that the if an operator A satisfies the condition $\mathrm{Tr}AP \otimes Q \geq 0$ for any projectors P and Q then it is Hermitian.

Remark 2.- As the conclusion from the Hahn-Banach theorem is valid for *any* Banach space our theorem can be generalized for infinitely dimensional Hilbert spaces. Namely, it can be seen that the condition is $\mathrm{Tr}A\varrho \geq 0$ for any bounded A such that $\mathrm{Tr}AP \otimes Q \geq 0$ for any projectors P and Q [3].

A good example of a nontrivial operator which satisfies the condition (9) is the one used by Werner [7], defined on $C^d \otimes C^d$ by $V\phi \otimes \tilde{\phi} = \tilde{\phi} \otimes \phi$. One can check that $\mathrm{Tr}VP \otimes Q = \mathrm{Tr}PQ \geq 0$. This operator, together with its $U_1 \otimes U_2$ transformations allowed to characterize the set of separable states within the class of two spin-$\frac{1}{2}$ mixtures with maximal entropies of the subsystems [6].

Now the main task is to translate the above theorem into the language of positive maps. For this purpose we will use the isomorphism between the positive maps and operators which are positive on the product projectors. Namely an arbitrarily established orthonormal basis in $\mathcal{A}_1$ defines an isomorphic map $\mathcal{S}$ from the set of linear maps $\Lambda : \mathcal{A}_1 \to \mathcal{A}_2$ into operators acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$:

$$\mathcal{L}(\mathcal{A}_1, \mathcal{A}_2) \ni \Lambda \to \mathcal{S}(\Lambda) = \sum_i E_i^\dagger \otimes \Lambda(E_i) \in \mathcal{A}_1 \otimes \mathcal{A}_2 \qquad (10)$$

According to [17] a transformation $\Lambda \in \mathcal{L}(\mathcal{A}_1, \mathcal{A}_2)$ is positive iff $\mathcal{S}(\Lambda)$ is Hermitian and $\mathrm{Tr}(\mathcal{S}(\Lambda)P \otimes Q) \geq 0$ for any projectors $P \in \mathcal{A}_1$ and $Q \in \mathcal{A}_2$.

Now, let us take as a basis in $\mathcal{A}_1$ the set of the operators $\{P_{ij}\}_{i,j=1}^{\dim \mathcal{H}_1}$ given by $P_{ij}e_l = \delta_{jl}e_i$ for any established basis $\{e_l\}$ in $\mathcal{H}_1$. Then the condition (9) is equivalent to the following

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[3]It follows from the fact that the set of the continuous functionals of the Banach space of trace class operators is isomorphic to the set of bounded operators.

one

$$\text{Tr}\left\{[(I \otimes \Lambda)\sum_{ij}P_{ji} \otimes P_{ij}]\,\varrho\right\} \geq 0 \tag{11}$$

or

$$\text{Tr}\left\{[(I \otimes \Lambda T)\sum_{ij}P_{ji} \otimes TP_{ij}]\,\varrho\right\} \geq 0 \tag{12}$$

where $T : \mathcal{A}_1 \to \mathcal{A}_1$ is given by $TP_{ij} = P_{ji}$ i.e. it is transposition of the operator written in a basis $\{e_i\}$. Of course T is a positive map and $T^2 = I$. Then any positive map $\tilde{\Lambda} : \mathcal{A}_1 \to \mathcal{A}_2$ is of the form $T\Lambda$ where Λ is a positive map. Putting $P_0 = \frac{1}{d}\sum_{ij} P_{ji} \otimes P_{ji}$ where $d = \dim\mathcal{H}_1$, and using the scalar product in Hilbert space $\mathcal{A}_1 \otimes \mathcal{A}_2$ the condition (12) can be rewritten in the form

$$\langle \varrho, (I \otimes \Lambda P_0)^\dagger \rangle \geq 0. \tag{13}$$

However, positive maps preserve Hermiticity hence the tensor product $I \otimes \Lambda$ also does. Then, as P_0 is Hermitian we obtain

$$\langle \varrho, I \otimes \Lambda P_0 \rangle \geq 0. \tag{14}$$

This is equivalent, by passing to the adjoint maps, to the condition

$$\langle I \otimes \Lambda\varrho, P_0 \rangle \equiv \text{Tr}[P_0(I \otimes \Lambda\varrho)] \geq 0. \tag{15}$$

for any positive maps $\Lambda : \mathcal{A}_2 \to \mathcal{A}_1$. Now, if a state is separable, then obviously the operator $I \otimes \Lambda\varrho$ is positive for any positive Λ. Conversly, if $I \otimes \Lambda\varrho$ is positive for any Λ, then as P_0 is a (one dimensional) projector, the condition (15) is satisfied hence the state is separable. In this way we have obtained the main result of this paper

Theorem 2 *Let ϱ act on Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then ϱ is separable iff for any positive map $\Lambda : \mathcal{A}_2 \to \mathcal{A}_1$ the operator $I \otimes \Lambda\varrho$ is positive.*

On the basis of the above theorem one can characterize the set of the separable states for 2×2 and 2×3 systems. Namely we have

