22 January 1996

PHYSICS LETTERS A

ELSEVIER

Physics Letters A 210 (1996) 377-381

Quantum α-entropy inequalities: independent condition for local realism?

R. Horodecki [a,1], P. Horodecki [b], M. Horodecki [b]

[a] *Institute of Theoretical Physics and Astrophysics, University of Gdansk, 80-952 Gdansk, Poland*

[b] *Departament of Mathematics and Physics, University of Gdansk, 80-952 Gdansk, Poland*

Received 3 August 1995; revised manuscript received 1 November 1995; accepted for publication 17 November 1995

Communicated by V.M. Agranovich

Abstract

Quantum α-entropy inequalities equivalent to Bell's inequality for pure states are considered in the context of the local hidden variable (LHV) model and compared with Bell's inequalities. For $\alpha = 1, 2$ they are shown to be satisfied by convex combinations of product states and Werner's mixtures admitting the model. The 2-entropy inequality is proven to be stronger than Bell's inequality in the two-spin-$\frac{1}{2}$ case. In the latter, the α-entropy inequalities taken as a joint condition exclude teleportation admitted in spite of the existence of the LHV model for the Werner-Popescu states.

1. Introduction

Since the work of Einstein-Podolsky-Rosen (EPR) [1], various aspects of quantum nonseparability have been investigated in connection with the concept of local realism. Bell first showed [2] that local realism implies constraints on the statistics of distant measurements on two separated systems, which can be violated by the statistical predictions of quantum mechanics. A broad class of these constraints may be written in terms of the expectation value of the Bell observable. The problem which pure states are local (nonlocal) has been solved completely by the study of Bell's inequality due to Clauser, Horne, Shimony and Holt (CHSH) [3]. Namely, it has been shown [4,5] that the only pure states which do not violate Bell's inequalities are direct product states and they are obviously local.

In the case of mixed states the relationship between local realism and violation of the Bell-CHSH inequality is not so obvious [6]. In fact, neither the states admitting the local hidden variable (LHV) model (called here LHV states) nor the ones satisfying the Bell type inequalities are completely characterized. However, some interesting results have been presented recently [6-11]. In particular, the necessary and sufficient condition for violating the Bell-CHSH inequality by mixed two-spin-$\frac{1}{2}$ states has been obtained [10] (see also Ref. [11]). It has been also pointed out [12] that the only two-spin-$\frac{1}{2}$ states violating the Bell-CHSH inequality maximally are the pure ones isomorphic (in the sense of product unitary transformations) to the singlet state.

It is interesting that the different behaviour of the pure and mixed states in the context of quantum nonseparability has been also shown within the information-theoretic approach to compound quantum systems [13,14]. In particular it has been pointed

[1] E-mail: fizrh@halina.univ.gda.pl.

Elsevier Science B.V.
SSDI 0375-9601(95)00930-2

out [13] that for two-component systems only pure states can reach the maximal value of the quantum index of correlation. This justifies a conjecture that there is a nontrivial connection between the notion of purity and quantum nonseparability. Then a natural question arises: are there any constraints involving a connection of this kind? In this paper we present such constraints (called α-entropy inequalities) in terms of the quantum counterpart of the Rényi α-entropies [15,16] and we consider them in the context of local realism and the Bell–CHSH inequality. In Section 2 we introduce the quantum α-entropy (α-E) inequalities and we point out that for the pure states they are equivalent to Bell's inequalities. We also analyse the behaviour of the α-E inequalities for known mixed LHV states. In Section 3, using the criterion for violating the Bell–CHSH inequality [10], we prove that the 2-E inequality is essentially stronger than the Bell–CHSH one in the two-spin-$\frac{1}{2}$ case.

