## Separability of a family of one parameter W and GHZ multiqubit states using Abe-Rajagopal q-conditional entropy approach

R. Prabhu, <sup>1</sup> A. R. Usha Devi, <sup>2, 3, \*</sup> and G. Padmanabha<sup>2</sup>

<sup>1</sup>Department of Physics, Kuvempu University, Shankaraghatta, Shimoga-577 451, India <sup>2</sup>Department of Physics, Bangalore University, Bangalore-560 056, India <sup>3</sup>Inspire Institute Inc., McLean, VA 22101, USA. (Dated: November 15, 2018)

We employ conditional Tsallis q entropies to study the separability of symmetric one parameter W and GHZ multiqubit mixed states. The strongest limitation on separability is realized in the limit  $q \to \infty$ , and is found to be much superior to the condition obtained using the von Neumann conditional entropy (q=1 case). Except for the example of two qubit and three qubit symmetric states of GHZ family, the q-conditional entropy method leads to sufficient - but not necessary conditions on separability.

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### I. INTRODUCTION

Quantum entanglement has evoked intense interest in recent years as it occupies a central position in quantum computation and information theory [1, 2]. Characterizing whether a given composite quantum state is separable or entangled is a key issue in this currently emerging field. Entropic characterization [3, 4, 5, 6, 7, 8, 9, 10] proves to be significant in this direction. One of the important observations is that the subsystems of a separable state  $\hat{\rho}_{AB}^{(\text{sep})} = \sum_i p_i \, \hat{\rho}_A^{(i)} \otimes \hat{\rho}_B^{(i)} \ \, (\text{with } 0 \leq p_i \leq 1, \ \, \sum_i p_i = 1),$  are more ordered than the whole system i.e.,  $S(\hat{\rho}_{AB}^{(\text{sep})}) \geq S(\hat{\rho}_A), \ \, S(\hat{\rho}_B),$  where  $S(\hat{\rho}) = -\text{Tr}[\hat{\rho}\log\hat{\rho}]$  denotes the von Neumann entropy. In contrast, an arbitrary pure entangled state satisfies the inequality

$$S(B|A) = S(\hat{\rho}_{AB}) - S(\hat{\rho}_{A}) \le 0, \tag{1}$$

reflecting the remarkable fact [11] that pure entangled states are more disordered locally than globally. Negative conditional entropies (implied by the inequality (1)) provide sufficient - but not necessary - criterion to characterize mixed entangled states. In the case of two qubit Werner state,  $\hat{\rho}_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|x + I_4(1-x)/4; \ 0 \le x \le 1, \ |\psi_{AB}\rangle = \frac{1}{2}\left(|\uparrow_A\uparrow_B\rangle + |\downarrow_A\downarrow_B\rangle\right)$ the conditional entropic criterion Eq. (1) leads to  $0 \le x \le 0.747$  as the range of separability ( the von Neumann conditional entropy is positive in this range of the parameter x), which is clearly weaker compared to that obtained through Peres' partial transpose criterion [12]:  $0 \le x \le \frac{1}{3}$ . This example brings out the limitation of the entropic inequality (1), in characterizing entanglement in mixed composite states. Generalized entropic measures [4, 5, 6, 7, 8, 9, 10] provide more sophisticated tools to explore global vs local disorder in mixed states and lead to more stringent limitation on separability than that obtained using positivity

of the conditional von Neumann entropy. In this context, the quantum counterparts of the Rényi entropy [4],  $S_q^{(R)}(\hat{\rho}) = \frac{1}{1-q} \log \operatorname{Tr} \left[\hat{\rho}^q\right]$ , and the Tsallis entropy [13],  $S_q^{(T)}(\hat{\rho}) = \frac{\operatorname{Tr} \left[\hat{\rho}^q\right] - 1}{1-q}$  have often been employed. In the limit  $q \to 1$  both these generalized entropic measures [14] reduce to the von Neumann entropy. Horodecki et. al. [4] recognized that  $S_q^{(R)}(\hat{\rho}_{AB}^{(\text{sep})}) \geq S_q^{(R)}(\hat{\rho}_A), S_q^{(R)}(\hat{\rho}_B)$  for separable states and thus negative values of the conditional Rényi entropy  $S_q^{(R)}(B|A) = S_q^{(R)}(\hat{\rho}_{AB}) - S_q^{(R)}(\hat{\rho}_A)$  (with  $S_q^{(R)}(\hat{\rho}_A)$  being the maximum of the subsystem Rényi entropies) is a signature of quantum entanglement. On the other hand, based on Tsallis entropy and the form invariant structures of Khinchin's axioms, Abe and Rajagopal [5] generalized the concept of conditional entropy as

