Some Features of the Conditional q-Entropies of Composite Quantum Systems

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Abstract

The study of conditional q-entropies in composite quantum systems has recently been the focus of considerable interest, particularly in connection with the problem of separability. The q-entropies depend on the density matrix ρ through the quantity $\omega_q = Tr\rho^q$, and admit as a particular instance the standard von Neumann entropy in the limit case $q \to 1$. A comprehensive numerical survey of the space of pure and mixed states of bipartite systems is here performed, in order to determine the volumes in state space occupied by those states exhibiting various special properties related to the signs of their conditional q-entropies and to their connections with other separability-related features, including the majorization condition. Different values of the entropic parameter q are considered, as well as different values of the dimensions N_1 and N_2 of the Hilbert spaces associated with the constituting subsystems. Special emphasis is paid to the analysis of the monotonicity properties, both as a function of q and as a function of N_1 and N_2, of the various entropic functionals considered.

I. INTRODUCTION

Some entangled states of quantum composite systems (in particular, all pure entangled states) exhibit the notable property of having an entropy smaller than the entropies of their subsystems. This feature of composite quantum systems, and its connections with other of their entanglement-related properties, has been recently investigated by several authors [1–11]. The phenomenon of entanglement is one of the most fundamental and non-classical features exhibited by quantum systems [12,13]. Quantum entanglement is the basic resource required to implement several of the most important processes studied by quantum information theory [13–19], such as quantum teleportation [20], superdense coding [21] and the exciting issue of quantum computation [18]. A state of a composite quantum system constituted by the two subsystems A and B is called "entangled" if it can not be represented as a convex linear combination of product states. In other words, the density matrix ρ^{AB} represents an entangled state if it can not be expressed as

$$\rho^{AB} = \sum_k p_k \, \rho_k^A \otimes \rho_k^B, \tag{1}$$

with $0 \leq p_k \leq 1$ and $\sum_k p_k = 1$. On the contrary, states of the form (1) are called separable. The above definition is physically meaningful because entangled states (unlike separable states) cannot be prepared locally by acting on each subsystem individually [22]. Due to the significance of quantum entanglement, it is important to survey the state space of composite quantum systems, in order to get a clear picture of the concomitant entanglement properties, and of the relationships between entanglement and other relevant features exhibited by the quantum states. Significant advances have been made by a program that attempts performing a systematic exploration of the space of arbitrary (pure or mixed) states of composite quantum systems [23–25] in order to determine the characteristic features shown by these states with regards to the phenomenon of quantum entanglement [23–30].

Separable quantum states share with classical composite systems the following basic property: the entropy of any of its subsystems is always equal or smaller than the entropy

characterizing the whole system [31]. In contrast, as already mentioned, a subsystem of a quantum system described by an entangled state may have an entropy greater than the entropy of the whole system, thus violating the concomitant classical entropic inequalities. This situation holds for the well known von Neumann entropy, as well as for the more general q-entropic (or q-information) measures [1–11], which incorporate both Rényi's [32] and Tsallis' [33–35] families of measures as special instances. These entropic functionals are characterized by a real parameter q.

The alluded to classical entropic inequalities constitute necessary and sufficient separability criteria for pure states. The situation is, however, more involved in the case of mixed states. In the latter case we can find entangled states that do not violate these inequalities. Consequently, the classical entropic inequalities provide only necessary separability criteria. As a matter of fact, the main motivation for studying the classical entropic inequalities (and their violation by some entangled states) is not any more the development of practical separability criteria. This is the case particularly since the introduction of the Positive Partial Transposition (PPT) criterion by Peres [36], and the related results obtained by the Horodeckis [37]. However, *the violation of the classical entropic inequalities is interesting in its own right, because they constitute, from the perspective of classical physics, a highly counterintuitive property exhibited by some entangled quantum states.* Moreover, this non-classical feature of certain entangled states is of a clear and direct information-theoretical nature.

The goal of the present work is to investigate further aspects of the relationship between quantum separability and the violation of the classical q-entropic inequalities. By performing a systematic numerical survey of the space of pure and mixed states of bipartite systems of any dimension we determine, for different values of the entropic parameter q, the volume in state space occupied by those states characterized by positive values of the conditional q-entropies. We pay particular attention to the monotonic tendency shown by these separability ratios as they evolve with q from finite values to the limiting case $q \to \infty$, for any Hilbert spaces dimension. The paper is organized as follows. In section II we briefly

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summarized some basic properties of both the q-entropies and the conditional q-entropies. Our main results are discussed in section III. Finally, some conclusions are drawn in section IV.