2. α-entropy inequalities and local realism

Let us consider the quantum counterpart of the Rényi α-entropy [15,16]

$$S_\alpha(\varrho) = \frac{1}{1-\alpha} \ln \mathrm{Tr}\, \varrho^\alpha, \quad \alpha > 1. \tag{1}$$

If α tends to 1 decreasingly, one obtains the von Neumann entropy $S_1(\varrho)$ as a limiting case,

$$S_1(\varrho) = -\mathrm{Tr}\, \varrho \ln \varrho. \tag{2}$$

As known, the entropy itself tells us only how close a given mixture is to some pure state [17]. However, it says nothing about how entangled that pure state is. Now, we observe that it is the entropy of the reductions of the pure state which can measure its entanglement [2]. Then it is natural to suppose that a given mixed state is nonclassical if it is more pure than any of its reductions. This means that the following inequality is violated,

$$S_\alpha(\varrho) \geqslant \max_{i=1,2} S_\alpha(\varrho_i), \tag{3}$$

where $\alpha \geqslant 1$, $S_\alpha(\varrho)$ denotes the entropy of the system and $S_\alpha(\varrho_i)$, $i = 1,2$, are the entropies of the subsystems. We find here a good correspondence with the fact that all the discrete classical states [3] satisfy the classical analogue of the α-E inequalities (3). It is remarkable that the set of quantum states satisfying the above inequality is $U_1 \otimes U_2$ invariant, i.e., it is invariant under the transformation $\varrho \to U_1 \otimes U_2 \varrho U_1^\dagger \otimes U_2^\dagger$ where U_1, U_2 are unitary operators. Note that both the set of the LHV states and that of those satisfying the Bell type inequalities possess this property.

Now, it seems reasonable to expect that the degree of violation of (3) will be a suitable measure of the entanglement for mixed states. Then, of course, the inequalities for α-entropies should behave correctly for the pure states. More precisely, we need the following lemma:

Lemma 1. A given pure state violates the α-E inequalities iff it is entangled.

The proof of Lemma 1 is obvious, as the α-entropy of any pure state vanishes while its reductions have positive entropies iff the state is entangled. The latter can be easily seen from the Schmidt decomposition of the state. Thus one can see that the α-E inequalities, Bell's inequalities and the LHV model are equivalent for the pure states. In other words, the inequalities are equivalent to the locality condition in this case.

Consider now the less obvious case of mixed states where neither the question of the existence of the LHV model nor of the violation of Bell's inequality is solved in general.

Maximal violation. It can be seen that for finite dimensional systems ($\mathcal{H} = \mathrm{C}^n \otimes \mathrm{C}^n$) the only states violating (3) maximally are the pure ones having the most disordered reductions (normalized identities), hence are isomorphic (in the sense of the unitary product transformations) to the singlet state

$$|\psi\rangle = \sum_{m=-s}^{s} \frac{(-1)^{s+m}}{\sqrt{2s+1}} e_m \otimes e_{-m}, \tag{4}$$

where e_m are basic vectors in C^n, $m = -s, -s+1, \ldots, s$, $s = \frac{1}{2}(n-1)$. In this case we have

$$S_\alpha(\varrho) - \max_{i=1,2} S_\alpha(\varrho_i) = -\ln n, \quad \alpha \geqslant 1. \tag{5}$$

[2] Note that the positive spectra of the reductions of the pure state coincide, hence their entropies are equal.

[3] By the classical discrete states we mean here the distributions $p_{ij} \geqslant 0$, $\sum_{ij} p_{ij} = 1$.

Now it is interesting to check inequalities (3) for known LHV states. In fact, the latter can be constructed on the basis of the Werner states and the classically correlated ones. Here we shall examine the behaviour of the α-E inequalities for those states.

Classically correlated states. The most classical LHV states are the states which can be approximated by convex combinations of product states in the trace norm (so-called classically correlated states) [6]. Then one can expect them to satisfy (3). We prove here the following theorem.

Theorem 1. For any classically correlated state ϱ on the finite dimensional Hilbert space inequality (3) is satisfied for $\alpha = 1, 2$.