$$S_{q}^{(T)}(B|A) = \frac{S_{q}^{(T)}(\hat{\rho}_{AB}) - S_{q}^{(T)}(A)}{1 + (1 - q)S_{q}^{(T)}(\hat{\rho}_{A})}$$
$$= \frac{1}{q - 1} \left[ 1 - \frac{\text{Tr}\left[\hat{\rho}_{AB}^{q}\right]}{\text{Tr}\left[\hat{\rho}_{A}^{q}\right]} \right], \tag{2}$$

where  $S_q^{(T)}(\hat{\rho}_{AB})$ ,  $S_q^{(T)}(\hat{\rho}_A)$ , respectively denote the Tsallis q-entropies associated with  $\hat{\rho}_{AB}$  and its subsystem density operator  $\hat{\rho}_A$ . The Abe-Rajagopal (AR) qconditional entropy, given by Eq. (2), is nonnegative for a separable state, but may assume negative value [15] in a quantum entangled state, suggesting its importance in the characterization of quantum entanglement in mixed composite states. In the particular case of two qubit Werner state, Abe and Rajagopal [5] recovered the necessary and sufficient condition for separability (i. e.,  $0 \le x \le \frac{1}{3}$ ), by examining the positivity of qconditional entropy in the limit  $q \to \infty$ . Employing the same approach, Tsallis et. al., [6] re-discovered the Peres criterion for separability in more general two qubit mixed states. As a further extension, Abe [7] showed that the negativity of q-conditional entropy gives the correct range of inseparability for generalized Werner states of N-qudits. Batle et. al. [9, 10] performed a comprehensive

 $<sup>{\</sup>rm *Electronic~address:~arutth@rediffmail.com}$ 

numerical survey of the space of bipartite systems and identified that the volume occupied by the states with positive conditional entropies  $S_q^{(R)}(B|A)$  and  $S_q^{(T)}(B|A)$  decreases monotonically as q increases (it is found that this monotonic behavior is more pronounced when Tsallis q-conditional entropies are investigated [10]). Moreover, the numerical investigation [10] indicated that the Peres criterion provides a much stronger condition on separability than that derived from the positive conditional entropies in the  $q \to \infty$  limit.

Positivity of conditional entropies,  $S_q^{(R,T)}(B|A)$ , is based entirely on the global and local spectra of the composite state and is basically, one of the implications of majorization [16, 17], which is the strongest spectral criterion of separability. It has been shown [17] that any criteria, based only on the eigenvalues of the state and its reductions, do not provide a complete characterization of entanglement. There are examples of entangled states, which can not be detected by any spectral criteria, as separable states with the same global and local spectra exist [16, 17]. So, the q-conditional entropic characterization does not lead, in general, to necessary and sufficient condition for separability. However, this method is fruitful in obtaining more stringent limitations on separability, than the one identified from the familiar q=1case (von Neumann conditional entropy). Further, negative q-conditional entropy implies that the composite state is distillable, as it signals violation of the reduction criterion [16].

In the present paper we investigate separability of one parameter symmetric multiqubit W and GHZ states using the AR q-conditional entropy approach. strongest limitation on separability, is obtained in the  $q \to \infty$  limit, and is found to agree with the Peres' criterion only for two and three qubit states of the GHZ family. We obtain the explicit N dependent range of separability based on the positivity of q-conditional entropies,  $S^{(T)}(B|A)$ , in the limit  $q \to \infty$ . In Sec. II, we discuss q-entropic characterization of separability in two and three gubit symmetric states and compare the results with that obtained from the Peres criterion. In Sec. III we derive the strongest constraints (obtained in the limit  $q \to \infty$ ) on separability for the one parameter family of W and GHZ multiqubit states. A summary of results is given in Sec. IV.

