

On the Correlations Between Quantum Entanglement and q -Information Measures

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Abstract

In the present study we revisit the application of the q -information measures R_q of Rényi's and S_q of Tsallis' to the discussion of special features of two qubits systems. More specifically, we study the correlations between the q -information measures and the entanglement of formation of a general (pure or mixed) state ρ describing a system of two qubits. The analysis uses a Monte Carlo procedure involving the 15-dimensional 2-qubits space of pure and mixed states, under the assumption that these states are uniformly distributed according to the product measure recently introduced by Zyczkowski *et al* [Phys. Rev. A **58** (1998) 883].

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Important tools have been developed in recent years for the systematic exploration of the entanglement properties of composite quantum systems [1,2]. Quantum entanglement is a physical resource associated with the peculiar nonclassical correlations that may exist between separated quantum systems [3,4]. Entanglement is indeed the basic resource required to implement quantum information processes [3–9]. A state of a composite quantum system is called “entangled” if it can not be represented as a mixture of factorizable pure states. Otherwise, the state is called separable. The above definition is physically meaningful because entangled states (unlike separable states) cannot be prepared locally by acting on each subsystem individually. In particular, for bipartite pure states $|\Psi\rangle_{AB}$ one finds that they are entangled if their Schmidt number is greater than one. Otherwise, they are separable and their associated reduced density matrices $\hat{\rho}_A$, $\hat{\rho}_B$ are projectors. Any bipartite pure state that cannot be expressed as the direct product $|\Psi\rangle_{AB} = |\phi\rangle_A |\chi\rangle_B$ is entangled and $\hat{\rho}_A$, $\hat{\rho}_B$ represent mixed states. Two-qubits systems are the simplest quantum mechanical systems exhibiting the entanglement phenomenon and play a fundamental role in quantum information theory. They also provide useful limit cases for testing the behaviour of more involved systems [10]. The concomitant space S of *mixed states* is 15-dimensional. Its entanglement properties being far from trivial, their complete characterization constitutes a currently active field of research [1,2,11–13].

Considerable attention has been paid in recent years to the application of q -entropies to the study of quantum entanglement [10,14–25]. These entropic measures incorporate both Rényi’s [26] and Tsallis’ [27–29] families of information measures as special instances (both admitting, in turn, Shannon’s measure as the particular case associated with the limit $q \rightarrow 1$). The early motivation for these studies was the development, on the basis of the conditional q -entropies, of practical separability criteria for density matrices. The discovery by Peres of the partial transpose criteria, which for two-qubits and qubit-qutrit systems turned out to be both necessary and sufficient, rendered that original motivation a bit weaker, once it was realized that it is not possible to find a necessary and sufficient separability criterium on the basis of just the eigenvalue spectra of the three density matrices $\hat{\rho}_{AB}$, $\hat{\rho}_A =$

$Tr_B[\hat{\rho}_{AB}]$, and $\hat{\rho}_B = Tr_A[\hat{\rho}_{AB}]$ associated with a composite system $A \otimes B$ [30]. It is clear, however, from the studies reported in [10,14–25], that q -entropies do play a significant role in the investigation of entanglement phenomena. It is our intention in this Communication to investigate the degree of correlation between (i) the amount of entanglement $E[\rho_{AB}]$ exhibited by a two-qubits state ρ_{AB} , and (ii) the q -entropies (or q -information measures) of ρ_{AB} (notice that we refer here to the total q -entropy of the density matrix ρ_{AB} describing the composite system as a whole. We shall not consider conditional q -entropies). It is well known that the amount entanglement and the degree mixture (as measured by the q -entropies) of a state ρ_{AB} are independent quantities. However, there is a certain degree of correlation among them. States with an increasing degree of mixture tend to be less entangled. In point of fact, all two-qubits states with a large enough degree of mixture are separable. We want to explore to what extent does the strength of the alluded to correlation depend upon the parameter q characterizing the q -entropy used to measure the degree of mixture. In particular, we want to find out if there is a special value of q yielding a better entropy-entanglement correlation than the entropy-entanglement correlations associated with other values of q . To such an end, and for the sake of completeness, a few words regarding measures of entanglement are necessary.

A physically motivated measure of entanglement is provided by the entanglement of formation $E[\hat{\rho}]$ [31]. This measure quantifies the resources needed to create a given entangled state $\hat{\rho}$. That is, $E[\hat{\rho}]$ is equal to the asymptotic limit (for large n) of the quotient m/n , where m is the number of singlet states needed to create n copies of the state $\hat{\rho}$ when the optimum procedure based on local operations is employed. For the particular case of two-qubits states Wootters obtained an explicit expression for $E[\hat{\rho}]$ in terms of the density matrix $\hat{\rho}$ [32]. Wootters' formula reads [32]

$$E[\hat{\rho}] = h\left(\frac{1 + \sqrt{1 - C^2}}{2}\right), \quad (1)$$

where $h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$, and C stands for the *concurrence* of the two-qubits state $\hat{\rho}$. The concurrence is given by $C = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$, λ_i , ($i =$

1, \dots, 4) being the square roots, in decreasing order, of the eigenvalues of the matrix $\hat{\rho}\tilde{\rho}$, with $\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$. The above expression has to be evaluated by recourse to the matrix elements of $\hat{\rho}$ computed with respect to the product basis.