Proof. For $\alpha = 1$ the statement follows from the result of Ref. [14] and the fact that the von Neumann entropy is continuous in the trace norm for the finite dimensional case. Then consider the case $\alpha = 2$. Given a state

$$\varrho = \sum_i p_i \sigma_i \otimes \tilde{\sigma}_i \tag{6}$$

on an arbitrary Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $\dim \mathcal{H} \leqslant \infty$, it is easy to check that for $\alpha = 2$ inequality (3) reads

$$S_2(\varrho) = -\ln \sum_{i,j} p_i p_j \operatorname{Tr}(\sigma_i \sigma_j) \operatorname{Tr}(\tilde{\sigma}_i \tilde{\sigma}_j) \geqslant S_2(\varrho_k), \quad k = 1, 2, \tag{7}$$

where

$$S_2(\varrho_1) = -\ln \sum_{i,j} p_i p_j \operatorname{Tr}(\sigma_i \sigma_j) \tag{8}$$

and similarly for $S_2(\varrho_2)$. The proof of inequality (7) follows directly from the fact that for any density matrices $\sigma, \tilde{\sigma}$ one has $0 \leqslant \operatorname{Tr}(\sigma\tilde{\sigma}) \leqslant 1$. Then, as $S_\alpha(\varrho)$ is continuous in the trace norm for $\alpha > 1$, the 2-E inequality holds for all classically correlated states.

Werner states. Much more sophisticated LHV states are the ones discovered by Werner [6]. They belong to a subclass of the family of $U \otimes U$ invariant states on $C^d \otimes C^d$, $d \geqslant 2$,

$$\varrho_W(\phi, d) = (d^3 - d)^{-1}[(d - \phi)I + (d\phi - 1)V], \tag{9}$$

where $-1 \leqslant \phi \leqslant 1$ and V is defined as $V\varphi \otimes \tilde{\varphi} = \tilde{\varphi} \otimes \varphi$. Only the $0 \leqslant \phi \leqslant 1$ region describes some classically correlated states (6). Contrary to intuition, Werner has proven the existence of the LHV model for

$$\phi = -1 + d^{-2}(d + 1). \tag{10}$$

Note that the spectrum of $\varrho_W(d, \phi)$ is highly degenerated as it has only two eigenvalues: $\lambda_1 = (d-1)(1+\phi)/(d^3 - d)$, $\lambda_2 = (d+1)(1-\phi)/(d^3 - d)$ with multiplicities $(d^2+d)/2$ and $(d^2-d)/2$, respectively. Then, for the Werner states, inequality (3) with $\alpha > 1$ is equivalent to the following one,

$$\frac{d+1}{2}\left(\frac{1+\phi}{d+1}\right)^\alpha + \frac{d-1}{2}\left(\frac{1-\phi}{d-1}\right)^\alpha \leqslant 1. \tag{11}$$

Hence for $d \geqslant 3$ the α-E inequality is satisfied for all $\alpha > 1$ independently of ϕ, thereby including the LHV states given by (10).

It is intriguing that the two-spin-$\frac{1}{2}$ case ($d = 2$) exhibits quite different features. Namely, we see from (11) that (3) is violated by any $\varrho_W(2, \phi < 0)$ for all sufficiently large α. Hence, if one puts (3) for all $\alpha > 1$ as a classical condition, it will cut off all of the considered states which are *not* classically correlated. This is an interesting result, as they are just the states which have been shown by Popescu [9] to be useful for teleportation [18]. Thus we see that the α-E inequalities grasp a nonlocality which occurs here despite the existence of the LHV model ($\phi = -\frac{1}{4}$).

Remark. It is easy to see that the set of LHV states is convex [6] and $U_1 \otimes U_2$ invariant. The set of the states satisfying the 1-E inequality has the same properties. In particular the convexity of the latter follows immediately from the concavity of the function $f(\varrho) \equiv S_1(\varrho) - S_1(\varrho_r)$ (ϱ_r denotes a reduction of ϱ) [19]. Moreover, if we restrict our attention to finite dimensional systems, the function $f(\varrho)$ appears to be continuous in the trace norm, hence the above set is closed in this norm. Then, we obtain the validity of the 1-E inequality for *all* known LHV states in $\mathcal{H} = C^d \otimes C^d$, $d \geqslant 2$, i.e., the convex hull of the classically correlated states and those of the form

$$\varrho = U_1^\dagger \otimes U_2^\dagger \varrho_W(d, \phi) U_1 \otimes U_2, \tag{12}$$

where $\phi = -1 + d^{-2}(d+1)$ and U_1, U_2 are unitary transformations[4].