II. Q-CONDITIONAL ENTROPIES

The "q-entropies" depend upon the eigenvalues p_i of the density matrix ρ of a quantum system through the quantity $\omega_q = \sum_i p_i^q$. More explicitly, we shall consider either the Rényi entropies [32],

$$S_q^{(R)} = \frac{1}{1-q} \ln (\omega_q), \tag{2}$$

or the Tsallis' entropies [33–35]

$$S_q^{(T)} = \frac{1}{q-1}(1 - \omega_q), \tag{3}$$

which have found many applications in many different fields of Physics. These entropic measures incorporate the important (because of its relationship with the standard thermodynamic entropy) instance of the von Neumann measure, as a particular limit $(q \to 1)$ situation

$$S_1 = -Tr\left(\hat{\rho}\ln\hat{\rho}\right). \tag{4}$$

We will be here rather more interested in *conditional $q-entropies$* than in total entropies, because of the former's relation with the issue of quantum separability. Conditional entropic measures are defined as

$$S_q^{(T)}(A|B) = \frac{S_q^{(T)}(\rho_{AB}) - S_q^{(T)}(\rho_B)}{1 + (1-q)S_q^{(T)}(\rho_B)} \tag{5}$$

for the Tsallis case, while its Rényi counterpart is

$$S_q^{(R)}(A|B) = S_q^{(R)}(\rho_{AB}) - S_q^{(R)}(\rho_B). \tag{6}$$

The matrix ρ_{AB} denotes an arbitrary quantum state of the composite system $A \otimes B$, not necessarily factorizable nor separable, and $\rho_B = Tr_A(\rho_{AB})$ (the conditional q-entropy $S_q^{(T)}(B|A)$ is defined in a similar way as (5), replacing ρ_B by $\rho_A = Tr_B(\rho_{AB})$). Interest in the conditional q-entropy (5) arises in view of its relevance with regards to the separability of density matrices describing composite quantum systems [5,6]. For separable states, we have [11]

$$S_q^{(T)}(A|B) \geq 0,$$
$$S_q^{(T)}(B|A) \geq 0. \tag{7}$$

As already mentioned, there are entangled states (for instance, all entangled pure states) characterized by negative conditional q-entropies. That is, for some entangled states one (or both) of the inequalities (7) are not verified. Since just the sign of the conditional entropy is important here, we can either use Tsallis' or Rényi's entropy, for (5) and (6) will always share the same sign. In what follows, when we speak of the positivity of either Tsallis' conditional entropy (5) or of Rényi's conditional entropy (6), we will make reference to the "classical q-entropic inequalities" issue.

III. VOLUMES IN STATE SPACE OCCUPIED BY STATES OF SPECIAL ENTROPIC PROPERTIES.

The systematic numerical study of pure and mixed states of a bipartite quantum system of arbitrary dimension $N = N_1 \times N_2$ requires the introduction of an appropriate measure μ defined over the corresponding space $\mathcal{S}$ of general quantum states. Such a measure is necessary in order to compute volumes within the space $\mathcal{S}$. The measure we are going to adopt in the present approach was introduced by Zyczkowski et $al.$ in several valuable contributions [23,24], and was later extensively used in the systematic exploration of the space of arbitrary (pure or mixed) states of composite quantum systems [28–30,38]. Any given arbitrary (pure or mixed) state ρ of a quantum system described by an N-dimensional Hilbert space can always be expressed as the product of three matrices

$$\rho = UD[\{\lambda_i\}]U^\dagger. \tag{8}$$

U stands for an $N \times N$ unitary matrix and $D[\{\lambda_i\}]$ is an $N \times N$ diagonal matrix whose diagonal elements are $\{\lambda_1, \ldots, \lambda_N\}$, with $0 \leq \lambda_i \leq 1$, and $\sum_i \lambda_i = 1$. The λ_i's are, of course, the eigenvalues of ρ. The Haar measure ν [39] yields a unique and uniform measure over the group of unitary matrices $U(N)$. On the other hand, the N-simplex Δ, defined by all the real N-uples $\{\lambda_1, \ldots, \lambda_N\}$ (Cf. Eq. (8)), is a subset of an $(N-1)$-dimensional hyperplane of $\mathcal{R}^N$. Consequently, the standard normalized Lebesgue measure $\mathcal{L}_{N-1}$ on $\mathcal{R}^{N-1}$ provides a natural measure for Δ. Thus, the Haar measure ν on $U(N)$ and Δ on the N-simplex lead then to a natural measure μ on the set $\mathcal{S}$ of all the states of our quantum system [23,24,39], namely,