Our investigations will be based upon a Monte Carlo exploration of \mathcal{S} : the set of *all states, pure and mixed* of a two-qubits system. To do this we need to define a proper measure on \mathcal{S} . The space of all (pure and mixed) states ρ of a quantum system described by an N -dimensional Hilbert space can be regarded as a product space $\mathcal{S} = \mathcal{P} \times \Delta$ [1,2,33,34]. Here \mathcal{P} stands for the family of all complete sets of orthonormal projectors $\{\hat{P}_i\}_{i=1}^N$, $\sum_i \hat{P}_i = I$ (I being the identity matrix). Δ is the set of all real N -uples $\{\lambda_1, \dots, \lambda_N\}$, with $0 \leq \lambda_i \leq 1$, and $\sum_i \lambda_i = 1$. The general state in \mathcal{S} is of the form $\rho = \sum_i \lambda_i P_i$. The Haar measure on the group of unitary matrices $U(N)$ induces a unique, uniform measure ν on the set \mathcal{P} [35]. On the other hand, since the simplex Δ is a subset of a $(N-1)$ -dimensional hyperplane of \mathcal{R}^N , the standard normalized Lebesgue measure \mathcal{L}_{N-1} on \mathcal{R}^{N-1} provides a natural measure for Δ . The aforementioned measures on \mathcal{P} and Δ lead to a natural measure,

$$\mu = \nu \times \mathcal{L}_{N-1}, \quad (2)$$

on the set \mathcal{S} of quantum states [1,2,36].

In the present investigation we deal with the case $N = 4$. Our present considerations are based on the assumption that the uniform distribution of states of a two-qubit system is the one determined by the measure μ . Thus, in our numerical computations we are going to randomly generate states of a two-qubits system according to the measure μ and study the relation between the entanglement properties of these states, on the one hand, and

- 1) the Tsallis q -entropy

$$S_q = \frac{1}{q-1}(1 - \omega_q), \text{ with } \omega_q = \text{Tr}(\hat{\rho}^q), \quad (3)$$

and

- 2) the Rényi q -entropy

$$R_q = \frac{1}{1-q} \ln [1 + (1-q) S_q], \quad (4)$$

on the other one.

Most recent research efforts dealing with the relationship between the degree of mixture and the amount of entanglement focus on the behaviour, as a function of the degree of mixture, of the entanglement properties exhibited by the set of states endowed with a given amount of mixedness. For instance, they consider the behaviour, as a function of the degree of mixture (as measured, for instance, by S_2), of the average entanglement of those states characterized by a given value of S_2 . Here we are going to adopt, in a sense, the reciprocal (and complementary) point of view. We are going to study the behaviour, as a function of C^2 , of the entropic properties associated with the set of states characterized by a given value of C^2 . This vantage point will enable us to clarify some aspects of the q -dependence of the entanglement-mixedness correlation. In particular, we want to assess, for different q -values, how sensitive are the average entropic properties to the value of the entanglement of formation (or, equivalently, to the value of the squared concurrence C^2).

First of all, we have randomly generated a large number of states (according to the measure μ given by expression (2)) and plotted them, for several q -values, in the (C^2, R_q) -plane. Each point in Fig. 1 corresponds to a particular state in the state space \mathcal{S} . The outcome is a series of “bands”, one for each q . The top band corresponds to $q = 0.5$. At the bottom we find that for $q = 10$. The remaining ones arrange themselves in between, in monotonic fashion. One appreciates the fact that, for the different q -values, we find, given a certain amount of entanglement as measured by C^2 , states of larger and larger entropies R_q as q diminishes. From the point of view of information theory this would entail that information about our states is lost as q decreases. If we now consider averages over all states that share a given concurrence we are led to consider Fig. 2.

We computed, as a function of C^2 , the average value of the Rényi entropy R_q associated with the set of states endowed with a given value of the squared concurrence C^2 . The results are exhibited in Fig. 2 (solid lines), where the mean value $\langle R_q \rangle$ is plotted against

C^2 , for $q = 0.5, 1, 2, 10$, and ∞ . As stated, the averages are taken over all the states $\hat{\rho} \in \mathcal{S}$ that are characterized by a fixed concurrence-value. For all q the average entropies diminish as C grows. This behaviour is consistent with the fact that states of increasing entropy tend to exhibit a decreasing amount of entanglement [1,33,34]. As q grows, the average entropy decreases, for any C^2 , although the decreasing tendency slows down for large q -values. Many recent efforts dealing with the relationship between q -entropies and entanglement were restricted to states ρ_{Bell} diagonal in the Bell basis. For such states, both the R_q entropy and the squared concurrence C^2 depend solely upon ρ_{Bell} 's largest eigenvalue, so that R_q can be expressed as a function of C^2 . The dashed line in Fig. 2 depicts the functional dependence of the R_∞ Rényi entropy, as a function of C^2 , for two-qubits states diagonal in the Bell basis. It is instructive to compare, in Fig. 2, the curve corresponding to states diagonal in the Bell basis with the curve corresponding (with $q = \infty$) to all two-qubit states. It can be appreciated that these two curves, even if sharing the same qualitative appearance, differ to a considerable extent.

For the sake of comparison, we plotted in Fig. 3 the mean value $\langle S_q \rangle$ of Tsallis' entropy, as a function of C^2 , for $q = 0.5, 1, 2$, and 10 . Again, for each value of C^2 , the entropy's average was computed over all those states characterized by that particular C^2 -value. Notice that for large q -values, the Tsallis entropy is approximately constant for all C^2 values, while the Rényi one seems to be much more sensitive in this respect. Entropies tend to vanish for $C^2 \rightarrow 1$, because only pure states can reach the maximum concurrence value. In the inset of Fig. 3 we depict the behaviour of $\langle S_q \rangle_{C^2}$ as a function of $1/q$ for a given value of the concurrence ($C^2 = 0.6$), thus illustrating the fact that the mean entropy is a monotonically decreasing function of q . For large q -values the Tsallis entropy cannot discriminate between different degrees of entanglement for states with $C^2 < 1$, while Rényi's measure can do it. This fact is related to an important difference between the behaviours, as a function of the parameter q , of Rényi's R_q and Tsallis' S_q entropies. The maximum value R_q^{max} attainable by Rényi's entropy (corresponding to the equi-probability distribution) is independent of q ,

$$R_q^{\max} = -\ln N, \quad (5)$$

where N is the total number of accesible states. On the contrary, the maximum value reachable by S_q does depend upon q ,

$$S_q^{\max} = \frac{1 - N^{1-q}}{(q-1)}. \quad (6)$$

Clearly, $S_q^{\max} \rightarrow 0$ for $q \rightarrow \infty$. One may think that the q -dependence of S_q^{\max} may be appropriately taken into account if one considers (instead of Tsallis' entropy itself), a *normalized* Tsallis' entropy (see Fig. 4),

$$S'_q = \frac{S_q}{S_q^{\max}}, \quad (7)$$

For instance, in the case of two qubits one has,

$$S_q^{\max} = \frac{1 - 4^{1-q}}{(q-1)}, \quad (8)$$

and we deal then with

$$S'_q = \frac{1 - \text{Tr}[\hat{\rho}^q]}{1 - 4^{1-q}} = \frac{1 - \{[\text{Tr}(\hat{\rho}^q)]^{1/q}\}^q}{1 - 4^{1-q}}. \quad (9)$$