### II. TWO AND THREE QUBIT SYMMETRIC STATES

Symmetric states of qubits [18] are those, which remain unaltered under permutations [19] of the qubits.

They have attracted a great deal of attention recently [20] due to mathematical elegance offered in characterizing them as well as for their experimental significance [21]. Symmetric multiqubit states get restricted - due to permutation symmetry - to a N+1 dimensional subspace  $\{|\frac{N}{2},M\rangle; -\frac{N}{2} \leq M \leq \frac{N}{2}\}$  of the entire  $2^N$  dimensional Hilbert space  $\mathcal{H}=(C^2)^{\otimes N}$  of N qubits. (Here  $|\frac{N}{2},M\rangle$  denote the simultaneous eigenstates of the squared total angular momentum operator  $\hat{J}^2$  and the z component  $\hat{J}_z$ , with  $\hat{J}=\frac{1}{2}\sum_{\alpha}\hat{\sigma}_{\alpha}$ ;  $\alpha=1,2,\ldots,N$ ;  $\hat{\sigma}_{\alpha}$  being the Pauli operator of the  $\alpha^{th}$  qubit).

In the following, we will be focusing on the AR qconditional entropy characterization of a simple one parameter mixed state of two qubits given by

$$\hat{\rho}_{AB} = \left(\frac{1-x}{3}\right) P_{N=2} + x |\Phi_{AB}^{+}\rangle\langle\Phi_{AB}^{+}|; \ 0 \le x \le 1 \ (3)$$

where  $P_2 = \sum_M |1, M\rangle\langle 1, M|; M = -1, 0, 1$ , corresponds to projection operator onto the symmetric subspaces of two qubit states characterized by the maximum value of total angular momentum J = N/2 = 1. In terms of the standard two qubit basis, the symmetric states  $\{|1, M\rangle\}$  are given by  $|1, 1\rangle = |\uparrow_A \uparrow_B\rangle, |1, -1\rangle = |\downarrow_A \downarrow_B\rangle$  and  $|1, 0\rangle = \frac{1}{\sqrt{2}} [|\uparrow_A \downarrow_B\rangle + |\downarrow_A \uparrow_B\rangle] = |\Phi_{AB}^+\rangle$ , one of the Bell states.

It is easy to identify the eigenvalues of  $\hat{\rho}_{AB}$ ;

$$p_1(AB) = p_2(AB) = \frac{1-x}{3},$$
  
 $p_3(AB) = \frac{1+2x}{3}, p_4(AB) = 0.$  (4)

The reduced subsystems density matrices of  $\hat{\rho}_{AB}$  are given by

$$\hat{\rho}_A = \operatorname{Tr}_B \left[ \hat{\rho}_{AB} \right] = \frac{1}{2} I_A$$

$$\hat{\rho}_B = \operatorname{Tr}_A \left[ \hat{\rho}_{AB} \right] = \frac{1}{2} I_B, \tag{5}$$

where  $I_{A(B)} = |\uparrow_{A(B)}\rangle\langle\uparrow_{A(B)}| + |\downarrow_{A(B)}\rangle\langle\downarrow_{A(B)}|$  denotes the identity operator in the subspace of individual qubits. Thus the eigenvalues of  $\hat{\rho}_A$   $(\hat{\rho}_B)$  are  $p_k(A \text{ or } B) = 1/2$ ; k = 1, 2.

We now construct the AR q-conditional entropy (see Eq. (2) for definition) for the two qubit symmetric state of Eq. (3) as,

$$S_q^{(T)}(B|A) = \frac{1}{q-1} \left[ 1 - \frac{\sum_{i=1}^4 \left[ p_i(AB) \right]^q}{\sum_{i=1}^2 \left[ p_k(A) \right]^q} \right] = \frac{1}{q-1} \left[ 1 - \frac{\left(\frac{1-x}{3}\right)^q + \frac{1}{2}\left(\frac{1+2x}{3}\right)^q}{\left(\frac{1}{2}\right)^q} \right]. \tag{6}$$