Theorem 3 *A state ϱ acting on $C^2 \otimes C^2$ or $C^2 \otimes C^3$ is separable iff its partial transposition is a positive operator.*

Here the partial transposition ϱ^{T_2} is given by

$$\varrho^{T_2} = I \otimes T \varrho \tag{16}$$

Proof.- If ϱ is separable then of course ϱ^{T_2} is positive[4]. To prove the converse statement we will use Strømer and Woronowicz results [18,19]. Namely the authors showed that any positive map $\Lambda : \mathcal{A}_1 \to \mathcal{A}_2$ with $\mathcal{H}_1 = \mathcal{H}_2 = C^2$ [18] or $\mathcal{H}_1 = C^3$, $\mathcal{H}_2 = C^2$ (equivalently $\mathcal{H}_1 = C^2$, $\mathcal{H}_2 = C^3$) [19] is of the form

$$\Lambda = \Lambda_1^{CP} + \Lambda_2^{CP} T \tag{17}$$

where Λ_i^{CP} are completely positive maps. Now due to the complete positivity of Λ_i^{CP} the map $\Lambda_i = I \otimes \Lambda_i^{CP}$ is a positive one. If ϱ^{T_2} is positive then $\Lambda_1 \varrho + \Lambda_2 \varrho^{T_2}$ also does hence from the Theorem 2 it follows that ϱ is separable.

Remark .- It is easy to see that in the Theorem 2 one can put $\tilde{\Lambda} \otimes \Lambda$ or $\Lambda \otimes I$ instead of $I \otimes \Lambda$ (involving any positive $\tilde{\Lambda} : A_1 \to A_2$, $\Lambda : A_2 \to A_1$). Consequently one can use the condition of positivity of partial transposition with respect to the first space [14]. The condition involving the second space is used here only for convenience.

The above theorem is an important result, as it allows us to determine unambigiously whether a given quantum state of 2×2 (2×3) system can be written as mixture of product states or not. It follows that for the considered cases Peres' conjecture *is valid*. Hence the necessary and sufficient condition is here surprisingly simple. However we will see that it is not true in general. Namely the positive maps [19] are characterized by the formula (17) only for the considered cases. In the Appendix we present the outline of the proof that the condition $\varrho^{T_2} \geq 0$ is, in general, only a necessary one.

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[4]This was first proved by Peres [14]. In our approach it is a consequence of the fact that T is a positive map.

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Examples .- We will use the following notation for matrix elements of any state of $M \times N$ system (i.e. of any density matrix acting on $C^M \otimes C^N$ space) (c.f. [14]):

$$\varrho_{m\mu,n\nu} = \langle e_m \otimes f_\mu | \varrho | e_n \otimes f_\nu \rangle, \tag{18}$$

where $\{e_m\}$ ($\{f_\mu\}$) denotes the arbitrary orhonormal basis in Hilbert space describing first (second) system. Hence the partial transposition of ϱ is defined as:

$$\varrho_{m\mu,n\nu}^{T_2} \equiv \varrho_{m\nu,n\mu}. \tag{19}$$

On the other hand the $M \times N$ state can be written as

$$\varrho = \begin{bmatrix} A_{11} & ... & A_{1M} \\ ... & ... & ... \\ A_{M1} & ... & A_{MM} \end{bmatrix}, \tag{20}$$

with $N \times N$ matrices A_{mn} acting on the second (C^N) space. They are defined by their matrix elements as $\{A_{mn}\}_{\mu\nu} \equiv \varrho_{m\nu,n\mu}$. Then the partial transposition will be realised simply by transposition of all of these matrices, namely

$$\varrho = \begin{bmatrix} A_{11}^T & ... & A_{1M}^T \\ ... & ... & ... \\ A_{M1}^T & ... & A_{MM}^T \end{bmatrix}. \tag{21}$$

Consider now the examples of the two spin-$\frac{1}{2}$ states. The following states has been introduced in the context of inseparability and Bell inequalities [20]

$$\varrho = p|\psi_1\rangle\langle\psi_1| + (1-p)|\psi_2\rangle\langle\psi_2| \tag{22}$$

where $|\psi_1\rangle = ae_1 \otimes e_1 + be_2 \otimes e_2$, $|\psi_2\rangle = ae_1 \otimes e_2 + be_2 \otimes e_1$ with $a, b > 0$, $\{e_i\}$ being standard basis in C^2. In this case we have

$$\varrho = \begin{bmatrix} pa^2 & 0 & 0 & pab \\ 0 & (1-p)a^2 & (1-p)ab & 0 \\ 0 & (1-p)ab & (1-p)b^2 & 0 \\ pab & 0 & 0 & pb^2 \end{bmatrix},$$

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$$\varrho^{T_2} = \begin{bmatrix} pa^2 & 0 & 0 & (1-p)ab \\ 0 & (1-p)a^2 & pab & 0 \\ 0 & pab & (1-p)b^2 & 0 \\ (1-p)ab & 0 & 0 & pb^2 \end{bmatrix}. \tag{23}$$