3. 2-entropy inequality and Bell's inequality

In this section we compare the Bell-CHSH inequality and the 2-E one in the two-spin-$\frac{1}{2}$ case. To investigate the possible relationship between the inequalities in this case, it is convenient to use the Hilbert-Schmidt space formalism. Consequently, we have the following representation of the 2×2 density matrix,

$$\varrho = \frac{1}{4}\left(I \otimes I + \boldsymbol{r} \cdot \boldsymbol{\sigma} \otimes I + I \otimes \boldsymbol{s} \cdot \boldsymbol{\sigma} + \sum_{n,m=1}^{3} t_{nm} \sigma_n \otimes \sigma_m\right), \tag{13}$$

where I stands for identity operator, $\{\sigma_n\}_{n=1}^{3}$ are the standard Pauli matrices, $\boldsymbol{r}$, $\boldsymbol{s}$ are vectors in R^3, $\boldsymbol{r} \cdot \boldsymbol{\sigma} = \sum_{i=1}^{3} r_i \sigma_i$.

It can be seen now that in the above representation the state parameters fall into two different classes: the first (matrix T) responsible for correlations and the second ($\boldsymbol{r}$ and $\boldsymbol{s}$) describing the local properties of the state. Indeed, the mean value of the Bell observable depends only on T [10] while $\boldsymbol{r}$ and $\boldsymbol{s}$ are simply the parameters of the reductions of the state ϱ. As we will see below, this allows one to derive the exact connection between the 2-E inequality and the Bell-CHSH one. Using the representation (13) we find the inequality

$$S_2(\varrho) \geqslant \max_{i=1,2} S_2(\varrho_i) \tag{14}$$

to be equivalent to the following one,

$$\|T\|^2 \leqslant 1 - \left|\|\boldsymbol{s}\|^2 - \|\boldsymbol{r}\|^2\right|, \tag{15}$$

where $\|T\|^2 \equiv \mathrm{Tr}(T^\dagger T)$ and $\|\boldsymbol{s}\|$ stands for the Euclidean norm. Now, we obtain the following theorem:

Theorem 2. Given an arbitrary two-spin-$\frac{1}{2}$ state ϱ, if the 2-E inequality holds then the Bell-CHSH inequality is satisfied and the maximal possible mean value of Bell observable is restricted by the inequality

$$\langle \mathcal{B}_{\max} \rangle_\varrho \leqslant 2\sqrt{1 - \left|\|\boldsymbol{r}\|^2 - \|\boldsymbol{s}\|^2\right|}. \tag{16}$$

Proof. The maximal possible mean value of the Bell observable in the state ϱ is [10]

$$|\langle \mathcal{B}_{\max} \rangle_\varrho| = \max_{\mathcal{B}_{\mathrm{CHSH}}} |\langle \mathcal{B}_{\mathrm{CHSH}} \rangle_\varrho| = 2\sqrt{M(\varrho)}, \tag{17}$$

where $M(\varrho) = \max_{i<j}(u_i + u_j)$ and u_i, $i = 1,2,3$, are eigenvalues of the matrix $U = T^\dagger T$. Then, the Bell-CHSH inequality is fulfilled iff $M(\varrho) \leqslant 1$ [10]. Hence, for each state satisfying (15) we make the following straightforward estimate,

$$M(\varrho) = \max_{i<j}(u_i + u_j) \leqslant \sum_{i=1}^{3} u_i = \mathrm{Tr}\, U = \|T\|^2 \leqslant 1 - \left|\|\boldsymbol{s}\|^2 - \|\boldsymbol{r}\|^2\right|. \tag{18}$$

Example. Consider the Werner mixtures $\varrho_W(\phi, 2)$. In this case the regions of validity of the two inequalities are

$$\tfrac{1}{2} - \tfrac{3}{4}\sqrt{2} \leqslant \phi \leqslant 1 \quad \text{(Bell-CHSH) [10]}, \tag{19}$$

$$\tfrac{1}{2} - \tfrac{1}{2}\sqrt{3} \leqslant \phi \leqslant 1 \quad \text{(2-E)}. \tag{20}$$