Non-positive value of  $S_q^{(T)}(B|A)$  necessarily implies that the state  $\hat{\rho}_{AB}$  of Eq. (3) is quantum entangled. To find the minimum value of x above (below) which the q-conditional entropy  $S_q^{(T)}(A|B)$  takes negative (positive) values, we plot  $S_q^{(T)}(B|A)$  as a function of x for different choices of the parameter q in Fig. (1). It is evident that the value of x for which  $S_q^{(T)}(B|A) \to 0$  keeps reducing with the increase of q. So, the strongest constraint on separability is obtained in the limit  $q \to \infty$ . An implicit plot of  $S_q^{(T)}(B|A) = 0$  (see Fig. 2) clearly indicates  $x \to \frac{1}{4}$  as  $q \to \infty$ . It is illuminating to note that the Peres Horodecki criterion also gives  $x \le \frac{1}{4}$  as necessary and sufficient condition for separability of the two qubit state  $\hat{\rho}_{AB}$  of Eq. (3).

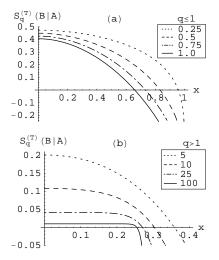


FIG. 1: The AR q-conditional entropy  $S_q^{(T)}(B|A)$  with (a)  $q \leq 1$  and (b) q > 1 for the two qubit mixed state  $\hat{\rho}_{AB}$  of Eq. (3) as a function of the parameter x. Note that  $\lim_{q \to 1} S_q^{(T)}(B|A) = 0$ , i.e., vanishing von Neumann conditional entropy, leads to x = 0.6593, which is clearly lower than the corresponding value obtained by solving the equation  $S_q^{(T)}(B|A) = 0$  with q > 1 justifying the superiority of the q-conditional entropy approach. All quantities are dimensionless.

We now consider a three qubit generalization of the example considered in Eq. (3):

$$\hat{\rho}_{ABC} = \left(\frac{1-x}{4}\right) P_{N=3} + x |W\rangle\langle W| \tag{7}$$

where

$$|W\rangle = \frac{1}{\sqrt{3}} (|\downarrow_A \uparrow_B \uparrow_C\rangle + |\uparrow_A \downarrow_B \uparrow_C\rangle + |\downarrow_A \downarrow_B \uparrow_C\rangle)(8)$$
$$= \left|\frac{3}{2}, \frac{1}{2}\right\rangle.$$

The projection operator is given by  $P_3 = \sum_M |3/2, M\rangle \langle 3/2, M|; M = 3/2, 1/2, -1/2, -3/2,$  and the symmetric states  $\{|3/2, M\rangle\}$  are given explicitly in terms of the three qubit basis states as follows:

$$\left| \frac{3}{2}, \frac{3}{2} \right\rangle = |\uparrow_A \uparrow_B \uparrow_c\rangle$$

$$\left| \frac{3}{2}, \frac{1}{2} \right\rangle = |W\rangle = \frac{1}{\sqrt{3}} (|\downarrow_A \uparrow_B \uparrow_C\rangle + |\uparrow_A \downarrow_B \uparrow_C\rangle + |\downarrow_A \downarrow_B \uparrow_C\rangle)$$

$$\left| \frac{3}{2}, -\frac{1}{2} \right\rangle = \frac{1}{\sqrt{3}} (|\uparrow_A \downarrow_B \downarrow_c\rangle + |\downarrow_A \uparrow_B \downarrow_c\rangle + |\downarrow_A \downarrow_B \uparrow_c\rangle)$$

$$\left| \frac{3}{2}, -\frac{3}{2} \right\rangle = |\downarrow_A \downarrow_B \downarrow_c\rangle.$$