We will investigate the condition of positivity of the partial transposition ϱ_2^T. For this purpose one needs to check the following determinants

$$W_1 = \varrho^{T_2}_{11,11}\varrho^{T_2}_{22,22} - \varrho^{T_2}_{11,22}\varrho^{T_2}_{22,11}$$

$$W_2 = \varrho^{T_2}_{12,12}\varrho^{T_2}_{21,21} - \varrho^{T_2}_{12,21}\varrho^{T_2}_{21,12} \tag{24}$$

We obtain $W_1 = a^2 b^2 (2p-1)$ and $W_2 = -W_1$. Thus for $ab \neq 0$, $p \neq \frac{1}{2}$ one of the determinants is negative hence by virtue of the necessity of the condition ([14], Theorem 3) we conclude that ϱ is inseparable for such parameters. For a=b both $|\psi_1\rangle$ and $|\psi_2\rangle$ become product states and then ϱ is separable. Finally, for $p = \frac{1}{2}$ it apears that $\varrho^{T_2} = \varrho$ being then positive. In this case, by virtue of sufficiency of the condition (Theorem 3) we obtain separability of the state (22) for $p = \frac{1}{2}$. According to [11] and [20], for the above states both the conditions: 2-entropy inequality [5] and positivity of the partial transposition appear to be equivalent.

Now we will show that the necessary and sufficient condition of the partial transposition is essentially stronger. Consider the states (c.f. [22]):

$$\varrho = p|\Psi_-\rangle\langle\Psi_-| + (1-p)|\uparrow\uparrow\rangle\langle\uparrow\uparrow| \tag{25}$$

with $|\Psi_-\rangle = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 - e_2 \otimes e_1)$, $|\uparrow\uparrow\rangle = e_1 \otimes e_1$. In this case we get

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[5]The α-entropy inequalities [11] are of the form $S_\alpha(\varrho) \leq \max_{i=1,2} S_\alpha(\varrho_i)$ for $1 \leq \alpha \leq \infty$. Here ϱ_i are the reductions of ϱ, and $S_\alpha(\varrho) = \frac{1}{1-\alpha} \ln \mathrm{Tr}\varrho^\alpha$ for $1 < \alpha < \infty$. S_1 is the usual von Neumann entropy and $S_\infty = -\ln \mathrm{Tr}||\varrho||$ [21].

$$\varrho = \begin{bmatrix} 1-p & 0 & 0 & 0 \\ 0 & \frac{p}{2} & -\frac{p}{2} & 0 \\ 0 & -\frac{p}{2} & \frac{p}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} , \quad \varrho^{T_2} = \begin{bmatrix} 1-p & 0 & 0 & -\frac{p}{2} \\ 0 & \frac{p}{2} & 0 & 0 \\ 0 & 0 & \frac{p}{2} & 0 \\ -\frac{p}{2} & 0 & 0 & 0 \end{bmatrix} . \tag{26}$$

Here the first of the determinants (24) is always positive but for the second one we have $W_1 = -\frac{p^2}{4} < 0$ for any $p \neq 0$. Hence the state (25) is separable only if it is a pure product state $| \uparrow\uparrow \rangle\langle \uparrow\uparrow |$. On the other hand it can be seen that 2-entropy inequality is fulfilled for $p \leq \frac{1}{3}$. Thus we see that the necessary and sufficient condition is essentially stronger than the 2-entropy one.

In conclusion, we have provided necessary and sufficient conditions for separability of mixed states. In particular, for 2×2 and 2×3 systems we have obtained an effective criterion of separability. It completes, in particular, Peres' result [14]. It follows however, that the positivity of the partial transposition of the state is not a sufficient condition in general. Finally, we belive that the above results will be useful for practical applications of quantum inseparability.

Appendix.- To prove that $\varrho^{T_2} \geq 0$ is not a sufficient condition for separability in higher dimensions it sufficies to show that the cone $S_T = \{A \in \mathcal{A}_1 \otimes \mathcal{A}_2 : A \geq 0, A^{T_2} \geq 0\}$ is not equal to the cone $S = \{\lambda \varrho : \lambda \geq 0; \varrho - \text{separable state}\}$ or equivalently to show that the dual cones (the cones W and W_{T_2} of functionals which are positive on elements from S and S_T respectively) are not equal. We know that W is isomporphic to the cone of all positive maps $\Lambda : \mathcal{A}_1 \to \mathcal{A}_2$. On the other hand it can be seen that

$$W_{T_2} = \{\text{Tr}[A \cdot \ ] : A = B + C^{T_2}, \ B, \ C - \text{positive operators}\}. \tag{27}$$

Hence W_{T_2} is isomorphic to the cone of all the positive maps of the form (17). However every positive map $\Lambda : \mathcal{A}_1 \to \mathcal{A}_2$ is of the form (17) only for the cases $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{M}_2$, $\mathcal{A}_1 = \mathcal{M}_2, \mathcal{A}_2 = \mathcal{M}_3$ or $\mathcal{A}_1 = \mathcal{M}_3, \mathcal{A}_2 = \mathcal{M}_2$ [19]. Thus, in general, $W \neq W_{T_2}$ hence the condition $\varrho^{T_2} \geq 0$ is not sufficient for separability of ϱ in higher dimensions.

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