$$\mu = \nu \times \mathcal{L}_{N-1}. \tag{9}$$

All our present considerations are based on the assumption that the uniform distribution of states of a quantum system is the one determined by the measure (9). Thus, in our numerical computations we are going to randomly generate states according to the measure (9). The situation encountered in [38] was the following one: the volume in phase space corresponding to those states complying with the classical q-entropic inequalities monotonically decreases as the entropic parameter q increases, adopting its minimum value in the limit case $q \to \infty$. In this limit case, the volume of states with positive conditional entropies adopts simultaneously: i) its lowest value and also ii) the one most closely resembling that of the set of states with positive partial transpose (PPT). The volume of states with positive conditional q-entropies is, however, even in the limit case $q \to \infty$, larger than the volume associated with states with a positive partial transpose. This means that, regarded as a separability criterion, the classical entropic inequality with $q = \infty$ is (among the conditional q-entropic criteria) the strongest one, though it is not as strong as the PPT criterion. In point of fact, it has been proven that there is no necessary and sufficient criteria for quantum separability based solely on the eigenvalues of ρ_{AB}, ρ_A, and ρ_B. Our main concern here is *not* the study of the classical inequalities *qua* separability criteria. Their study is interesting *per se* because

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it provides us with additional insight into the issue of quantum separability, on account of their intuitive information-theoretical nature. We want to survey the state-space in order to obtain a picture, as detailed as possible, of i) how the signs of the q-conditional entropies are correlated with *other* entanglement-related features of quantum states, and ii) how these correlations depend both on the value of q and on the dimensionality of the systems under consideration.

As reported in [38], the volume occupied by states with positive values of the conditional q-entropies decreases with q in a monotonous fashion as the entropic parameter grows from finite q-values to $q = \infty$. It is to be remarked that some authors had previously conjectured [5] that the conditional q-entropy $S_q(A|B)[\rho]$, evaluated in each particular density matrix ρ, is a monotonous decreasing function of q. This conjecture implies that it should be enough to consider the value $q \to \infty$ in order to decide on the positivity of the conditional q-entropies for all q. If this conjecture were true it would lead, as an immediate consequence, to the monotonous behavior (as a function of q) of the volume of states with positive values of the conditional q-entropies.

Alas, one can find several low-rank counterexamples to the monotonicity of the conditional Tsallis or Rényi entropies with q (a particularly interesting case of non-monotonicity with q of Tsallis' conditional entropies has been recently discussed by Tsallis, Prato, and Anteneodo in [8]). A rather surprising situation ensues: the volumes associated with positive valued conditional q-entropies behave in a monotonous way *in spite of the fact that the alluded to conjecture is not valid.* One of the aims of the present effort is precisely to investigate this point in more detail. By recourse to a Monte Carlo calculation we have determined numerically (both for two-qubits and qubit-qutrit systems) the proportion of states which behave monotonously as q changes. This involves exploring either the 15-dimensional space of two-qubits ($N = 4$) or the 35-dimensional space of one qubit-one qutrit mixed states. Table I shows the results for different ranks, dimensions, and entropies used for the mixed state ρ. In each case (that is, for each set of values for q, total Hilbert Space dimension $N = N_1 \times N_2$, and rank of ρ) we have randomly generated 10^7 density matrices. This implies

that the relative numerical error associated with the values reported in Table I is less than 10^{-3}. We consider it remarkable that most of the states have a conditional entropy that behaves monotonically with q, this fact being more pronounced for the case of the Tsallis entropy. The proportion of these states diminishes as the rank of the state ρ decreases, regardless of the dimension and the conditional entropy used. The general trend suggested by Table I is that the percentage of states with monotonous conditional q-entropies increases with the total (Hilbert space's) dimension of the system and, for a given total dimension, increases with the rank of the density operator. This is fully consistent with the monotonic behavior (as a function of q) exhibited by the total volume corresponding to states with positive conditional q-entropies.