Expressing  $\hat{\rho}_{ABC}$  in the total angular momentum basis states  $\{|3/2, M\rangle, M = 3/2, 1/2, -1/2, -3/2\}$ , we obtain

$$\hat{\rho}_{ABC} = \begin{pmatrix} \frac{1-x}{4} & 0 & 0 & 0\\ 0 & \frac{1+3x}{4} & 0 & 0\\ 0 & 0 & \frac{1-x}{4} & 0\\ 0 & 0 & 0 & \frac{1-x}{4} \end{pmatrix}$$
(9)

from which the nonzero eigenvalues of the three qubit state  $\hat{\rho}_{ABC}$  are easily found to be

$$\frac{1+3x}{4}, \frac{1-x}{4} \text{ (three times)}. \tag{10}$$

The two qubit marginal density matrix  $\hat{\rho}_{AB}$  is given by [22]

$$\hat{\rho}_{AB} = \left(\frac{1-x}{3}\right) P_2 + \frac{x}{3} [|1,1\rangle\langle 1,1| + 2|\Phi_{AB}^+\rangle\langle \Phi_{AB}^+|]$$
 (11)

the spectrum of which is given by,

$$\frac{1}{3}, \frac{1+x}{3}, \frac{1-x}{3}. \tag{12}$$

We also find that the corresponding single qubit density matrix is the identity operator:  $\hat{\rho}_A = \frac{1}{2} (\mid \uparrow_A \rangle \langle \uparrow_A \mid + \mid \downarrow_A \rangle \langle \downarrow_A \mid) = \frac{1}{2} I_A$ .

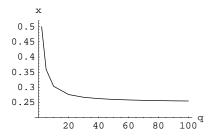


FIG. 2: Implicit plot of  $S_q^{(T)}(B|A) = 0$  as a function of q, for the two qubit state  $\hat{\rho}_{AB}$  of Eq. (3). In the limit  $q \to \infty$  it may be seen that  $x \to 0.25$ . All quantities are dimensionless.

There are two independent q-conditional entropies associated with three qubit states [7], which are given by

$$S_q^{(T)}(A|BC) = \frac{S_q^{(T)}(\hat{\rho}_{ABC}) - S_q^{(T)}(\hat{\rho}_{BC})}{1 + (1 - q)S_q^{(T)}(\hat{\rho}_{BC})}$$
$$S_q^{(T)}(AB|C) = \frac{S_q^{(T)}(\hat{\rho}_{ABC}) - S_q^{(T)}(\hat{\rho}_C)}{1 + (1 - q)S_q^{(T)}(\hat{\rho}_C)}$$
(13)

and it was found [7], in the case of three qubit Werner state, that  $S_q^{(T)}(A|BC)$  leads to stronger conditions on separability than that identified from  $S_q^{(T)}(AB|C)$ . We examine here if this result holds good for the three qubit symmetric states  $\hat{\rho}_{ABC}$  also. The q-conditional entropies  $S_q^{(T)}(A|BC)$ ,  $S_q^{(T)}(AB|C)$  are evaluated explicitly, by using Eqs. (2), (10), (12) and (13)

$$S_q^{(T)}(A|BC) = \frac{1}{q-1} \left[ 1 - \frac{\left(\frac{1+3x}{4}\right)^q + 3\left(\frac{1-x}{4}\right)^q}{\left(\frac{1}{3}\right)^q + \left(\frac{1+x}{3}\right)^q + \left(\frac{1-x}{3}\right)^q} \right]$$

$$S_q^{(T)}(AB|C) = \frac{1}{q-1} \left[ 1 - \frac{\left(\frac{1+3x}{4}\right)^q + 3\left(\frac{1-x}{4}\right)^q}{2\left(\frac{1}{2}\right)^q} \right]$$
(14)

In Fig. 3(a) (3(b)) we plot the limiting values of the parameter x for which  $S_q^{(T)}(A|BC)=0$  ( $S_q^{(T)}(AB|C)=0$ ) by varying q. From these implicit plots it is clearly seen that

$$x \to \frac{1}{5}$$
 as  $\lim_{q \to \infty} S_q^{(T)}(A|BC) = 0$   
 $x \to \frac{1}{3}$  as  $\lim_{q \to \infty} S_q^{(T)}(AB|C) = 0$ .

Thus the conditional entropy  $S_q^{(T)}(A|BC)$  leads (in the limit  $q \to \infty$ ) to

$$0 \le x \le \frac{1}{5} \tag{15}$$

as the range of separability for the state  $\hat{\rho}_{ABC}$  which is obviously stronger than the separability domain  $0 \le x \le \frac{1}{3}$  inferred by  $S_q^{(T)}(AB|C)$ .