Consider now the limit $q \rightarrow \infty$ for a density matrix $\hat{\rho}$ corresponding to a state of fixed concurrence C . In such a process one immediately appreciates the fact that $[\text{Tr}(\hat{\rho}^q)]^{1/q} \rightarrow \lambda_{\max}$, where λ_{\max} is the largest eigenvalue of $\hat{\rho}^q$. Thus, the limiting value we reach is

$$S'_q \rightarrow [1 - (\lambda_{\max})^q], \quad (10)$$

and we see that this is always equal to unity for all $C^2 < 1$ and vanishes exactly if $C^2 = 1$ (see Fig. 4). Consequently, even employing the normalized S'_q , the information concerning the entropy-entanglement correlation tends to disappear in the $q \rightarrow \infty$ limit.

We see that, with regards to the analysis of the entropy-entanglement correlations of quantum bipartite systems (remember that here we are talking about the total entropy of the whole composite system), Tsallis' and Rényi's entropies behave in different ways. It is instructive to compare this behaviour of the q -entropies, to their behaviour with regards

to the relationship between separability and conditional q -entropies. What matters in this last case *is the sign of the conditional q -entropies*. If a classical composite system $A \times B$ is described by a suitable probability distribution $\{p_{AB}\}$ (defined over the concomitant phase space), the q -information measure (or q -entropy) associated to the whole system is always equal or greater than that pertaining to its subsystems, i.e., those associated to $\{p_A\}$ or $\{p_B\}$. In other words, the conditional entropy

$$S_q[A|B] = S_q[p_{AB}] - S_q[p_B], \quad (11)$$

is always positive. This is also the case for *separable* states $\hat{\rho}_{AB}$ of a composite quantum system $A \otimes B$ [30,23]. In contrast, a subsystem of a quantum system described by an entangled state may have an entropy greater than the entropy of the whole system. That is to say, *entangled states may have negative conditional q -entropies*. Now, since Rényi's entropy is a monotonically increasing function of Tsallis' measure, it is clear that the sign of the conditional Rényi's entropy associated with a quantum state $\hat{\rho}_{AB}$, $R_q[A|B] = R_q[\hat{\rho}_{AB}] - R_q[\hat{\rho}_B]$, will be the same as the sign of the conditional Tsallis' entropy, $S_q[A|B] = S_q[\hat{\rho}_{AB}] - S_q[\hat{\rho}_B]$. Hence, as far as the relationship between conditional entropies and separability (which depends only upon the sign of the conditional q -entropies) is concerned, Tsallis' and Rényi's measures are equivalent.

Returning to our discussion of the connection between entanglement and (total) q -entropies of bipartite quantum systems, we have seen that Rényi's entropy is particularly well suited for (i) discussing the $q \rightarrow \infty$ limit and (ii) studying the q -dependence of the entropy-entanglement correlations. For these reasons, in the rest of the present contribution we will focus upon Rényi entropy.

We tackle now the question of the dispersion around these entropic averages. Fig. 5 is a graph of the dispersions

$$\sigma_q^{(R)} = \left[\langle R_q^2 \rangle - \langle R_q \rangle^2 \right]^{1/2}, \quad (12)$$

as a function of C^2 , for the same q -values of Fig. 2. We see that the size of the dispersions diminishes rather rapidly as C^2 increases towards unity. Also, dispersions tend to become

smaller as q grows. This suggests that, as q increases, the correlation between $\langle R_q \rangle$ and entanglement improves. A similar tendency, but in the case of S_q , was detected in [25].

In order to estimate in a quantitative the sensitiveness of the average q -entropy to changes in the value of C^2 , we computed the derivatives with respect to C^2 of the average value of Rényi's entropy associated with states of given C^2 ,

$$\frac{d\langle R_q \rangle}{d(C^2)}. \quad (13)$$

In Fig. 6 we plot the above derivatives, against C^2 , for $q = 0.5, 1, 2, 10$, and ∞ . These derivatives fall abruptly to zero, in the vicinity of the origin, as C^2 diminishes. As a counterpart, for all q , the derivatives exhibit a rapid growth with C^2 for small values of the concurrence. This growing tendency stabilizes itself and, for q large enough, saturation is reached.

Now let us assume that we know the value of the entropy $R_q[\rho]$ of certain state ρ . How useful is this knowledge in order to infer the value of C^2 ?. In other words, how good is R_q as an “indicator” of entanglement? It has been suggested that $q = \infty$ provides a better “indicator” of entanglement than other values of q [24,25]. There are two ingredients that must be taken into account in order to determine the q -value yielding the best entropic “indicator” of entanglement. On the one hand, the sensitivity of the entropic mean value $\langle R_q \rangle$ to increments in C^2 , as measured by the derivative $d\langle R_q \rangle/d(C^2)$. On the other hand, the dispersion $\sigma_q^{(R)}$, given by (12). A given q -value would lead to a good entropic “indicator” if it corresponds to (i) a large value of $d\langle R_q \rangle/d(C^2)$, and (ii) a small value of $\sigma_q^{(R)}$. These two factors are appropriately taken into account if we compute the ratio

$$r = \left| \frac{\sigma_q^{(R)}}{d\langle R_q \rangle/d(C^2)} \right|, \quad (14)$$