Examples of non-monotonous behavior of the conditional q-entropy are depicted in Fig. 1, for a pair of two-qubits states of range four. The dashed line corresponds to a state whose conditional entropy, although non-monotonous, remains always positive. The continuous line refers to an entangled state such that $S_q^{(T)}(A|B) < 0$ for large enough q-values. The q-interval in which the monotonicity of the last state is broken is depicted in the inset. One gathers form these results that it seems correct to regard $q \to \infty$ as the right value to ascertain positivity for a single given state ρ, as was recently suggested by Abe [10] on the basis of his analysis of a mono-parametric family of mixed states for multi-qudit systems.

To further explore the issue of monotonicity we have computed the fraction of the total state space volume occupied by (that is, the probabilities of finding) states with positive conditional q-entropies (for both (i) different finite values of q and (ii) $q = \infty$), in the case of bipartite quantum systems described by Hilbert spaces of increasing dimensionality. Let i) N_1 and N_2 stand for the dimension of the Hilbert space associated with each subsystem, and ii) $N = N_1 \times N_2$ be the dimension of the Hilbert space associated with the concomitant composite system. We have considered two sets of systems: (1) systems with $N_1 = 2$ and increasing values of N_2, and (2) systems with $N_1 = 3$ and increasing dimensionality. The computed probabilities for the first set of systems are depicted in Fig. 2, as a function of the total dimension N. The case of the second set is depicted in Fig. 3. In order to obtain

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each point in Figures 2 and 3 (as well as to obtain each of the points appearing in the subsequent Figures of this article) we have randomly generated 10^7 density matrices. This leads to Monte Carlo results with a relative, numerical error less than 10^{-3}. In Fig. 2 one plots different values of the probabilities associated with positive conditional q-entropy for (a) $q = 2, 4, 8, 16$, and ∞ and (b) different values of the total dimension N of the system.

With respect to the behavior of these probabilities, one is to focus attention upon two aspects: i) evolution with q for a given N and ii) evolution with the dimension for fixed q. In the first instance one clearly sees a common behavior for all N. As q increases, the probabilities of finding states that comply with the classical entropic inequalities decreases, with different rates, down to the saturating value corresponding to $q \to \infty$. This tendency is universal for any dimension and answers the query about the monotonicity of the "q-volumes" occupied by states behaving "classically" in what regards to their conditional q-entropy. With respect to the second aspect, i.e., evolution with N for fixed q, one sees that for any value of q, and for $N \leq 6$, all the curves of Fig. 3 behave in an approximately linear fashion (sure enough, this linear behavior can not continue for arbitrarily large values of N). There is also a sort of "transition" in the behavior of the probabilities, depending on the value of q. For small q values, as the total dimension $N = 2 \times N_2$ grows, the conditional q-entropies tend to behave classically: the probabilities of positive conditional entropies increase in a monotonous way with N and approach 1. This "classical behavior" is ruled out beyond a certain value of q, when the system, as its dimension increases, exhibits the quantum feature given by negative conditional entropies. This behavior is more pronounced for higher q-values. Interestingly, these two behaviors seem to be "separated" by a certain "critical" value $q = q*$. The probabilities of finding states with positive conditional $(q = q^*)$-entropies are (when keeping N_1 constant) rather insensitive to changes in N_2. In the case of Fig. 1 we have $q* \in [2, 4]$.

We pass now to the consideration of systems for which the former qubit is replaced by a qutrit (Fig. 3). This figure exhibits the features already encountered in Fig. 2 (for the same values of q). For a fixed dimension, all probabilities are monotonous with q and, again, the

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curves exhibit two types of qualitative behavior. As q grows, one seems to pass from one of them to the other at a certain critical $q = q^*$-value. This special q-value discriminates between i) the region where the "classical" behavior of the conditional entropies becomes more important with increasing N, from ii) the region where negative conditional entropies (which can not occur classically) are predominant for large N. In this case, q^* lies, as before, between the values 2 and 4. It is interesting to notice, after glancing at both Figs. 2 and 3, that the probabilities of finding states with positive conditional q-entropies are not just a function of the *total* dimension $N = N_1 \times N_2$, as is the case, with good approximation, for the probability of having a positive partial transpose (this was already noted in [38]). The probabilities of having positive conditional q-entropies depend on the *individual dimensions* (N_1 and N_2) of both subsystems.