It may be noted that Eq. (15) obtained from qconditional entropies serves only as a sufficient condition for separability of the state  $\hat{\rho}_{ABC}$ . The necessary

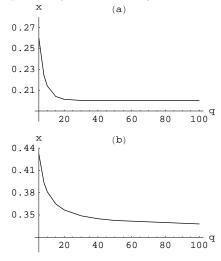


FIG. 3: (a) Implicit plot of  $S_q^{(T)}(A|BC)=0$  as a function of q, corresponding to the three qubit state  $\hat{\rho}_{ABC}$  of Eq. (7). In the limit  $q\to\infty$  it may be seen that  $x\to\frac15$ . (b) Implicit plot of  $S_q^{(T)}(AB|C)=0$  as a function of q. Here  $x\to\frac13$  in the limit  $q\to\infty$ . All quantities are dimensionless.

and sufficient condition resulting from the Peres criterion:  $0 \le x \le 0.1547$  is obviously a strongest limitation on the separability.

# III. SEPARABILITY OF ONE PARAMETER SYMMETRIC MULTI-QUBIT W AND GHZ STATES

In this section, we investigate quantum correlations in a N-particle generalization of the one parameter mixed states.

### A. One parameter family of W states

The symmetric one parameter mixed multiqubit states, involving a W-state, of our interest are

$$\hat{\rho}_{A_1, A_2, A_3, \dots, A_N}^W = \left(\frac{1-x}{N+1}\right) P_N + x |W\rangle_N \,_N \langle W| \quad (16)$$

and the corresponding reduced N-n qubit density operator is given by,

$$\hat{\rho}_{A_{1},A_{2},A_{3},...,A_{N-n}}^{W} = \left(\frac{1-x}{N-n+1}\right) P_{N-n} + \frac{x}{N} \left[ (N-n) |W\rangle_{N-n} |N-n| \langle W| + n |\downarrow_{A_{1}}, ..., \downarrow_{A_{N-n}} \rangle \langle \downarrow_{A_{1}}, ..., \downarrow_{A_{N-n}} | \right],$$

$$n = 0, 1, 2, ..., N-2. \tag{17}$$

Here

$$|W\rangle_{N} = |N/2, N/2 - 1\rangle = \frac{1}{\sqrt{N}} [|\downarrow_{A_{1}}, \uparrow_{A_{2}}, \dots, \uparrow_{A_{N}}\rangle + |\uparrow_{A_{1}}, \downarrow_{A_{2}}, \uparrow_{A_{3}}, \dots, \uparrow_{A_{N}}\rangle + \text{permutations}]$$

and 
$$P_N = \sum_{M=-N/2}^{N/2} |N/2, M\rangle \langle N/2, M|$$
.

From the discussions of Sec. II, it follows that the conditional entropy

$$S_q^{(T)}(A_1|A_2, A_3, \dots, A_{N-n}) = \frac{S_q^{(T)}(\hat{\rho}_{A_1, A_2, A_3, \dots, A_{N-n}}) - S_q^{(T)}(\hat{\rho}_{A_2, A_3, \dots, A_{N-n}})}{1 + (1 - q)S_q^{(T)}(\hat{\rho}_{A_2, A_3, \dots, A_{N-n}})}; \quad n = 0, 1, 2, \dots, N - 2$$

$$(18)$$

provides a stronger condition, than that obtained from  $S_q^{(T)}(A_1, \ldots, A_m | A_{m+1}, \ldots, A_{N-n}),$   $m=2,3,\ldots,N-n-1,$  on the separability of multiparticle states  $\hat{\rho}_{A_1,A_2,A_3,\ldots,A_{N-n}}^W$ . In order to examine the asymptotic negativity of  $S_q^{(T)}(A_1 | A_2, A_3, \ldots, A_{N-n})$ we now proceed to evaluate the eigenvalues of  $\hat{\rho}^W_{A_1,A_2,A_3,\dots,A_{N-n}}$  and its one qubit reduced marginal density matrix [22]  $\hat{\rho}_{A_2,A_3,A_4,\dots,A_{N-n}}^W \equiv \hat{\rho}_{A_1,A_2,A_3,\dots,A_{N-n-1}}^W$ . The nonzero eigenvalues of  $\hat{\rho}_{A_1,A_2,A_3,\dots,A_{N-n}}^W$  are easily

evaluated by expressing the state in the symmetric basis

of total angular momentum  $J = \frac{N-n}{2}$ :

$$\frac{1-x}{N-n+1} \ ((N-n-1) \text{ fold degenerate}),$$

$$\frac{1-x}{N-n+1} + \frac{nx}{N}, \ \frac{1-x}{N-n+1} + \frac{(N-n)x}{N}. \ (19)$$