between the dispersions depicted in Fig. 5 and the derivative of Fig. 6. The ratio r provides a quantitative measure for the strength of the entropic-entanglement correlations. The quantity r constitutes an estimate of the smallest increment ΔC^2 in the squared concurrence which is associated with an appreciable change in R_q . In order to clarify this last assertion, an

analogy with the uncertainty associated with the measurement of time in quantum mechanics can be established. Let us assume that we can measure an observable \hat{A} . Then, the time uncertainty Δt depends upon two quantities, (i) the time derivative of the expectation value of the observable, $d\langle\hat{A}\rangle/dt$, and (ii) the uncertainty of the observable, $\Delta\hat{A} = [\langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2]^{1/2}$. The time uncertainty is given by [41]

$$\Delta t = \frac{\Delta\hat{A}}{d\langle\hat{A}\rangle/dt} \quad (15)$$

The above expression for Δt gives an estimation of the smallest time interval that can be detected from measurements of the observable \hat{A} . In the analogy we want to establish, C^2 plays the role of t , and R_q plays the role of the observable A . The ratio r is depicted in Fig. 7, as a function of C^2 , for $q = 1$ and $q = \infty$. The two upper curves in Fig. 7 correspond to the r -values obtained when all the states in the two-qubits state-space \mathcal{S} are considered. On the other hand, the lower curves are the ones obtained when the computation of r is restricted to states diagonal in the Bell basis. When all states in \mathcal{S} are considered, the values of r associated with $q = \infty$ are seen to be smaller than the values corresponding to $q = 1$, which can be construed as meaning that the q -entropies with $q = \infty$ can indeed be regarded as better “indicators” of entanglement than the q -entropies associated with finite values of q , as was previously suggested in [24,25]. Alas, the results depicted in Fig. 7 indicate that this improvement of the entropy-entanglement correlation associated with $q = \infty$ is not considerable. The usefulness of q -entropies with $(q \rightarrow \infty)$ as an “indicator” of entanglement was proposed in [24] on the basis of the behaviour of states diagonal in the Bell basis. As already mentioned, the squared concurrence C^2 of states ρ_{Bell} diagonal in the Bell basis can be expressed as a function of R_∞ , since both these quantities depend solely on the largest eigenvalue λ_m of ρ_{Bell} (in particular, $R_\infty = -\ln \lambda_m$). This means that, as pointed out in [24,25], for states diagonal on the Bell basis there is a perfect correlation between C^2 and $(q = \infty)$ -entropies (and, consequently, r vanishes). This implies that, when restricting our considerations *only* to states diagonal in the Bell basis, the entropy-entanglement correlation is much more strong for $q = \infty$ than for other values of q . States diagonal in the Bell basis

are important for many reasons, but their properties are by no means typical of the totality of the state-space \mathcal{S} . See for instance, as depicted in Fig.7, the behavior of r (for $q = 1$) associated with (i) all states in \mathcal{S} and (ii) states diagonal in the Bell basis. There are remarkable differences between the two cases.

We thus find ourselves in a position to assert that the relationship between the q -entropies and the amount of entanglement exhibited by the family of states diagonal in the Bell basis does not constitute a reliable guide to infer the typical behavior of states in the two-qubits state-space \mathcal{S} . When considering the complete state-space \mathcal{S} , the $q = \infty$ -entropies turn out to be only a slightly better, as entanglement “indicators”, than the entropies associated with other values of q .

We summarize now our present considerations. By recourse to a Monte Carlo procedure we have studied the q -dependence of the correlations exhibited by two-qubits states between (i) the amount of entanglement and (ii) the q -entropies. It was previously conjectured by other researchers, on the basis of the study of states diagonal in the Bell basis, that the q -entropies associated with $q = \infty$ are better “indicators” of entanglement than the entropies corresponding to finite values of q . In other words, it was suggested that the q -entropy with $q = \infty$ exhibits a stronger correlation with entanglement than the other q entropies. By a comprehensive numerical survey of the complete (pure and mixed) state-space of two-qubits, we have shown here that the alluded to conjecture is indeed correct. However, when globally considering the whole state-space the advantage, as an entanglement indicator, of $(q = \infty)$ -entropy turns out to be much smaller than what can be inferred from the sole study of states diagonal in the Bell basis. This constitutes an instructive example of the perils that entails trying to infer typical properties of general two-qubits states from the study of just a particular family of states, such as those diagonal in the Bell basis.

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FIGURE CAPTIONS

Fig.1- Rényi entropy R_q vs. the squared concurrence C^2 for all two-qubits states and several q values. All depicted quantities are dimensionless.

Fig.2- Average value of the Rényi entropy $\langle R_q \rangle$ of all states with a given squared concurrence C^2 , as a function of C^2 , and for several q -values (solid lines). The dashed line depicts the functional dependence of the R_∞ Rényi entropy, as a function of C^2 , for two-qubits states diagonal in the Bell basis. All depicted quantities are dimensionless.

Fig.3- Average value of the Tsallis' entropy $\langle S_q \rangle$ of all states with a given squared concurrence C^2 , as a function of C^2 , and for several q -values. The inset shows $\langle S_q \rangle$ vs. $1/q$ for the particular value of the squared concurrence $C^2 = 0.6$. All depicted quantities are dimensionless.

Fig.4- Average value of the normalized Tsallis entropy $\langle S_q \rangle / S_{q\max}$ vs. C^2 , for several q -values. All depicted quantities are dimensionless.

Fig.5- Dispersion of the Rényi entropy $\sigma_q^{(R)} = [\langle R_q^2 \rangle - \langle R_q \rangle^2]^{1/2}$ for all qubits states with a given C^2 , as a function of C^2 , and for several q -values. All depicted quantities are dimensionless.

Fig.6- The derivative $d\langle R_q \rangle / d(C^2)$, as a function of the squared concurrence C^2 , for several values of the q -parameter. All depicted quantities are dimensionless.