A better insight into the monotonicity issue (how the probabilities of having positive conditional entropies change with q) is provided by Figs. 4 and 5. In Fig. 4 we depict, for $N = 2 \times N_2$ systems, the evolution of those probabilities with q, for fixed values of the total dimension N. A similar evolution is plotted in Fig. 5 for $N = 3 \times N_2$ systems. The curves in these two figures behave in similar fashion. For given values of N_1 and N_2, the probabilities decrease in a monotonous way with q. On the other hand, for a fixed q-value, the probabilities behave in a monotonous fashion with N_2. Again (as was already mentioned in connection with Figures 2 and 3), there seem to be a special q-value, q^*, such that above q^* the probabilities decrease with N_2, while below q^*, the opposite behavior is observed.

Thus far we have considered specific systems for which one of the parties has fixed dimension while that of its partner augments. But what if we consider the case of composite systems with $N_1 = N_2 = D$ (that is, two-*qudits* systems)?. It was already shown in [38], for the case $q = \infty$, that the concomitant probabilities of finding states complying with the classical entropic inequalities (that is, having positive both conditional q-entropies) exhibit a behavior that is quite different from the one previously discussed. Indeed, the numerical evidence gathered for $q = \infty$ in [38] suggests that, for an $N_1 \times N_2$-composite system of increasing dimensionality, the probability trends that interest us here are clearly different

if one considers either (i) increasing dimension for one of the subsystems and constant dimension for the other, or (ii) increasing dimension for both subsystems. In case (i) we have that, for $q = \infty$, the probabilities of finding states with positive conditional q-entropies diminish as N grows. In the present effort we have extended the study of case (i) to finite values of q, obtaining a similar type of behavior for q-values above a certain special value q^*. In case (ii) the probability of finding states complying with the classic entropic inequalities steadily grows with N and approaches unity as $N \to \infty$. The reader is referred to Fig. 10 of [38]. The evolution of the probabilities for systems with $N = D \times D$ for finite q does not qualitatively differ from that pertaining to the limit case $q \to \infty$. As far as monotonicity is concerned, these probabilities share the monotonic behavior (with q) so far discussed for a fixed dimension.

We will now look at two-qudits systems from the following, different perspective: instead of considering the probability of states having positive conditional entropies for both parties, consider the behavior, as a function of the entropic parameter q, of the *global probability that an arbitrary state of a two-qudit systems* exhibits either (i) both a positive conditional q-entropy and a positive partial transpose, or, (ii) both a negative conditional q-entropy and a non positive partial transpose. That is, we now focus attention on the probability that i) Peres' PPT criterion and ii) the signs of the conditional q-entropies (regarded as the basis of a separability criterion), *both* lead to the same answer as far as separability or entanglement are concerned.

Figs. 6 and 7 illustrate the cases $D = 3$ ($N = 3 \times 3$) and $D = 4$ ($N = 4 \times 4$), respectively. We depict there the referred to probabilities as a function of $1/q$, for values of $q \in [2, 20]$. Keeping also in mind the results plotted in Fig. 5 of Ref. [38] (for $D = 2$), we conclude that (i) agreement with Peres' criterion becomes larger in all cases as q increases up to $q = \infty$, and (ii) the largest degree of agreement, achieved in this limit case, rapidly decreases as D augments from its $D = 2$-amount (nearly 75 per cent [38]) to the $D = 3$-one (Fig. 6) of nearly 22 per cent, and further down to the $D = 4$-instance (Fig. 7) of 4.5 per cent.

We also computes, for composite systems of (Hilbert Space) dimensions $2 \times N_2$ and $3 \times N_3$,

the volumes occupied by those states complying with the majorization separability criterion [31]. The results are depicted in Fig. 8, where the alluded to volumes are compared with the volumes associated with states endowed with positive $(q = \infty)$-conditional entropies. It can be seen in Fig. 8 that the qualitative behaviour of these volumes (as a function of N_2) is similar. In particular, for states of dimension $3 \times N_2$, the volumes associated with the majorization condition are very close to those associted with positive $(q = \infty)$-conditional entropy.