Using these eigenvalues the AR conditional entropy of Eq. (18) is obtained as

$$S_q^{(T)}(A_1|A_2, A_3, \dots, A_{N-n}) = \frac{1}{q-1} \left[ 1 - \frac{(N-n-1)\left(\frac{1-x}{N-n+1}\right)^q + \left(\frac{1-x}{N-n+1} + \frac{nx}{N}\right)^q + \left(\frac{1-x}{N-n+1} + \frac{(N-n)x}{N}\right)^q}{(N-n-2)\left(\frac{1-x}{N-n}\right)^q + \left(\frac{1-x}{N-n} + \frac{(N-n)x}{N}\right)^q + \left(\frac{1-x}{N-n} + \frac{(N-n-1)x}{N}\right)^q} \right]$$
(20)

The limiting value of the parameter x satisfying  $\lim_{q\to\infty} S_q^{(T)}(A_1|A_2,A_3,\ldots,A_{N-n})=0$  is determined by noting that only the maximum eigenvalue  $p_{N-n}^{(\max)}$  of  $\hat{\rho}_{A_1,A_2,A_3,\ldots,A_{N-n}}^{W}$  and  $p_{N-n-1}^{(\max)}$  of  $\hat{\rho}_{A_2,A_3,\ldots,A_{N-n}}^{W}$  contribute in Eq. (20) in the limit  $q\to\infty$  [7]. We thus find from the asymptotic negativity of  $S_q^{(T)}(A_1|A_2,A_3,\ldots,A_{N-n})$  that the state  $\hat{\rho}_{A_1,A_2,A_3,\ldots,A_{N-n}}^W$  is separable if

$$0 \le x < \frac{N}{(N-n)^2 + 2N - n}. (21)$$

For  $n=0,\,N=2$  we recover the separability condition  $0\leq x\leq \frac{1}{4}$  for the two qubit state of Eq.(3) from the general result given in Eq. (21).

### One parameter N-qubit GHZ states

We now proceed to find the q-entropic inference on the separability of one parameter family of symmetric states containing the maximally entangled GHZ state:

$$\hat{\rho}_{A_1,A_2,A_3,\dots,A_N}^{\text{GHZ}} = \left(\frac{1-x}{N+1}\right) P_N + x |\text{GHZ}\rangle_N \sqrt{\text{GHZ}}|$$
(22)

where

$$|\mathrm{GHZ}\rangle_N = \frac{1}{2} (|\uparrow_{A_1}, \uparrow_{A_2}, \dots, \uparrow_{A_N}\rangle + |\downarrow_{A_1}, \downarrow_{A_2}, \dots, \downarrow_{A_N}\rangle).$$

In order to determine  $S_q^{(T)}(A_1|A_2,A_3,\ldots,A_N)$  of the state  $\hat{\rho}_{A_1,A_2,A_3,\ldots,A_N}^{\text{GHZ}}$  we now proceed to evaluate the

eigenvalues of the given state and its one qubit reduced

marginal density matrix, which is given by,

$$\hat{\rho}_{A_{2},A_{3},...,A_{N}}^{\text{GHZ}} = \left(\frac{1-x}{N}\right) P_{N-1} + \frac{x}{2} \left[ \left| \frac{N-1}{2}, \frac{N-1}{2} \right\rangle \left\langle \frac{N-1}{2}, \frac{N-1}{2} \right| + \left| \frac{N-1}{2}, -\frac{N-1}{2} \right\rangle \left\langle \frac{N-1}{2}, -\frac{N-1}{2} \right| \right]. \tag{23}$$

non-vanishing eigenvalues state  $\hat{\rho}_{A_1,A_2,A_3,...,A_N}^{\mathrm{GHZ}}$  are evaluated as,