Fig.7- The absolute value of the quotient $r = \left| \frac{\sigma_q^{(R)}}{d\langle R_q \rangle / d(C^2)} \right|$, as a function of the squared concurrence C^2 , for $q = 1$ and $q = \infty$. The two upper curves correspond to all states in the two-qubit state-space \mathcal{S} . The lower curves correspond to states diagonal in the Bell basis. All depicted quantities are dimensionless.

fig 1

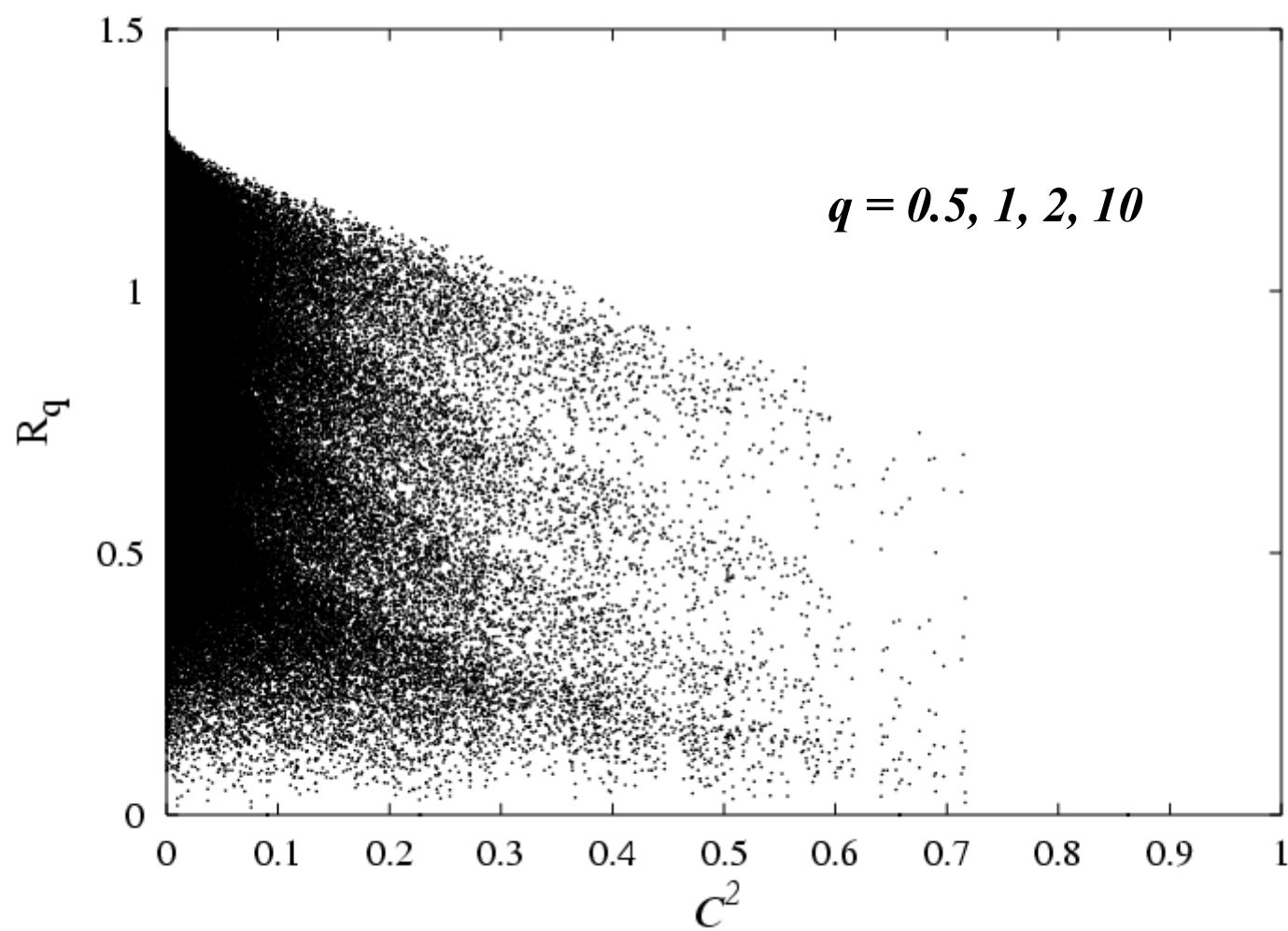


fig2

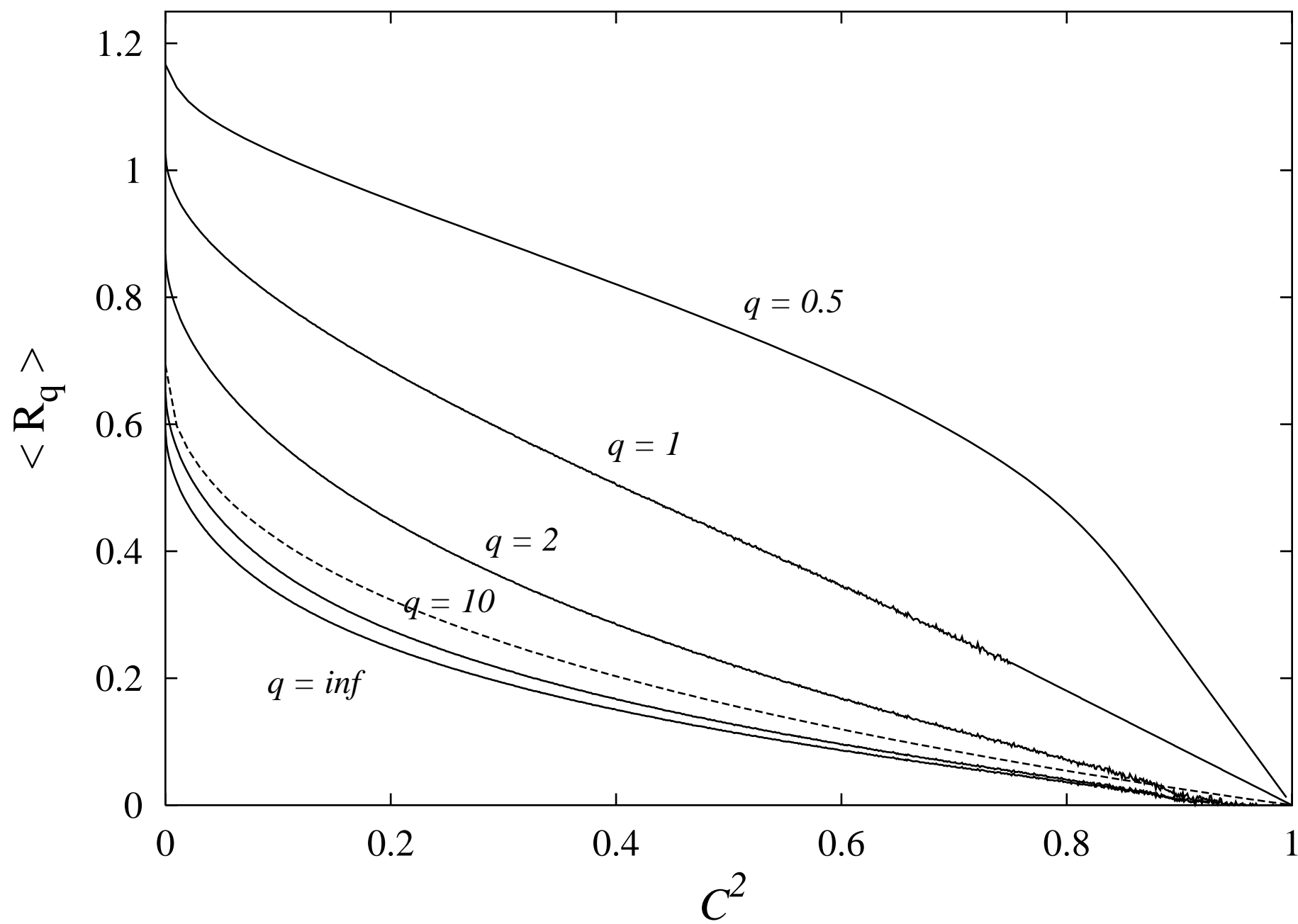


fig.3

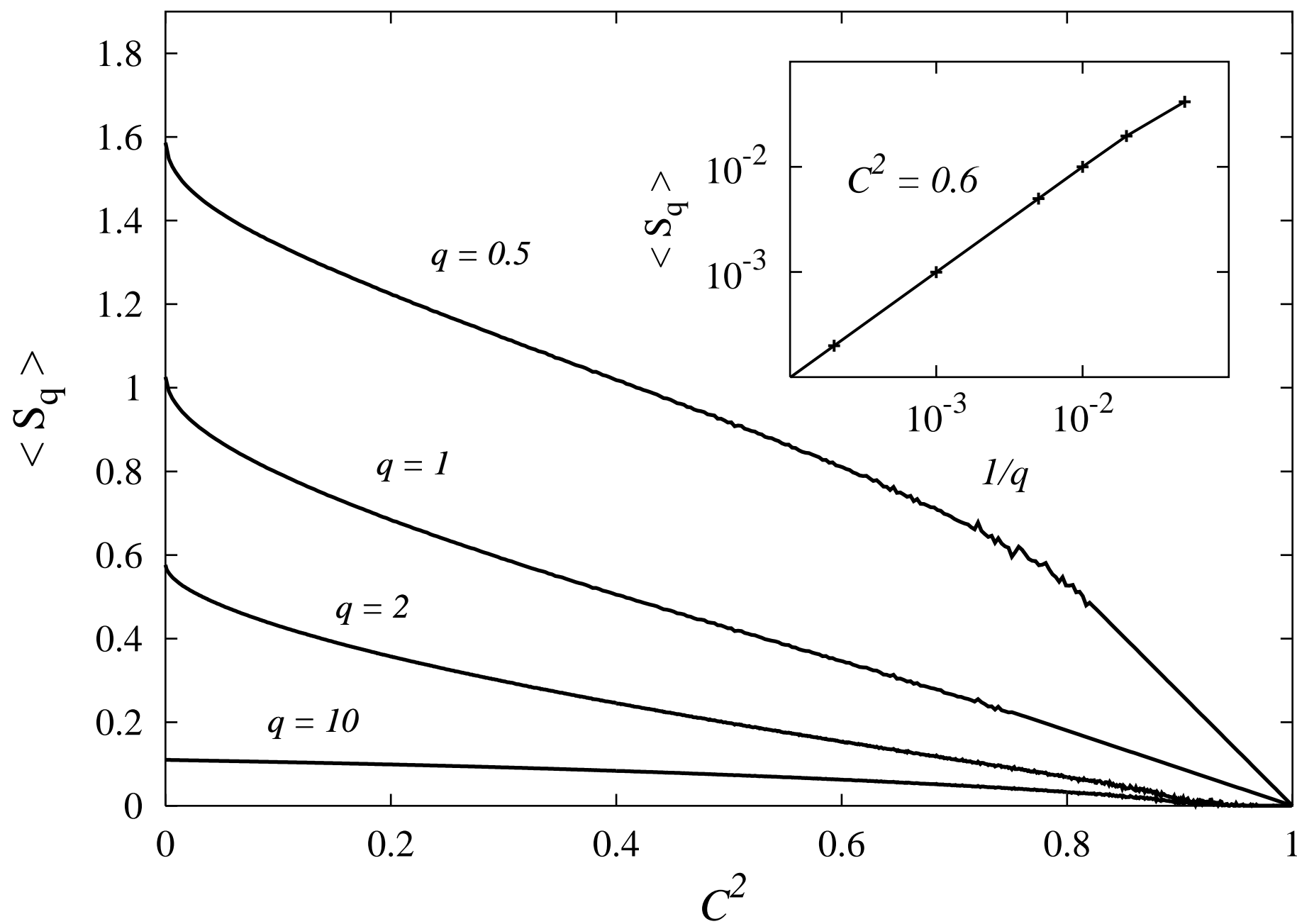