In particular, for $-1 \leqslant \phi \leqslant \frac{1}{2}$, putting $\beta = \frac{1}{3}(1 - 2\phi)$ one obtains the Werner-Popescu states [11] $\varrho = \frac{1}{4}(1-\beta)I + \beta|\psi\rangle\langle\psi|$, where $|\psi\rangle$ is the singlet state. In this case inequalities (20) and (20) restricted to the region $-1 \leqslant \phi \leqslant \frac{1}{2}$ take the form

$$0 \leqslant \beta \leqslant \frac{1}{\sqrt{2}} \quad \text{(Bell-CHSH)}, \tag{21}$$

$$0 \leqslant \beta \leqslant \frac{1}{\sqrt{3}} \quad \text{(2-E)}. \tag{22}$$

Remark. The fact that the 2-E inequality is stronger than the Bell-CHSH one for the spin-$\frac{1}{2}$ case does not mean the same for higher dimensions. To see this, consider the states [7] defined on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2 \equiv \mathrm{C}^{2n} \otimes \mathrm{C}^{2n} = \oplus_{i,j=1}^{n} \mathcal{H}_{ij}$, $\mathcal{H}_{ij} = \mathrm{C}^2 \otimes \mathrm{C}^2$ being any $U_1 \otimes U_2$ transformations of arbitrary mixtures of the singlet states $|\phi_{ij}\rangle\langle\phi_{ij}|$ defined on $\mathcal{H}_{ij}$, i.e.

$$\varrho = \sum_{ij} p_{ij} |\psi_{ij}\rangle\langle\psi_{ij}|, \tag{23}$$

[4] Note that although the Werner LHV state with $d = 2$ violates the inequalities for all large α, it satisfies the α-E inequalities for $\alpha = 1, 2$.

where $|\psi_{ij}\rangle = U_1 \otimes U_2 |\phi_{ij}\rangle$ and the usual singlet states $|\phi_{ij}\rangle = (1/\sqrt{2})(|\uparrow\rangle_i \otimes |\downarrow\rangle_j - |\downarrow\rangle_i \otimes |\uparrow\rangle_j)$. Any state of the form (23) violates the Bell-CHSH inequality maximally [7]. The 2-E inequality reads in this case

$$S_2(\varrho) = -\ln \sum_{ij} p_{ij}^2$$
$$\geqslant -\max\left\{ \ln\left[\tfrac{1}{2}\sum_i \left(\sum_j p_{ij}\right)^2\right], \ln\left[\tfrac{1}{2}\sum_j \left(\sum_i p_{ij}\right)^2\right]\right\}. \tag{24}$$

Now, taking $p_{ij} = 1/n^2$ we obtain that the inequality is satisfied for all $n > 2$.

4. Conclusions

We have considered the quantum α-entropy inequalities in the context of local realism and Bell's inequalities. We have shown that the α-E inequalities are violated by any pure entangled state. For the mixed states the properties depend both on the parameter α and the dimension of the Hilbert space ($\mathcal{H} = C^d \otimes C^d$, $d < \infty$). In particular, we have shown that the 1-E inequality is satisfied by all known LHV states. For $\alpha > 1$ and $d \geqslant 3$, the inequality is satisfied by the whole Werner family (9). The above results include the ones obtained earlier within the information-theoretic approach to the compound quantum systems [14].

We have proven, for $d = 2$, that the α-E inequalities, if taken jointly, appear to be a very strong condition. Namely, all the states (9) which are not classically correlated do not satisfy such a condition. The latter excludes, in this way, all the Werner states which are useful for teleportation, even though they admit the LHV model.

Subsequently, we have pointed out that the 2-E inequality holds for all classically correlated states for $2 \leqslant d \leqslant \infty$. Moreover, in the two-spin-$\frac{1}{2}$ case it appears to be stronger than the Bell–CHSH one. Thus, it can be a useful tool for seeking the states which cannot be easily classified as local (nonlocal) ones.

Finally it should be emphasized that, in general, the presented inequalities appear to be rather an independent than a stronger condition in comparison with the Bell–CHSH inequality. However, the theorem establishing the relationship between the Bell–CHSH inequality and the 2-E one for 2×2-systems confirms earlier speculations [13,14,20] concerning a possible connection between Bell's inequalities and the information-theoretic approach.

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