So, we obtain the q-conditional entropy associated with the N-qubit state  $\hat{\rho}_{A_1,A_2,A_3,...,A_N}^{\text{GHZ}}$  as,

$$\frac{1-x}{N+1}$$
 (N fold degenerate),  $\frac{1+Nx}{N+1}$  (24)

and that of its one qubit reduced subsystem  $\hat{\rho}_{A_1,A_2,A_3,...,A_{N-1}}^{\rm GHZ}$  are,

$$\frac{1-x}{N}$$
 ((N-2) times),  $\frac{2+x(N-2)}{2N}$  (twice). (25)

$$S_q^{(T)}(A_1|A_2, A_3, \dots, A_N) = \frac{1}{q-1} \left[ 1 - \frac{N\left(\frac{1-x}{N+1}\right)^q + \left(\frac{1+Nx}{N+1}\right)^q}{\left(N-2\right)\left(\frac{1-x}{N}\right)^q + 2\left(\frac{2+x(N-2)}{2N}\right)^q} \right]$$
(26)

As discussed earlier, we find the value of the parameter x, that marks the border of separability of the quantum state  $\hat{\rho}_{A_1,A_2,A_3,...,A_N}^{\rm GHZ}$ , by solving the equation  $\lim_{q\to\infty} S_q^{(T)}(A_1|A_2,A_3,...,A_N)=0$ , which leads to  $\left(1-\frac{p_N^{(\max)}}{p_{N-1}^{(\max)}}\right)=0$ , in terms of the maximum eigenvalues  $p_N^{(\text{max})}$ ,  $p_{N-1}^{(\text{max})}$  of  $\hat{\rho}_{A_1,A_2,A_3,...,A_N}^{\text{GHZ}}$  and  $\hat{\rho}_{A_2,A_3,...,A_{N-1}}^{\text{GHZ}}$ respectively. So, we obtain the range of separability as

$$0 \le x < \frac{2}{N^2 + N + 2} \tag{27}$$

for the state  $\hat{\rho}_{A_1,A_2,A_3,...,A_N}^{\rm GHZ}$ . Note that when N=3, the necessary and sufficient condition for separability  $0 \le x \le \frac{1}{7}$ , inferred from the Peres criterion, agrees with that obtained from the qconditional entropy method (see Eq. (27)). But for N =4, we obtain 0 < x < 0.0625 from the Peres criterion whereas Eq. (27) gives  $0 \le x \le 0.0909$ . In other words, the q-entropy result is weaker compared to that obtained from the Peres' criterion in one parameter family of Nqubit GHZ states for N > 3.

#### IV. CONCLUSION

We have employed the Abe-Rajagopal q-conditional entropy method to characterize separability in one parameter family of symmetric N-qubit W and GHZ mixed states. It is identified that positivity of the q-conditional entropy,  $\lim_{q\to\infty} S_q^{(T)}(A_1|A_2,A_2\ldots)$ , gives rise to strongest limitations on separability, compared to that obtained from the positivity of von Neumann conditional entropy (q = 1 limit). In the case of symmetric one parameter family of two qubit states  $\hat{\rho}_{AB}$ , given by Eq. (3), and the three qubit state  $\hat{\rho}_{A_1,A_2,A_3}^{GHZ}$ , we have recovered, using the q-entropy approach, the necessary and sufficient conditions of separability on the parameter x. However, for the mixed W states,  $\hat{\rho}_{A_1,A_2,A_3,...,A_N}^W$  n = 0,1,..., N-2 and the GHZ states  $\hat{\rho}_{A_1,A_2,A_3,...,A_N}^{GHZ}$  with N>3, the range of separability, identified from the q-conditional entropy approach, is found to be weaker compared to that obtained from the Peres' criterion. These general observations, concerning the separability of one-parameter family of symmetric states are in confirmation with the results revealed by the detailed numerical investigations performed on the full state space of arbitrary bipartite mixed states [9, 10]. It would be illuminating to numerically investigate the q-conditional entropic behavior of more general multiqubit states.

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