IV. CONCLUSIONS

A systematic survey of the space of pure and mixed states of bipartite systems of arbitrary dimension has been performed, in order to study in detail the behavior of the state space-volume occupied by those states endowed with positive conditional q-entropies, as a function of both the parameter q and the dimensions N_1 and N_2 of the constituting subsystems. The monotonicity with q of both the Tsallis and Rényi entropies has been analyzed for two-qubits and a qubit-qutrit system, for different values of the rank of the pertinent (mixed state) statistical operator ρ. In spite of the fact that most states have a Tsallis or Rényi conditional entropy behaving in a monotonic fashion with q, the proportion of these states always diminishes as the rank of the state ρ decreases, regardless of the dimension of the system and the conditional entropy used. The proportion of states with a monotonous conditional entropy is larger for the case of the Tsallis information measure.

Concerning the volumes in state-space associated with states complying with the "classical" entropic inequalities, we have presented results for states of dimensions 2×2 up to 2×10 and for states ranging from 3×3 to 3×7. In general, the volume occupied by states with positive conditional q-entropies (for a given q) is not a function solely of the total dimension $N = N_1 \times N_2$. Instead, it depends on both subsystems' dimensions, N_1 and N_2. For a given fixed value of $N_1 = 2, 3$, and for q-values above a special value q^* (which itself depends upon N_1), the alluded to volume decreases in a monotonous way with N_2.

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In addition, the behavior of two-qudits systems of dimension 3×3 and 4×4 has also been taken into account. In all these cases, our numerical results indicate that the probability of finding states endowed either with (i) positive conditional q-entropies and a positive partial transpose, or (ii) negative conditional q-entropies and a non positive partial transpose, increase in a monotonic way with q. However, the largest value of this probability (corresponding to $q = \infty$) diminishes in a very fast fashion with D.

Finally, we computed the volumes (for composite systems with Hilbert space dimensions $2 \times N_2$ and $3 \times N_2$) occupied by states complying with the majorization separability criterion, and compared them with the volumes corresponding to states endowed with positive $(q = \infty)$-conditional entropies. The qualitative behaviour (as a function of N_2) of the volumes associated with states complying (i) with the majorization condition and (ii) with the classical, $(q = \infty)$-conditional entropic inequalities, turned out to be qualitatively alike (and very close to each other in the case of systems of dimension $3 \times N_2$).

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TABLES

TABLE I. Proportion of states which behave monotonously as q changes. Both Tsallis' and Rényi's conditional entropies, for two-qubits and one qubit-one qutrit systems, are considered. For a given dimensionality one is to notice how the system evolves with the rank of the pertinent state ρ.

	Tsallis	Rényi
$2 \times 2.Rank,\ 4$	0.972	0.719
$Rank,\ 3$	0.850	0.434
$Rank,\ 2$	0.204	0.003
$2 \times 3.Rank,\ 6$	0.996	0.888
$Rank,\ 5$	0.99	0.79
$Rank,\ 4$	0.96	0.64
$Rank,\ 3$	0.84	0.38
$Rank,\ 2$	0.32	0.003

FIGURE CAPTIONS

Fig. 1- Conditional Tsallis entropy $S_q(B|A)$ for two sample states ρ of a two-qubits system (with rank 4) which do not change in monotonous fashion when q grows. The dashed line corresponds to a state whose conditional entropy remains positive for all q-values. The solid line corresponds to a state whose conditional entropy eventually becomes negative (and, consequently, the state becomes entangled) for large values of q. The inset depicts, for the last case, details of the rather tiny region where monotonicity is broken. All quantities depicted are dimensionless.

Fig. 2- Probability of finding a state ρ for systems in $2 \times N_2$ dimensions which, for different values of q, has its two conditional q-entropies positive. Different curves are assigned to various values of q. These curves "saturate" when the limit $q \to \infty$ is reached. Also, two regimes of growth with the dimension are to be noticed. See text for details. The lines are just a guide for the eye. All quantities depicted are dimensionless.

Fig. 3- Same as in Fig. 2 for systems of $3 \times N_2$ dimensions. Values of probabilities are higher and the rate of saturation is different. All quantities depicted are dimensionless.

Fig. 4- Probability of finding a state ρ (for systems of $2 \times N_2$ dimensions) which has its two conditional q-entropies of a positive nature vs. $1/q$. Different curves correspond to different dimensions. The monotonic decreasing behavior of these probabilities is apparent. The lines are just a guide for the eye. All quantities depicted are dimensionless.