fig.4

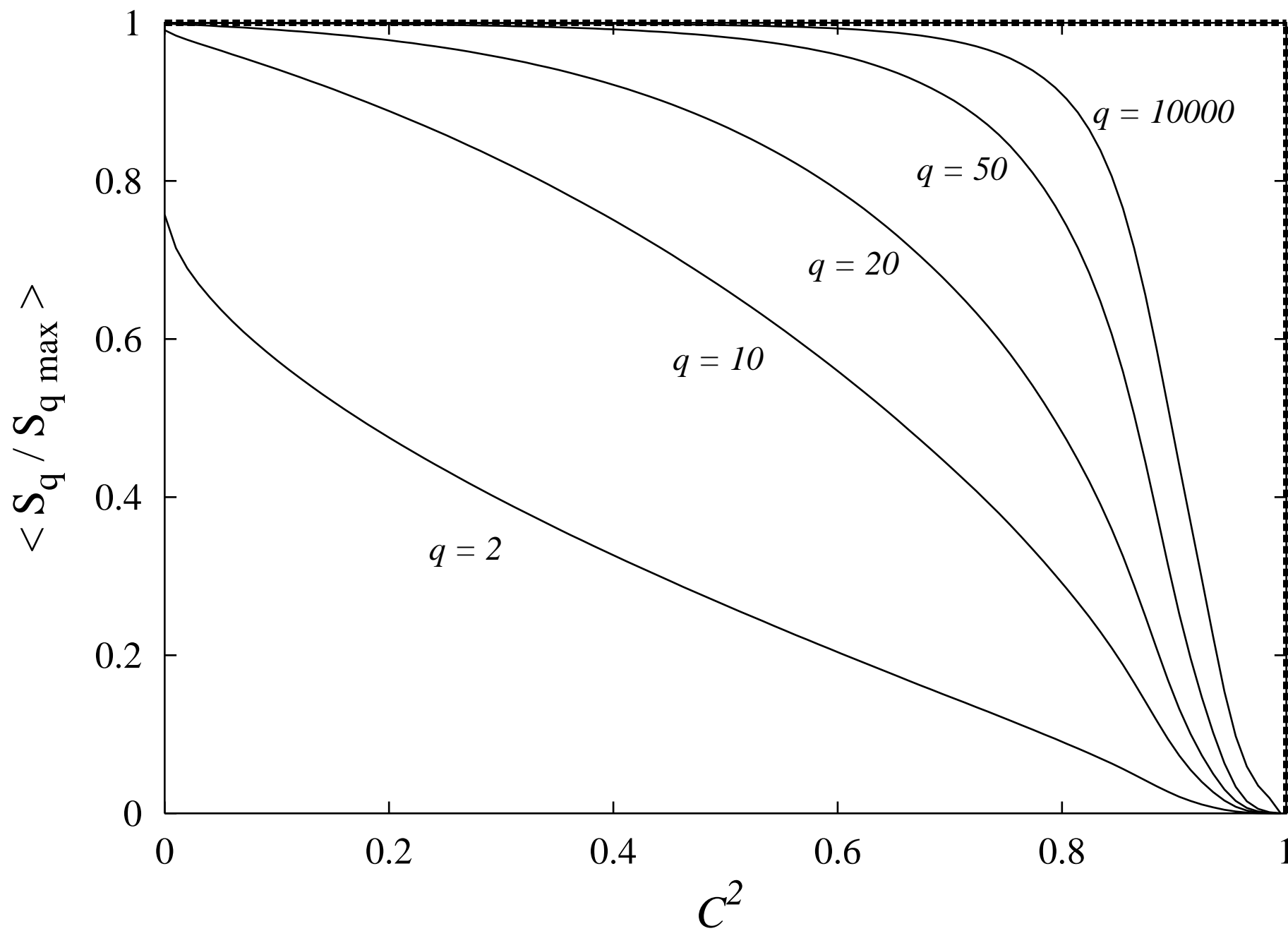


fig.5

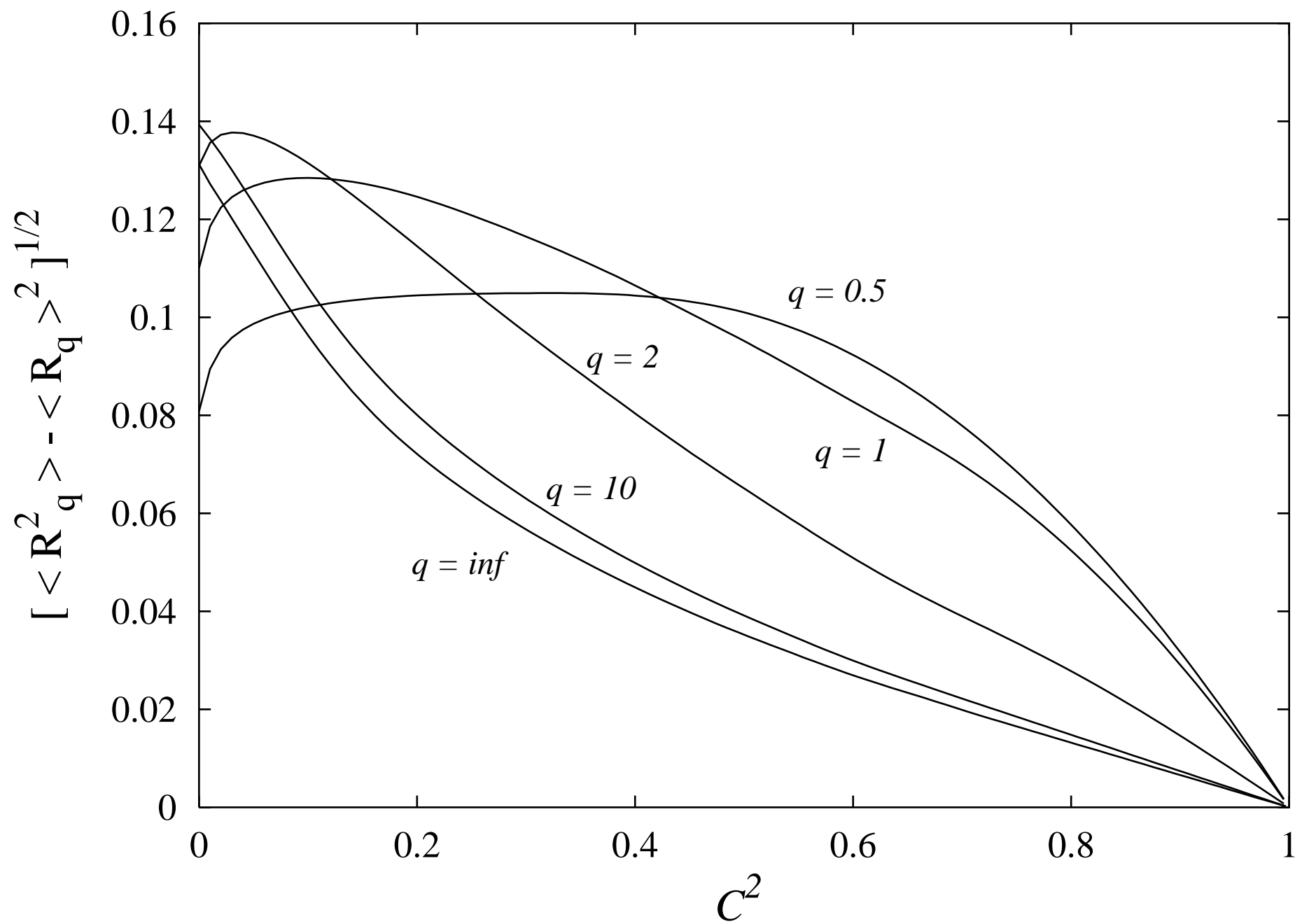


fig.6

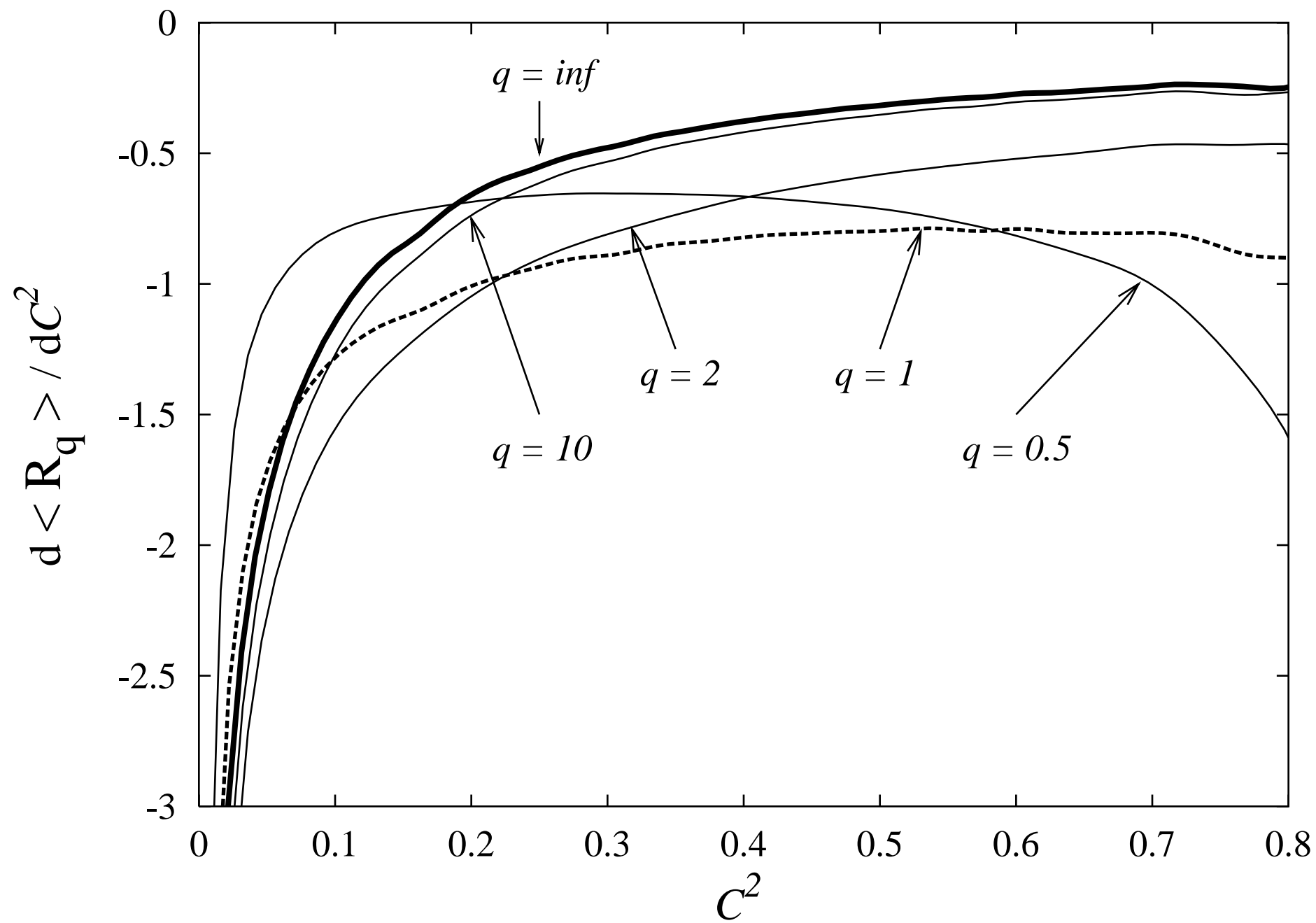


fig.7