Fig. 5- Same as in Fig. 5, but for systems in $3 \times N_2$ dimensions. All quantities depicted are dimensionless.

Fig. 6- Probability (as a function of q) of finding a two-qudits state ($D \times D, D = 3$) which is characterized by either i) positive conditional q-entropy and positive partial transpose, or ii) a negative conditional q-entropy and a non positive partial transpose. As q grows so does

the degree of agreement with the PPT-criterion, from the von Neumman ($q = 1$) case to the "best" $q \to \infty$ improves. The lines are just a guide for the eye. All quantities depicted are dimensionless.

Fig. 7- Same as in Fig. 7 for a system of $D = 4$ two-qudits. Notice that, as compared to the $D = 3$ case, the values of the pertinent probabilities are considerably smaller here. All quantities depicted are dimensionless.

Fig. 8- The volumes (for composite systems with Hilbert space dimensions $2 \times N_2$ and $3 \times N_2$) occupied by (i) states complying with the majorization separability criterion and (ii) states endowed with positive ($q = \infty$)-conditional entropies. The lines are just a guide for the eye. All quantities depicted are dimensionless.
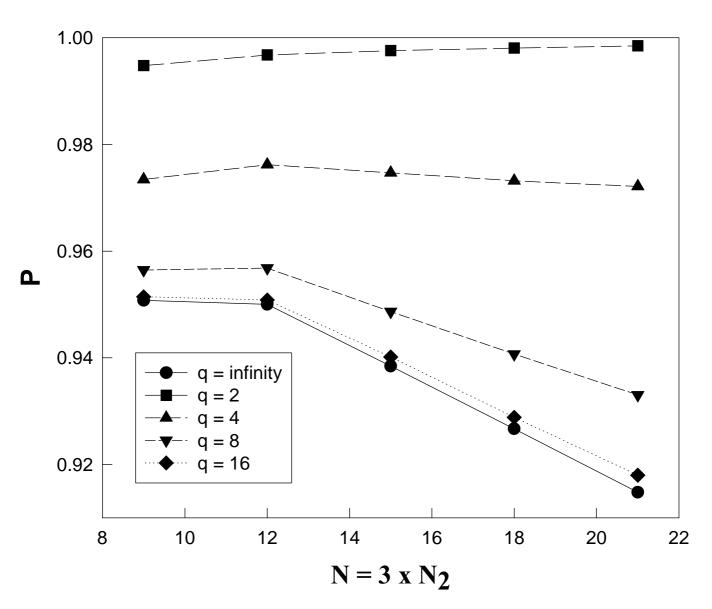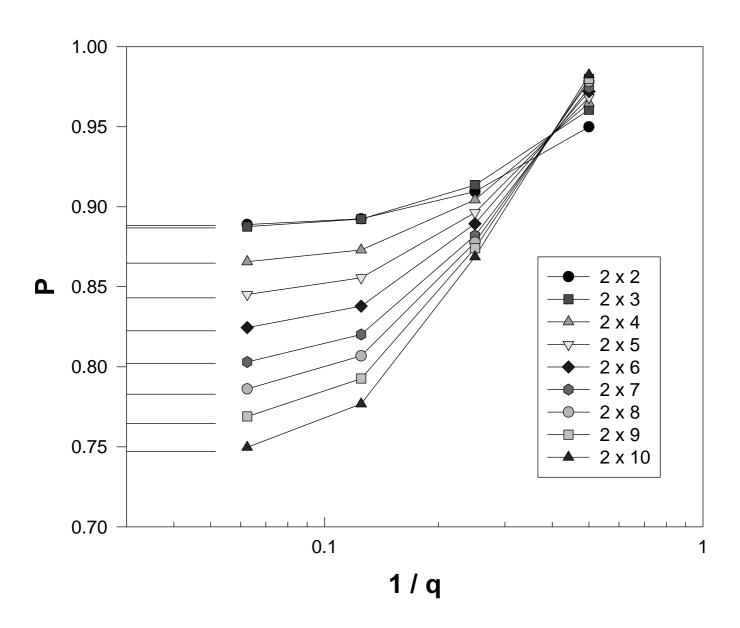
fig 1

fig 2

fig 3

fig 4

fig 5

fig 6

fig 7

fig 8